

Program Reasoning

7. Hoare Logic

Kihong Heo



The Story So Far

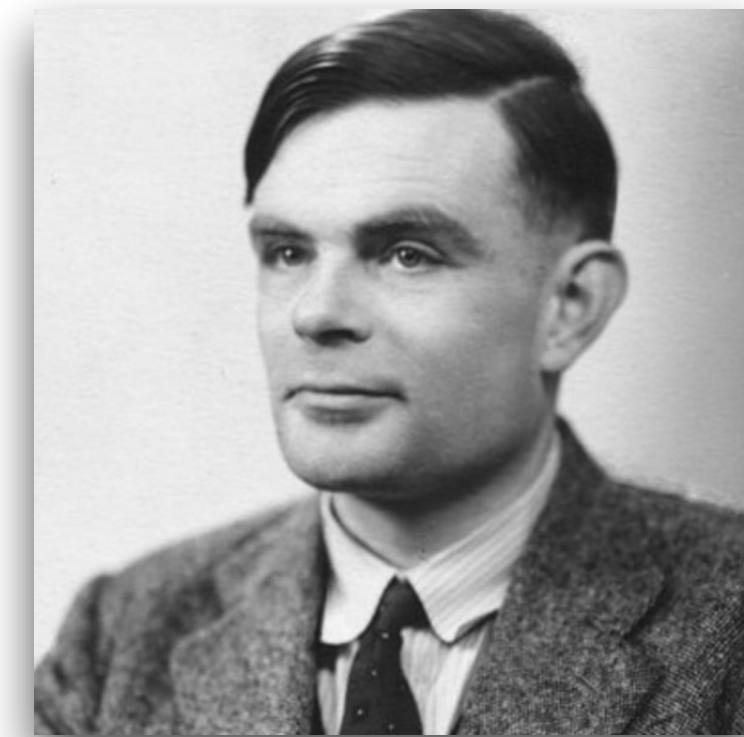
- Mechanization of logic and mathematics



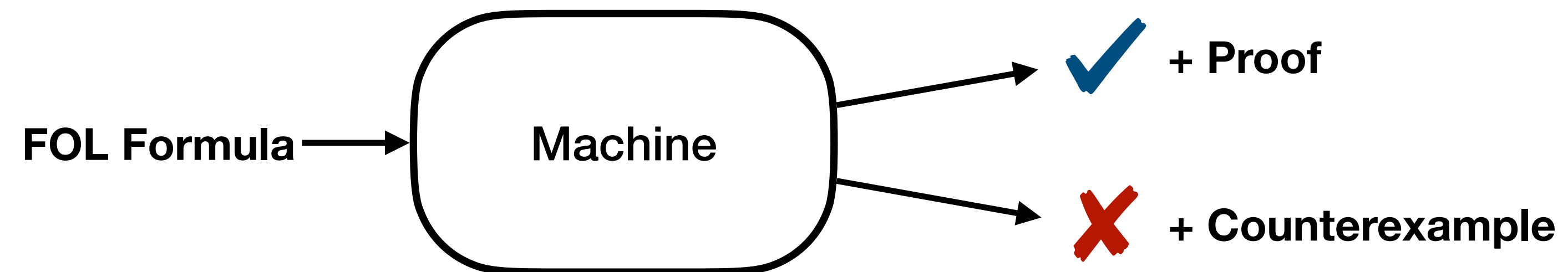
“Is the following machine possible?”
- D. Hilbert, 1930



“NO!”
-K. Gödel, 1931

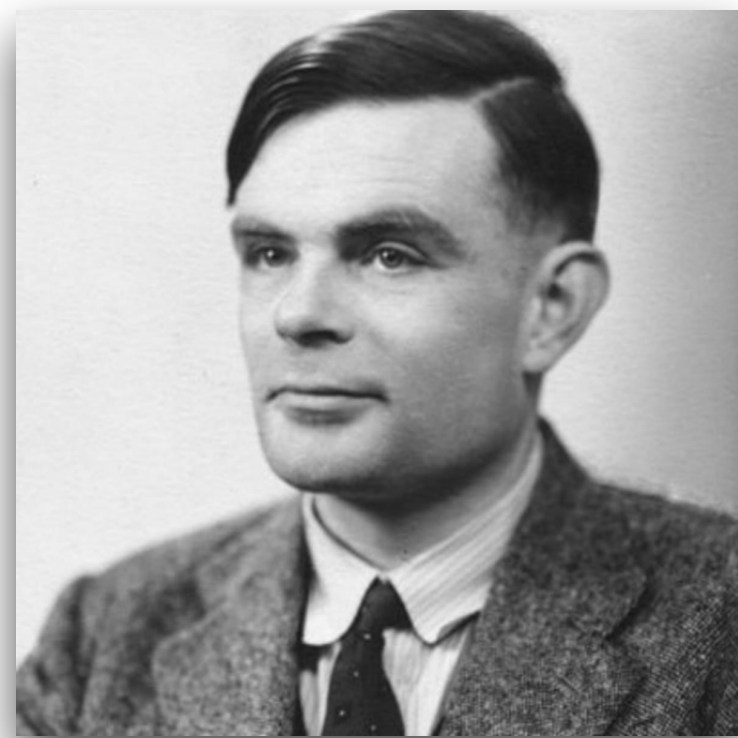


“NO!”
- A. Turing, 1936

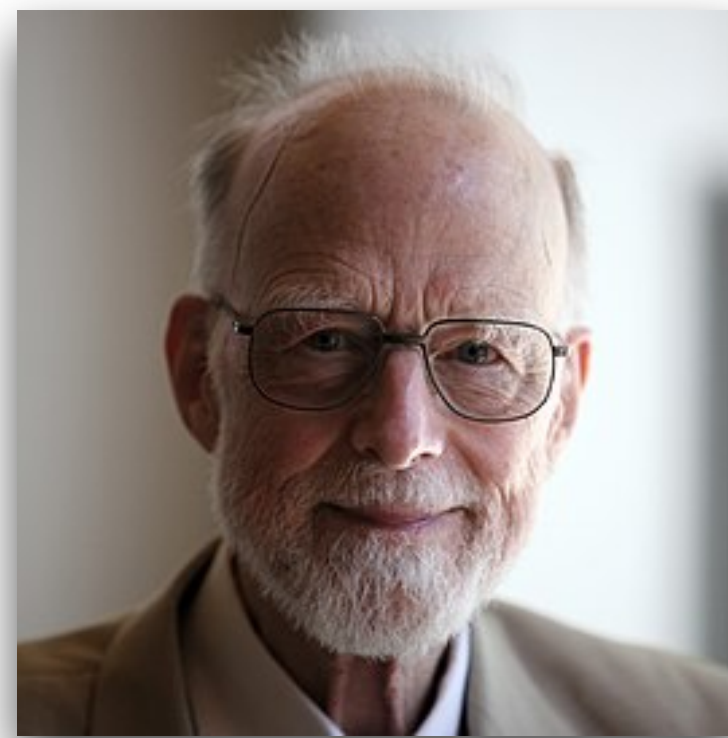


Never Ending Story

- Reasoning about programs



“How to check a program is correct?”
- A. Turing, 1949



“Hoare Logic”
- T. Hoare, 1969



1980



“Weakest Precondition Calculus”
- E. Dijkstra, 1975

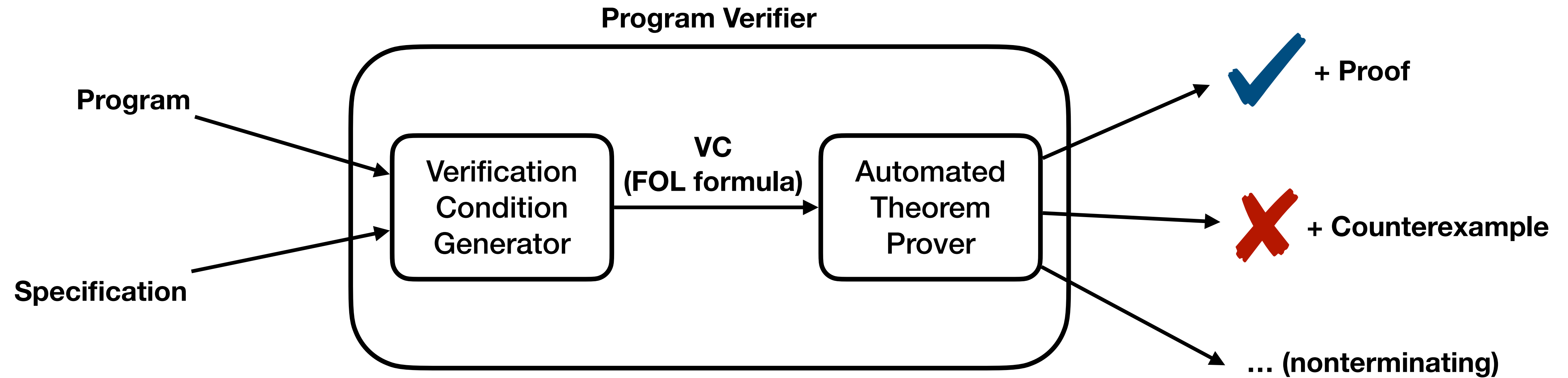


1972

Program Verification

- Specifying and proving properties of programs
- Specification: precise statement of properties that a program should exhibit in FOL
- Partial correctness properties: certain states cannot ever occur during the execution
 - “Bad things never happen” (e.g., integer overflow, buffer overflow, deadlock, etc)
 - Proof by inductive assertion method
- Total correctness properties: certain states are eventually reached during the execution
 - “Good things will eventually happen” (e.g., termination, fairness)
 - Proof by ranking function method

Overview



Specification

- Typically embedded into program text as program annotations
- An annotation is a FOL formula F
- An annotation F at location L asserts that F is true whenever program control reaches L
- Three types of annotations:
 - Function specification
 - Loop invariant
 - Assertion

Function Specification

- A pair of annotations: precondition and postcondition
- Precondition (전제조건): a formula whose free variables include only the formal parameters
 - “What should be true upon entering the function?”
- Postcondition (사후조건): a formula whose free variables include only the formal parameters and the return value
 - “What is the relationship between the input and output?”

Example: Linear Search

- What would be the pre and post conditions?

@pre: $0 \leq l \wedge u < |a|$

@post: $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    for (int i := l; i <= u; i := i + 1) {  
        if (a[i] = e) return true;  
    }  
    return false;  
}
```

- BTW, is this nontrivial precondition (a formula other than \top) is always acceptable?
 - In terms of the software engineering practice (e.g., public API)

Example: More Robust Linear Search

- What would be the pre and post conditions?

@pre: \top

@post: $rv \leftrightarrow \exists i. 0 \leq l \leq i \leq u < |a| \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    if (l < 0 || u >= |a|) return false;  
    for (int i := l; i <= u; i := i + 1) {  
        if (a[i] = e) return true;  
    }  
    return false;  
}
```

Example: Binary Search

- What would be the pre and post conditions?

@pre: $0 \leq l \wedge u < |a| \wedge \text{sorted}(a, l, u)$

@post: $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$

```
bool BinarySearch(int[] a, int l, int u, int e) {  
    if (l > u) return false;  
    int m := (l + u) / 2;  
    if (a[m] = e) return true;  
    else if (a[m] < e) return BinarySearch(a, m + 1, u, e)  
    else return BinarySearch(a, l, m - 1, e)  
}
```

- The sorted predicate is defined in the combined theory of integers and arrays:

$$\text{sorted}(a, l, u) \iff \forall i, j. l \leq i \leq j \leq u \rightarrow a[i] \leq a[j]$$

Example: Bubble Sort

- What would be the pre and post conditions?

@pre: T

@post: $\text{sorted}(rv, 0, |rv| - 1)$

```
bool BubbleSort(int[] a0) {  
    int a[] := a0;  
    for (int i := |a| - 1; i > 0; i := i - 1) {  
        for (int j := 0; j < i; j := j + 1) {  
            if (a[j] > a[j + 1]) {  
                int t := a[j];  
                a[j] := a[j + 1];  
                a[j + 1] := t;  
            }  
        }  
    }  
    return a;  
}
```



Loop Invariants

- Each loop has an annotation called loop invariant

```
while
  @ $F$ 
  ( $\langle condition \rangle$ ) {
     $\langle body \rangle$ 
  }
```

- The assertion F must hold at the beginning of every iteration
 - $F \wedge \langle condition \rangle$ holds on entering the body
 - $F \wedge \neg \langle condition \rangle$ holds when exiting the loop
- Why are loop invariants needed?

Example: Linear Search

- What would be the loop invariant?

@pre: $0 \leq l \wedge u < |a|$

@post: $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    int i := l;  
    while  
        @L :  $l \leq i \wedge (\forall j. l \leq j < i \rightarrow a[j] \neq e)$   
        (i <= u) {  
        if (a[i] = e) return true;  
        i := i + 1;  
    }  
    return false;  
}
```

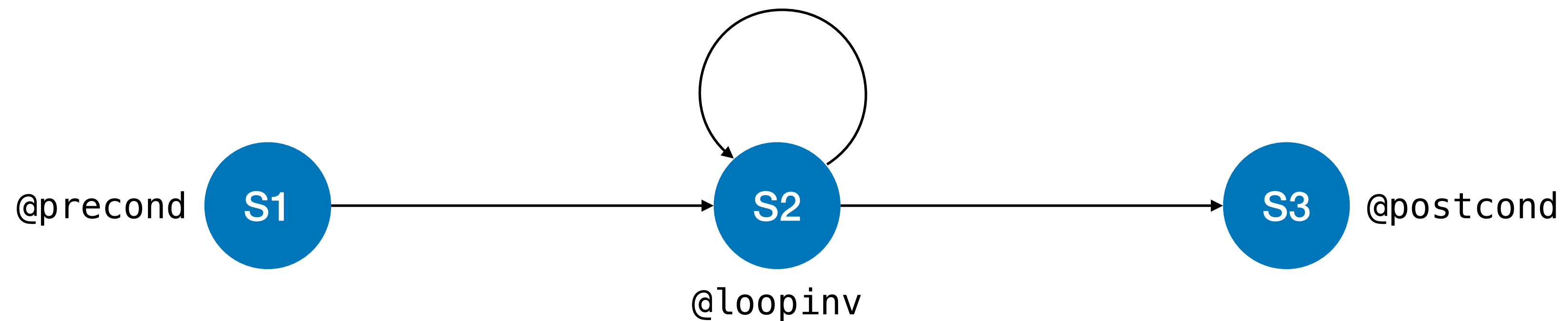
Assertions

- Allows programmers to provide a formal comment
- Runtime assertions: a special class of assertions
 - E.g., division by 0, null dereference, etc
- Example: linear search with runtime assertions

```
bool LinearSearch(int[] a, int l, int u, int e) {  
    int i := l;  
    while (i <= u) {  
        @ 0 ≤ i < |a|  
        if (a[i] = e) return true;  
        i := i + 1;  
    }  
    return false;  
}
```


Inductive Assertion Method

- Proof technique for partial correctness of programs
- Idea: derive verification conditions from a function given annotations
- Example:



Hoare Triple

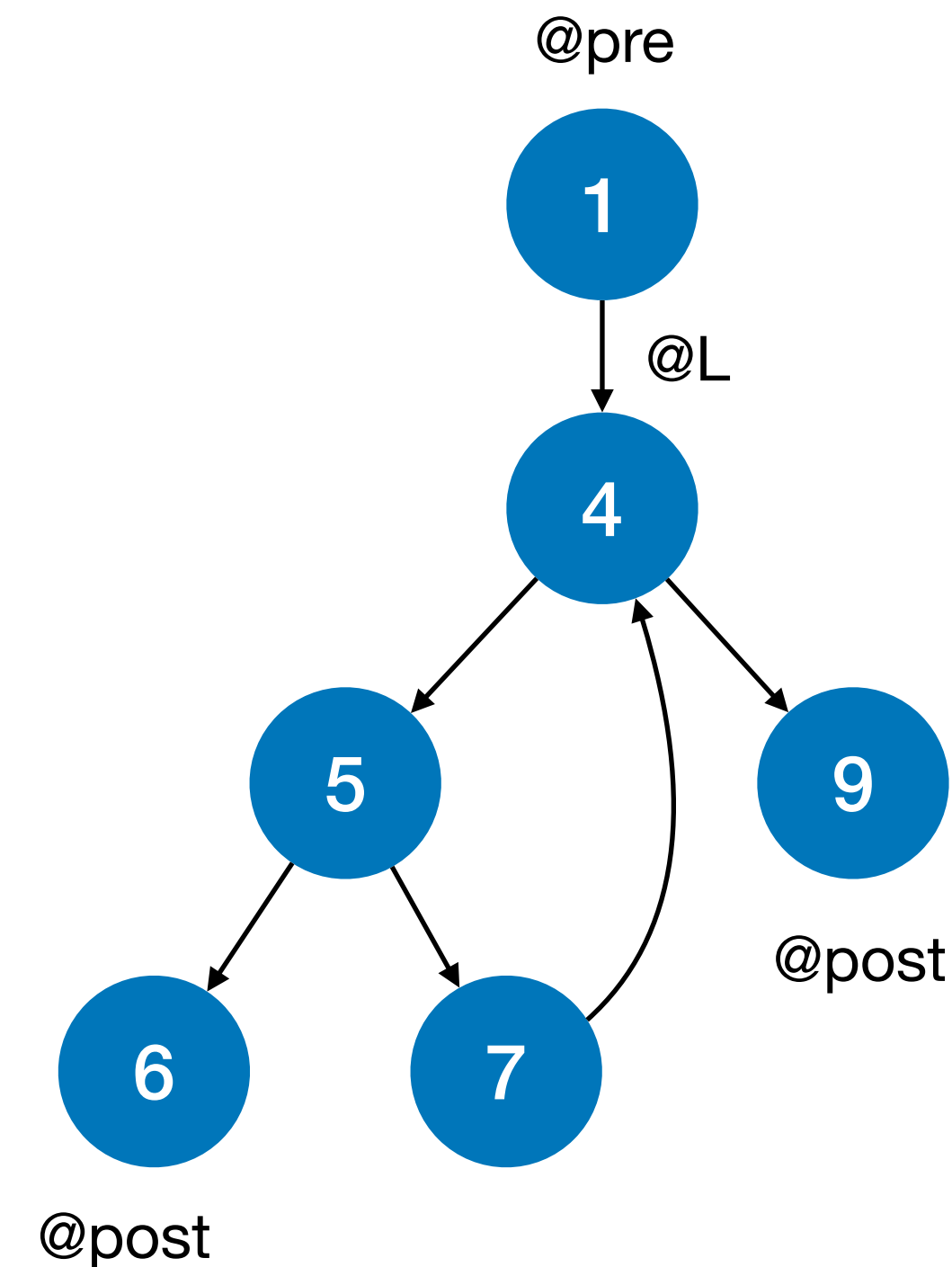
- Partial correctness specified using Hoare triple: $\{P\} S \{Q\}$
 - S : program fragment
 - P : precondition
 - Q : postcondition
- Meaning of Hoare triple:
 - If S is executed in a state satisfying P and if the execution of S terminates
 - Then, the program state after S terminates satisfies Q

Example

```


```
@pre: $0 \leq l \wedge u < |a|$
@post: $rv \leftrightarrow \exists i. l \leq i \leq u \wedge a[i] = e$
bool LinearSearch(int[] a, int l, int u, int e) {
1: int i := l;
2: while
3: @L: $l \leq i \wedge (\forall j. l \leq j < i \rightarrow a[j] \neq e)$
4: (i <= u) {
5: if (a[i] = e)
6: return true;
7: i := i + 1
8: }
9: return false;
}
```


```



- What conditions should be proven?

$\{\text{@pre}\} S1 \{\text{@L}\} \quad \{\text{@L}\} S4; S5; S6 \{\text{@post}\} \quad \{\text{@L}\} S4; S5; S7 \{\text{@L}\} \quad \{\text{@L}\} S4; S9 \{\text{@post}\}$

- What about $\{\text{@L}\} S4; S5; S7; S4; S5; S6 \{\text{@post}\}$? Why?

Hoare Logic

- A logic to prove the validity of Hoare triple
- A set of logical rules for reasoning about the partial correctness of programs
- In this lecture, we assume the following simple imperative language

$$\begin{array}{l} S \rightarrow \text{skip} \\ | \quad x := E \\ | \quad S; S \\ | \quad \text{if } E \text{ then } S \text{ else } S \\ | \quad \text{while } E \text{ do } S \end{array}$$

Example

- Which one is valid?
 - $\{x = 0\} \ x \ := \ x + 1 \ \{x = 1\}$
 - $\{x = 0 \wedge y = 1\} \ x \ := \ x + 1 \ \{x = 1 \wedge y = 2\}$
 - $\{x = 0\} \ x \ := \ x + 1 \ \{x = 1 \vee y = 2\}$
 - $\{x = 0\} \ \text{while true do } x \ := \ 0 \ \{x = 1\}$

Hoare Rules (1)

- Rule for skip $\frac{}{\{P\} \text{ skip } \{P\}}$
- Rule for assignment $\frac{}{\{Q[E/x]\} x := E \{Q\}}$
 - Intuition: revert to the state before the assignment
 - Example:

$$\{\text{true}\} x := 1 \{x = 1\}$$

$$\{x + 1 > 0\} x := x + 1 \{x > 0\}$$

$$\{y = 1\} x := y \{x = 1\}$$

$$\{\text{false}\} x := y + 3 \{y = 0 \wedge x = 12\}$$

Hoare Rules (2)

- Rule for precondition strengthening
$$\frac{\{P'\} S \{Q\} \quad P \implies P'}{\{P\} S \{Q\}}$$

- Example:

$$\frac{\frac{\{y > 0[x/y]\} y := x \{y > 0\}}{\{x > 0\} y := x \{y > 0\}} \quad x = 2 \implies x > 0}{\{x = 2\} y := x \{y > 0\}}$$

- Rule for postcondition weakening
$$\frac{\{P\} S \{Q'\} \quad Q' \implies Q}{\{P\} S \{Q\}}$$

- Example:

$$\frac{\frac{\dots}{\{true\} S \{x = y \wedge z = 2\}} \quad x = y \wedge z = 2 \implies x = y}{\{true\} S \{x = y\}}$$

Hoare Rules (3)

- Rule for composition
$$\frac{\{P\} S_1 \{Q\} \quad \{Q\} S_2 \{R\}}{\{P\} S_1; S_2 \{R\}}$$

$$\frac{\frac{\{x = 2[2/x]\} x := 2 \{x = 2\}}{\{\text{true}\} x := 2 \{x = 2\}} \quad \frac{\{x = 2 \wedge y = 2[x/y]\} y := x \{x = 2 \wedge y = 2\}}{\{x = 2\} y := x \{x = 2 \wedge y = 2\}}}{\{\text{true}\} x := 2; y := x \{x = 2 \wedge y = 2\}}$$

- Rule for if statement
$$\frac{\{P \wedge E\} S_1 \{Q\} \quad \{P \wedge \neg E\} S_2 \{Q\}}{\{P\} \text{ if } E \text{ then } S_1 \text{ else } S_2 \{Q\}}$$

$$\frac{\frac{\{y \geq 0[x/y]\} y := x \{y \geq 0\} \quad x > 0 \implies x \geq 0}{\{x > 0\} y := x \{y \geq 0\}} \quad \frac{\{y \geq 0[-x/y]\} y := -x \{y \geq 0\}}{\{x \leq 0\} y := -x \{y \geq 0\}}}{\{\text{true}\} \text{ if } x > 0 \text{ then } y := x \text{ else } y := -x \{y \geq 0\}}$$

Hoare Rules (4)

- Rule for loop
$$\frac{\{P \wedge E\} S \{P\}}{\{P\} \text{ while } E \text{ do } S \{P \wedge \neg E\}}$$
- $$\dots$$
$$\frac{\{x \leq n \wedge x < n\} x := x + 1 \{x \leq n\}}{\{x \leq n\} \text{ while } x < n \text{ do } x := x + 1 \{x = n\}}$$

Loop Invariant

- Challenge: impossible to know how many times a given loop iterates
- How to prove the partial correctness of a loop within finite time?
- Analogy: mathematical induction
- Loop invariant I satisfies the following properties:
 - I holds initially before the loop
 - I holds after each iteration of the loop

- Example

```
i := 0; sum := 0; n := 10;
while (i < n) { // loop invariants?
    i := i + 1;
    sum := sum + i;
}
```

Inductive Invariant

- Not all invariants are provable
- Example:

```
i := 5;  
while (i > 1) { // invariant: i > 0  
    i := i - 2;  
}  
assert(i = 1);
```

$$\frac{\{P \wedge E\} S \{P\}}{\{P\} \text{ while } E \text{ do } S \{P \wedge \neg E\}}$$

- Inductive invariant: invariant we can prove using induction
- Challenge: finding inductive loop invariants
 - Practice: human, static analysis, machine learning, etc

Automatically Proving Partial Correctness

- $\{P\} S \{Q\}$: Given the precondition satisfied, the postcondition is satisfied after the execution (if it terminates)
- Assumption: loop invariants are given by an oracle
 - Oracle: human, static analysis, machine learning, etc
- How to automatically prove correctness?
- Idea: deriving verification conditions (VCs) and check the validity

Verification Condition

- A FOL formula F such that the program is correct iff F is valid
- Automatically proving partial correctness
 - Generating VCs from a program + checking the validity of VCs by a theorem prover
- Two ways to generate verification conditions
 - Forward: starting from prediction, generate formulas to prove postcondition (strongest postconditions)
 - Backward: starting from postcondition, generate formulas to prove precondition (weakest preconditions)

Weakest Liberal Preconditions

- Goal: verify Hoar triple $\{P\} S \{Q\}$
- Weakest liberal precondition $wlp(S, Q)$ [Dijkstra75]
 - Weakest: most general condition that guarantees Q will hold after S in any execution
 - Liberal: we do not care about termination
- Proof of the Hoar triple $\{P\} S \{Q\}$: $P \rightarrow wlp(S, Q)$
- Example: $\{y \geq 10\} x := y + 1 \{x \geq 0\}$

Weakest Precondition Calculus

- Inductively define wlp following Hoare rules
- $wlp(x := E, Q) = Q[E/x]$
- $wlp(s_1; s_2, Q) = wlp(s_1, wlp(s_2, Q))$
- $wlp(\text{if } E \text{ then } s_1 \text{ else } s_2, Q) = E \rightarrow wlp(s_1, Q) \wedge \neg E \rightarrow wlp(s_2, Q)$
- $wlp(\text{while } E \text{ do } S, Q) = I \wedge \forall y_1, \dots, y_k. ((E \wedge I \rightarrow wlp(S, I)) \wedge (\neg E \wedge I \rightarrow Q))[y_i/x_i]$
 - Assumption: an inductive invariant I is provided
 - x_i is an assigned variable in S and y_i is a fresh variable

Example (1)

- $S : x := y + 1; \text{if } x > 0 \text{ then } z := 1 \text{ else } z := -1$
 - $wlp(S, z > 0)?$
 - $wlp(S, z \leq 0)?$
 - $\{y > -1\} S \{z > 0\}?$
 - $\{y = -2\} S \{z < 0\}?$

Example (2)

- Prove the assertion when $I : i \geq 0 \wedge \text{odd}(i)$ using
 $wlp(\text{while } E \text{ do } S, Q) = I \wedge \forall y_1, \dots, y_k. ((E \wedge I \rightarrow wlp(S, I)) \wedge (\neg E \wedge I \rightarrow Q))[y_i/x_i]$

```
// @pre: T
i := 5;
while (i > 1) {
    i := i - 2;
}
assert(i = 1);
```

(1): $i \geq 0 \wedge \text{odd}(i)$

(2): $\forall i'. (i' > 1) \wedge (i' \geq 0 \wedge \text{odd}(i')) \rightarrow (i' \geq 2) \wedge \text{odd}(i')$

(3): $\forall i'. (i' \leq 1) \wedge (i' \geq 0 \wedge \text{odd}(i')) \rightarrow i' = 1$

Example (3)

- Prove the assertion when $I : x \geq 0 \wedge y = 5$ using
 $wlp(\text{while } E \text{ do } S, Q) = I \wedge \forall y_1, \dots, y_k. ((E \wedge I \rightarrow wlp(S, I)) \wedge (\neg E \wedge I \rightarrow Q))[y_i/x_i]$

```
// @pre: T
x := 0;
y := 5;
while (x < 10) {
    x := x + 1;
}
assert(y = 5);
```

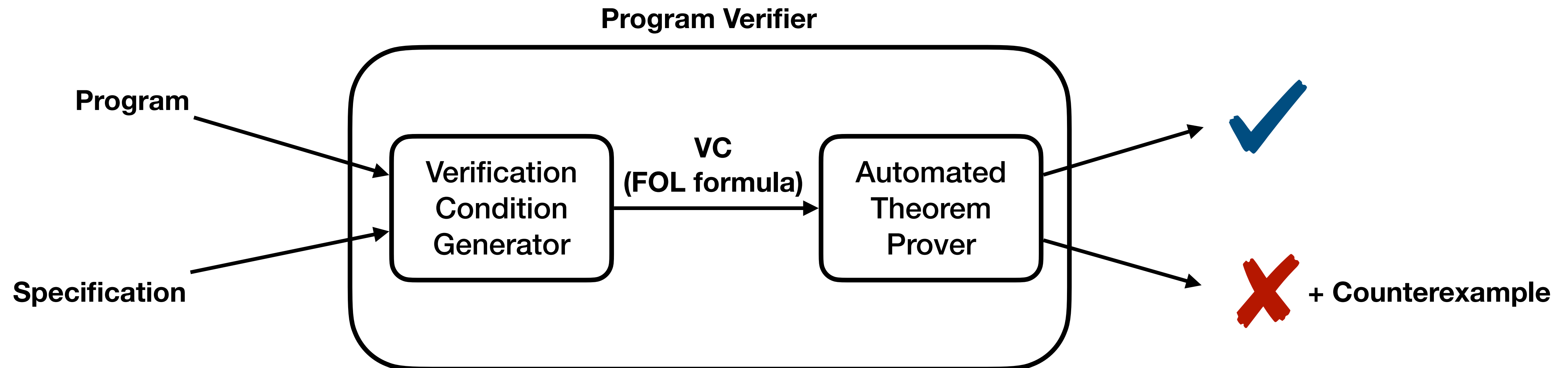
(1): $x \geq 0 \wedge y = 5$

(2): $\forall x'. (0 \leq x' < 10) \wedge (y = 5) \rightarrow x' \geq -1 \wedge y = 5$

(3): $\forall x'. (x' \geq 10) \rightarrow y = 5$

Verification of Hoare Triple

- Validity of $\{P\} S \{Q\}$
- Verification condition: $P \rightarrow wlp(S, Q)$



Summary

- Hoare triple: specifications for partial correctness $\{P\} S \{Q\}$
- Hoare logic: a logic to prove the validity of Hoare triple
 - Proof rules for each program command
- Verification condition is valid iff the Hoare triple is valid
- Automated program verification: check whether the VC is valid using theorem provers