ECON 432 Homework 5

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March 11, 2020

Review Questions

1.

(a)

The stylized facts on the series are:volatility clustering(periods of high and low volatility), no (or weak) autocorrelation, nearly martingale difference series and heavier tail than normal distribution.

(b)

i MDS

$$E[r_t|\mathcal{F}_{t-1}] = E[\sigma_t e_t|\mathcal{F}_{t-1}] = \sigma_t E[e_t|\mathcal{F}_{t-1}] = 0 \tag{1}$$

ii No (or weak) autocorrelation Since MDS is a uncorrelated process, so from i, we can get no autocorrelation.

iii Volatility Clustering From a small trick, we get $r_t^2 = r_t^2 + \sigma_t^2 - \sigma_t^2$, then we define $V_t = r_t^2 - \sigma_t^2$. First, we should get σ_t^2, since:

$$\begin{aligned} Var(r_t|\mathcal{F}_{t-1}) &= E[r_t^2|\mathcal{F}_{t-1}] - (E[r_t|\mathcal{F}_{t-1}])^2 \\ &= E[r_t^2|\mathcal{F}_{t-1}] = E[\sigma_t^2 e_t^2|\mathcal{F}_{t-1}] = \sigma_t^2 E[e_t^2|\mathcal{F}_{t-1}] \\ &= \sigma_t^2 E[e_t^2] = \sigma_t^2 \end{aligned} \tag{2}$$

Hence, $\sigma_t^2 = E[r_t^2|\mathcal{F}_{t-1}]$, $E[V_t|\mathcal{F}_{t-1}]] = E[r_t^2 - \sigma_t^2|\mathcal{F}_{t-1}]$, V_t is MDS.
Therefore, $r_t^2 = \sigma_t^2 + V_t = \omega + \alpha_1 r_{t-1}^2 + V_t$ is a AR(1) process for r_t^2, from the property of AR process, we know that Cor($\mathrm{r}_t^2, r_{t-1}^2) = \alpha_1^{|j|}, so it has a volatility clustering.$

iv Heavy Tailedness To show this, we need to prove $Kurt(r_t) \geq 3$, below is the poof(remember we have $E[r_t] = E[\sigma_t e_t] = \sigma_t E[e_t] = 0$):

$$\begin{aligned} Kurt[r_t] &= E[\frac{r_t - E[r_t]}{SD[r_t]}]^4 = E[\frac{r_t}{SD[r_t]}]^4 \\ &= E[\frac{r_t^4}{(E[r_t^2])^2}] = E[\frac{\sigma_t^4 e_t^4}{(E[\sigma_t^2 e_t^2])^2}] \end{aligned} \tag{3}$$

Since $e_t \sim N(0,1), E[e_t^4] = 3$, from Jensen inequality, we have

$$Kurt[r_t] = 3\frac{E[\sigma^4]}{(E[\sigma_t^2])^2} \quad \geq 3 * \frac{(E[\sigma_t^2])^2}{(E[\sigma_t^2])^2} \geq 3 \tag{4}$$

Q.E.D

(c)

Using the same trick as above question,

$$\begin{aligned} r_t^2 &= r_t^2 + \sigma_t^2 - \sigma_t^2 = \sigma_t^2 + V_t \\ &= \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + V_t \\ &= \omega + \alpha_1 r_{t-1}^2 + \beta_1 (r_{t-1}^2 - V_{t-1}) + V_t \\ &= \omega + (\alpha_1 + \beta_1) r_{t-1}^2 + V_t - \beta_1 V_{t-1} \end{aligned}$$

Similarly we get that V_t is a MDS process, so from definition, this model can be interpreted as ARMA(1,1) for r_t^2