Problem Set 2: CAPM

UCLA MAE Core Finance

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In this homework you will explicitly calculate a market equilibrium when investors have mean variance preferences. The formal approach is similar to the one we developped at the beginning of Class 05, with one important difference: we will calculate equilibrium prices explicitly first, before expected returns.

As in Class, 05, we assume that there are two time periods, $t = 0$ and $t = 1$: investors make portfolio choice at $t = 0$, and they collect payoffs at $t = 1$. We will assume that there are in total 5 risky assets and one risk-free asset. We fix the risk-free interest rate to be equal to $r_f = 1\%$. The risky assets are indexed by $i \in \{1, 2, 3, 4, 5\}$. Asset i has price P_i at time $t = 0$, and pays out dividend D_i at time $t = 1$. Since the economy ends at time $t = 1$, assets have a price of zero at $t = 1$. The expected dividend of all assets are normalized[1] to 1. The variance covariance matrix of asset dividends is:

$$\Sigma = \begin{pmatrix} 0.040 & 0.064 & 0.08 & 0.016 & 0.048 \\ 0.064 & 0.160 & 0.16 & 0.032 & 0.096 \\ 0.080 & 0.160 & 0.25 & 0.040 & 0.120 \\ 0.016 & 0.032 & 0.04 & 0.010 & 0.024 \\ 0.048 & 0.096 & 0.12 & 0.024 & 0.090 \end{pmatrix}.$$

The column vector of asset supplies is $\bar{N} \equiv \begin{pmatrix} 5 & 10 & 20 & 30 & 40 \end{pmatrix}^T$, where we use the subscript "T" for transpose. There is one "representative" investor[2] who starts with wealth A_0 at $t = 0$. The budget constraint of such investor is:

$$A_0 = \sum_{i=1}^{5} N_i P_i + N_f, \tag{1}$$

where N_i is the number of risky asset i purchased by the investor, and N_f is the amount invested in the risk-free asset. The payoff of the investor at time $t = 1$ is

$$A_1 = \sum_{i=1}^{5} N_i D_i + N_f(1 + r_f). \tag{2}$$

[1] The reason it is a normalization is because we can always scale up the supply and scale down the dividend by the same factor. The price would be evidently scaled down but nothing else would change.

[2] That is we imagine that there is just one investor who takes price as given. It can be shown that, in the present model, the results would not change in the more realistic case with many investors.

Finally, we assume that the investor chooses N_i and N_f to maximize the mean variance objective $\mathrm{E}_0[A_1] - \frac{\gamma}{2}\mathrm{var}_0[A_1]$ where γ is a coefficient parameterizing risk-aversion. In what follows, we will let N denote the column vector of asset demands:

$$N \equiv \begin{pmatrix} N_1 & N_2 & \dots & N_5 \end{pmatrix}^T.$$

Likewise, we will let P denote the column vector of asset prices:

$$P \equiv \begin{pmatrix} P_1 & P_2 & \dots & P_5 \end{pmatrix}^T,$$

and D denote the column vector of risky asset (random) dividends:

$$D \equiv \begin{pmatrix} D_1 & D_2 & \dots & D_5 \end{pmatrix}^T.$$

1. Use the two budget constraints (1) and (2) to show that:

$$A_1 = N^T \left(D - (1 + r_f)P\right) + A_0(1 + r_f)$$

2. Show that the investor's mean-variance objective is to maximize:

$$N^T \left(\mathrm{E}_0[D] - (1 + r_f)P\right) - \frac{\gamma}{2}N^T\Sigma N.$$

3. Take derivative with respect to N and find the first-order condition of the investor.

4. Argue that, in a market equilibrium, we must have that $N = \bar{N}$.

5. Show that, in a market equilibrium:

$$P = \frac{1}{1 + r_f}\left(\mathrm{E}_0[D] - \gamma\Sigma\bar{N}\right)$$

6. Interpret the formula: what are the main economic determinants of asset prices?

7. Recall that the return of asset i is equal to $r_i = D_i/P_i - 1$. Recall that we have normalized expected dividend to one. Assume that $\gamma = 0.01$. Write a program in R to calculate

   (a) Asset prices;

   (b) Asset expected returns;

   (c) The portfolio weight of each asset in the market portfolio;

(d) The beta of each stock return with respect to the return of the market portfolio;

8. Use your R program to verify that expected returns lie on the security market line, consistent with the theory we studied in class.

hints for questions 7 and 8:

- The variance covariance matrix of dividend is not the same as the variance covariance matrix of returns. Indeed, recall that $r_i = D_i/P_i - 1$. This means that the covariance between r_i and r_j can be written:

$$\text{cov}(r_i, r_j) = \text{cov}(D_i/P_i, D_j/P_j) = 1/(P_iP_j)\text{cov}(D_i, D_j).$$

  This formula allows you to calculate every entry in the variance covariance matrix of return, given that you know the prices, and the entries of the variance covariance matrix of dividends.

- Let V denote the variance covariance matrix of returns. Then, we have seen in class that, given any vector w of portfolio weight, the variance of the return of the portfolio is equal to $w^T V w$. If we consider two different portfolios, with weight w and $\hat{w}$, then the covariance between the returns on these portfolio is equal to $w^T V \hat{w}$. Note that this formula can be used to calculate the covariance between the market portfolio and any asset i: indeed, one just need to replace w by w_M (the portfolio weights for the market portfolio), and $\hat{w}$ by $\begin{pmatrix} 0 & \dots & 1 & 0 & \dots \end{pmatrix}^T$, that is, a vector with a 1 in position i and zero everywhere else (why?).