

Program Analysis

3. Denotational Semantics

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Different Styles of Semantics (Revisited)

- **Operational Semantics**: “How to compute the execution result”
 - so-called transitional style
 - $3 * (2 + 1) : \mathbf{3 * (2 + 1)} \rightarrow \mathbf{3 * 3} \rightarrow 9$
- **Denotational Semantics**: “What the execution result is”
 - so-called compositional style
 - $3 * (2 + 1) : \mathbf{9}$
- ...

Different approaches for different purposes and languages

Denotational Semantics

- **Mathematical** meaning of a program (i.e., no state or transition)
- Program semantics is a **function** from input states to output states
- The semantics of a program is determined by that of each subcomponent (i.e., **compositional**)
- Notation: $\llbracket P \rrbracket : \mathbb{M} \rightarrow \mathbb{M}$

A Simple Imperative Language

$E ::=$		arithmetic expressions
	n	integer constants
	x	variable
	$E \odot E$	binary operation
$B ::=$		boolean expression
	$\text{true} \mid \text{false}$	boolean constants
	$E \oslash E$	comparison expressions
$C ::=$		commands
	<code>skip</code>	command that does nothing
	$C; C$	sequence
	$x := E$	assignment
	<code>input(x)</code>	command reading of a value
	<code>if B then C else C</code>	conditional command
	<code>while B C</code>	loop command

Semantic Domains

- Sets of semantic objects
- Memory is a mapping $\mathbb{M} = \mathbb{X} \rightarrow \mathbb{V}$
 - \mathbb{X} : the set of variables
 - \mathbb{V} : the set of integers (\mathbb{Z}) and booleans (\mathbb{B})
- Example: $\llbracket x := 7 ; y := 3 \rrbracket \{\} = \{x \mapsto 7, y \mapsto 3\}$

Semantics of Expressions

$$\llbracket E \rrbracket : \mathbb{M} \rightarrow \mathbb{Z}$$

$$\llbracket n \rrbracket(m) = n$$

$$\llbracket x \rrbracket(m) = m(x)$$

$$\llbracket E_1 \odot E_2 \rrbracket(m) = \llbracket E_1 \rrbracket(m) \odot \llbracket E_2 \rrbracket(m)$$

$$\llbracket B \rrbracket : \mathbb{M} \rightarrow \mathbb{B}$$

$$\llbracket \text{true} \rrbracket(m) = \text{true}$$

$$\llbracket \text{false} \rrbracket(m) = \text{false}$$

$$\llbracket B_1 \oslash B_2 \rrbracket(m) = \llbracket B_1 \rrbracket(m) \oslash \llbracket B_2 \rrbracket(m)$$

Semantics of Commands (1)

$$\llbracket C \rrbracket : \mathbb{M} \rightarrow \mathbb{M}$$

$$\llbracket \text{skip} \rrbracket(m) = m$$

$$\llbracket C_0 ; C_1 \rrbracket(m) = \llbracket C_1 \rrbracket(\llbracket C_0 \rrbracket(m))$$

$$\llbracket x := E \rrbracket(m) = m\{x \mapsto \llbracket E \rrbracket(m)\}$$

$$\llbracket \text{input}(x) \rrbracket(m) = m\{x \mapsto n\}$$

$$\llbracket \text{if } B \text{ then } C_1 \text{ else } C_2 \rrbracket(m) = \begin{cases} \llbracket C_1 \rrbracket(m) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ \llbracket C_2 \rrbracket(m) & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases}$$

Compositional? Yes!

Semantics of Commands (2)

- Semantics of While loop

$$\llbracket \text{while } B \ C \rrbracket(m) = \begin{cases} \llbracket \text{while } B \ C \rrbracket(\llbracket C \rrbracket(m)) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases}$$

Compositional? **NO!**

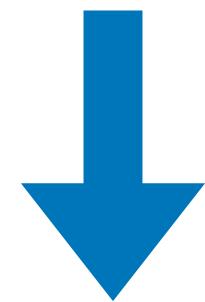
- C.f) inductive vs compositional definition of the Fibonacci function

$$fib(n) = fib(n - 1) + fib(n - 2) \quad \text{vs} \quad fib(n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right)$$

Semantics as Fixed Point (1)

- Consider this inductive definition as an equation

$$\llbracket \text{while } B \ C \rrbracket(m) = \begin{cases} \llbracket \text{while } B \ C \rrbracket(\llbracket C \rrbracket(m)) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases}$$



$$\llbracket \text{while } B \ C \rrbracket = \mathcal{F}_{B,C}(\llbracket \text{while } B \ C \rrbracket)$$

where $\mathcal{F}_{B,C}(X) = \lambda m. \begin{cases} X(\llbracket C \rrbracket(m)) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases}$

$$F(X) = \dots$$

$$\iff F = \lambda X. \dots$$

Semantics as Fixed Point (2)

- What is the semantics of loops? **A solution of this equation!**

$$\llbracket \text{while } B \text{ } C \rrbracket = \mathcal{F}_{B,C}(\llbracket \text{while } B \text{ } C \rrbracket) \quad \longleftrightarrow \quad X = \mathcal{F}_{B,C}(X) \text{ where } X = \llbracket \text{while } B \text{ } C \rrbracket$$

- Solution: a fixed point of $\mathcal{F}_{B,C}$
- Especially, we define the semantics as the least fixed point

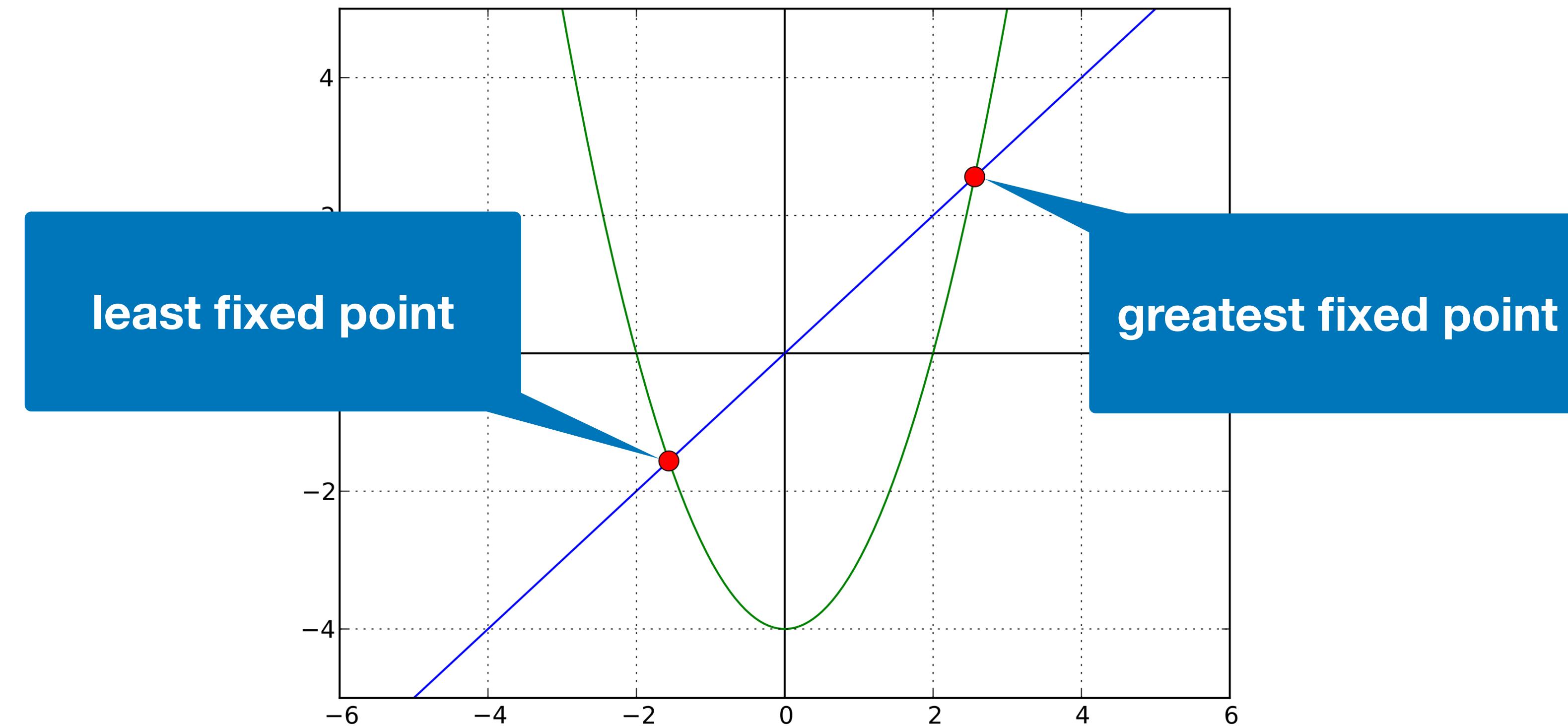
$$\llbracket \text{while } B \text{ } C \rrbracket = \text{lfp} \mathcal{F}_{B,C}$$

$$\text{where } \mathcal{F}_{B,C}(X)(m) = \begin{cases} X(\llbracket C \rrbracket(m)) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases}$$

Compositional? Yes!

Exercise: Fixed Point

$$f(x) = x^2 - 4$$



*https://en.wikipedia.org/wiki/Least_fixed_point

Exercise: Fixed Point

$$\mathbb{N} = \{0\} \cup \{n + 1 \mid n \in \mathbb{N}\}$$

\mathbb{N} is the least fixed point of F where $F(X) = \{0\} \cup \{n + 1 \mid n \in X\}$

$$\therefore \mathbb{N} = fix(\lambda X. \{0\} \cup \{n + 1 \mid n \in X\})$$

Exercise: Fixed Point

$$\mathbb{L} = \{\text{nil}\} \cup \{0 :: l \mid l \in \mathbb{L}\}$$

\mathbb{L} is the least fixed point of F where $F(X) = \{\text{nil}\} \cup \{0 :: l \mid l \in X\}$

$$\therefore \mathbb{L} = \text{fix}(\lambda X. \{\text{nil}\} \cup \{0 :: l \mid l \in X\})$$

Exercise: Fixed Point

$$\text{reach}(N) = N \cup \text{reach}(\text{next}(N))$$

$$\text{reach} = \lambda N. N \cup \text{reach}(\text{next}(N))$$

`reach` is the least fixed point of F where $F(X) = \lambda N. N \cup X(\text{next}(N))$

$$\therefore \text{reach} = \text{fix}(\lambda X. (\lambda N. N \cup X(\text{next}(N))))$$

Exercise: Fixed Point

```
fact(N) = if N = 0 ? 1 : N * fact(N - 1)
```

$$\text{fact} = \lambda N. \text{if } N = 0 ? 1 : N * \text{fact}(N - 1)$$

fact is the least fixed point of F where $F(X) = \lambda N. \text{if } N = 0 ? 1 : N * X(N - 1)$

$$\therefore \text{fact} = \text{fix}(\lambda X. (\lambda N. \text{if } N = 0 ? 1 : N * X(N - 1)))$$

Semantics as Fixed Point (Revisited)

- What is the semantics of loops? **A solution of this equation!**

$$\llbracket \text{while } B \text{ } C \rrbracket = \mathcal{F}_{B,C}(\llbracket \text{while } B \text{ } C \rrbracket) \quad \longleftrightarrow \quad X = \mathcal{F}_{B,C}(X) \text{ where } X = \llbracket \text{while } B \text{ } C \rrbracket$$

- Solution: a fixed point of $\mathcal{F}_{B,C}$
- Especially, we define the semantics as the least fixed point

$$\llbracket \text{while } B \text{ } C \rrbracket = \text{lfp} \mathcal{F}_{B,C}$$

$$\text{where } \mathcal{F}_{B,C}(X)(m) = \begin{cases} X(\llbracket C \rrbracket(m)) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases}$$

Compositional? Yes!

Questions

- Why does the semantic equation have a solution?
- Does the equation have a unique solution?
- How to compute the solution?



Domain Theory

- Semantics of a program is an element of a domain called **CPO** (complete partial order set)
 - Example: $\llbracket C \rrbracket \in \mathbb{M} \rightarrow \mathbb{M}$
- Semantics of a program is the **least fixed point** of a **continuous function**
 - Example: $\mathcal{F}_{B,C} : (\mathbb{M} \rightarrow \mathbb{M}) \rightarrow (\mathbb{M} \rightarrow \mathbb{M})$
- Established by Dana Scott
 - *Outline of a Mathematical Theory of Computation*, 1970
 - *Mathematical Concepts in Programming Language Semantics*, 1972

Partial Order

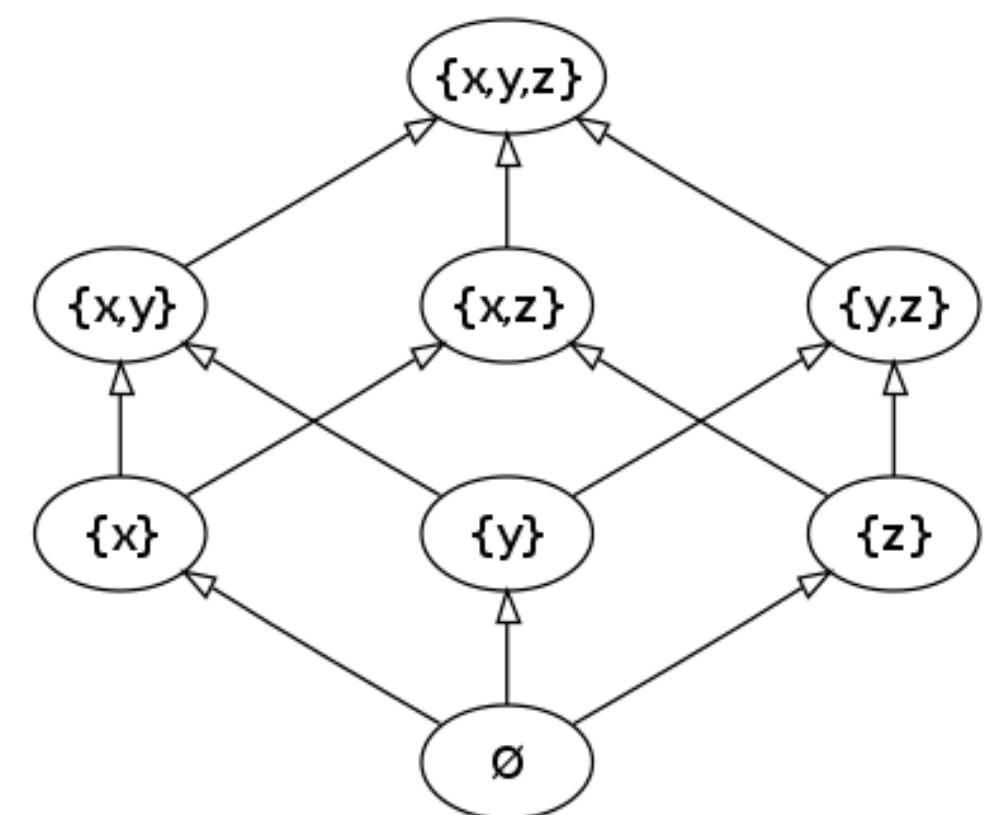
Definition (Partial Order). A binary relation \sqsubseteq is a **partial order** on a set D if it has:

1. Reflexivity: $a \sqsubseteq a$ for all $a \in D$
2. Antisymmetry: $a \sqsubseteq b$ and $b \sqsubseteq a$ implies $a = b$
3. Transitivity: $a \sqsubseteq b$ and $b \sqsubseteq c$ implies $a \sqsubseteq c$

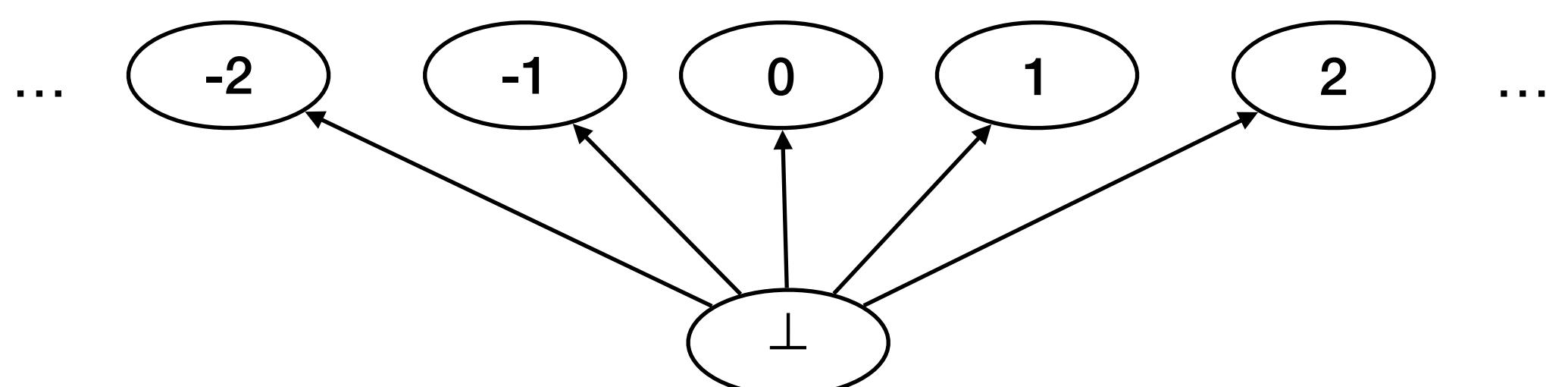
A set D with a partial order \sqsubseteq is called a **partially ordered set** (D, \sqsubseteq) , or simply **poset**.

Partial Order

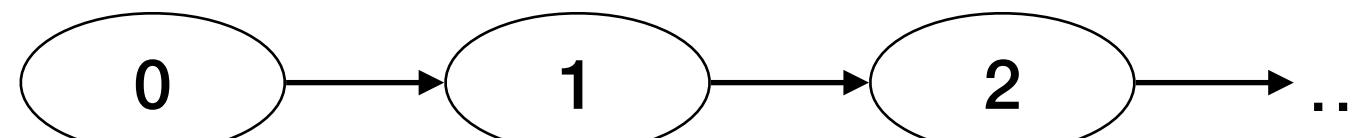
- Example 1: $(\wp(\{x, y, z\}), \subseteq)$



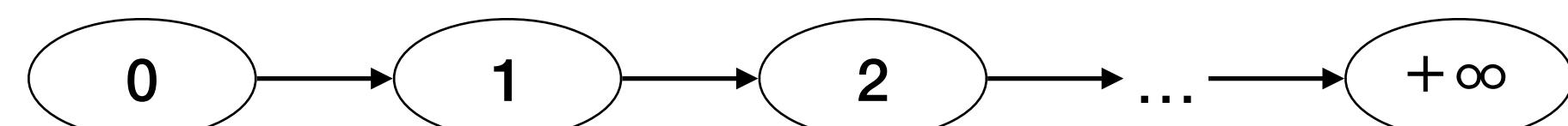
- Example 2: $(\mathbb{Z}_\perp, \sqsubseteq)$



- Example 3: (\mathbb{N}, \leq)



- Example 4: $(\mathbb{N} + \{+\infty\}, \leq)$



Least Upper Bound (Join)

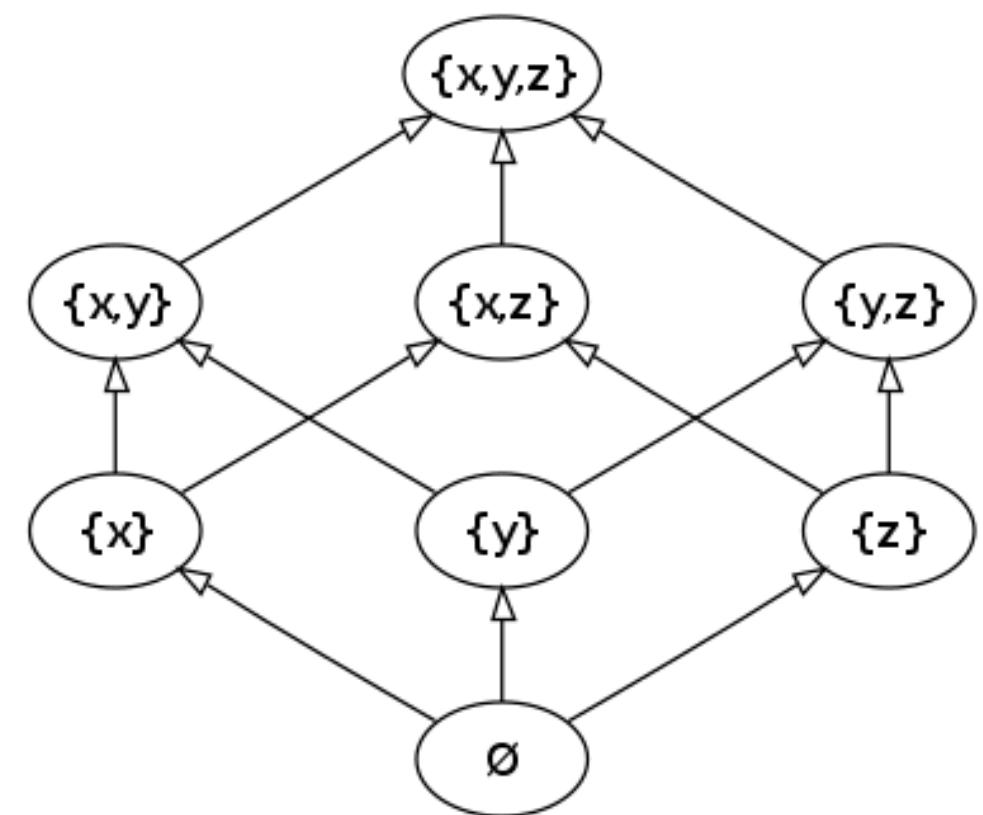
Definition (Least Upper Bound). For a partially ordered set (D, \sqsubseteq) and subset $X \subseteq D$, $d \in D$ is an **upper bound** of X iff

$$\forall x \in X. x \sqsubseteq d.$$

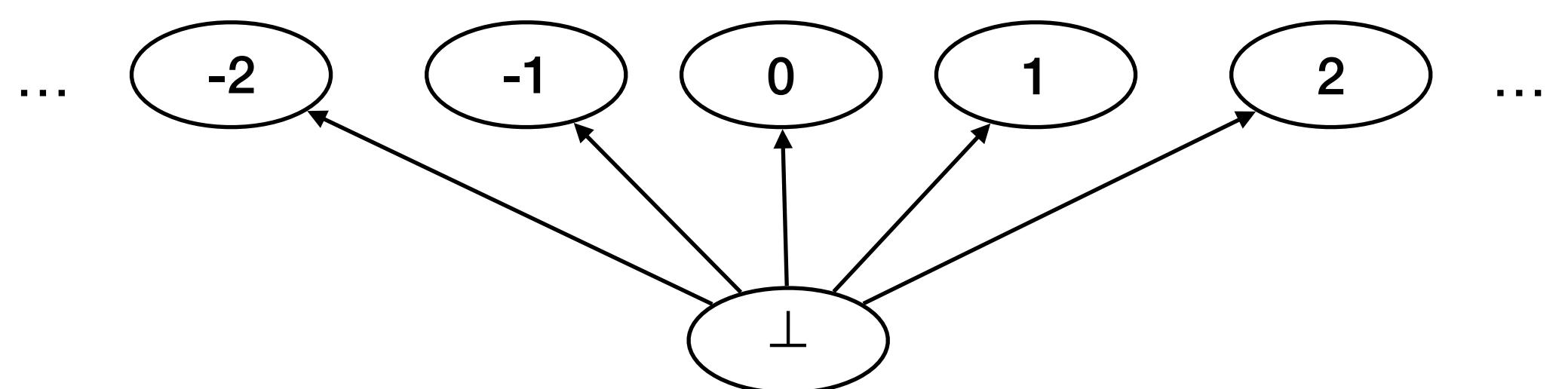
An upper bound d is the **least upper bound** of X iff for all upper bounds y of X , $d \sqsubseteq y$.
The least upper bound of X is denoted by $\sqcup X$.

Least Upper Bound (Join)

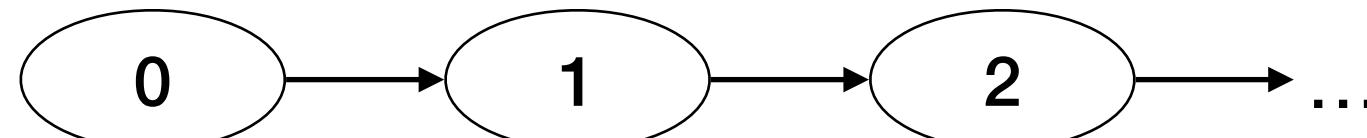
- Example 1: $(\wp(\{x, y, z\}), \subseteq)$



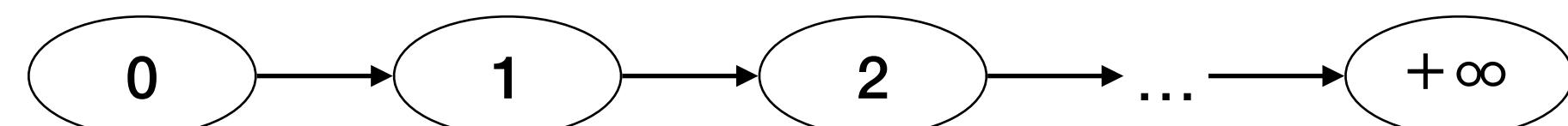
- Example 2: $(\mathbb{Z}_\perp, \sqsubseteq)$



- Example 3: (\mathbb{N}, \leq)



- Example 4: $(\mathbb{N} + \{+\infty\}, \leq)$



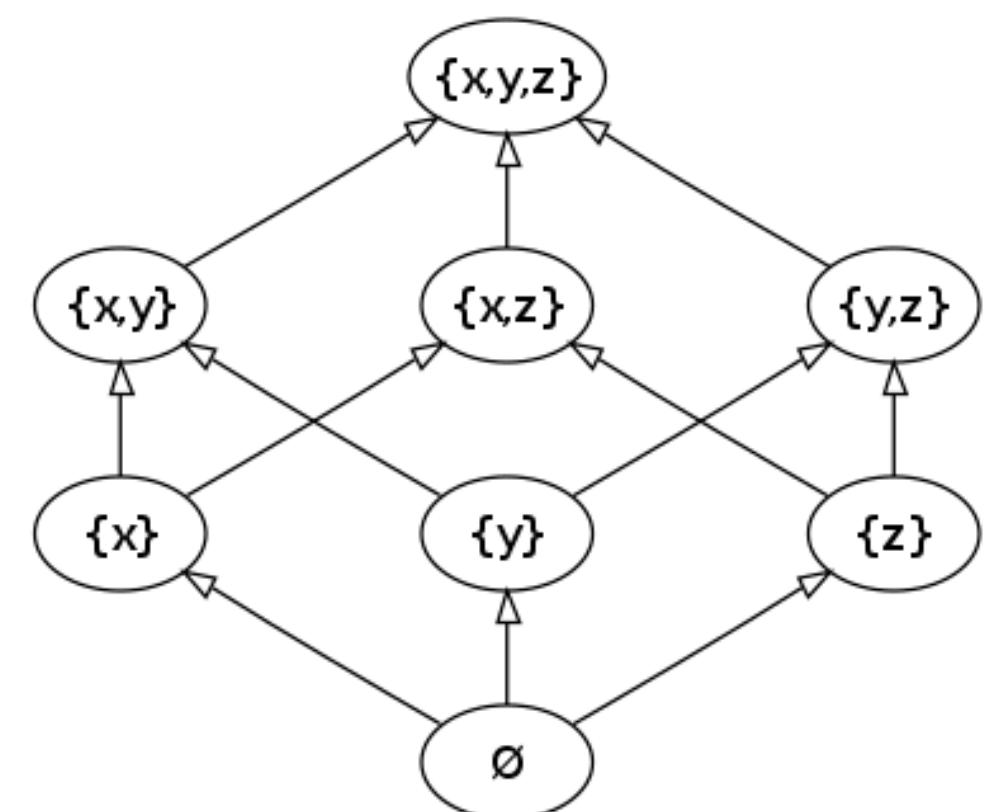
Chain

Definition (Chain). Let (D, \sqsubseteq) be a partial ordered set. A subset $X \subseteq D$ is called **chain** if X is totally ordered:

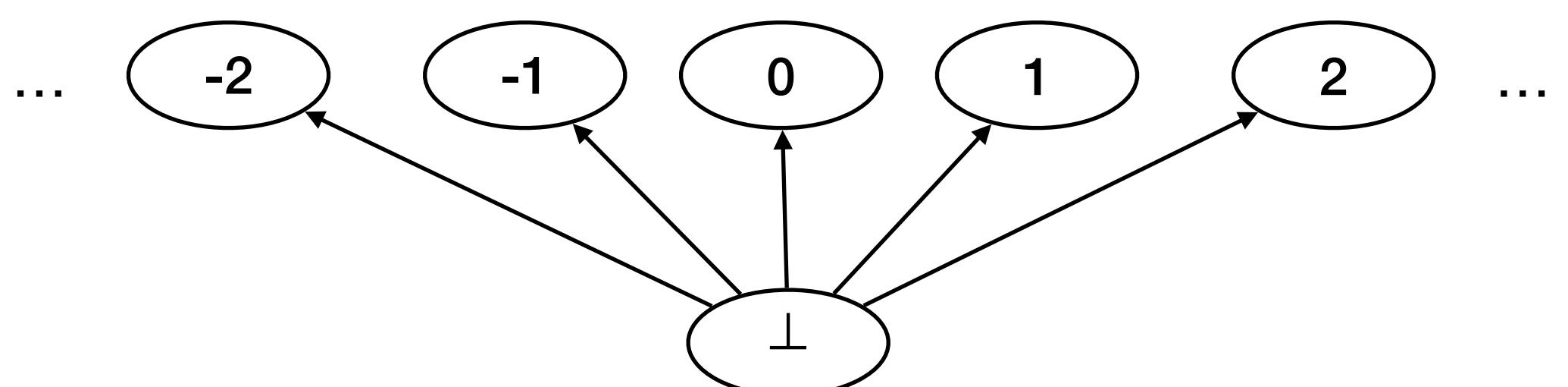
$$\forall x_1, x_2 \in X. x_1 \sqsubseteq x_2 \text{ or } x_2 \sqsubseteq x_1$$

Chain

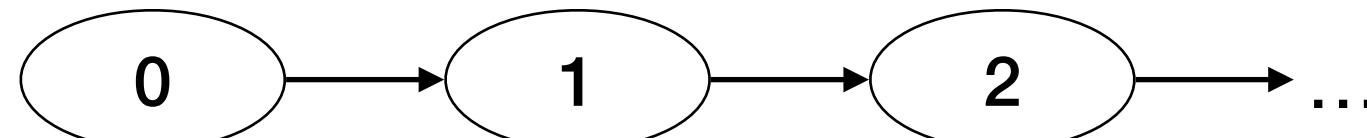
- Example 1: $(\wp(\{x, y, z\}), \subseteq)$



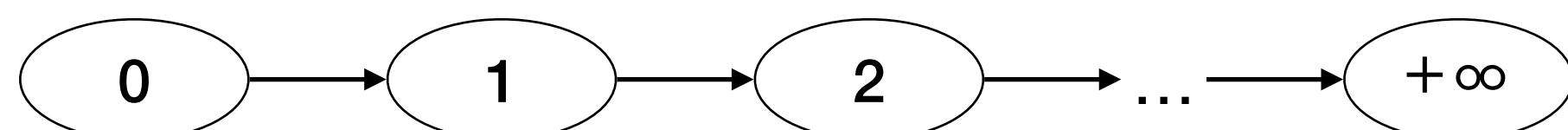
- Example 2: $(\mathbb{Z}_\perp, \sqsubseteq)$



- Example 3: (\mathbb{N}, \leq)



- Example 4: $(\mathbb{N} + \{+\infty\}, \leq)$



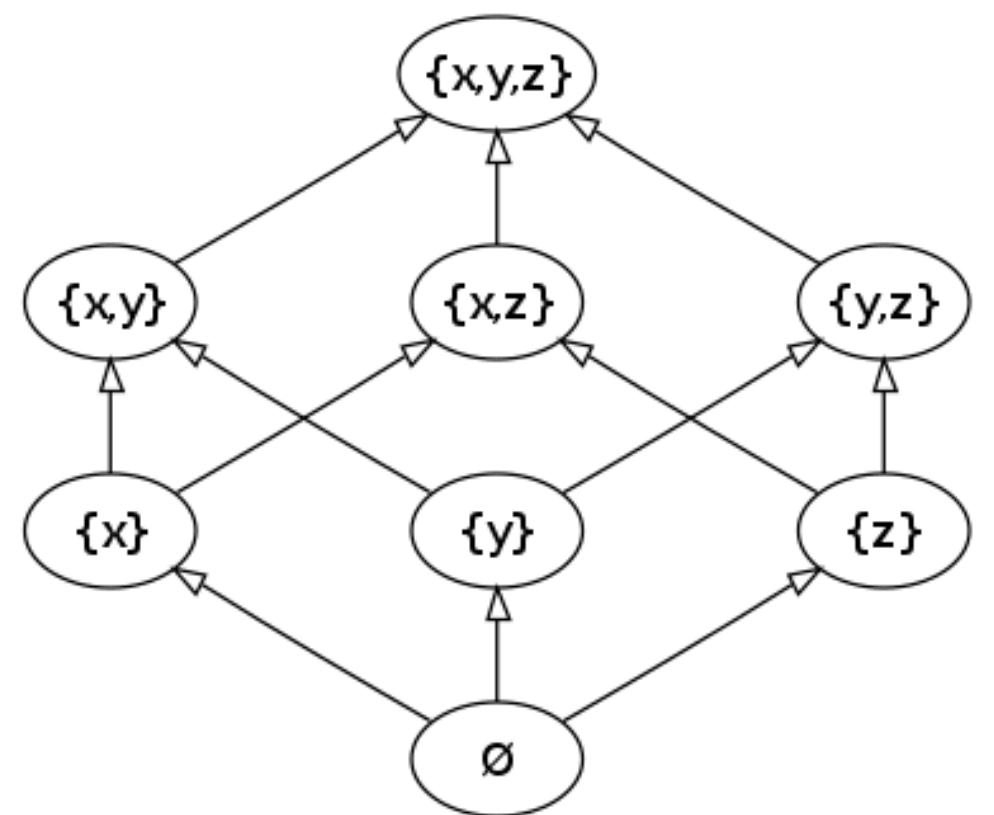
CPO

Definition (CPO). A poset (D, \sqsubseteq) is a **CPO** (complete partial order) if every chain X of D has $\bigsqcup X \in D$.

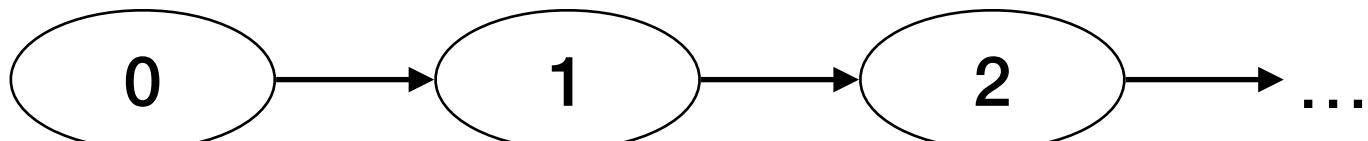
Lemma. If poset (D, \sqsubseteq) is a CPO, it has the **least element** $\perp = \bigsqcup \emptyset$.

CPO

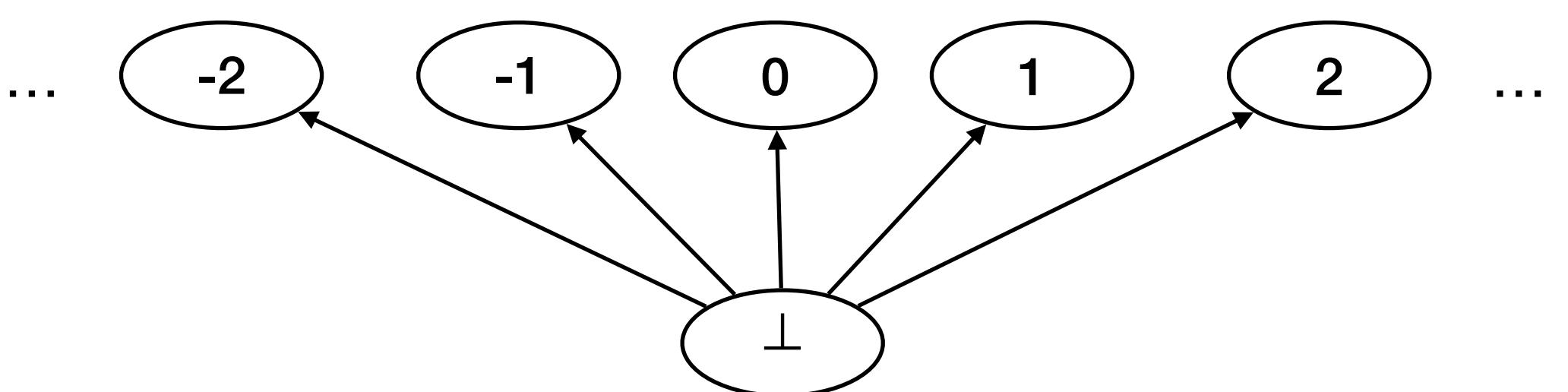
- Example 1: $(\wp(\{x, y, z\}), \subseteq)$



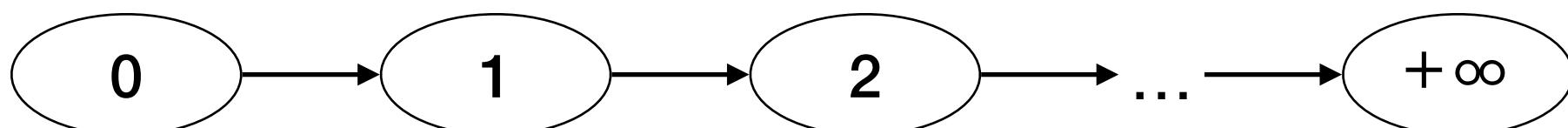
- Example 3: (\mathbb{N}, \leq)



- Example 2: $(\mathbb{Z}_\perp, \sqsubseteq)$



- Example 4: $(\mathbb{N} + \{+\infty\}, \leq)$

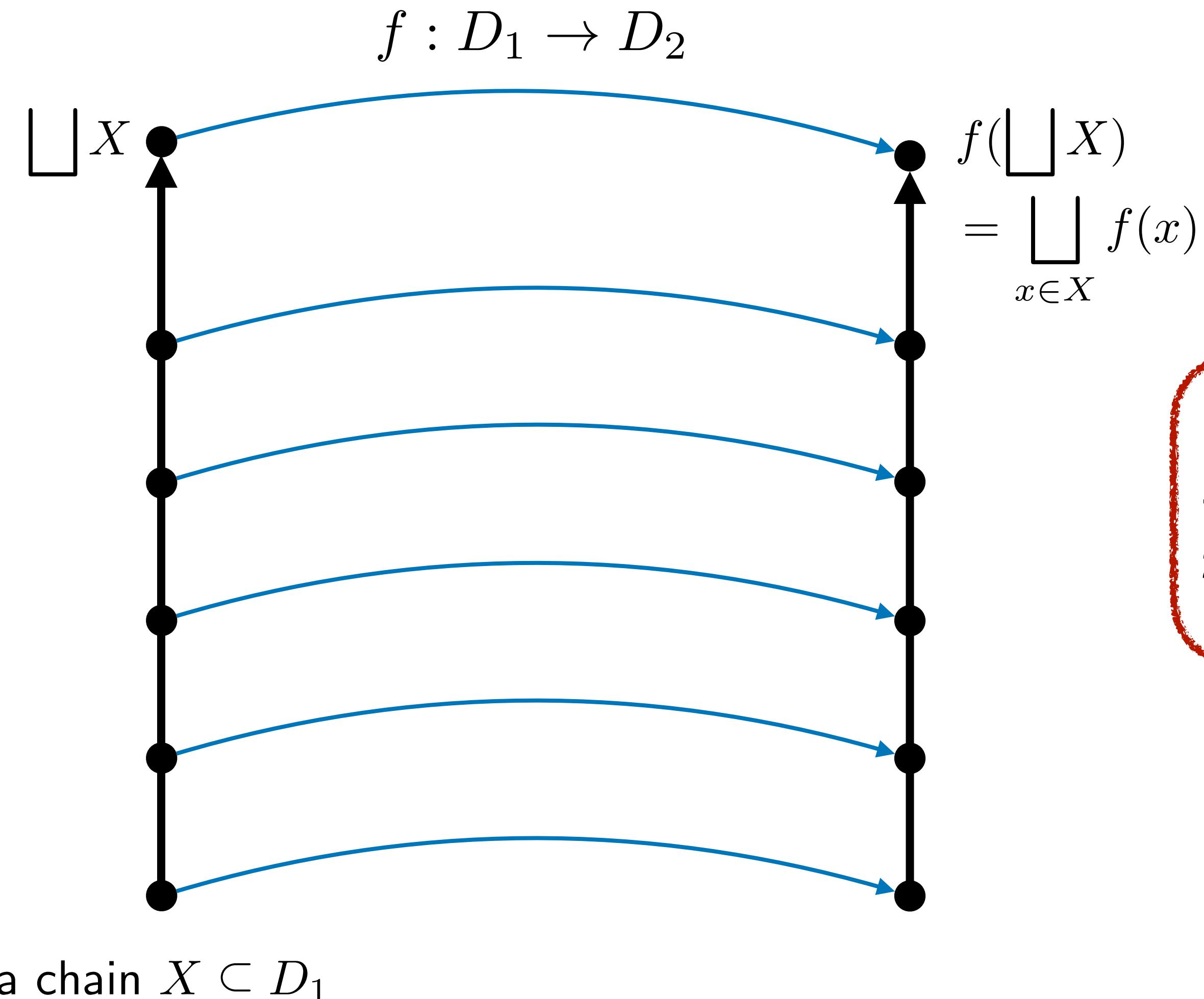


Continuous Function

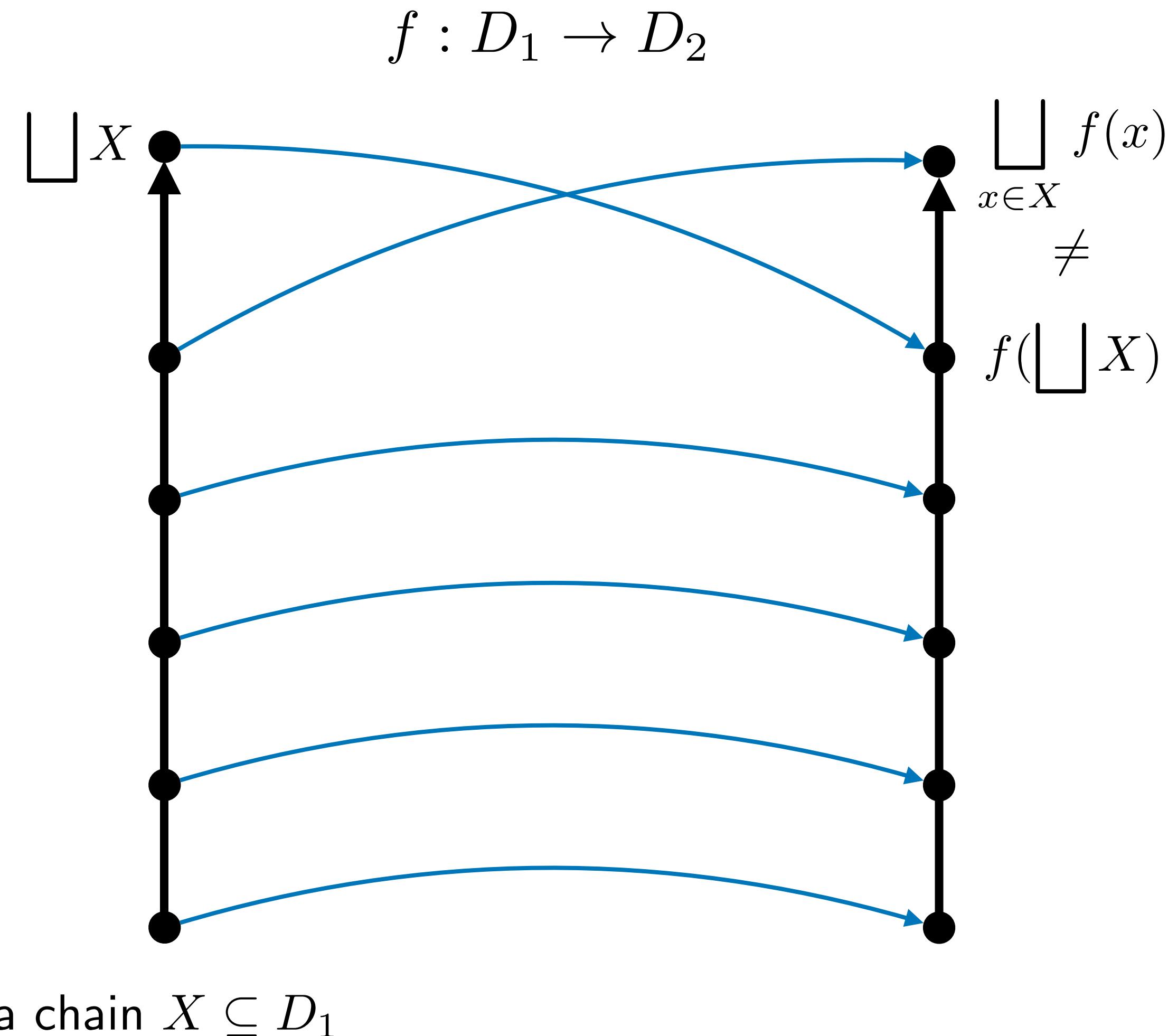
Definition (Continuous Function). Given two partially ordered sets D_1 and D_2 , a function $f: D_1 \rightarrow D_2$ is **continuous** if it preserves least upper bounds of chains:

$$\forall \text{chain } X \subseteq D_1. \bigcup_{x \in X} f(x) = f(\bigcup X).$$

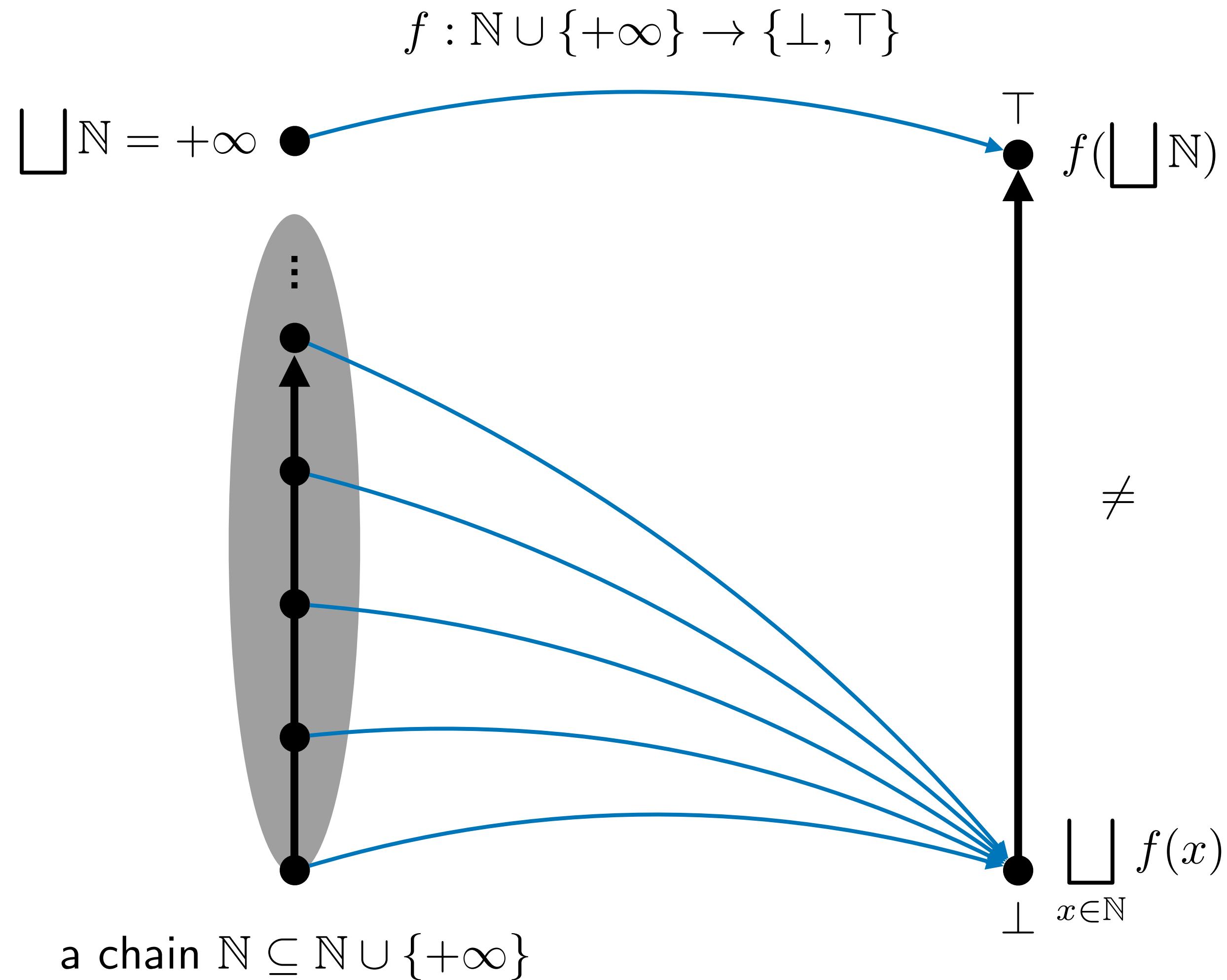
Continuous Function



Non-Continuous Function (1)



Non-Continuous Function (2)



Continuous Function

Lemma. If a function f is continuous, then f is monotone.

Proof. Suppose that f is not monotone, i.e., for some $a \sqsubseteq b, f(a) \sqsupset f(b)$. Then,

$$\begin{aligned} f(a) &\sqsubseteq f(a) \sqcup f(b) && (\text{by definition of } \sqcup) \\ &= f(a \sqcup b) && (\text{by continuity of } f) \\ &= f(b) && (\text{because } a \sqsubseteq b) \end{aligned}$$

This is a contradiction. Therefore, $f(a) \sqsubseteq f(b)$.

Fixed Point

Definition (Fixed Point). Let (D, \sqsubseteq) be a partially ordered set. A **fixed point** of a function $f: D \rightarrow D$ is an element x such that $f(x) = x$. We write $\text{lfp } f$ for the **least fixed point** of such that

$$f(\text{lfp } f) = \text{lfp } f \quad \text{and} \quad \forall d \in D . f(d) = d \implies \text{lfp } f \sqsubseteq d$$

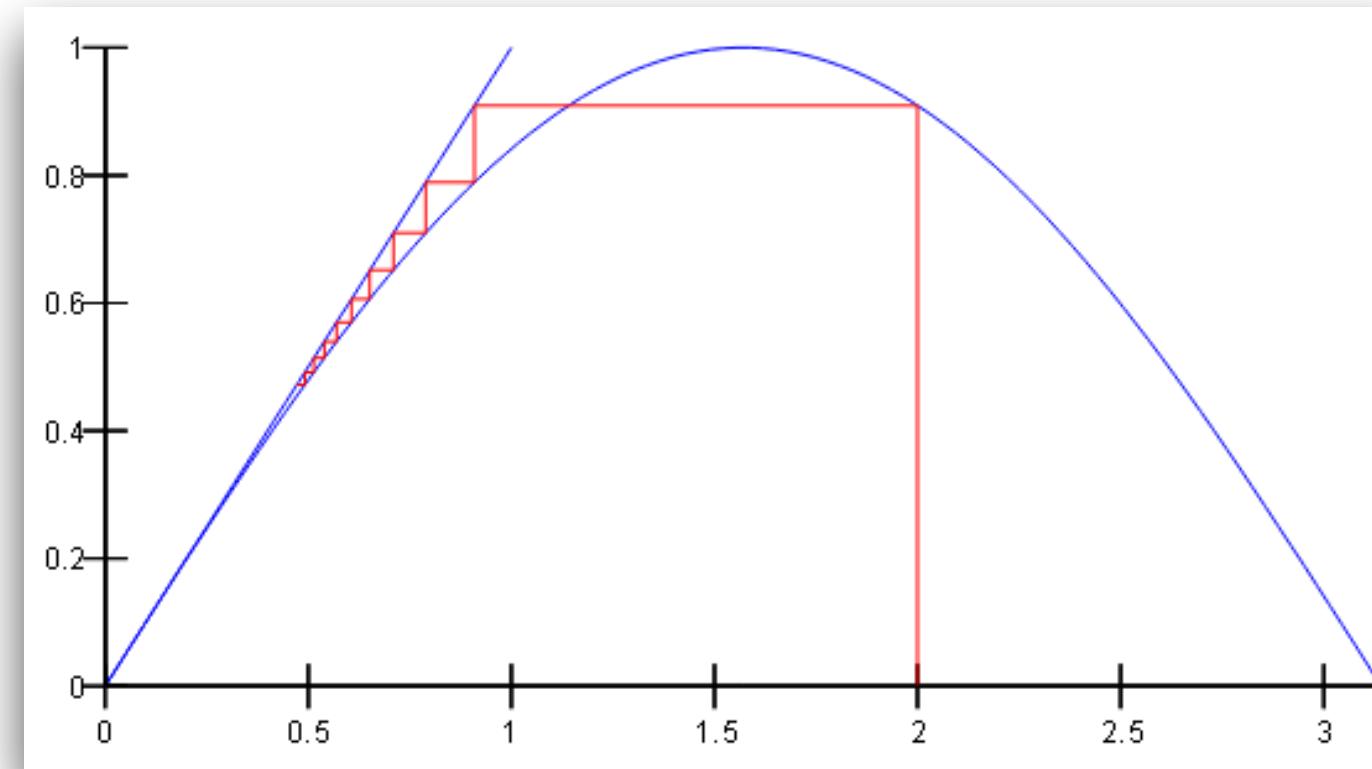
Theorem (Kleene Fixed Point). Let $f: D \rightarrow D$ be a continuous function on a CPO D . Then, f has the **least fixed point** $\text{lfp } f$ and

$$\text{lfp } f = \bigsqcup_{i \geq 0} f^i(\perp)$$

Fixed-point Iteration

- How to compute the fixed point?
- C.f., how to compute a fixed point of $x = \sin x$?*

```
x = bot;  
while (x != f(x)) {  
    x = x ∪ f(x);  
}  
return x;
```



*https://en.wikipedia.org/wiki/Fixed-point_iteration

Proof

$$\text{lfp } f = \bigsqcup_{i \geq 0} f^i(\perp)$$

- Plans: It is enough to show the following two things:

(1) There exists the chain $\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \dots$ and its least upper bound $\bigsqcup_{i \geq 0} f^i(\perp)$ in D .

(2) The least upper bound $\bigsqcup_{i \geq 0} f^i(\perp)$ is the least fixed point of f .

Proof of (1)

(1) There exists the chain $\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \dots$ and its least upper bound $\bigsqcup_{i \geq 0} f^i(\perp)$ in D .

Proof. We show by induction that $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq f^{n+1}(\perp)$:

- $\perp \sqsubseteq f(\perp)$ (\perp is the least element of the CPO)
- $f^n(\perp) \sqsubseteq f^{n+1}(\perp) \implies f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$ (by monotonicity of f)

By definition of CPO, least upper bounds of all chains are also in the CPO. Therefore, the least upper bound $\bigsqcup_{i \geq 0} f^i(\perp)$ of the above chain is in D .

Proof of (2)

(2) The least upper bound $\bigsqcup_{i \geq 0} f^i(\perp)$ is the least fixed point of f .

The proof consists of two parts:

(2-1) $\bigsqcup_{i \geq 0} f^i(\perp)$ is a fixed point of f

(2-2) $\bigsqcup_{i \geq 0} f^i(\perp)$ is smaller than all the other fixed points

Proof of (2-1)

(2-1) $\bigsqcup_{i \geq 0} f^i(\perp)$ is a fixed point of f .

Proof.

$$\begin{aligned} f\left(\bigsqcup_{n \geq 0} f^n(\perp)\right) &= \bigsqcup_{n \geq 0} f(f^n(\perp)) && (\text{by continuity of } f) \\ &= \bigsqcup_{n \geq 0} f^{n+1}(\perp) \\ &= \bigsqcup_{n \geq 0} f^n(\perp) \end{aligned}$$

Proof of (2-2)

(2-2) $\bigsqcup_{i \geq 0} f^i(\perp)$ is smaller than all the other fixed points.

Proof. Suppose d is a fixed point, i.e., $d = f(d)$. We show that any element $f^i(\perp)$ is smaller than d by induction:

$$\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq d.$$

- $\perp \sqsubseteq d$ (\perp is the least element of the CPO)
- $f^n(\perp) \sqsubseteq d \implies f^{n+1}(\perp) \sqsubseteq f(d) = d$ (by monotonicity of f)

Because all the elements $f^i(\perp)$ are smaller than d , their least upper bound

$\bigsqcup_{i \geq 0} f^i(\perp)$ is also smaller than d . Therefore

$$\bigsqcup_{i \geq 0} f^i(\perp) = \text{lfp } f$$

Constructions on CPOs

- If S is a set, and D_1 and D_2 are CPOs, then the followings are CPOs
 - Lifted set : $D = S_\perp$
 - Cartesian product : $D = D_1 \times D_2$
 - Separated sum : $D = D_1 + D_2$
 - Function : $D = D_1 \rightarrow D_2$



Lifted Set

- $D = S_{\perp}$

For any set S , let $D = S + \{\perp\}$ where \perp is an element not in S . Then (D, \sqsubseteq) is a CPO where

$$d \sqsubseteq d' \iff (d = d') \vee (d = \perp)$$

- Why CPO?

Cartesian Product

- $D = D_1 \times D_2$

Given two CPOs (D_1, \sqsubseteq_1) and (D_2, \sqsubseteq_2) , (D, \sqsubseteq) is a CPO where

$$D = D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$

$$(d_1, d_2) \sqsubseteq (d'_1, d'_2) \iff (d_1 \sqsubseteq_1 d'_1) \wedge (d_2 \sqsubseteq_2 d'_2)$$

- Why CPO?

Separated Sum

- $D = D_1 + D_2$

Given two CPOs (D_1, \sqsubseteq_1) and (D_2, \sqsubseteq_2) , (D, \sqsubseteq) is a CPO where

$$D = D_1 + D_2 = \{(d_1, 1) \mid d_1 \in D_1\} \cup \{(d_2, 2) \mid d_2 \in D_2\} \cup \{\perp\}$$

$$(d_1, 1) \sqsubseteq (d'_1, 1) \iff d_1 \sqsubseteq_1 d'_1$$

$$(d_2, 2) \sqsubseteq (d'_2, 2) \iff d_2 \sqsubseteq_2 d'_2$$

- Why CPO?

Function

- $D = D_1 \rightarrow D_2$

Given two CPOs (D_1, \sqsubseteq_1) and (D_2, \sqsubseteq_2) , (D, \sqsubseteq) is a CPO where

$$D = D_1 \rightarrow D_2 = \{f \mid f : D_1 \rightarrow D_2 \text{ is a continuous function}\}$$

$$f \sqsubseteq f' \iff \forall d_1 \in D_1. f(d_1) \sqsubseteq_2 f'(d_1)$$

- Why CPO?

Proof

- According to the definition $(f \sqcup f')(x) = f(x) \sqcup f'(x)$, so $(\bigsqcup f_i)(x) = \bigsqcup f_i(x)$
- We show that the LUB of chain $f_0 \sqsubseteq f_1 \sqsubseteq \dots$ exists in D .
- It is enough to show the LUB $\bigsqcup f_i$ is a continuous function.
- For all $x_0 \sqsubseteq x_1 \sqsubseteq \dots$ in D_1 ,

$$\begin{aligned} (\bigsqcup_{i \geq 0} f_i)(\bigsqcup_{j \geq 0} x_j) &= \bigsqcup_{i \geq 0} (f_i(\bigsqcup_{j \geq 0} x_j)) \quad (\text{by definition of } \bigsqcup f_i) \\ &= \bigsqcup_{i \geq 0} (\bigsqcup_{j \geq 0} f_i(x_j)) \quad (\text{by continuity of each } f_i) \\ &= \bigsqcup_{j \geq 0} (\bigsqcup_{i \geq 0} f_i(x_j)) \quad (\text{by interchange law}) \quad (\text{why? exercise!}) \\ &= \bigsqcup_{j \geq 0} (\bigsqcup_{i \geq 0} f_i)(x_j) \quad (\text{by definition of } \bigsqcup f_i) \end{aligned}$$

Semantic Domains (Revisited)

- Domains of memories, variables, and values: CPOs

$$\mathbb{M} = \mathbb{X} \rightarrow \mathbb{V}$$

$$\mathbb{X} = Var_{\perp}$$

$$\mathbb{V} = \mathbb{Z}_{\perp} + \mathbb{B}_{\perp}$$

- Domain of commands: function CPO

$$[C] \in \mathbb{M} \rightarrow \mathbb{M}$$

- Domain of expressions: function CPO

$$[E] \in \mathbb{M} \rightarrow \mathbb{Z}_{\perp}$$

$$[B] \in \mathbb{M} \rightarrow \mathbb{B}_{\perp}$$

Semantics of while (Revisited)

- Semantics of while is defined as the least fixed point as follows: (i.e., the lfp exists)
 - All the sets are CPOs
 - All the functions are continuous

$$\llbracket \text{while } B \text{ } C \rrbracket = \text{lfp} \mathcal{F}_{B,C}$$

where $\mathcal{F}_{B,C}(X)(m) = \begin{cases} X(\llbracket C \rrbracket(m)) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases}$

Example

- `while (x < 10) x := x + 1`

$$\llbracket \text{while } (x < 10) \ x := x + 1 \rrbracket = \lambda m. \begin{cases} \llbracket \text{while } (x < 10) \ x := x + 1 \rrbracket (\llbracket x := x + 1 \rrbracket(m)) & \text{if } \llbracket x < 10 \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket x < 10 \rrbracket(m) = \text{false} \end{cases}$$

$$\llbracket \text{while } (x < 10) \ x := x + 1 \rrbracket = \text{lfp } \mathcal{F} \text{ where } \mathcal{F}(X) = \lambda m. \begin{cases} X(\llbracket x := x + 1 \rrbracket(m)) & \text{if } \llbracket x < 10 \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket x < 10 \rrbracket(m) = \text{false} \end{cases}$$

$$\text{lfp } \mathcal{F} = \perp \sqcup \mathcal{F}(\perp) \sqcup \mathcal{F}^2(\perp) \sqcup \dots$$

Example

$$\mathcal{F}(X) = \lambda m. \begin{cases} X(\llbracket x := x + 1 \rrbracket(m)) & \text{if } \llbracket x < 10 \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket x < 10 \rrbracket(m) = \text{false} \end{cases}$$

\perp

0 iter

$$\mathcal{F}(\perp) = \lambda m. \begin{cases} \perp(\llbracket x := x + 1 \rrbracket(m)) & \text{if } \llbracket x < 10 \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket x < 10 \rrbracket(m) = \text{false} \end{cases}$$

0, 1 iter

$$\mathcal{F}^2(\perp) = \lambda m. \begin{cases} \mathcal{F}(\perp)(\llbracket x := x + 1 \rrbracket(m)) & \text{if } \llbracket x < 10 \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket x < 10 \rrbracket(m) = \text{false} \end{cases}$$

$$= \lambda m. \begin{cases} \begin{cases} \perp(\llbracket x := x + 1 \rrbracket^2(m)) & \text{if } \llbracket x < 10 \rrbracket(\llbracket x := x + 1 \rrbracket(m)) = \text{true} \\ \llbracket x := x + 1 \rrbracket(m) & \text{if } \llbracket x < 10 \rrbracket(\llbracket x := x + 1 \rrbracket(m)) = \text{false} \end{cases} & \text{if } \llbracket x < 10 \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket x < 10 \rrbracket(m) = \text{false} \end{cases}$$

0,1,2 iter

$$\mathcal{F}^3(\perp) = \dots$$

Semantics of Commands (Revisited)

$$\begin{aligned}\llbracket C \rrbracket &: \mathbb{M} \rightarrow \mathbb{M} \\ \llbracket \text{skip} \rrbracket &= \lambda m. m \\ \llbracket C_0 ; C_1 \rrbracket &= \lambda m. \llbracket C_1 \rrbracket(\llbracket C_0 \rrbracket(m)) \\ \llbracket x := E \rrbracket &= \lambda m. m\{x \mapsto \llbracket E \rrbracket(m)\} \\ \llbracket \text{input}(x) \rrbracket &= \lambda m. m\{x \mapsto n\} \\ \llbracket \text{if } B \text{ then } C_1 \text{ else } C_2 \rrbracket &= \lambda m. \begin{cases} \llbracket C_1 \rrbracket(m) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ \llbracket C_2 \rrbracket(m) & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases} \\ \llbracket \text{while } B \text{ } C \rrbracket &= \text{lfp} \lambda X. \left(\lambda m. \begin{cases} X(\llbracket C \rrbracket(m)) & \text{if } \llbracket B \rrbracket(m) = \text{true} \\ m & \text{if } \llbracket B \rrbracket(m) = \text{false} \end{cases} \right)\end{aligned}$$

Summary

- Denotational semantics describes mathematical meaning of programs
 - semantics of a program is an element of a **CPO**: $\llbracket P \rrbracket \in D$
 - semantics is the **least fixed point** of a **continuous** function: $\mathcal{F} \in D \rightarrow D$
 - **compositionally** defined by the semantics of subcomponents
 - the least fixed point is the **least upper bound** of the following chain:

$$\begin{aligned}\llbracket P \rrbracket &= \mathcal{F}(\llbracket P \rrbracket) \\ &= \text{lfp } \mathcal{F} \\ &= \bigsqcup_{i \geq 0} \mathcal{F}^i(\perp)\end{aligned}$$