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Lecture 1
Introduction to probability

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Aims

- Understand the concepts of probability
- Recognise terms used in probability
- Enumerate simple events
- Construct sample spaces
- Summing and product rules
- Consider methods of counting

Definition

- **Probability**
- A scaled measurement of the likelihood of an event happening
- Scale 0-1
  - 0: never happens
  - 0.5: equally as likely not to happen as to happen
  - 1: certain to happen
- A probability is calculated as follows:

  $$\text{P(event)} = \frac{Number\ of\ outcomse\ that\ result\ in\ a\ favorable\ event}{Total\ number\ of\ outcomes}$$
- e.g. 0.05
- Alternative expressions:
  - Percentage 5%
  - Fraction e.g. 1/20

Terms

- **Statistical Experiment**

Any procedure which produces a random outcome

- **Elementary event**

One of the possible outcomes of an experiment

- **Sample Space (S / Ω)**

The set of all elementary events

- **Event**

A set of elementary events which have a common feature

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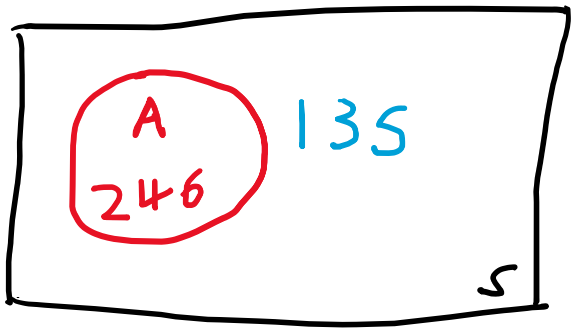
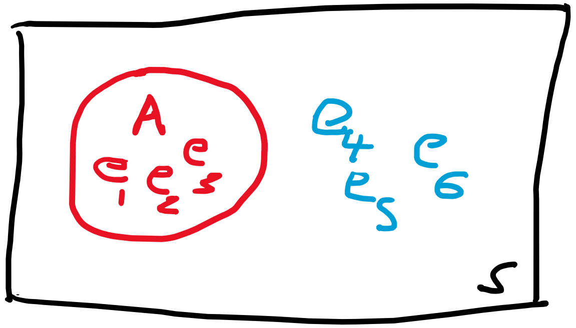
Events / elementary events / sample space

- Event A = {throw an even number}
- = 2,4,6
- = e_1, e_2, e_3
- In sample space S
- P(A) = $\frac{3}{6} = \frac{1}{2}$
- $\sum_{e_i \; in \; A} = P(e_1) + P(e_2) + P(e_3)$



- Events not A are A^c or $\bar{A}$
- $A + A^c = S = 1$



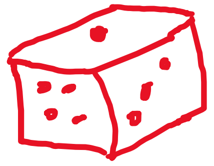
Terms

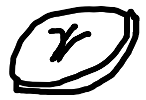
- **Mutually exclusive events**

If one event happens the others cannot happen

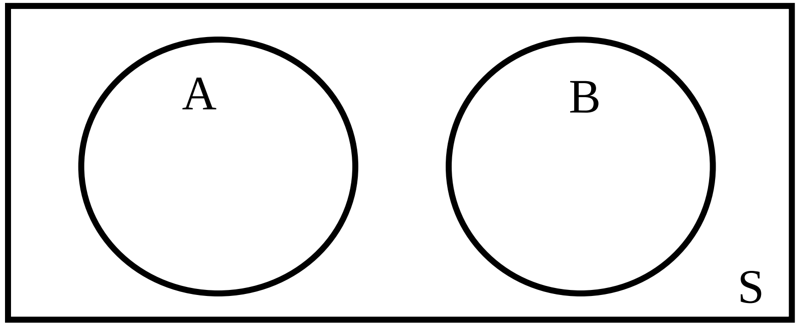
- **Independent events**

One event does not affect the other





Summing rule (or)



- If we have A ways of doing something and B ways of doing something else…
- …and can not do both at the same time (**mutually exclusive**).
- A and B are disjoint
- $P(A \cap B) = 0$
- A or B can be done in A + B ways
- $P(A \cup B) = P(A) + P(B)$

Summing rule

Example

The probability of being blood group A is 40%, the probability of blood group B is 11%.

What is the probability of being blood group A **or** B

- A = p(blood group A) = 0.40
- B = p(blood group B) = 0.11
- A+B = 0.51

Product rule (and)

- Two events are independent i.e. A does not affect B
- The probity of getting A and B is A.B

Example

What is the probability of rolling > 4 on a 6 sided die **and** flipping a tail

- $2/6 \,.\, 1/2 = 2/12 = 1/6$
- $2/6 = (P(5) = 1/6) + (P(6) = 1/6)$

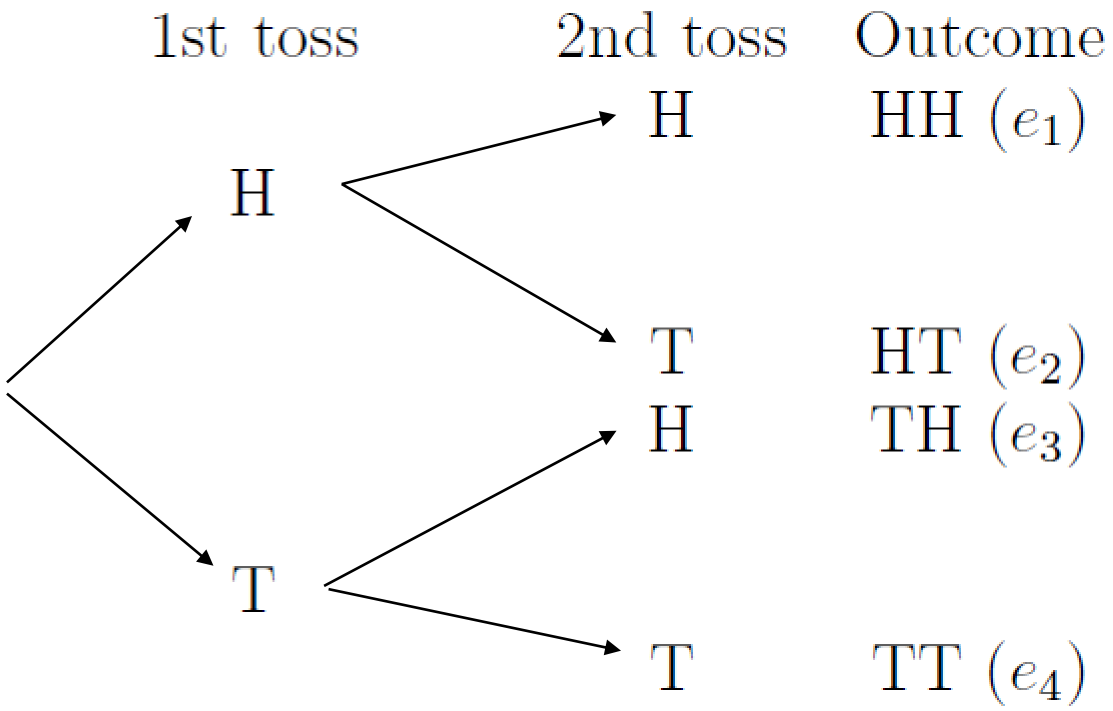
Coin toss example

- Likelihood of getting a head?
  - 0.5
- Therefore, Likelihood of getting tail
  - 1 – 0.5 = 0.5
- Each event is independent and mutually exclusive

Coin toss example

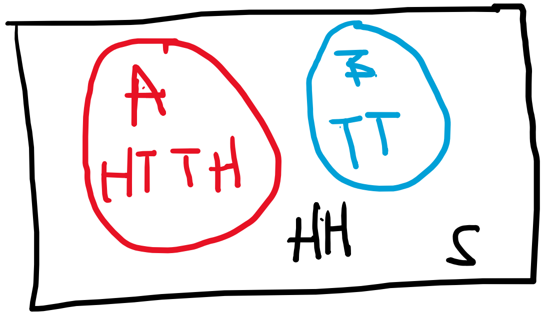
- For considering a string of events, a tree diagrams are useful
- E.g. Model the tossing of a fair coin twice
- S = {HH, HT, TH, TT}
- S = $\{e_1, e_2, e_3, e_4\}$

Tree Diagram

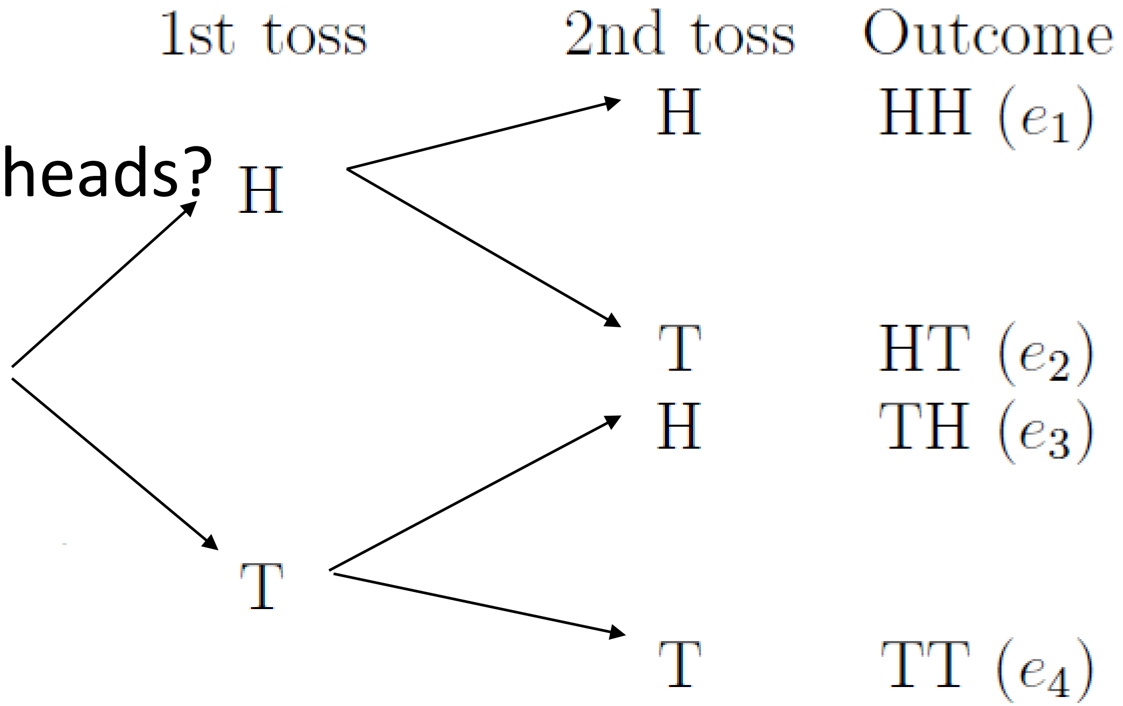


Coin toss example

- What is the probability of getting exactly one head?
    - A = {exactly one head}
    - P(A) = $\dfrac{number\ of\ elementary\ events\ in\ A}{number\ of\ elementry\ events\ in\ S}$
    - A = {HT, TH} = {e_2, e_3} = $\frac{2}{4} = \frac{1}{2}$
- What is the probability of getting no heads?
    - B = {no heads}
    - B = {TT} = {e_4} = $\frac{1}{4}$

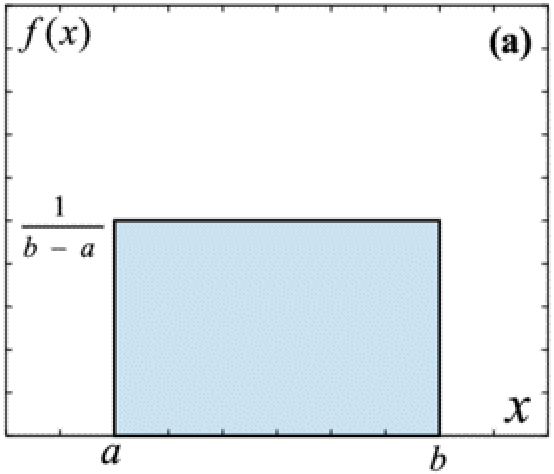


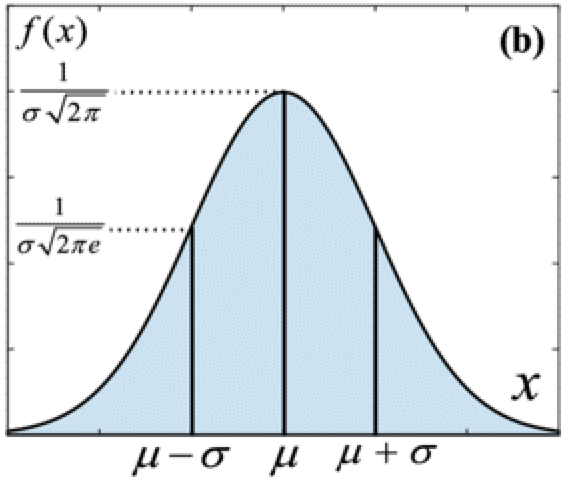
Tree Diagram



Use what we know

- $P(A) \; = \dfrac{number\ of\ elementary\ events\ in\ A}{number\ of\ elementry\ events\ in\ S}$
- $0 \; \leq \; P(A) \; \leq \; 1\, for\ all\ events\ A$
- Uniform distribution





Use what we know

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- A fair die should have a uniform distribution
- i.e. every outcome equally as likely
- $S = \{1,2,3,4,5,6\} = \{e_{1,}\ e_{2,}\ e_{3,}\ e_{4,}\ e_{5,}\ e_6\}$
- $P(e_i) = \mathbf{\frac{1}{6}} = \mathbf{0.167}$
- $A = \{\text{Throw even number}\} = \{e_{2,}\ e_{4,}\ e_6\} = p(A) = \frac{3}{6} = \mathbf{\frac{1}{2}}$
- $B = \{\text{Throw number} > 4\} = \{e_{5,}\ e_6\} = p(B) = \frac{2}{6} = \mathbf{\frac{1}{3}}$

Test for loaded die

- Expect a uniform distribution
- S = {1,2,3,4,5,6} = $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ = $p(e_i)$ = $\frac{1}{6}$
- Empirically we test the die
- We find even numbers are twice as likely as odd numbers
- A = {Throw even number} = $\{e_2, e_4, e_6\}$

= p(A) = $\frac{2}{9} + \frac{2}{9} + \frac{2}{9} + = \frac{6}{9} = \mathbf{\frac{2}{3}} \neq \mathbf{\frac{1}{2}}$

- B = {Throw number > 4} = $\{e_5, e_6\}$

= p(B) = $\frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \mathbf{\frac{1}{3}} = \mathbf{\frac{1}{3}}$

Number	Occurrence
1	I
2	II
3	I
4	II
5	I
6	II
Total	9

Complimentary events

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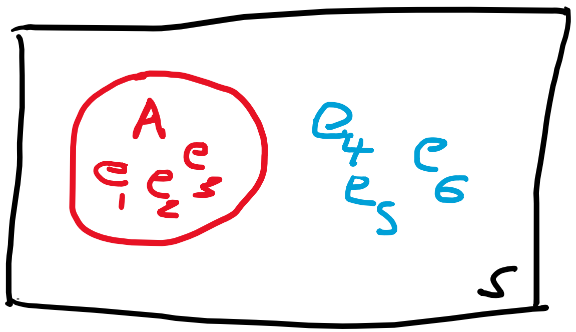
- $A^c = 1 - A \therefore P(A^c) = 1 - P(A)$
- $A = 1 - A^c \therefore P(A) = 1 - P(A^c)$
- Complement can provide a quick shortcut
- Q. a multiple choice test three questions, each with two choices, what is the probability of getting at least one correct; P(at lest one correct)
- let **c** = correct and **w** = wrong
  a) Could count up all correct answers (frequent event)
  $\frac{7}{8}$
  b) Or use the compliment event (less frequent)
  P(at lest one correct) = 1 - P(all wrong)($\frac{1}{8}$)
  $= 1 - \frac{1}{8} = \frac{7}{8}$



Q1	Q2	Q3
W	W	W
W	W	C
W	C	C
C	C	C
C	C	W
C	W	W
W	C	W
C	W	C

Contingency table

- Census taken to ascertain modes of travel to work, classified on where they live
- P(live in Rural area)?
  - $\frac{35}{100} = 0.35$
- P(travel by bus)?
  - $\frac{30}{100} = 0.3$
- P(travel by car and live in town)?
  - $\frac{40}{100} = 0.4$
- P(travel by bus and live in the countryside)?
  - $\frac{5}{100} = 0.05$

Contingency Table

	Live		
	Town	Rural	**Total**
Car	40	30	**70**
Bus	25	5	**30**
Total	**65**	**35**	100

Contingency table

Summing rule

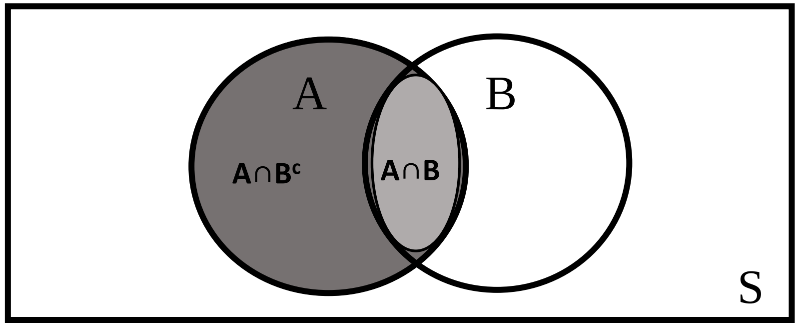
$P(A + B) = P(A) + P(B) - P(A \cap B)$

$P(A) + P(A^c) = 1$
$P(B) + P(B^c) = 1$

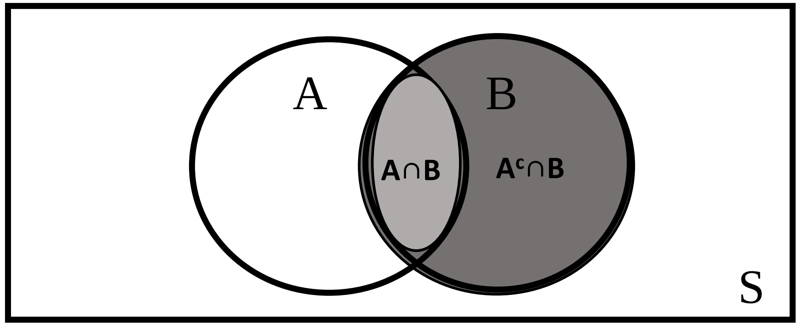
$P(A) = 1 - P(A^c)$
$P(B) = 1 - P(B^c)$

	A	A^c	Total
B	$P(A \cap B)$	$P(Ac \cap B)$	$P(B)$
B^c	$P(A \cap Bc)$	$P(Ac \cap Bc)$	$P(B^c)$
Total	$P(A)$	$P(Ac)$	1

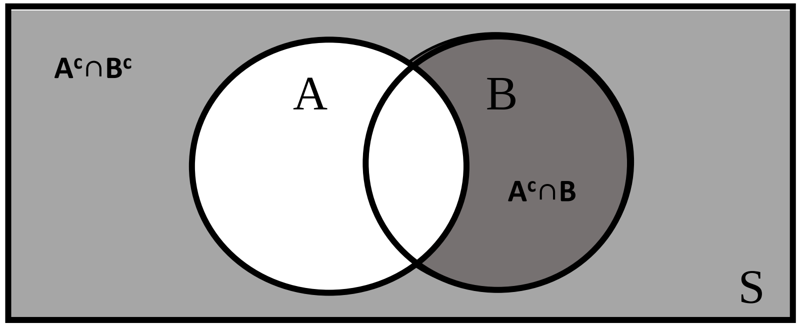
P(A)



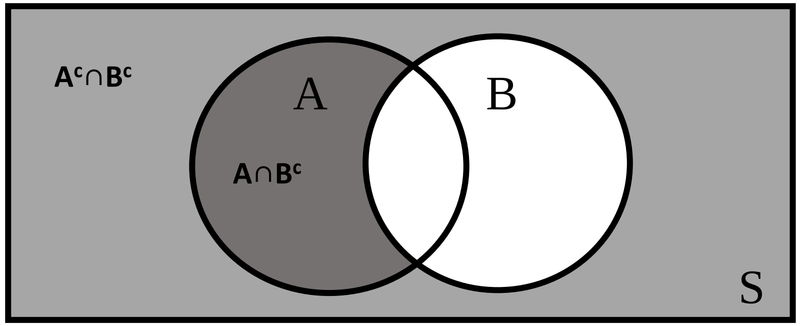
P(B)



P(Aᶜ)



P(Bᶜ)



Counting methods

- Up until now we have taken the manual approach to counting.
- This is fine for small numbers, e.g. how many ways can you combine 3 letters?
- But falls down when we get to larger numbers
- **Powers**
- **Factorial**
- **Permutations**
- **Combinations**

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Counting methods

- Easiest way to think of this is drawing from a bag, with or without replacement
- Q. How many ways of arranging the 3 letters {DAN}, with replacement?
- 3.3.3 = 3^3 = 27
- 6^6 = 46,656

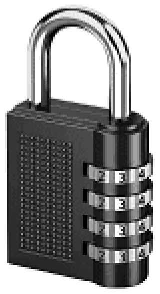
- Q. How many ways of arranging the 3 letters {DAN} without replacement?
- 3
- 2
- 1
- 3.2.1 = 6
- = 3!

D D D	D D D
D D A	D D A
D D N	D D N
D A D	D A D
D A A	D A A
D A N	D A N
D N D	D N D
D N A	D N A
D N N	D N N
A D D	A D D
A D A	A D A
A D N	A D N
A A D	A A D
A A A	A A A
A A N	A A N
A N D	A N D
A N A	A N A
A N N	A N N
N D D	N D D
N D A	N D A
N D N	N D N
N A D	N A D
N A A	N A A
N A N	N A N
N N D	N N D
N N A	N N A
N N N	N N N

Factorial

- 3! = 3.2.1 = 6
- 4! = 4.3.2.1 = 24
- 0! = 1
- $n! \; = \; n.(n-1).(n-2)\ldots 1$
- 4! = 4.3! = 4.6 = 24
- $\dfrac{4!}{2!} = \dfrac{4.3.\cancel{2.1}}{\cancel{2.1}} = \dfrac{4.3}{} = 12$
- You can find ! on your scientific calculator

Permutations (order is important)



- Q. How many ways of arranging {permutation}?
- 11^{11} = 285,311,670,611 (replacement)
- or 11! = 39,916,800 (without replacement)
- Q. How many ways of choosing 2 letters from {permutation} where the order is important?
- with replacement?
- ${}^{n}P_{r} = n^{r} = 11^{2} = 121$ Where n = total, r = # sample
- without replacement?
- ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$
- ${}^{11}P_{2} = \dfrac{11!}{(11-2)!} = \dfrac{11!}{9!} = \dfrac{11.10.\cancel{9.8.7.6.5.4.3.2.1}}{\cancel{9.8.7.6.5.4.3.2.1}} = \dfrac{11.10}{} = 110$

pp	pp
pe	pe
pr	pr
pm	pm
pu	pu
pt	pt
pa	pa
pt	pt
pi	pi
po	po
pn	pn
ep	ep
ee	ee
er	er
em	em
eu	eu
et	et
ea	ea
et	et
ei	ei
eo	eo
en	en
rp	rp
re	re
rr	rr
rm	rm
ru	ru
rt	rt
ra	ra
rt	rt
ri	ri
ro	ro

Combinations (order is not important)

- Q. How many ways of choosing 2 letters from {permutation} where the order is not important, without replacement?
- $^{n}P_{r} = \dfrac{n!}{(n-r)!}$
- $^{n}C_{r} = \dfrac{n!}{(n-r)!r!}$

Ways of ranging r item

- $^{11}C_{2} = \dfrac{11!}{(11-2)!.2!} = \dfrac{11!}{9!.2!}$
- $= \dfrac{11.10.\cancel{9.8.7.6.5.4.3.2.1}}{(\cancel{9.8.7.6.5.4.3.2.1}).(2.1)} = \dfrac{11.10}{2} = 55$
- $^{11}P_{2} = \dfrac{11!}{(11-2)!} = \dfrac{11!}{9!} = \dfrac{11.10.\cancel{9.8.7.6.5.4.3.2.1}}{\cancel{9.8.7.6.5.4.3.2.1}} = \dfrac{11.10}{} = 110$
- **N.B if *n* and *r* are the same then the answer is 1**

pe	pe
pr	pr
pm	pm
pu	pu
pt	pt
pa	pa
pt	pt
pi	pi
po	po
pn	pn
ep	
er	er
em	em
eu	eu
et	et
ea	ea
et	et
ei	ei
eo	eo
en	en
rp	
re	
rm	rm
ru	ru
rt	rt
ra	ra
rt	rt
ri	ri

Combinations (order is not important)

- Q. How many ways of choosing 2 letters from {permutation} where the order is not important, with replacement?
- ${}^{n}C_{r} = \dfrac{n!}{(n-r)!r!}$
- ${}^{n}C_{r} = \dfrac{(n+r-1)!}{(n-r)!r!}$
- ${}^{11}C_{2} = \dfrac{(11+2-1)!}{(11-1)!2!} = \dfrac{12!}{9!.2!}$
- $= \dfrac{12.11.\cancel{10.9.8.7.6.5.4.3.2.1}}{(\cancel{10.9.8.7.6.5.4.3.2.1}).(2.1)} = \dfrac{12.11}{2} = 66$
- $= \dfrac{11.10.\cancel{9.8.7.6.5.4.3.2.1}}{(\cancel{9.8.7.6.5.4.3.2.1}).(2.1)} = \dfrac{11.10}{2} = 55$

	p p
p e	p e
p r	p r
p m	p m
p u	p u
p t	p t
p a	p a
p t	p t
p i	p i
p o	p o
p n	p n
	e e
e r	e r
e m	e m
e u	e u
e t	e t
e a	e a
e t	e t
e i	e i
e o	e o
e n	e n
	r r
r m	r m
r u	r u
r t	r t
r a	r a
r t	r t
r i	r i

Summing up

- **If you want to know the number of ways to arrange n items?**
  - Power - if you are replacing the items
  - Factorial - if you are not relacing the items
- **If you want to know the arrangement of a subset of n items**
  - Permutations – if the order is important i.e. "DAN" $\neq$ "AND"
  - Combinations – if the order is not important i.e. "DAN" = "AND"

Examples

a) A supermarket stocks 4 brands of beans and 3 brands of peas. In how many ways can they be arranged on a self?

ii. How many ways of arranging **In any arrangement?**

7 6 5 4 3 2 1

_ _ _ _ _ _ _

4 + 3 = 7!

= 5040

ii. How many ways of arranging **beans and peas in contiguous blocks?**

4 3 2 1   3 2 1

_ _ _ _   _ _ _

The beans can be ordered 4!, the peas 3!. However, the positions can swap so

4!.3!.2

= 288

Examples

b) In how many ways can 6 people be seated on a sofa if there are only 3 seats?

- The arrangement of people is distinct, i.e. 1,2,3 $\neq$ 3,2,1 $\therefore$ we use a permutation
- ${}^{6}P_{3}$
- $= 120$

Examples

c) In the main National lottery draw, six numbers are chosen from 49. What is the probability of winning the jackpot?

The arrangement of numbers is not important

i.e. 1,2,3,4,5,6 = 6,5,4,3,2,1 $\therefore$ we use a combination

- ${}^{49}C_6 = 13983816 = \frac{1}{13983816} = 0.0000000715112384$
- What is the probability that you get 3 winning numbers?
- The 3 winning numbers can be any of the 6 numbers drawn
- $= {}^{6}C_3 = 20$
- The non-winning number can be any of the other numbers
- ${}^{49-6}C_3 = {}^{43}C_3$
- $\therefore P(3 \text{ winning numbers}) = \frac{{}^{6}C_3 * {}^{43}C_3}{{}^{49}C_6}$
- $= \frac{20 * 12341}{13983816} = 0.0177 = \frac{1}{56}$