ECON 432 Homework 4

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Review Questions

1.

(a)

We know that p_t is a martingale sequence, so $E[p_t|\mathcal{F}_{t-1}] = p_{t-1}$, thus:

$$E[r_t|\mathcal{F}_{t-1}] = E[p_t - p_{t-1}|\mathcal{F}_{t-1}] = p_{t-1} - p_{t-1} = 0 \tag{1}$$

From definition, we can conclude that continuously compounded return for stock market price is an martingale difference sequence.

(b)

i. If $r_t \sim iid(0, \sigma^2)$, r_t is independent with each other and has the same distribution. So,

$$\begin{aligned} E[r_t|\mathcal{F}_{t-1}] &= 0 \\ Var(r_t) &= \sigma^2 \end{aligned} \tag{2}$$

Hence, $r_t \sim mds(0, \sigma^2)$.

ii. If $r_t \sim mds(0, \sigma^2)$, we have $E[r_t|\mathcal{F}_{t-1}] = 0, Var(r_t) = \sigma^2$, from these, we can get:

$$\begin{aligned} E[r_t] &= E[E[r_t|\mathcal{F}_{t-1}]] = 0 \\ cov(r_t, r_{t-j}) &= E[r_t r_{t-j}] - E[r_t]E[r_{t-j}] \\ &= E[r_t r_{t-j}] \\ &= E[E[r_t r_{t-j}|\mathcal{F}_{t-j}]] \\ &= E[r_{t-j}E[r_t|\mathcal{F}_{t-j}]] \\ &= E[r_{t-j}E[E[r_t|\mathcal{F}_{t-1}]|\mathcal{F}_{t-j}]] \\ &= 0 \end{aligned} \tag{3}$$

Hence, $r_t \sim WN(0, \sigma^2)$

(c)

i. From r_t is independent, we can directly get:

$$\begin{aligned} F(r_1, r_2, \cdots, r_j) &= F(r_1) * F(r_2) * \cdots * F(r_j) \\ F(r_{1+s}, r_{2+s}, \cdots, r_{j+s}) &= F(r_{1+s}) * F(r_{2+s}) * \cdots * F(r_{j+s}) \end{aligned} \tag{4}$$

From r_t is identical, we can get $\forall \alpha \in R$, $F(r_\alpha)$ is the same, so $F(r_1, r_2, \cdots, r_j) = F(r_{1+s}, r_{2+s}, \cdots, r_{j+s})$, hence r_t is strictly stationary.

ii. - First we need to proof that mean, variance and covariance exist.
Since $E[|r_t|^2] < \infty$, we have $(E[r_t])^2 \leq E[|r_t|^2] < \infty$, so $E[r_t]$ exists. Then $Var[r_t] = E[|r_t|^2] - (E[r_t])^2 < \infty$, so $Var[r_t]$ exists. Last, since $E[r_t r_s] \leq (E[r_t^2])^{\frac{1}{2}}(E[r_s^2])^{\frac{1}{2}} < \infty$, so $Cov(r_t, r_s) = E[r_t r_s] - E[r_t]E[r_s] < \infty$, hence $Cov(r_t, r_s)$ exists.

- Next we need to proof mean and variance are constant, covariance only depends time shift.
For any $t \neq s$, we have $F_{r_t}(r) = F_{r_s}(r)$:

$$E[r_t] = \int_{-\infty}^{\infty} r dF_{r_t} = \int_{-\infty}^{\infty} r dF_{r_s} = E[r_s] \tag{5}$$

which shows constant mean.
Also, for any $t \neq s$,

$$E[r_t^2] = \int_{-\infty}^{\infty} r^2 dF_{r_t} = \int_{-\infty}^{\infty} r dF_{r_s} = E[r_s^2] \tag{6}$$

Thus, $Var(r_t) = E[r_t^2] - (E[r_t])^2 = E[r_s^2] - (E[r_s])^2 = Var(r_s)$, which shows constant variance.
Also, For any shift j,

$$E[r_t r_s] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_1, r_2 dF_{r_t, r_s}(r_1, r_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_1, r_2 dF_{r_{t+j}, r_{s+j}}(r_1, r_2) = E[r_{t+j} r_{s+j}] \tag{7}$$

Thus, $Cov(r_t, r_s) = E[r_t r_s] - E[r_t]E[r_s] = E[r_{t+j} r_{s+j}] - E[r_{t+j}]E[r_{s+j}] = Cov(r_{t+j}, r_{s+j})$, which shows covariance only depends on time shift.
Q.E.D

2.

(a)

This process is stationary, since $|\phi| = 0.55 < 1$

(b)

Performing expectation to both sides, we get

$$\begin{aligned} E(Y_t) &= 5 - 0.55E(Y_{t-1}) + E(\epsilon_t) \\ \mu &= 5 - 0.55\mu \\ \mu &\approx 3.23 \end{aligned} \tag{8}$$

(c)

We know that $cov(Y_{t-1}, \epsilon_t = 0$. Thus,

$$\begin{aligned} VAR(Y_t) &= 0.55^2 VAR(Y_{t-1}) + VAR(\epsilon_t) \\ \sigma_Y^2 &= \frac{\sigma_\epsilon^2}{1-\theta^2} \\ &= \frac{1.2}{1-0.55^2} \\ &\approx 1.72 \end{aligned} \tag{9}$$

(d)

We start from a general AR(1) model $Y_t = c + \phi Y_{t-1} + \epsilon_t$. Similary to (a), we get

$$\mu = \frac{c}{1-\phi} \tag{10}$$

Hence, $c = (1-\phi)\mu$. Insert this into original model and substract mu on both sides:

$$Y_t - \mu = c + \phi(Y_{t-1}) - \mu - \epsilon_t \quad = \phi(Y_{t-1} - \mu) + \epsilon_t \tag{11}$$

Using Yule-Walker equation, we multiple $Y_{t-s} - \mu$ and perform expectations on both sides to get:

$$\gamma_s = \phi \gamma_{s-1} \tag{12}$$

From iteration, we get $\gamma_s = \phi^s * \gamma_0 = \ \phi^s * \sigma_Y^2 = (0.55)^s * 1.72$, where s is the time of lag.