

# Introduction to Program Analysis

## 7. Abstract Interpretation (3): Widening and Narrowing

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# Design of Static Analysis

- Goal: **conservative** and **terminating** static analysis
- Design principles:
  - Define **concrete semantics**
  - Define **abstract semantics** (sound w.r.t the concrete semantics)
- Computation & implementation:
  - Abstract semantics of a program: **the least fixed point** of the semantic function
  - Static analyzer: **compute** the least fixed point within **finite time**

# Computing Abstract Semantics

- If the abstract domain  $\mathbb{D}^\sharp$  has **finite** height (i.e., all chains are finite)

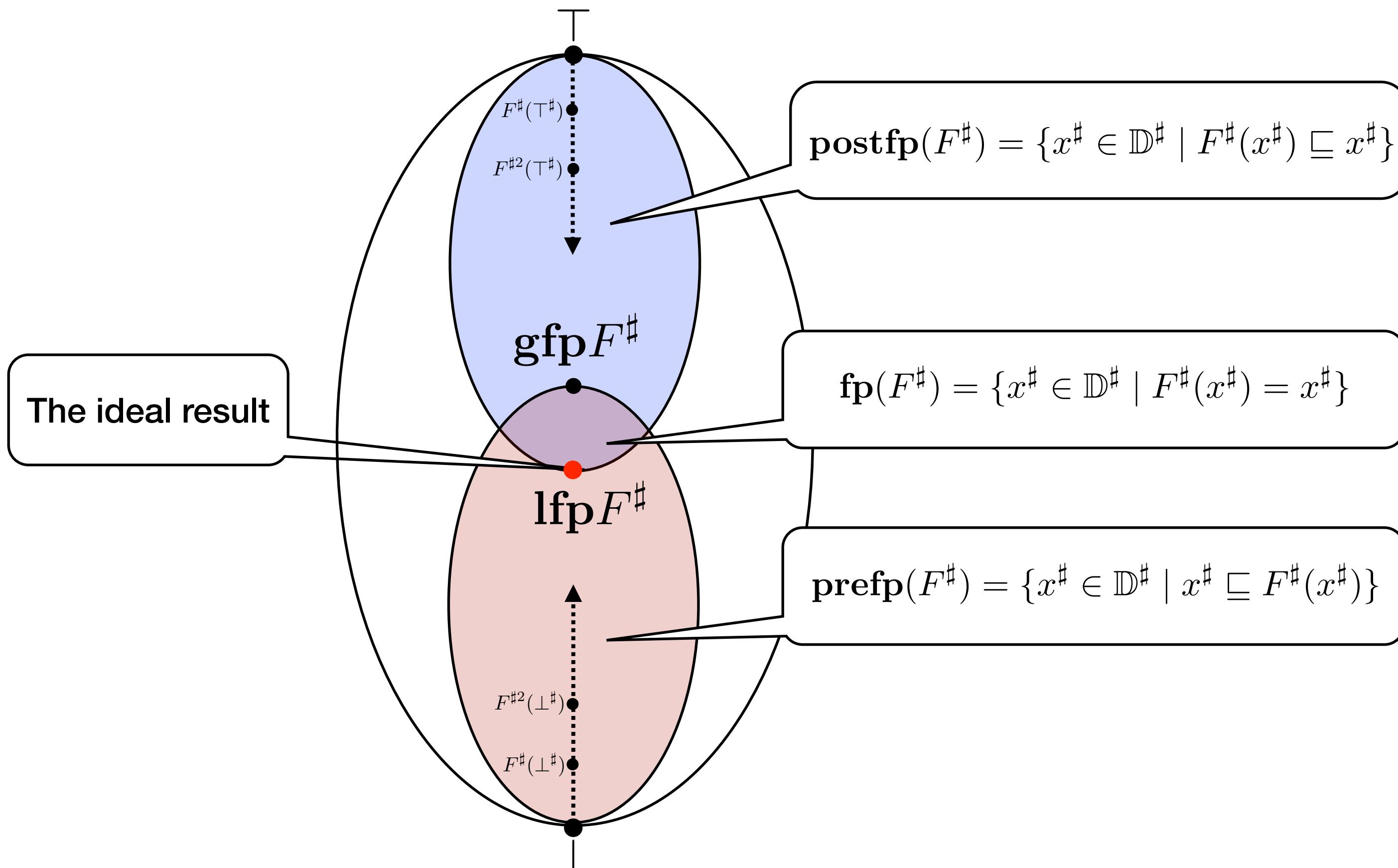
$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp)$$

- If the abstract domain  $\mathbb{D}^\sharp$  has **infinite** height, we compute a finite chain

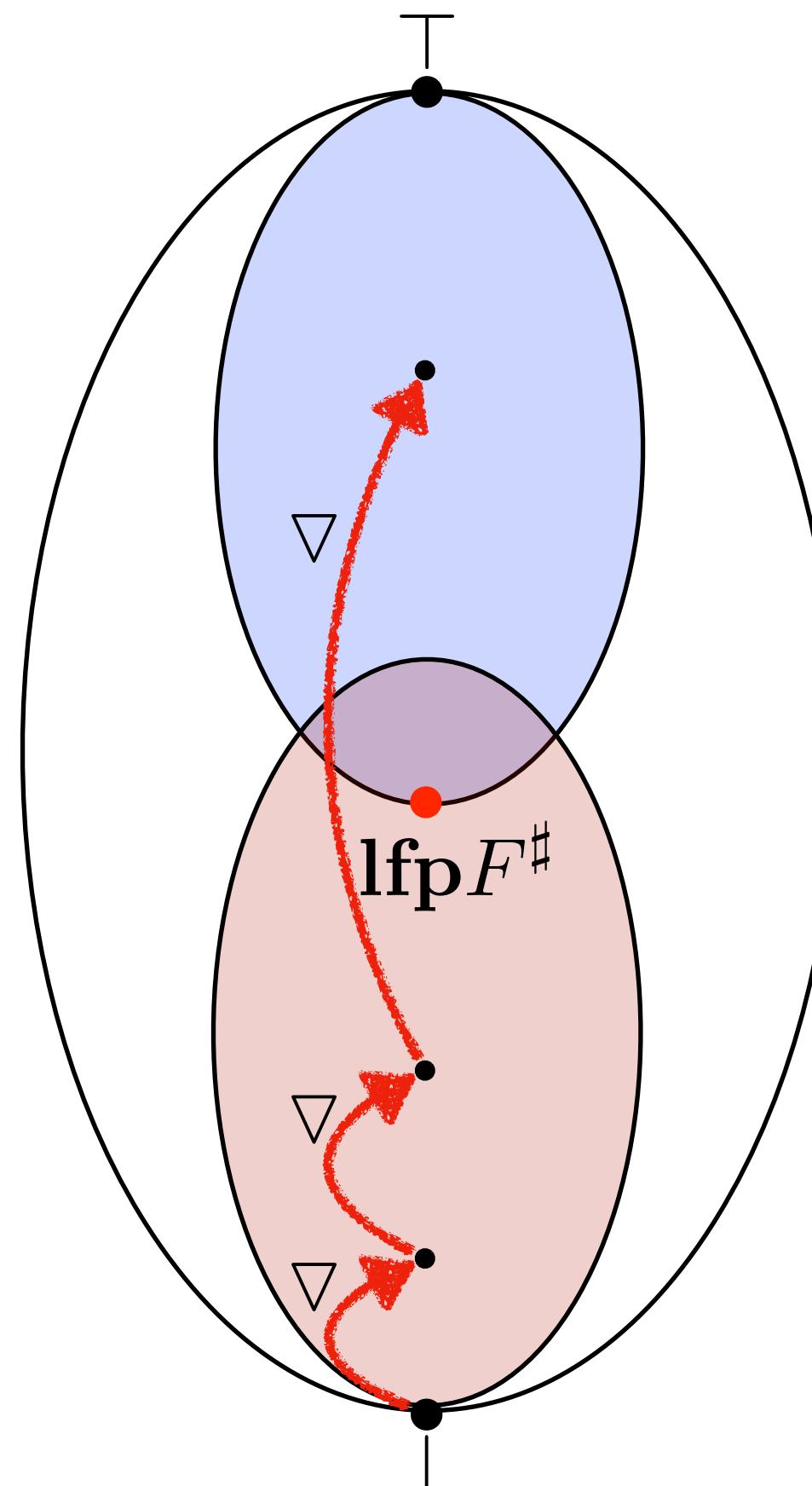
$X_0^\sharp \sqsubseteq X_1^\sharp \sqsubseteq X_2^\sharp \sqsubseteq \dots \sqsubseteq X_{\text{lim}}^\sharp$  such that

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq X_{\text{lim}}^\sharp$$

# Fixed Points



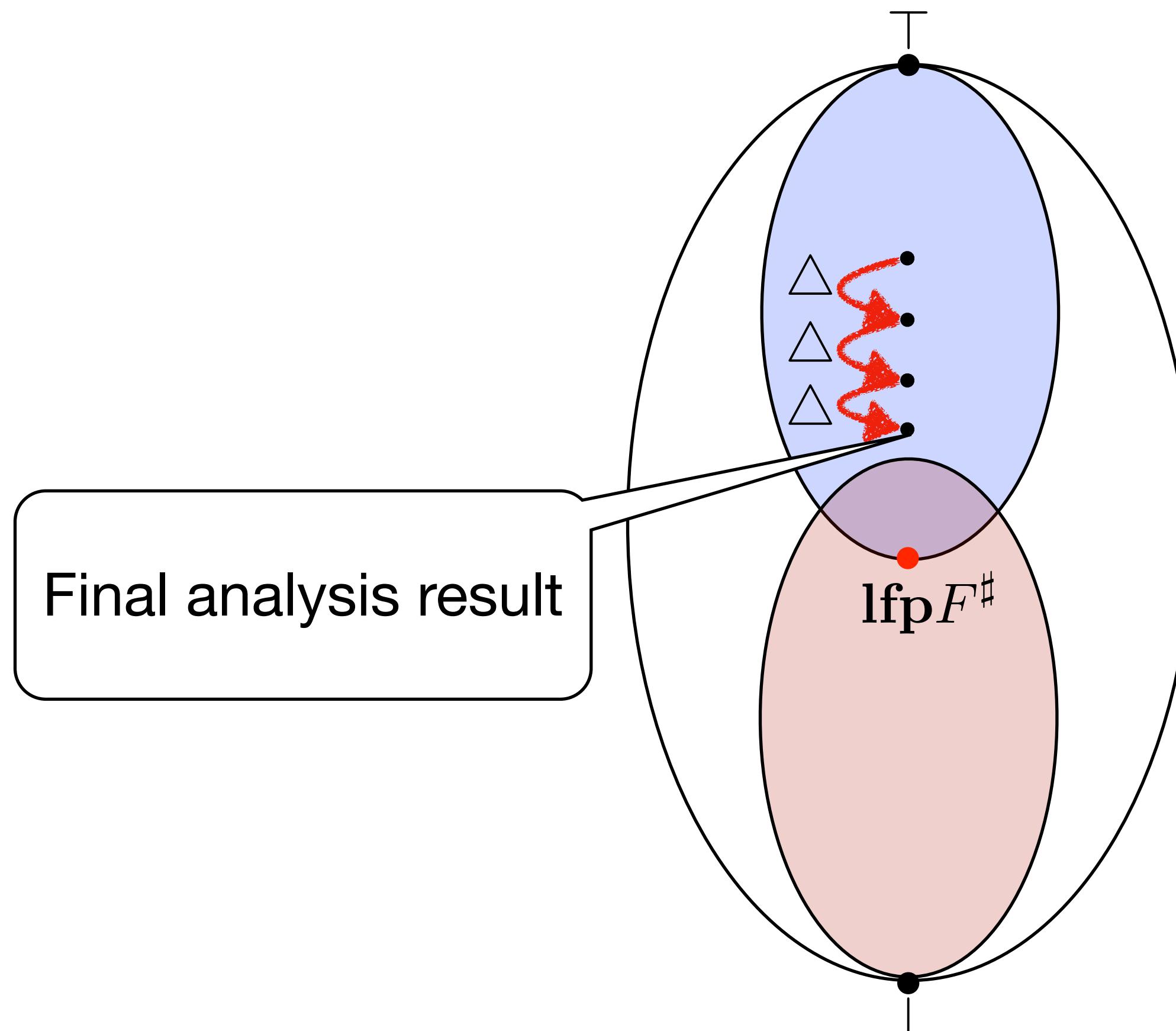
# Widening



$$\nabla : \mathbb{D}^\# \times \mathbb{D}^\# \rightarrow \mathbb{D}^\#$$

**Widening: enforcing the convergence of fix point iterations**

# Narrowing



$$\Delta : \mathbb{D}^\# \times \mathbb{D}^\# \rightarrow \mathbb{D}^\#$$

**Narrowing: refining the analysis results  
with widening**

# Example

- What is the value of x at the end of this program?
- Recall:

$$\llbracket \text{while } B \text{ } C \rrbracket^\sharp(m^\sharp) = \llbracket \neg B \rrbracket^\sharp \left( \bigsqcup_{i \geq 0} F^{\sharp i}(\perp) \right)$$

where  $F^\sharp(X) = m^\sharp \sqcup \llbracket C \rrbracket^\sharp \circ \llbracket B \rrbracket^\sharp(X)$

```
x = 0;
while(*) {
    x++
}
```

- Computation:  $[0, 0] \sqcup [0, 1] \sqcup [0, 2] \sqcup \dots$

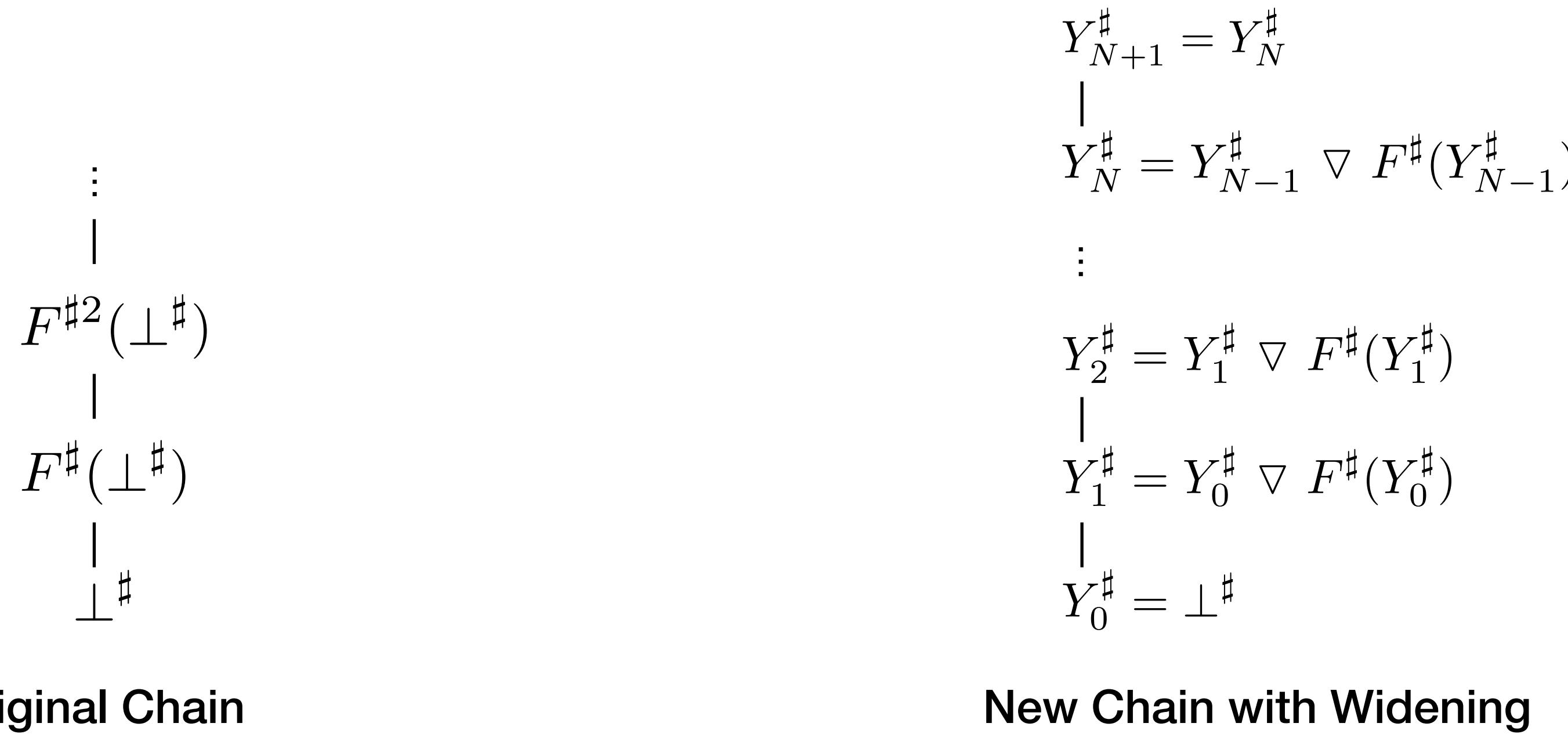
# Overshooting by Widening

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^{\sharp}) \sqsubseteq Y_{\lim}^{\sharp}$$

- Define finite chain  $\{Y_i\}_i$  by an widening operator  $\nabla \in \mathbb{D}^{\sharp} \times \mathbb{D}^{\sharp} \rightarrow \mathbb{D}^{\sharp}$ :

$$Y_0^{\sharp} = \perp^{\sharp}$$
$$Y_{i+1}^{\sharp} = \begin{cases} Y_i^{\sharp} & \text{if } F^{\sharp}(Y_i^{\sharp}) \sqsubseteq Y_i^{\sharp} \\ Y_i^{\sharp} \nabla F^{\sharp}(Y_i^{\sharp}) & \text{otherwise} \end{cases}$$

# Finite Increasing Chain with Widening



Original Chain

New Chain with Widening

**Q. What conditions are required to ensure**

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Y_{\lim}^\sharp$$

# Example

- A simple widening operator for the interval domain

$$\begin{array}{lll} [a, b] \quad \nabla \quad \perp & = & [a, b] \\ \perp \quad \nabla \quad [c, d] & = & [c, d] \\ [a, b] \quad \nabla \quad [c, d] & = & [(c < a? -\infty : a), (b < d? +\infty : b)] \end{array}$$

**Check:** Safety conditions  
for widening

- What is the value of  $x$  at the end of this program?

```
x = 0;  
while(*) {  
    x++  
}
```

$$[0, 0] \quad \nabla \quad [0, 1] = [0, +\infty]$$

# Safety of Widening Operator

- Conditions on widening operator:
  - $\forall a, b \in \mathbb{D}^\sharp. (a \sqsubseteq a \vee b) \wedge (b \sqsubseteq a \vee b)$
  - $\forall$  increasing chain  $\{x_i\}_i$ , the following increasing chain  $\{y_i\}_i$  is finite:

$$y_0 = x_0$$

$$y_{i+1} = y_i \vee x_{i+1}$$

- Then,
  - Chain  $\{Y_i^\sharp\}_i$  is finite
  - $\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Y_{\lim}^\sharp$

# Proof

**Theorem** (Widening's Safety). *Let  $\mathbb{D}^\sharp$  be a CPO,  $F^\sharp : \mathbb{D}^\sharp \rightarrow \mathbb{D}^\sharp$  be a monotone function, and  $\nabla : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \rightarrow \mathbb{D}^\sharp$  be a widening operator. Then, chain  $\{Y_i^\sharp\}_i$  eventually stabilizes and*

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Y_{\lim}^\sharp$$

*where  $Y_{\lim}^\sharp$  is the greatest element of the chain.*

**Proof.** First we prove chain  $\{Y_i^\sharp\}_i$  is finite. According to the second condition on widening operator, it is enough to show that chain  $\{F^\sharp(Y_i^\sharp)\}_i$  is increasing. The chain is increasing because 1)  $F^\sharp(Y_{i+1}^\sharp)$  is either  $F^\sharp(Y_i^\sharp)$  or  $F^\sharp(Y_i^\sharp \nabla F^\sharp(Y_i^\sharp))$ , 2)  $Y_i^\sharp \sqsubseteq Y_i^\sharp \nabla F^\sharp(Y_i^\sharp)$  according to the first condition on widening, and 3)  $F^\sharp$  is monotone.

Second, we prove  $\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Y_{\lim}^\sharp$ . It is enough to show that  $\forall i \in \mathbb{N}. F^{\sharp i}(\perp^\sharp) \sqsubseteq Y_i^\sharp$  that can be proven by induction. The base case is trivial. The inductive case is as follows:

$$\begin{aligned} F^{\sharp i+1}(\perp^\sharp) &= F^\sharp(F^{\sharp i}(\perp^\sharp)) \\ &\sqsubseteq F^\sharp(Y_i^\sharp) \quad (\text{by induction hypothesis and monotonicity of } F^\sharp) \end{aligned}$$

If  $F^\sharp(Y_i^\sharp) \sqsubseteq Y_i^\sharp$ , then  $Y_{i+1}^\sharp = Y_i^\sharp$  by definition. Therefore,  $F^{\sharp i+1}(\perp^\sharp) \sqsubseteq Y_{i+1}^\sharp$ .

If  $F^\sharp(Y_i^\sharp) \sqsupset Y_i^\sharp$ , then  $Y_{i+1}^\sharp = Y_i^\sharp \nabla F^\sharp(Y_i^\sharp)$  by definition. According to the first condition on widening,  $F^\sharp(Y_i^\sharp) \sqsubseteq Y_i^\sharp \nabla F^\sharp(Y_i^\sharp)$ . Therefore,  $F^{\sharp i+1}(\perp^\sharp) \sqsubseteq Y_{i+1}^\sharp$ .

# Example (Revisited)

- A simple widening operator for the interval domain

$$\begin{array}{lcl} [a, b] \quad \nabla \quad \perp & = & [a, b] \\ \perp \quad \nabla \quad [c, d] & = & [c, d] \\ [a, b] \quad \nabla \quad [c, d] & = & [(c < a? -\infty : a), (b < d? +\infty : b)] \end{array}$$

**Check:** Safety conditions  
for widening

- What is the value of  $x$  at the end of this program?

```
x = 0;  
while(*) {  
    x++  
}
```

$$[0, 0] \quad \nabla \quad [0, 1] = [0, +\infty]$$

- How about this?

```
x = 0;  
while(x < 10) {  
    x++  
}
```

$$[\![x \geq 10]\!]( [0, 0] \quad \nabla \quad [0, 1] = [0, +\infty] ) = [10, +\infty]$$

Good  
enough?

# Refinement by Narrowing

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^{\sharp}) \sqsubseteq Z_{\text{lim}}^{\sharp}$$

- Define finite chain  $\{Z_i^{\sharp}\}_i$  by an narrowing operator  $\Delta \in \mathbb{D}^{\sharp} \times \mathbb{D}^{\sharp} \rightarrow \mathbb{D}^{\sharp}$  :

$$\begin{aligned} Z_0^{\sharp} &= Y_{\text{lim}}^{\sharp} \\ Z_{i+1}^{\sharp} &= Z_i^{\sharp} \triangle F^{\sharp}(Z_i^{\sharp}) \end{aligned}$$

# Finite Decreasing Chain with Narrowing

$$\begin{array}{c} Y_{\lim}^{\sharp} \\ | \\ F^{\sharp}(Y_{\lim}^{\sharp}) \\ | \\ F^{\sharp 2}(Y_{\lim}^{\sharp}) \\ \vdots \end{array}$$

Original Chain

$$\begin{array}{c} Z_0^{\sharp} = Y_{\lim}^{\sharp} \\ | \\ Z_1^{\sharp} = Z_0^{\sharp} \triangle F^{\sharp}(Z_0^{\sharp}) \\ | \\ Z_2^{\sharp} = Z_1^{\sharp} \triangle F^{\sharp}(Z_1^{\sharp}) \\ \vdots \\ Z_N^{\sharp} = Z_{N-1}^{\sharp} \triangle F^{\sharp}(Z_{N-1}^{\sharp}) \\ | \\ Z_{N+1}^{\sharp} = Z_N^{\sharp} \end{array}$$

New Chain with Narrowing

**Q. What conditions are required to ensure**

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^{\sharp}) \sqsubseteq Z_{\lim}^{\sharp}$$

# Example

- A simple narrowing operator for the interval domain

$$\begin{array}{lcl} [a, b] \triangle \perp & = & \perp \\ \perp \triangle [c, d] & = & \perp \\ [a, b] \triangle [c, d] & = & [(a = -\infty?c : a), (b = +\infty?d : b)] \end{array}$$

**Check:** Safety conditions  
for narrowing

- What is the value of  $x$  at the end of this program?

```
x = 0;  
while(x < 10) {  
    x++  
}
```

(Widening)	$[0, 0] \triangledown [0, 1] = [0, +\infty]$
(Narrowing)	$[0, +\infty] \triangle [1, 10] = [0, 10]$
(Termination condition)	$\llbracket x \geq 10 \rrbracket([0, 10]) = [10, 10]$

Good enough!

# Safety of Narrowing Operator

- Conditions on narrowing operator:

- $\forall a, b \in \mathbb{D}^\sharp. a \sqsupseteq b \implies a \sqsupseteq (a \triangle b) \sqsupseteq b$
- For all decreasing chain  $\{y_i\}_i$ , the following decreasing chain  $\{z_i\}_i$  is finite

$$z_0 = y_0$$

$$z_{i+1} = z_i \triangle y_{i+1}$$

- Then,

- Decreasing chain  $\{Z_i^\sharp\}_i$  is finite
- $\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Z_{\lim}$

# Proof

**Theorem** (Narrowing's Safety). *Let  $\mathbb{D}^\sharp$  be a CPO,  $F^\sharp : \mathbb{D}^\sharp \rightarrow \mathbb{D}^\sharp$  be a monotone function, and  $\Delta : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \rightarrow \mathbb{D}^\sharp$  be a narrowing operator. Then, chain  $\{Z_i^\sharp\}_i$  eventually stabilizes and*

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Z_{\lim}^\sharp$$

where  $Z_{\lim}^\sharp$  is the least element of the chain.

**Proof.** First we prove chain  $\{Z_i^\sharp\}_i$  is finite. According to the condition on narrowing operator, it is enough to show that chain  $\{F^\sharp(Z_i^\sharp)\}_i$  is decreasing. The chain is decreasing if  $\forall i \in \mathbb{N}. Z_i^\sharp \sqsupseteq F^\sharp(Z_i^\sharp)$  because  $Z_i^\sharp \sqsupseteq Z_i^\sharp \Delta F^\sharp(Z^\sharp) \sqsupseteq F^\sharp(Z^\sharp)$  and  $F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_i^\sharp \Delta F^\sharp(Z^\sharp)) = F^\sharp(Z_{i+1}^\sharp)$ . We prove  $\forall i \in \mathbb{N}. Z_i^\sharp \sqsupseteq F^\sharp(Z_i^\sharp)$  by induction. The base case is true by definition of  $Z_{\lim}^\sharp$  from the increasing chain by widening. The inductive case is as follows:

$$\begin{aligned} Z_i^\sharp &\sqsupseteq F^\sharp(Z_i^\sharp) && \text{(by induction hypothesis)} \\ \implies Z_i^\sharp &\sqsupseteq Z_i^\sharp \Delta F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_i^\sharp) && \text{(by the first condition on narrowing)} \\ \implies Z_i^\sharp &\sqsupseteq Z_{i+1}^\sharp \sqsupseteq F^\sharp(Z_i^\sharp) && \text{(by definition)} \\ \implies F^\sharp(Z_i^\sharp) &\sqsupseteq F^\sharp(Z_{i+1}^\sharp) && \text{(by monotonicity of } F^\sharp \text{ and } Z_i^\sharp \sqsupseteq Z_{i+1}^\sharp) \\ \implies Z_{i+1}^\sharp &\sqsupseteq F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_{i+1}^\sharp) \end{aligned}$$

Second we prove chain  $\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Z_{\lim}^\sharp$ . It is enough to show that  $\forall i \in \mathbb{N}. F^{\sharp i}(\perp^\sharp) \sqsubseteq Z_i^\sharp$  that can be proven by induction. The base case is trivial. The inductive case is as follows:

$$\begin{aligned} F^{\sharp i+1}(\perp^\sharp) &= F^\sharp \circ F^{\sharp i}(\perp^\sharp) \\ &\sqsubseteq F^\sharp(Z_i^\sharp) && \text{(by induction hypothesis and monotonicity of } F^\sharp) \\ &\sqsubseteq Z_i^\sharp \Delta F^\sharp(Z_i^\sharp) && \text{(by condition } \forall i \in \mathbb{N}. Z_i^\sharp \sqsupseteq F^\sharp(Z_i^\sharp)) \\ &= Z_{i+1}^\sharp && \text{(by definition)} \end{aligned}$$

# Computable Abstract Semantics

$$\llbracket C \rrbracket^\# : \mathbb{M}^\# \rightarrow \mathbb{M}^\#$$

$$\llbracket \text{skip} \rrbracket^\# = \lambda m^\#.m^\#$$

$$\llbracket C_0 ; C_1 \rrbracket^\# = \lambda m^\#. \llbracket C_1 \rrbracket^\# \circ \llbracket C_0 \rrbracket^\#(m^\#)$$

$$\llbracket x := E \rrbracket^\# = \lambda m^\#.m^\#\{x \mapsto \llbracket E \rrbracket^\#(m^\#)\}$$

$$\llbracket \text{input}(x) \rrbracket^\# = \lambda m^\#.m^\#\{x \mapsto \alpha(\mathbb{Z})\}$$

$$\llbracket \text{if } B \text{ then } C_1 \text{ else } C_2 \rrbracket^\# = \lambda m^\#. \llbracket C_1 \rrbracket^\# \circ \llbracket B \rrbracket^\#(m^\#) \sqcup \llbracket C_2 \rrbracket^\# \circ \llbracket \neg B \rrbracket^\#(m^\#)$$

$$\llbracket \text{while } B \text{ } C \rrbracket^\# = \lambda m^\#. \llbracket \neg B \rrbracket^\# \circ \text{Narrow} \circ \text{Widen} \circ (\lambda X.m^\# \sqcup \llbracket C \rrbracket^\# \circ \llbracket B \rrbracket^\#(X))$$

$$Widen(F^\#) = \lim_{i \in \mathbb{N}} \begin{cases} Y_0^\# = \perp \\ Y_{i+1}^\# = \begin{cases} Y_i^\# & \text{if } F^\#(Y_i^\#) \sqsubseteq Y_i^\# \\ Y_i^\# \bigtriangledown F^\#(Y_i^\#) & \text{o.w.} \end{cases} \end{cases}$$

$$\text{Narrow}(m^\#) = \lim_{i \in \mathbb{N}} \begin{cases} Z_0^\# = m^\# \\ Z_{i+1}^\# = Z_i^\# \bigtriangleup F^\#(Z_i^\#) \end{cases}$$

# Summary

- Computing a sound approximation of the concrete semantics
  - Finite height: directly compute the fixed point by iteration
  - Infinite height: fixpoint iteration with widening and narrowing
- Widening: termination guarantee
- Narrowing: refinement of widening results