Program Analysis

7. Abstract Interpretation (3): Widening and Narrowing

Kihong Heo

KAIST

Design of Static Analysis

- Goal: conservative and terminating static analysis
- Design principles:
  - Define concrete semantics
  - Define abstract semantics (sound w.r.t the concrete semantics)
- Computation & implementation:
  - Abstract semantics of a program: the least fixed point of the semantic function
  - Static analyzer: compute the least fixed point within finite time

Computing Abstract Semantics

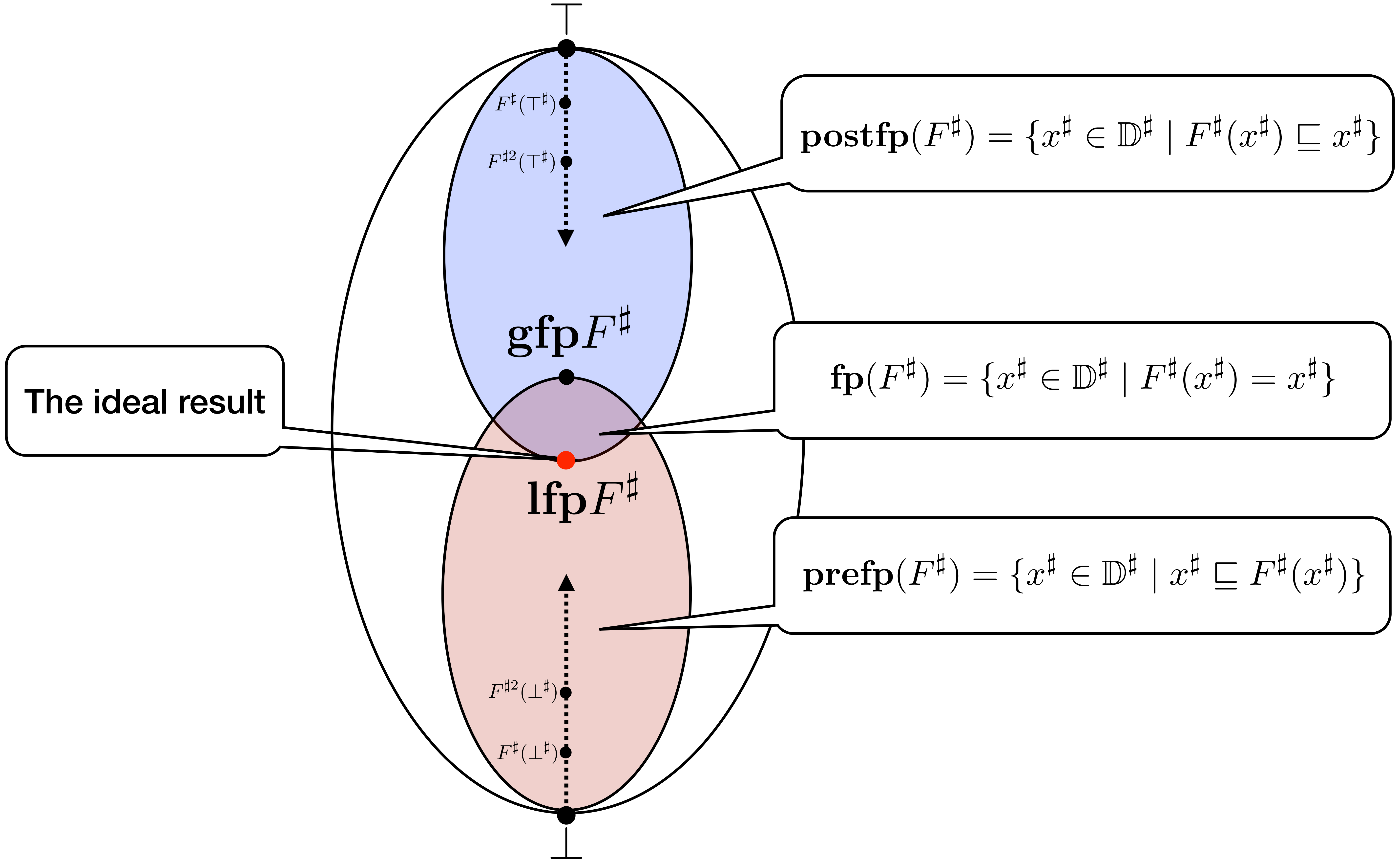
- If the abstract domain $\mathbb{D}^\sharp$ has **finite** height (i.e., all chains are finite)

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp)$$

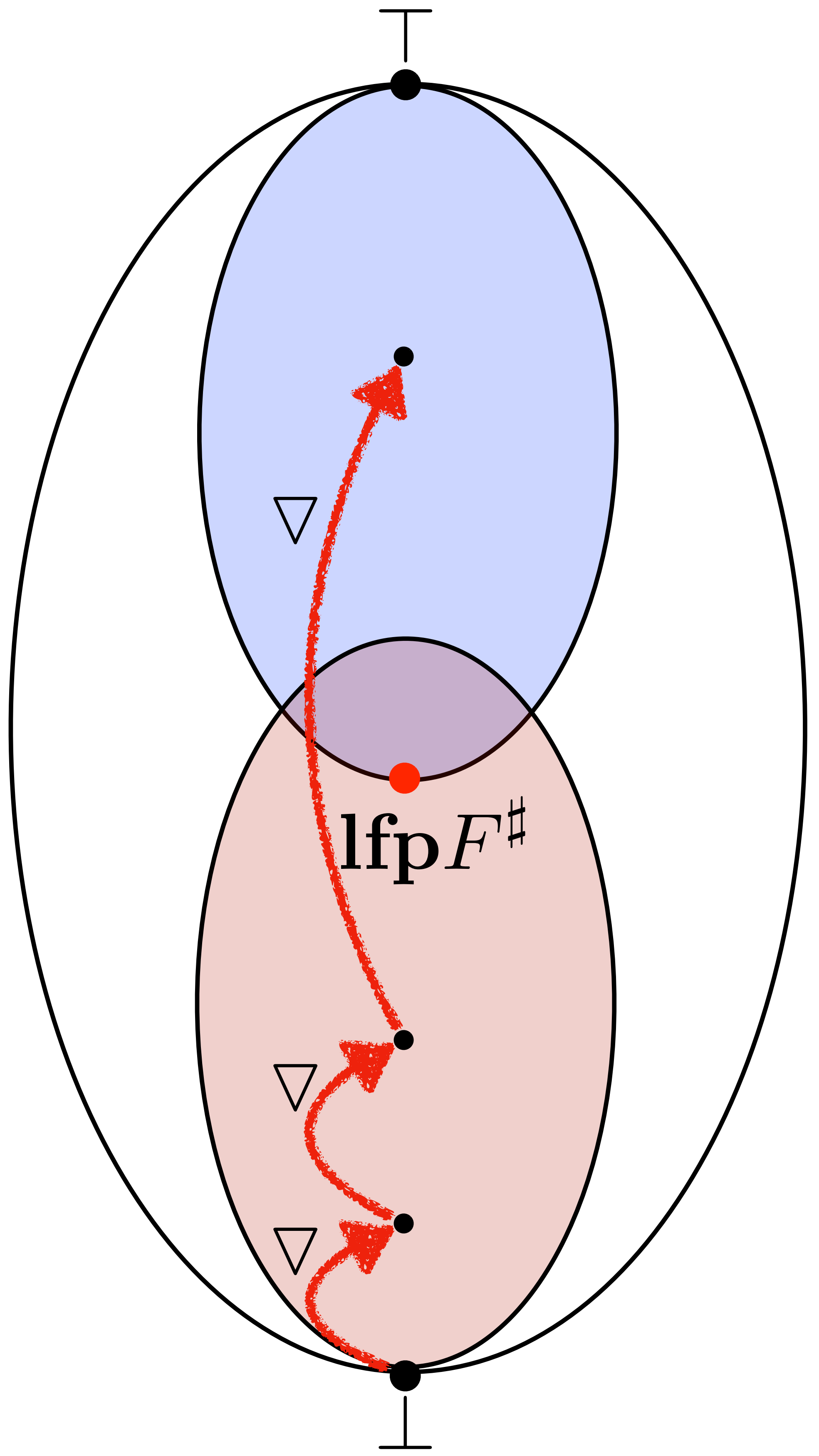
- If the abstract domain $\mathbb{D}^\sharp$ has **infinite** height, we compute a finite chain $X_0^\sharp \sqsubseteq X_1^\sharp \sqsubseteq X_2^\sharp \sqsubseteq \ldots \sqsubseteq X_{\lim}^\sharp$ such that

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp) \sqsubseteq X_{\lim}^\sharp$$

Fixed Points



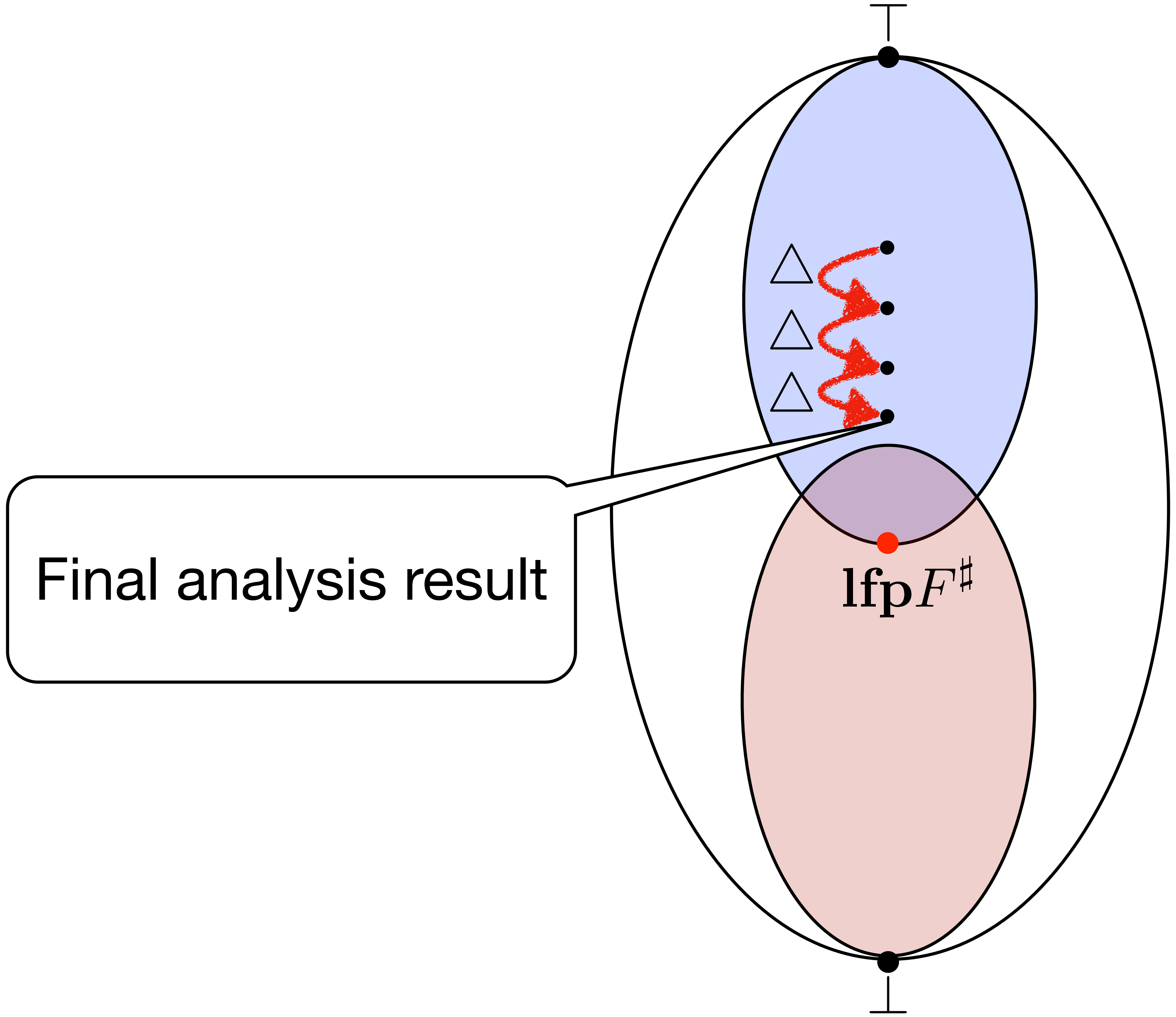
Widening



$$\nabla : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \to \mathbb{D}^\sharp$$

Widening: enforcing the convergence of fix point iterations

Narrowing



$$\triangle : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \to \mathbb{D}^\sharp$$

Narrowing: refining the analysis results with widening

Example

- What is the value of x at the end of this program?

```
x = 0;
while(*) {
  x++
}
```

- Recall:

$$[\![\texttt{while}\ B\ C]\!]^\sharp(m^\sharp) \;=\; [\![\neg B]\!]^\sharp\Big(\bigsqcup_{i \geq 0} F^{\sharp i}(\bot)\Big)$$

$$\text{where } F^\sharp(X) = m^\sharp \sqcup [\![C]\!]^\sharp \circ [\![B]\!]^\sharp(X)$$

- Computation: $[0,0] \sqcup [0,1] \sqcup [0,2] \sqcup \cdots$

Overshooting by Widening

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^{\sharp}) \sqsubseteq Y^{\sharp}_{\text{lim}}$$

- Define finite chain $\{Y_i\}_i$ by an widening operator $\triangledown \in \mathbb{D}^{\sharp} \times \mathbb{D}^{\sharp} \to \mathbb{D}^{\sharp}$:

$$\begin{array}{rcl} Y_0^{\sharp} & = & \bot^{\sharp} \\ Y_{i+1}^{\sharp} & = & \begin{cases} Y_i^{\sharp} & \text{if } F^{\sharp}(Y_i^{\sharp}) \sqsubseteq Y_i^{\sharp} \\ Y_i^{\sharp} \ \triangledown \ F^{\sharp}(Y_i^{\sharp}) & \text{otherwise} \end{cases} \end{array}$$

Finite Increasing Chain with Widening

$$\begin{array}{c} \vdots \\ | \\ F^{\sharp 2}(\bot^\sharp) \\ | \\ F^\sharp(\bot^\sharp) \\ | \\ \bot^\sharp \end{array}$$

Original Chain

$$\begin{array}{l} Y^\sharp_{N+1} = Y^\sharp_N \\ | \\ Y^\sharp_N = Y^\sharp_{N-1} \triangledown F^\sharp(Y^\sharp_{N-1}) \\ \vdots \\ Y^\sharp_2 = Y^\sharp_1 \triangledown F^\sharp(Y^\sharp_1) \\ | \\ Y^\sharp_1 = Y^\sharp_0 \triangledown F^\sharp(Y^\sharp_0) \\ | \\ Y^\sharp_0 = \bot^\sharp \end{array}$$

New Chain with Widening

Q. What conditions are required to ensure

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp) \sqsubseteq Y^\sharp_{\lim}$$

Example

- A simple widening operator for the interval domain

$$
\begin{array}{rcccl}
[a,b] & \triangledown & \bot & = & [a,b] \\
\bot & \triangledown & [c,d] & = & [c,d] \\
[a,b] & \triangledown & [c,d] & = & [(c<a?-\infty:a),(b<d?+\infty:b)]
\end{array}
$$

Check: Safety conditions for widening

- What is the value of x at the end of this program?

```
x = 0;
while(*) {
  x++
}
```

$$[0,0] \;\triangledown\; [0,1] = [0,+\infty]$$

$$\llbracket \texttt{while}\ B\ C \rrbracket^\sharp(m^\sharp) \;=\; \llbracket \neg B \rrbracket^\sharp\Big(\bigsqcup_{i\geq 0} F^{\sharp i}(\bot)\Big)$$

where $F^\sharp(X) = m^\sharp \sqcup \llbracket C \rrbracket^\sharp \circ \llbracket B \rrbracket^\sharp(X)$

Safety of Widening Operator

- Conditions on widening operator:
  - $\forall a, b \in \mathbb{D}^\sharp.\ (a \sqsubseteq a \ \triangledown \ b) \ \wedge \ (b \sqsubseteq a \ \triangledown \ b)$
  - $\forall$increasing chain $\{x_i\}_i$, the following increasing chain $\{y_i\}_i$ is finite:

$$y_0 = x_0$$
$$y_{i+1} = y_i \ \triangledown \ x_{i+1}$$

- Then,
  - Chain $\{Y_i^\sharp\}_i$ is finite
  - $\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp) \sqsubseteq Y^\sharp_{\lim}$

Proof (1)

Theorem (Widening's Safety). *Let $\mathbb{D}^\sharp$ be a CPO, $F^\sharp : \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a monotone function, and $\triangledown : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a widening operator. Then, chain $\{Y_i^\sharp\}_i$ eventually stabilizes and*

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Y^\sharp_{\lim}$$

where $Y^\sharp_{\lim}$ is the greatest element of the chain.

Proof.

1. We prove chain $\{Y_i^\sharp\}_i$ is finite. According to the second condition on widening operator, it is enough to show that chain $\{F^\sharp(Y_i^\sharp)\}_i$ is increasing. The chain is increasing because

   1) $Y^\sharp_{i+1}$ is either $Y_i^\sharp$ if $F^\sharp(Y_i^\sharp) \sqsubseteq Y_i^\sharp$ or $Y_i^\sharp \triangledown F^\sharp(Y_i^\sharp)$ if $F^\sharp(Y_i^\sharp) \sqsupset Y_i^\sharp$

   2) If $F^\sharp(Y_i^\sharp) \sqsubseteq Y_i^\sharp$ then $F^\sharp(Y_i^\sharp) = F^\sharp(Y^\sharp_{i+1})$. Therefore, $F^\sharp(Y_i^\sharp) \sqsubseteq F^\sharp(Y^\sharp_{i+1})$

   3) If $F^\sharp(Y_i) \sqsupset Y_i$ then $F^\sharp(Y^\sharp_{i+1}) = F^\sharp(Y_i^\sharp \triangledown F^\sharp(Y_i^\sharp))$

      3-1) $Y_i^\sharp \sqsubseteq Y_i^\sharp \triangledown F^\sharp(Y_i^\sharp)$ according to the first condition on widening, and

      3-2) $F^\sharp$ is monotone.

      3-3) Therefore, $F^\sharp(Y_i^\sharp) \sqsubseteq F^\sharp(Y^\sharp_{i+1})$

Proof (2)

Theorem (Widening's Safety). *Let $\mathbb{D}^\sharp$ be a CPO, $F^\sharp : \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a monotone function, and $\triangledown : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a widening operator. Then, chain $\{Y_i^\sharp\}_i$ eventually stabilizes and*

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp) \sqsubseteq Y^\sharp_{\lim}$$

where $Y^\sharp_{\lim}$ is the greatest element of the chain.

Proof (cont'd).

2. We prove $\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp) \sqsubseteq Y^\sharp_{\lim}$. It is enough to show that $\forall i \in \mathbb{N}.\ F^{\sharp i}(\bot^\sharp) \sqsubseteq Y_i^\sharp$ that can be proven by induction. The base case is trivial. The inductive case is as follows:

$$\begin{aligned} F^{\sharp i+1}(\bot^\sharp) &= F^\sharp(F^{\sharp i}(\bot^\sharp)) \\ &\sqsubseteq F^\sharp(Y_i^\sharp) \qquad \text{(by induction hypothesis and monotonicity of } F^\sharp\text{)} \end{aligned}$$

If $F^\sharp(Y_i^\sharp) \sqsubseteq Y_i^\sharp$, then $Y_{i+1}^\sharp = Y_i^\sharp$ by definition. Therefore, $F^{\sharp i+1}(\bot^\sharp) \sqsubseteq Y_{i+1}^\sharp$.

If $F^\sharp(Y_i^\sharp) \sqsupset Y_i^\sharp$, then $Y_{i+1}^\sharp = Y_i^\sharp \ \triangledown\ F^\sharp(Y_i^\sharp)$ by definition. According to the first condition on widening, $F^\sharp(Y_i^\sharp) \sqsubseteq Y_i^\sharp \ \triangledown\ F^\sharp(Y_i^\sharp)$. Therefore, $F^{\sharp i+1}(\bot^\sharp) \sqsubseteq Y_{i+1}^\sharp$.

Example (Revisited)

- A simple widening operator for the interval domain

$$\begin{array}{rcccl} [a,b] & \triangledown & \bot & = & [a,b] \\ \bot & \triangledown & [c,d] & = & [c,d] \\ [a,b] & \triangledown & [c,d] & = & [(c<a?-\infty:a),(b<d?+\infty:b)] \end{array}$$

Check: Safety conditions for widening

- What is the value of x at the end of this program?

```
x = 0;
while(*) {
  x++
}
```

$$[0,0] \ \triangledown \ [0,1] = [0,+\infty]$$

- How about this?

```
x = 0;
while(x < 10) {
  x++
}
```

$$[\![x \geq 10]\!]([0,0] \ \triangledown \ [0,1] = [0,+\infty]) = [10,+\infty]$$

Good enough?

Refinement by Narrowing

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Z^\sharp_{\text{lim}}$$

- Define finite chain $\{Z^\sharp_i\}_i$ by an narrowing operator $\triangle \in \mathbb{D}^\sharp \times \mathbb{D}^\sharp \rightarrow \mathbb{D}^\sharp$:

$$\begin{array}{rcl} Z^\sharp_0 & = & Y^\sharp_{\text{lim}} \\ Z^\sharp_{i+1} & = & Z^\sharp_i \ \triangle \ F^\sharp(Z^\sharp_i) \end{array}$$

Finite Decreasing Chain with Narrowing

$$
\begin{array}{c}
Y^\sharp_{\text{lim}} \\
| \\
F^\sharp(Y^\sharp_{\text{lim}}) \\
| \\
F^{\sharp 2}(Y^\sharp_{\text{lim}}) \\
\vdots
\end{array}
$$

Original Chain

$$
\begin{array}{l}
Z^\sharp_0 = Y^\sharp_{\text{lim}} \\
| \\
Z^\sharp_1 = Z^\sharp_0 \mathbin{\triangle} F^\sharp(Z^\sharp_0) \\
| \\
Z^\sharp_2 = Z^\sharp_1 \mathbin{\triangle} F^\sharp(Z^\sharp_1) \\
\vdots \\
Z^\sharp_N = Z^\sharp_{N-1} \mathbin{\triangle} F^\sharp(Z^\sharp_{N-1}) \\
| \\
Z^\sharp_{N+1} = Z^\sharp_N
\end{array}
$$

New Chain with Narrowing

Q. What conditions are required to ensure

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Z^\sharp_{\text{lim}}$$

Example

- A simple narrowing operator for the interval domain

$$\begin{array}{rcccl} [a,b] & \triangle & \bot & = & \bot \\ \bot & \triangle & [c,d] & = & \bot \\ [a,b] & \triangle & [c,d] & = & [(a = -\infty ? c : a), (b = +\infty ? d : b)] \end{array}$$

Check: Safety conditions for narrowing

- What is the value of x at the end of this program?

```
x = 0;
while(x < 10) {
  x++
}
```

$$[\![\texttt{while}\ B\ C]\!]^\sharp(m^\sharp) = [\![\neg B]\!]^\sharp\Big(\bigsqcup_{i \geq 0} F^{\sharp i}(\bot)\Big)$$

where $F^\sharp(X) = m^\sharp \sqcup [\![C]\!]^\sharp \circ [\![B]\!]^\sharp(X)$

(Widening) $[0,0] \ \triangledown\ [0,1] = [0,+\infty]$

(Narrowing) $[0,+\infty] \triangle [0,10] = [0,10]$

(Termination condition) $[\![x \geq 10]\!]([0,10]) = [10,10]$

Good enough!

Safety of Narrowing Operator

- Conditions on narrowing operator:
  - $\forall a, b \in \mathbb{D}^\sharp.\ a \sqsupseteq b \implies a \sqsupseteq (a \bigtriangleup b) \sqsupseteq b$
  - For all decreasing chain $\{y_i\}_i$, the following decreasing chain $\{z_i\}_i$ is finite

$$z_0 = y_0$$
$$z_{i+1} = z_i \bigtriangleup y_{i+1}$$

- Then,
  - Decreasing chain $\{Z_i^\sharp\}_i$ is finite
  - $\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp) \sqsubseteq Z_{\lim}$

Proof (1)

Theorem (Narrowing's Safety). *Let $\mathbb{D}^\sharp$ be a CPO, $F^\sharp : \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a monotone function, and $\triangle : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a narrowing operator. Then, chain $\{Z_i^\sharp\}_i$ eventually stabilizes and*

$$\bigsqcup_{i \geq 0} F^{\sharp i}(\bot^\sharp) \sqsubseteq Z^\sharp_{\lim}$$

where $Z^\sharp_{\lim}$ is the least element of the chain.

Proof.

1. We prove chain $\{Z_i^\sharp\}_i$ is finite. According to the condition on narrowing operator, it is enough to show that chain $\{F^\sharp(Z_i^\sharp)\}_i$ is decreasing. The chain is decreasing if $\forall i \in \mathbb{N}.\ Z_i^\sharp \sqsupseteq F^\sharp(Z_i^\sharp)$ because

   1) $Z_i^\sharp \sqsupseteq Z_i^\sharp \triangle F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_i^\sharp)$ by the first condition on narrowing, and

   2) $F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_i^\sharp \triangle F^\sharp(Z_i^\sharp)) = F^\sharp(Z^\sharp_{i+1})$ by monotonicity of $F^\sharp$.

We prove $\forall i \in \mathbb{N}.\ Z_i^\sharp \sqsupseteq F^\sharp(Z_i^\sharp)$ by induction. The base case is true by definition of $Y^\sharp_{\lim}$ from the increasing chain by widening. The inductive case is as follows:

$$\begin{array}{llr}
 & Z_i^\sharp \sqsupseteq F^\sharp(Z_i^\sharp) & \text{(by induction hypothesis)} \\
\Longrightarrow & Z_i^\sharp \sqsupseteq Z_i^\sharp \triangle F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_i^\sharp) & \text{(by the first condition on narrowing)} \\
\Longrightarrow & Z_i^\sharp \sqsupseteq Z_{i+1}^\sharp \sqsupseteq F^\sharp(Z_i^\sharp) & \text{(by definition)} \\
\Longrightarrow & F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_{i+1}^\sharp) & \text{(by monotonicity of } F^\sharp \text{ and } Z_i^\sharp \sqsupseteq Z_{i+1}^\sharp) \\
\Longrightarrow & Z_{i+1}^\sharp \sqsupseteq F^\sharp(Z_i^\sharp) \sqsupseteq F^\sharp(Z_{i+1}^\sharp) &
\end{array}$$

Proof (2)

> **Theorem** (Narrowing's Safety). *Let $\mathbb{D}^\sharp$ be a CPO, $F^\sharp : \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a monotone function, and $\triangle : \mathbb{D}^\sharp \times \mathbb{D}^\sharp \to \mathbb{D}^\sharp$ be a narrowing operator. Then, chain $\{Z_i^\sharp\}_i$ eventually stabilizes and*
>
> $$\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Z_{\lim}^\sharp$$
>
> *where $Z_{\lim}^\sharp$ is the least element of the chain.*

Proof (cont'd).

2. we prove $\bigsqcup_{i \geq 0} F^{\sharp i}(\perp^\sharp) \sqsubseteq Z_{\lim}^\sharp$. It is enough to show that $\forall i \in \mathbb{N}.\ F^{\sharp i}(\perp^\sharp) \sqsubseteq Z_i^\sharp$ that can be proven by induction. The base case is trivial. The inductive case is as follows:

$$\begin{aligned}
F^{\sharp i+1}(\perp^\sharp) &= F^\sharp \circ F^{\sharp i}(\perp^\sharp) \\
&\sqsubseteq F^\sharp(Z_i^\sharp) && \text{(by induction hypothesis and monotonicity of } F^\sharp) \\
&\sqsubseteq Z_i^\sharp \triangle F^\sharp(Z_i^\sharp) && (\text{by condition } \forall i \in \mathbb{N}.\ Z_i^\sharp \sqsupseteq F^\sharp(Z_i^\sharp)) \\
&= Z_{i+1}^\sharp && \text{(by definition)}
\end{aligned}$$

Computable Abstract Semantics

$$
\begin{array}{rcl}
[\![C]\!]^\sharp & : & \mathbb{M}^\sharp \rightarrow \mathbb{M}^\sharp \\
\\
[\![\texttt{skip}]\!]^\sharp & = & \lambda m^\sharp . m^\sharp \\
[\![C_0\,;C_1]\!]^\sharp & = & \lambda m^\sharp . [\![C_1]\!]^\sharp \circ [\![C_0]\!]^\sharp(m^\sharp) \\
[\![x := E]\!]^\sharp & = & \lambda m^\sharp . m^\sharp\{x \mapsto [\![E]\!]^\sharp(m^\sharp)\} \\
[\![\texttt{input}(x)]\!]^\sharp & = & \lambda m^\sharp . m^\sharp\{x \mapsto \alpha(\mathbb{Z})\} \\
[\![\texttt{if}\ B\ \texttt{then}\ C_1\ \texttt{else}\ C_2]\!]^\sharp & = & \lambda m^\sharp . [\![C_1]\!]^\sharp \circ [\![B]\!]^\sharp(m^\sharp) \sqcup [\![C_2]\!]^\sharp \circ [\![\neg B]\!]^\sharp(m^\sharp) \\
[\![\texttt{while}\ B\ C]\!]^\sharp & = & \lambda m^\sharp . [\![\neg B]\!]^\sharp \circ \mathit{Narrow} \circ \mathit{Widen} \circ \left(\lambda X . m^\sharp \sqcup [\![C]\!]^\sharp \circ [\![B]\!]^\sharp(X)\right)
\end{array}
$$

$$
\mathit{Widen}(F^\sharp) = \lim_{i \in \mathbb{N}} \begin{cases} Y_0^\sharp = \bot \\ Y_{i+1}^\sharp = \begin{cases} Y_i^\sharp & \text{if } F^\sharp(Y_i^\sharp) \sqsubseteq Y_i^\sharp \\ Y_i^\sharp \mathbin{\triangledown} F^\sharp(Y_i^\sharp) & \text{o.w.} \end{cases} \end{cases}
\qquad
\mathit{Narrow}(Y_{\lim}^\sharp) = \lim_{i \in \mathbb{N}} \begin{cases} Z_0^\sharp = Y_{\lim}^\sharp \\ Z_{i+1}^\sharp = Z_i^\sharp \mathbin{\vartriangle} F^\sharp(Z_i^\sharp) \end{cases}
$$

Summary

- **Computing** a sound approximation of the concrete semantics
  - Finite height: directly compute the fixed point by iteration
  - Infinite height: fixpoint iteration with widening and narrowing
- Widening: **termination** guarantee
- Narrowing: **refinement** of widening results