Method to solve equations for atmospheric gravity waves with very small vertical-to-horizontal wavelength ratios

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Equations for GWs with very small aspect ratios

- $W_p = \delta^{3/2} U_p = O(10^{-2})$ or $W_p = \delta^2 U_p = O(10^{-3})$ for $U_p = O(10)$ and $\delta = O(10^{-2})$

$$\frac{\bar{D}_h u'}{Dt} + u'\frac{\partial \bar{u}}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} - fv' + \frac{\partial}{\partial x}\left(\frac{p'}{\bar{\rho}}\right) = \frac{\rho'}{\bar{\rho}}\left(\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}\right),$$
$$\frac{\bar{D}_h v'}{Dt} + u'\frac{\partial \bar{v}}{\partial x} + v'\frac{\partial \bar{v}}{\partial y} + fu' + \frac{\partial}{\partial y}\left(\frac{p'}{\bar{\rho}}\right) = \frac{\rho'}{\bar{\rho}}\left(\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y}\right),$$
$$\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) - g\frac{\theta'}{\bar{\theta}} = 0,$$
$$\frac{\bar{D}_h}{Dt}\left(\frac{p'}{\bar{\rho}c_s^2}\right) + \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) = 0,$$
$$\frac{\bar{D}_h}{Dt}\left(\frac{\theta'}{\bar{\theta}}\right) + \frac{N^2}{g}w' = 0.$$

- Energy equation with an invariant form with respect to approximation

$$\frac{\bar{D}_h}{Dt}\left(E_k + E_p + E_e\right) = -\boldsymbol{\nabla}\cdot\left(p'\boldsymbol{v}'\right) + T_{hs} + T_{hp},$$

where $E_k = (1/2)\bar{\rho}\left(u'^2 + v'^2\right)$, $E_p = (1/2)(g^2/N^2)(\theta'/\bar{\theta})^2$, $E_e = (1/2)p'^2/(\bar{\rho}c_s^2)$, $T_{hs} = -\bar{\rho}u'u'\partial\bar{u}/\partial x$ - $\bar{\rho}u'v'\partial\bar{u}/\partial y$ - $\bar{\rho}u'v'\partial\bar{v}/\partial x$ - $\bar{\rho}v'v'\partial\bar{v}/\partial y$, and $T_{hp} = (\rho'u')/\bar{\rho}\partial\bar{p}/\partial x + (\rho'v')/\bar{\rho}\partial\bar{p}/\partial y$.

Large-scale flow - I

- Isothermal atmosphere
  - N and c_s are constants.
- Steady jet flows

$$
\begin{aligned}
\frac{\bar{D}_h}{Dt} &= \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y} \\
f_0 \bar{v} &= \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} \\
f_0 \bar{u} &= -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y} \\
\bar{p} &= \bar{p}(x, y, z)
\end{aligned}
$$

Large-scale flow - II

- $f_0 \partial \bar{u}/\partial z = (g/\bar{\rho})\partial\bar{\rho}/\partial y \neq 0$, but $\bar{\rho}$ is considered as a constant ($\bar{\rho}_0$) in the horizontal pressure gradient term since the horizontal variation of $\bar{\rho}$ is found to be two order of magnitude smaller than the horizontal variation of wave phases in the scale analysis [$\alpha_1^{-1} = O(10)$ and $\varepsilon = O(10^{-1})$]:

$$\frac{\partial}{\partial x}\left(\frac{p'}{\bar{\rho}}\right) = \underset{O(\alpha_1^{-1})}{\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}} - \underset{O(\varepsilon)}{\frac{p'}{\bar{\rho}^2}\frac{\partial \bar{\rho}}{\partial x}} \approx \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} \equiv \frac{\partial}{\partial x}\left(\frac{p'}{\bar{\rho}_0}\right)$$

For consistency, we let

$$-\frac{\rho'}{\bar{\rho}_0} = \frac{\theta'}{\bar{\theta}_0} - \frac{p'}{\bar{\rho}_0 c_s^2}$$

- The vertical variation of $\bar{\rho}$ is ignored in the vertical pressure gradient term as a result of the scale analysis:

$$\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) - \frac{p'}{\bar{\rho}H} \approx \frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) \equiv \frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}_0}\right) = \frac{1}{\bar{\rho}_0}\frac{\partial p'}{\partial z}$$

Large-scale flow - III

- Scale analysis indicates that the horizontal variations of the large-scale flow is one order of magnitude smaller than those of wave phases [$\alpha_1^{-1} = O(10)$]:

$$\frac{\partial}{\partial y}\left(\bar{u}\frac{\partial v'}{\partial x}\right) = \underset{O(\alpha_1^{-1})}{\frac{\partial \bar{u}}{\partial y}\frac{\partial v'}{\partial x}} + \underset{O(\alpha_1^{-2})}{\bar{u}\frac{\partial^2 v'}{\partial x \partial y}} \approx \bar{u}\frac{\partial^2 v'}{\partial x \partial y}$$

- The terms such as $v'\partial\bar{u}/\partial y$ in the momentum equations will be retained to see wave responses to initial horizontal wind perturbations, but their horizontal and vertical gradients are approximated by

$$\frac{\partial}{\partial x}\left(v'\frac{\partial \bar{u}}{\partial y}\right) = \frac{\partial v'}{\partial x}\frac{\partial \bar{u}}{\partial y} + v'\frac{\partial^2 \bar{v}}{\partial x \partial y} \approx \frac{\partial v'}{\partial x}\frac{\partial \bar{u}}{\partial y}$$

$$\frac{\partial}{\partial z}\left(v'\frac{\partial \bar{u}}{\partial y}\right) = \frac{\partial v'}{\partial z}\frac{\partial \bar{u}}{\partial y} + v'\frac{\partial^2 \bar{v}}{\partial z \partial y} \approx \frac{\partial v'}{\partial z}\frac{\partial \bar{u}}{\partial y}$$

Governing equations

- Perturbation equations on the f-plane.

$$\frac{\bar{D}_h u'}{Dt} + u'\frac{\partial \bar{u}}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} - f_0 v' + \frac{\partial \pi'}{\partial x} = \frac{d'}{\bar{\rho}_0}\frac{\partial \bar{p}}{\partial x}, \tag{1}$$

$$\frac{\bar{D}_h v'}{Dt} + u'\frac{\partial \bar{v}}{\partial x} + v'\frac{\partial \bar{v}}{\partial y} + f_0 u' + \frac{\partial \pi'}{\partial y} = \frac{d'}{\bar{\rho}_0}\frac{\partial \bar{p}}{\partial y}, \tag{2}$$

$$\frac{\partial \pi'}{\partial z} = b', \tag{3}$$

$$\frac{1}{c_s^2}\frac{\bar{D}_h \pi'}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{v}' = 0, \tag{4}$$

$$\frac{\bar{D}_h b'}{Dt} + N^2 w' = 0. \tag{5}$$

- $\pi' = p'/\bar{\rho}_0$, $b' = g\theta'/\bar{\theta}_0$, and $d' = \rho'/\bar{\rho}_0$.

Mathematical manipulations - I

- $\partial/\partial x$ (1) + $\partial/\partial y$ (2) + $\partial/\partial z$ (3):

$$\frac{\bar{D}_h}{Dt}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)+\nabla^2\pi'-\frac{\partial b'}{\partial z}\approx R \tag{6}$$

$$R=f_0\zeta'-\left(\frac{\partial u'}{\partial x}\frac{\partial \bar{u}}{\partial x}+\frac{\partial v'}{\partial x}\frac{\partial \bar{u}}{\partial y}+\frac{\partial u'}{\partial y}\frac{\partial \bar{v}}{\partial x}+\frac{\partial v'}{\partial y}\frac{\partial \bar{v}}{\partial y}\right)+\frac{1}{\bar{\rho}_0}\left(\frac{\partial d'}{\partial x}\frac{\partial \bar{p}}{\partial x}+\frac{\partial d'}{\partial y}\frac{\partial \bar{p}}{\partial y}\right) \tag{7}$$

  where ζ' is the perturbation vorticity given by $\partial v'/\partial x - \partial u'/\partial y$.

- (4) $\Longrightarrow$ (7):

$$-\frac{\bar{D}_h}{Dt}\left(\frac{\partial w'}{\partial z}\right)-\frac{1}{c_s^2}\frac{\bar{D}_h^2\pi'}{Dt^2}+\nabla^2\pi'-\frac{\partial b'}{\partial z}\approx R \tag{8}$$

- $\partial/\partial z(\bar{D}_h/Dt)$ (7):

$$\frac{\bar{D}_h^2}{Dt^2}\left(\frac{\partial^2 w'}{\partial z^2}-\frac{N^2}{c_s^2}w'\right)+N^2\nabla_H^2 w'\approx-\frac{\partial}{\partial z}\frac{\bar{D}_h R}{Dt}\approx-\frac{\bar{D}_h}{Dt}\frac{\partial R}{\partial z} \tag{9}$$

Mathematical manipulation - II

- Approximation of RHS of (9)

$$
\begin{aligned}
-\frac{\bar{D}_h}{Dt}\frac{\partial R}{\partial z} \approx -f_0\frac{\bar{D}_h}{Dt}\frac{\partial \zeta'}{\partial z} & \\
+\frac{\bar{D}_h}{Dt}\left(\frac{\partial^2 u'}{\partial x \partial z}\frac{\partial \bar{u}}{\partial x}+\frac{\partial^2 v'}{\partial x \partial z}\frac{\partial \bar{u}}{\partial y}\right) & \\
+\frac{\bar{D}_h}{Dt}\left(\frac{\partial^2 u'}{\partial y \partial z}\frac{\partial \bar{v}}{\partial x}+\frac{\partial^2 v'}{\partial y \partial z}\frac{\partial \bar{v}}{\partial y}\right) & \\
-\frac{1}{\bar{\rho}_0}\frac{\bar{D}_h}{Dt}\left(\frac{\partial^2 d'}{\partial x \partial z}\frac{\partial \bar{p}}{\partial x}+\frac{\partial^2 d'}{\partial y \partial z}\frac{\partial \bar{p}}{\partial y}\right) = RHS & \qquad (10)
\end{aligned}
$$

- For zonal jet flow,

$$
\bar{v}=\frac{1}{f_0}\frac{\partial \bar{p}}{\partial x}=0, \quad \frac{\partial \bar{u}}{\partial x}=-\frac{1}{f_0}\frac{\partial^2 \bar{p}}{\partial x \partial y}=0, \quad \frac{\bar{D}_h}{Dt}=\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}
$$

$$
\therefore\ RHS=-f_0\frac{\bar{D}_h}{Dt}\frac{\partial \zeta'}{\partial z}+\frac{\bar{D}_h}{Dt}\left(\frac{\partial^2 v'}{\partial x \partial z}\frac{\partial \bar{u}}{\partial y}\right)-\frac{1}{\bar{\rho}_0}\frac{\bar{D}_h}{Dt}\left(\frac{\partial^2 d'}{\partial y \partial z}\frac{\partial \bar{p}}{\partial y}\right) \qquad (11)
$$

Fourier transform

- Fourier transform in the horizontal direction:

$$FT(\mathcal{L}(w')) = FT(RHS) \tag{12}$$

where $FT(\psi') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi' e^{i(kx+ly)} dxdy$, and $FT(\partial^n \psi'/\partial x^n) = (-ik)^n FT(\psi')$. This FT is equivalent to setting $w'(x, y, z, t)$ equal to $\hat{w}(z, t)e^{-i(kx+ly)}$.

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial t^2} - 2i(k\bar{u} + l\bar{v})\frac{\partial}{\partial t} - (k\bar{u} + l\bar{v})^2\right]\left(\frac{\partial^2 \hat{w}}{\partial z^2} - \frac{N^2}{c_s^2}\hat{w}\right) - N^2(k^2 + l^2)\hat{w} \\
&= -\left[\frac{\partial}{\partial t} - i(k\bar{u} + l\bar{v})\right]\left[f_0\frac{\partial \hat{\zeta}}{\partial z} + ik\left(\frac{\partial \bar{u}}{\partial x}\frac{\partial \hat{u}}{\partial z} + \frac{\partial \bar{u}}{\partial y}\frac{\partial \hat{v}}{\partial z}\right)\right. \\
&\qquad \left. + il\left(\frac{\partial \bar{v}}{\partial x}\frac{\partial \hat{u}}{\partial z} + \frac{\partial \bar{v}}{\partial y}\frac{\partial \hat{v}}{\partial z}\right) - \frac{ik}{\bar{\rho}_0}\frac{\partial \bar{p}}{\partial x}\frac{\partial \hat{d}}{\partial z} - \frac{il}{\bar{\rho}_0}\frac{\partial \bar{p}}{\partial y}\frac{\partial \hat{d}}{\partial z}\right], \\
&= -i\left[\frac{\partial}{\partial t} - i(k\bar{u} + l\bar{v})\right]\left[X\frac{\partial \hat{u}}{\partial z} + Y\frac{\partial \hat{v}}{\partial z} - Z\frac{\partial \hat{d}}{\partial z}\right]
\end{aligned} \tag{13}$$

where $X = lf_0 + k\dfrac{\partial \bar{u}}{\partial x} + l\dfrac{\partial \bar{v}}{\partial x}$, $Y = -kf_0 + k\dfrac{\partial \bar{u}}{\partial y} + l\dfrac{\partial \bar{v}}{\partial y}$, and $Z = \dfrac{k}{\bar{\rho}_0}\dfrac{\partial \bar{p}}{\partial x} + \dfrac{l}{\bar{\rho}_0}\dfrac{\partial \bar{p}}{\partial y}$.

Laplace transform of the Fourier transform

- Laplace transform in the time domain:

$$LT(FT(\mathcal{L}(w'))) = LT(FT(RHS)) \tag{14}$$

where $LT(\psi') = \int_0^\infty e^{-st}\psi' dt$ (where $\mathrm{Re}(s) > 0$) and $LT(\partial\psi'/\partial t) = -\psi'|_{t=0} + s_r LT(\psi')$.

$$\begin{aligned}(s - ik_h\bar{u}_h)^2 \left(\frac{\partial^2 \tilde{w}}{\partial z^2} - \frac{N^2}{c_s^2}\tilde{w}\right) - N^2(k^2 + l^2)\tilde{w} \\ = -i\,(s - ik_h\bar{u}_h)\left(X\frac{\partial \tilde{u}}{\partial z} + Y\frac{\partial \tilde{v}}{\partial z} - Z\frac{\partial \tilde{d}}{\partial z}\right) \\ + iX\left.\frac{\partial \hat{u}}{\partial z}\right|_{t=0} + iY\left.\frac{\partial \hat{v}}{\partial z}\right|_{t=0} - iZ\left.\frac{\partial \hat{d}}{\partial z}\right|_{t=0}.\end{aligned} \tag{15}$$

where $k_h\bar{u}_h \equiv k\bar{u} + l\bar{v}$, $\left.\partial\hat{w}/\partial t\right|_{t=0} = 0$ and $\left.\hat{w}\right|_{t=0} = 0$ are used.

Differential equation and possible experiments

- Transformed differential equation

$$(s - ik_h\bar{u}_h)^2 \left(\frac{\partial^2 \tilde{w}}{\partial z^2} - \frac{N^2}{c_s^2}\tilde{w} \right) - N^2 k_h^2 \tilde{w} =$$
$$-i(s - ik_h\bar{u}_h) \left(X\frac{\partial \tilde{u}}{\partial z} + Y\frac{\partial \tilde{v}}{\partial z} - Z\frac{\partial \tilde{d}}{\partial z} \right) + iX\frac{\partial \hat{u}_0}{\partial z} + iY\frac{\partial \hat{v}_0}{\partial z} - iZ\frac{\partial \hat{d}_0}{\partial z}, \quad (16)$$

  where $k_h^2 = k^2 + l^2$, and $\hat{u}_0$, $\hat{v}_0$, and $\hat{d}_0$ are Fourier transforms of initial u', v', and $\rho'/\bar{\rho}_0$, respectively; $\tilde{u}$, $\tilde{v}$, and $\tilde{d}$ are Laplace transforms of $\hat{u}$, $\hat{v}$, and $\hat{d}$.
- X, Y, and Z

$$X = lf_0 + k\frac{\partial \bar{u}}{\partial x} + l\frac{\partial \bar{v}}{\partial x}, \quad Y = -kf_0 + k\frac{\partial \bar{u}}{\partial y} + l\frac{\partial \bar{v}}{\partial y}, \quad Z = \frac{k}{\bar{\rho}_0}\frac{\partial \bar{p}}{\partial x} + \frac{l}{\bar{\rho}_0}\frac{\partial \bar{p}}{\partial y}. \quad (17)$$

- Three experiments are possible.
  - Case 1: Initially, $\rho'/\bar{\rho}_0$ with nonzero vertical derivative is only given.
  - Case 2: Initially, $\boldsymbol{v}'$ with nonzero vertical derivative is only given.
  - Case 3: Both $\rho'/\bar{\rho}_0$ and $\boldsymbol{v}'$ are initially given.

Differential equation in the vertical - I

- For given s, k_h, and $\bar{u}_h$,

$$\frac{d^2\tilde{w}}{dz^2} - a\tilde{w} - \frac{b}{(s - ik_h\bar{u}_h)^2}\tilde{w} = -\frac{\tilde{F}}{s - ik_h\bar{u}_h} + \frac{\hat{F}_0}{(s - ik_h\bar{u}_h)^2} \tag{18}$$

  where $a = N^2/c_s^2$ and $b = N^2 k_h^2$, which can be assumed to be constants.

- This 2nd-order differential equation be converted into a system of two 1st-order differential equations:

$$\frac{d\xi}{dz} = a\psi + \frac{b}{(s - ik_h\bar{u}_h)^2}\psi - \frac{\tilde{F}}{s - ik_h\bar{u}_h} + \frac{\hat{F}_0}{(s - ik_h\bar{u}_h)^2}, \tag{19}$$

$$\frac{d\psi}{dz} = \xi \tag{20}$$

  where $\xi = 0$ and $\psi = 0$ at $z = \pm z_t$ for rigid-lid boundary conditions.

- This conversion is done because most of numerical algorithms for two-points boundary value problems are implemented such that they solve a system of the 1st-order equations rather than handle directly the 2nd-order equations.

Differential equation in the vertical - II

- Discrete values of k and l can be given in a way that fast Fourier transform algorithms assume.
  - For example, $k = 0, \dfrac{1}{N\Delta x}, \dfrac{2}{N\Delta x}, ..., \dfrac{N/2}{N\Delta x}, -\dfrac{N/2}{N\Delta x}, ... -\dfrac{1}{N\Delta x}$.
  - The angular wavenumber (with the factor of 2π) is used in the mathematical formulation of the Fourier transform, but nonangular wavenumber can often be used in the FFT algorithms. This difference should be considered in actual computation.
- If $\bar{u}$ and $\bar{v}$ are assumed to be constants, $\bar{u}_h$ can be defined based on values of k and l given above.
- Discrete values of s should be defined in advance.
  - Specification of s depends on how the inverse Laplace transform will be numerically computed.
  - Numerical inverse Laplace transform is achieved by computing numerically the Bromwich integral given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{st}F(s)ds \tag{21}$$

where c is a positive real constant, $F(s)$ is the Laplace transform of $f(t)$.

Differential equation in the vertical - III

- Bromwich integral is computed along a vertical line $[Re(s) = c]$ on the complex plane, and singularities of $F(s)$ should be located on the left of the vertical line.
- Specification of the imaginary part of s
  - Note $s - ik_h\bar{u}_h = c + i(s_i - k_h\bar{u}_h)$.
  - If we think of s_i as the observed frequency ω, $s_i - k_h\bar{u}_h$ becomes the intrinsic frequency.
  - Since we are interested only in vertically propagating gravity waves, $|s_i - k_h\bar{u}_h| > |f_0|$.
  - It seems reasonable to set s_i, for example, as follows:

$$s_i - k_h\bar{u}_h = \pm 1.1|f_0|, \pm 1.2|f_0|, \pm 1.3|f_0| \cdots \tag{22}$$

- Specification of the real part of s (i.e., c)
  - c should be a positive real number.
  - Considering that $F(s) = (s+\alpha)/[(s+\alpha)^2 + \omega^2]$ for $f(t) = e^{-\alpha t}cos(\omega t)$, the singularities of $F(s)$ is located on the left of $Re(s) = -\alpha$ in the complex plane.
  - If $f(t)$ grows with time (i.e., $\alpha < 0$), c should be larger than $-\alpha(> 0)$.
  - Therefore, we need to choose a sufficiently large value for c so that we can obtain numerical solutions for the growing modes (However, too large c may damage the accuracy of numerical inverse Laplace transform).

Iterative method to obtain solutions - I

- First iteration

$$(s - ik_h\bar{u}_h)^2 \left(\frac{\partial^2 \tilde{w}^{(1)}}{\partial z^2} - \frac{N^2}{c_s^2}\tilde{w}^{(1)} \right) - N^2 k_h^2 \tilde{w}^{(1)}$$

$$= iX\frac{\partial \hat{u}_0}{\partial z} + iY\frac{\partial \hat{v}_0}{\partial z} - iZ\frac{\partial \hat{d}_0}{\partial z}. \quad (23)$$

where $\tilde{w}^{(1)}$ can be numerically obtained by using appropriate top and bottom boundary conditions for $\tilde{w}$ and $\partial\tilde{w}/\partial z$.

- Top and bottom boundary conditions
  - Initial horizontal wind and density perturbations are confined near $z = 0$.
  - Rigid lids: $\tilde{w} = 0$, and $\dfrac{\partial \tilde{w}}{\partial z} = 0$ at $z = z_t$ and $-z_t$.
- Determination of the other variables for the first iteration
  - Thermodynamic energy equation : $\tilde{w}^{(1)} \rightarrow \tilde{b}^{(1)}$.
  - Hydrostatic balance : $\tilde{b}^{(1)}$ and boundary conditions for $\tilde{\pi} \rightarrow \tilde{\pi}^{(1)}$.
  - Equation of state : $\tilde{b}^{(1)}$ and $\tilde{\pi}^{(1)} \rightarrow \tilde{d}^{(1)}$.
  - Momentum equations : $\tilde{\pi}^{(1)}$ and $\tilde{d}^{(1)} \rightarrow \tilde{u}^{(1)}$ and $\tilde{v}^{(1)}$.

Iterative method to obtain solutions - II

- Subsequent iterations

$$
\begin{aligned}
(s - ik_h\bar{u}_h)^2 \left(\frac{\partial^2 \tilde{w}^{(n)}}{\partial z^2} - \frac{N^2}{c_s^2} \tilde{w}^{(n)} \right) - N^2 k_h^2 \tilde{w}^{(n)} = \\
- i(s - ik_h\bar{u}_h) \left(X \frac{\partial \tilde{u}^{(n-1)}}{\partial z} + Y \frac{\partial \tilde{v}^{(n-1)}}{\partial z} - Z \frac{\partial \tilde{d}^{(n-1)}}{\partial z} \right) \\
+ iX \frac{\partial \hat{u}_0}{\partial z} + iY \frac{\partial \hat{v}_0}{\partial z} - iZ \frac{\partial \hat{d}_0}{\partial z}.
\end{aligned}
\tag{24}
$$

- Criteria to stop iterations should be found.
- Numerical inverse Laplace and Fourier transforms of $\tilde{w}^{(n)}$ will give the w' in the 4-dimensional domain and thus b', π', d', u', and v' can be sequentially obtained.
- Reliable numerical algorithms for vertical differential equations with two boundary conditions and Laplace transforms for complex variables should be found.

Numerical algorithms - I

- Two-points boundary value problems for 1-D differential equation
  - Cash, J. R., and F. Mazzia, 2005: A new mesh selection algorithm, based on conditioning, for two-point boundary value codes. J. Comput. Appl. Math., 184, 362–381.
  - Cash, J. R., and F. Mazzia, 2006: Hybrid mesh selection algorithms based on conditioning or two-point boundary value problems. J. Numer. Analy. Industr. Appl. Math., 1, 81–90.
  - These algorithms automatically generate an appropriate grid (meshes) for a given accuracy (or tolerance) limit.
  - Terms in differential equation should be given in an analytic form.
  - Fortran-77 source codes are provided through http://www.imperial.ac.uk/∼jcash/BVP_software/readme.php
  - For Fortran-95 modules for these two algorithms, see `BVPC.F90`, `BVPL.F90`, `BVPShared.F90`, and `BVPExtern.F90` in `TwoptsBVP.tar`.
  - Various examples are also included in `TwoptsBVP.tar`.

Numerical algorithms - II

- Inverse Laplace transform
  - Dubner, H., and J. Abate, 1968: Numerical inversion of Laplace transforms and the finite Fourier transform. JACM, 371–376 (http://dl.acm.org/citation.cfm?id=321446).
  - Cohen, A. M., 2007: *Numerical Methods for Laplace Transform*, Springer-Verlag, New York (see Chapter 4.3.2: The Sidi mW-Transformation of the Bromwich integral).
  - Cohen (2007) demonstrated that Fourier-based algorithms like Sidi's modification give the most robust results for various functions when compared to the other algorithms.
  - C source codes for Sidi's algorithm are available at https://www.cs.hs-rm.de/∼weber/lapinv/sidi/sidi.htm
  - For Fortran-95 modules for this algorithm, see `InvLaplace.F90` and `InvLaplaceExtern.F90` in `InvLaplace.tar`.
  - Various examples are included in `InvLaplace.tar`.

Numerical algorithms - III

- Issue in inverse Laplace transform
  - Discrete inverse Laplace transform algorithms could not be found.
  - Algorithms, we have searched for so far, compute the numerical inversion of $F(s)$ given in an analytic form.
- Workaround
  - An analytic function can be fit to results obtained for a sequence of s_i.
  - Considering that the Laplace transforms of sine and cosine functions are given by rational functions, and various kinds of functions can be modeled using rational functions as in the Padé approximation, rational functions are used for the analytic function.

$$F(s) = \frac{a_0 + a_1 s^1 + \cdots + a_N s^N}{b_0 + b_1 s^1 + \cdots + b_M s^M} \tag{25}$$

    where N and M are integers, and note that M should be much larger than N for rapid decay of F at large s.
  - The coefficients a_0, ..., a_N and b_0, ..., b_M are determined using nonlinear least-square algorithm.

Numerical algorithms - IV

- Nonlinear least-square method
  - Levenberg-Marquardt algorithm is used.
  - Fortran-77 source codes are included in MINPACK, and Fortran-90 style routines are found at http://jblevins.org/mirror/amiller.
  - For Fortran-95 modules for this algorithm, see `LMNonFit.F90` and `LMExtern.F90` in `LMNonFit.tar`.
- Modification for fitting to complex values
  - As mentioned regarding inverse Laplace transform, s is not a real number.
  - However, most of minimization (or optimization) algorithms work for real numbers. Therefore, the rational function for fitting is modified, for example, as follows:

$$\frac{a_0}{b_0 + b_1 s} \rightarrow \frac{a_0(b_0 + b_1 s_r) - i a_0 b_1 s_i}{(b_0 + b_1 s_r)^2 + (b_1 s_i)^2}. \tag{26}$$

    where coefficients are assumed to be real numbers.
  - Real (Imaginary) part of the function can be fitted to real (imaginary) parts of numerical results.
  - Note that the real and imaginary parts of the fitting function includes all the required three coefficients (a_0, b_0, and b_1).
  - Fitting results for real and imaginary parts should be consistent within a certain error criteria.

Thank you for your attention

Questions?