How do we update estimates based on current estimates?

Let's start with 2 "batches":

$$\tilde{y}_1 = [\tilde{y}_{11} \;\; \tilde{y}_{12} \;\; \cdots \;\; \tilde{y}_{1m_1}]^T$$

$$\tilde{y}_2 = [\tilde{y}_{21} \;\; \tilde{y}_{22} \;\; \cdots \;\; \tilde{y}_{2m_2}]^T$$

$$\tilde{y}_1 = H_1 \underline{x} + \underline{v}_1$$

$$\tilde{y}_2 = H_2 \underline{x} + v_2$$

trying to estimate the $\underline{x}$

For $\tilde{y}_1$, from LLS before we have:

$$\hat{\underline{x}}_1 = (H_1^T W_1 H_1)^{-1} H_1^T W_1 \tilde{y}_1$$

If we had both measurements together:

$$\underset{\sim}{\tilde{y}} = H \underline{x} + \underline{v}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x + \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix}$$

Let's assume:

$$W = \left[\begin{array}{c|c} W_1 & 0 \\ \hline 0 & W_2 \end{array}\right]$$

ie. there is no correlation of noise btwn first & second set of measurements

Given both measurement sets, we have:

$$\hat{\underline{x}}_2 = (H^T W H)^{-1} H^T W \tilde{y}$$

$\hat{x}_2$ takes into account BOTH measurement sets

Since W is block diagonal, we can write:

$$\hat{\underline{x}}_2 = \left[H_1^T W_1 H_1 + H_2^T W_2 H_2\right]^{-1} \left(H_1^T W_1 \tilde{y}_1 + H_2^T W_2 \tilde{y}_2\right)$$

But we don't want to recalculate everything every time - we want efficient use of previous calculations.

define: $$P_1 = \left[H_1^T W_1 H_1\right]^{-1}$$

$$P_2 = \left[H_1^T W_1 H_1 + H_2^T W_2 H_2\right]^{-1}$$

thus:

$$P_2^{-1} = P_1^{-1} + H_2^T W_2 H_2$$

Now we can rewrite:

$$\hat{\underline{x}}_1 = P_1 H_1^T W_1 \tilde{y}_1$$

$$\hat{\underline{x}}_2 = P_2 \left(\underbrace{H_1^T W_1 \tilde{y}_1}_{} + H_2^T W_2 \tilde{y}_2\right)$$

We want to replace this with $\hat{x}_1$, because we already did those calculations

$$H_1^T W_1 \tilde{y}_1 = P_1^{-1} \hat{\underline{x}}_1$$

Substituting that in:

$$\hat{\underline{x}}_2 = P_2 \left(P_1^{-1} \hat{\underline{x}}_1 + H_2^T W_2 \tilde{y}_2 \right)$$

$$\hat{\underline{x}}_2 = P_2 P_1^{-1} \hat{\underline{x}}_1 + P_2 H_2^T W_2 \tilde{y}_2$$

inverses are expensive, so use:

$$P_2^{-1} = P_1^{-1} + H_2^T W_2 H_2$$ from earlier

$$\hat{\underline{x}}_2 = P_2 \left[P_2^{-1} - H_2^T W_2 H_2 \right] \hat{\underline{x}}_1 + P_2 H_2^T W_2 \tilde{y}_2$$

$$= \left[I - P_2 H_2^T W_2 H_2 \right] \hat{\underline{x}}_1 + P_2 H_2^T W_2 \tilde{y}_2$$

$$= \hat{\underline{x}}_1 - \underline{P_2 H_2^T W_2} H_2 \hat{\underline{x}}_1 + \underline{P_2 H_2^T W_2} \tilde{y}_2$$

$$= \hat{\underline{x}}_1 + \underbrace{P_2 H_2^T W_2}_{\text{"}K\text{"}} [\underbrace{\tilde{y}_2 - H_2 \hat{\underline{x}}_1}]$$

previous "a priori" estimate of $\hat{\underline{x}}$

"gain" matrix

"error" between new measurements and expectation based on previous estimate

Now we can generalize:

$$\boxed{\begin{aligned} &\underline{\text{Kalman update equations}} \\ &\hat{\underline{x}}_{k+1} = \hat{\underline{x}}_k + K_{k+1}(\tilde{y}_{k+1} - H_{k+1}\hat{\underline{x}}_k) \\ &K_{k+1} = P_{k+1} H^T_{k+1} W_{k+1} \\ &P^{-1}_{k+1} = P^{-1}_k + H^T_{k+1} W_{k+1} H_{k+1} \end{aligned}}$$

$$\hat{\underline{x}}_{k+1} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \quad 3\times1$$

$$\tilde{\underline{y}}_{k+1} = [\quad] \quad 1\times1$$

$$H_{k+1} = [\qquad] \quad 1\times3$$

$$K_{k+1} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad 3\times1$$

$$w_{k+1} = [\quad] \quad 1\times1$$

$$P_{k+1} = [\qquad] \quad 3\times3$$

$\boxed{\texttt{np.atleast\_2d(x)}}$

This is a time-varying dynamic system - rearranging yields:

$$\hat{\underline{x}}_{k+1} = \underbrace{\left[I - K_{k+1} H_{k+1} \right]}_{\substack{\text{you can} \\ \text{check stability,} \\ \text{response time,} \\ \text{etc.}}} \hat{\underline{x}}_k + \left[K_{k+1} \right] \tilde{\underline{y}}_{k+1}$$

This calculation requires an inverse (expensive)

$$P_{k+1} = \left[P_k^{-1} + H_{k+1}^T W_{k+1} H_{k+1} \right]^{-1}$$

But, if the new measurements being added are few compared to the prev. ones we can write:

$$P_{k+1} = P_k - P_k H_{k+1}^T \left(H_{k+1} P_k H_{k+1}^T + W_{k+1}^{-1} \right)^{-1} H_{k+1} P_k$$

There are other ways to reorganize as well.