Lecture 5

The Binomial Distribution and The Poisson Distribution

By
Dr Sean Maudsley-Barton and Abdul Ali

Manchester Metropolitan University

Aims

- Recognise situations in which the Binomial distribution should be applied.
- Calculate simple Binomial probabilities using formulae.
- Use Binomial tables to look up probabilities.
- Recognise situations in which the Poisson distribution should be applied.
- Understand the role of the mean of the Poisson distribution.
- Combine Poisson distributions.
- Use Poisson tables to look up probabilities.

The Binomial Distribution (Chapter 7)

The Binomial Distribution

- One of the most **important discrete** distributions
- Finds application in a wide number of areas.

The Binomial distribution can be used to find probabilities whenever the following **three conditions** are met:

1. **Trials:** A **fixed** number of **independent trials**, **n say**, is conducted with **each trial** resulting in a ”***success***” or a ”***failure***”.
   - The exact definition of ”success” and ”failure” will depend on the area of application, but the important idea is that of ***two outcomes*** - hence the ”**Bi**” in Bi-nomial.

2. **Success probability:** The probability of observing a ”**success**” on **each trial is a fixed quantity**, π, say, i.e. **P(success)=** π.

3. **Random variable:** The random variable of interest counts the number of successes observed in the n trials.
   - If, for example, we let the random variable $\mathbf{X}$ **be the number of successes**, then $\mathbf{X}$ could potentially **take on any value in the range** $0, 1, \ldots, n$.

The Binomial Distribution

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Example 7.1 (*Coin tossing*)

The simplest example of a Binomial distribution involves tossing a coin. Suppose we toss a **(fair) coin 10 times** and count the number of heads we observe. Does this situation conform to that of the Binomial distribution?

The **three conditions** we need to check are:

1. **Trials** (fixed, independent and 2 outcomes on each trial)**:**
   - The coin is tossed 10 times, each toss being independent of any other and each can result in one of two outcomes, **H** or **T**.
2. **Success probability** (fixed on each trial)**:**
   - If we define **"*success*" ≡ H**, we have **P("*success*") = P(H) =** $\frac{1}{2}$, for each trial, i.e. each toss of the coin.
3. **Random variable** (takes any value in the range *0, 1, . . . ,* n.)**:**
   - Let the random variable, X = *number of Heads in 10 tosses*, clearly a **discrete** random variable which can take on values, in this case, **0, 1, . . . , 10**.

The Binomial Distribution

Note: The easiest way in which to decide whether it is appropriate to apply the Binomial distribution in a given situation is to try and **make the analogy with the coin tossing example**.

The only **things that might vary** are:

- The number of tosses (can be any number **≥ 1** as long as it's fixed).
- The definition of "***success***" and "***failure***" and the, probability of getting a "*success*" - just imagine tossing a biased coin.

The Binomial Distribution

The Binomial Mass Function

We can derive the **Binomial mass function** from first principles as follows.

Theorem 7.1 (*The Binomial Probability Distribution*)

Suppose the random variable $\mathbf{X}$ satisfies the conditions of a Binomial distribution, i.e with $\boldsymbol{n}$ trials and **success probability** $\boldsymbol{\pi}$, then,

$$\boxed{P(X = x) = {}^{n}_{x}C\pi^{x}(1-\pi)^{n-x}, \qquad \text{for } x = 0,1,2,\dots,n;\ and\ \ 0 < \pi < 1}$$

Where, n = Number of trials (Fixed and independent).

x = successes observed.

π = Probability of success on each trial.

The Binomial Distribution

The Binomial Mass Function

Proof 7.1

- If our $\boldsymbol{n}$ trials result in $\boldsymbol{x}$ **success** each with probability $\boldsymbol{\pi}$, there must also have been $\boldsymbol{n-x}$ **failures** each with probability $(\boldsymbol{1-\pi})$ (the law of complements).
- Using independence, the probability of this happening is,

Successes | Failures

$$\underbrace{\pi \times \pi \times \cdots \times \pi}_{x \text{ times}} \times \underbrace{(1-\pi) \times (1-\pi) \times \cdots \times (1-\pi)}_{n-x \text{ times}} = \pi^x (1-\pi)^{n-x}$$

- The **number of ways** we could have observed x *"successes"* (and thus $n-x$ *"failures"*) from n trials is ${}^{n}_{x}\boldsymbol{C}$.

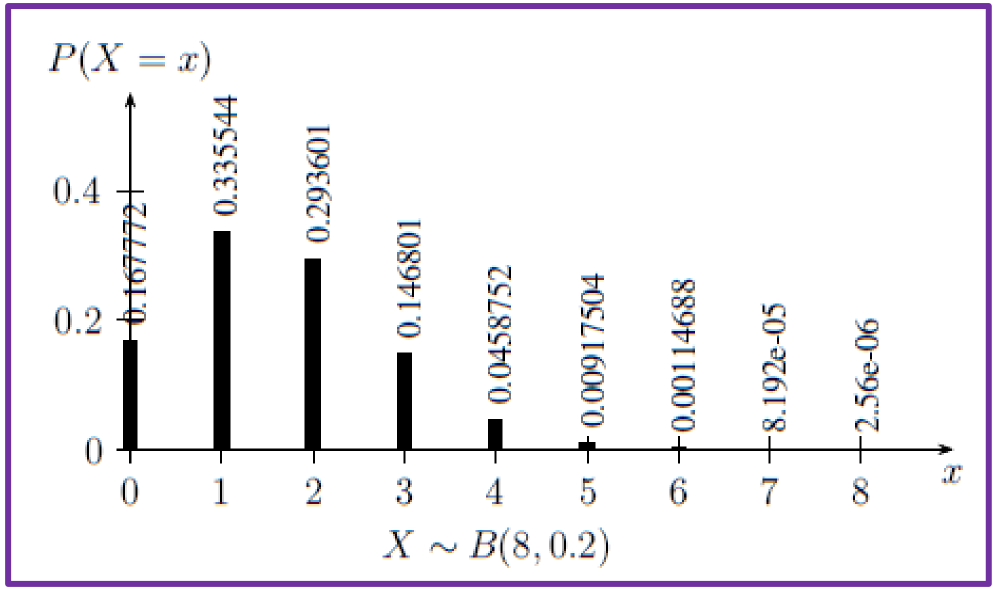
Hence the result,

$$\boxed{P(X=x) = {}^{n}_{x}C\pi^x(1-\pi)^{n-x}, \qquad \text{for } x = 0,1,2,\ldots,n;\ and\ \ 0 < \pi < 1}$$

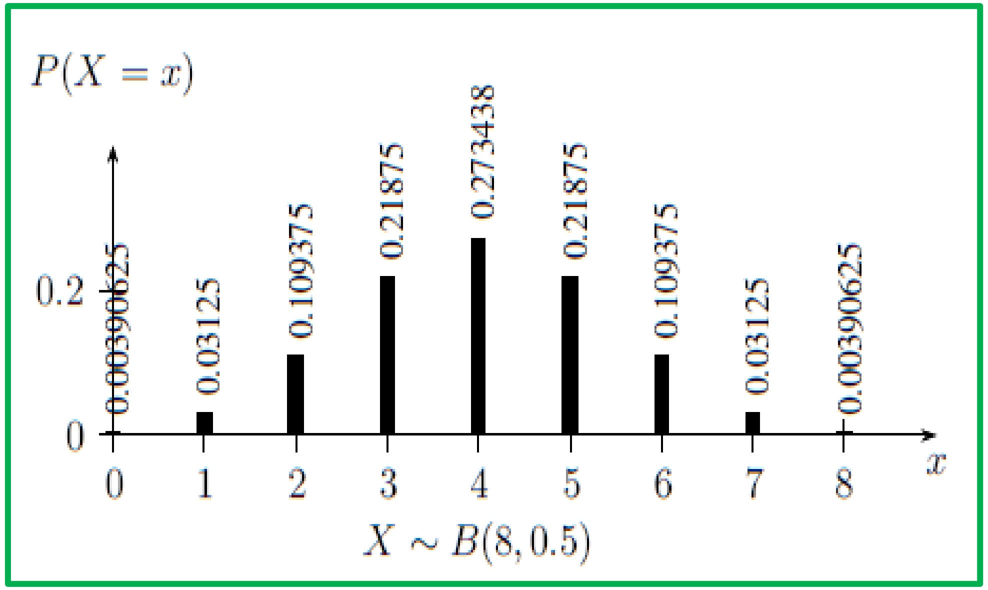
The Binomial Distribution

The Binomial Mass Function

Some examples of Binomial probability distributions (mass functions) are shown in Figure below.



Binomial Mass Function for n=8, π=0.2



Binomial Mass Function for n=8, π=0.5

Note:

- The difference in the shape of the distributions - can you explain why?
- Make sure that you can calculate the values of the probabilities that are displayed.

The Binomial Distribution

Example 7.2 (*Use of **Formula***)

Suppose a **die** is rolled 4 times. What is the probability of getting,

(a) exactly 1 six?

(b) at most 1 six

Firstly, is the Binomial distribution appropriate for this situation?

- The number of (independent) trials is $\boldsymbol{n}$ **= 4**.
- Each trial can result in one of two outcomes, a six (labelled "*success*"), or not a six (labelled "*failure*").
- The probability of a **six** on each roll of the die is 1/6 (⇒ probability not six = 5/6).

Hence all the conditions for the application of the Binomial distribution apply.

The Binomial Distribution

Example 7.2 (*Use of **Formula***)

Here $\boldsymbol{n}$**=4**; $\boldsymbol{\pi}$**=** $\frac{\mathbf{1}}{\mathbf{6}}$,

$$P(X = x) = {}^{n}_{x}C\pi^{x}(1-\pi)^{n-x}, \qquad \text{for } x = 0,1,2,\ldots,n;\ and\ \ 0 < \pi < 1$$

(a)

$$P(X = 1) = {}^{4}_{1}C\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{4-1}$$

$$= 4 \times \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{3} = \mathbf{0.3858}$$

(b)

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= {}^{4}_{0}C\left(\frac{1}{6}\right)^{0}\left(\frac{1}{6}\right)^{4-0} + {}^{4}_{1}C\left(\frac{1}{6}\right)^{1}\left(\frac{1}{6}\right)^{4-1}$$

$$= \left(\frac{1}{6}\right)^{4} + 0.3858 = \mathbf{0.8681}$$

The Binomial Distribution

Example 7.2 (*Use of **Formula***)

A train station has **5** self-service ticket machines. The probability of a **machine not working** at any time is **0.15**. Find the probability that the number of **machines not working** is,

(a) exactly 2 $\longrightarrow$ $\{=\}$

(b) at least 4 $\longrightarrow$ $\{\geq\}$

(c) at most 2 $\longrightarrow$ $\{\leq\}$

Firstly, we need to check whether we have a Binomial distribution.

- The number of machines is **fixed at 5**. Each machine is either **not working** with probability **0.15**, or working with probability **0.85** - **two outcomes**.
- The only assumption we need to make is that the machines **operate independently**.

Thus, we have a Binomial distribution for the number of machines not working, $\mathbf{X}$, with $\boldsymbol{\pi}$ **= 0.15** and $\boldsymbol{n}$ **= 5**. Using the formula, we have

The Binomial Distribution

Example 7.2 (***Use of Formula***)

$\boldsymbol{n}$**=5;** $\boldsymbol{\pi}$**=0.15**

$$\boxed{P(X=x) = {}^{n}_{x}C\pi^{x}(1-\pi)^{n-x}, \qquad \text{for } x = 0,1,2,\ldots,n;\ and\ \ 0<\pi<1}$$

(a)

$$P(X{=}2) = {}^{5}_{2}C \times 0.15^{2}(1-0.15)^{5-2}$$

$$= 10 \times 0.15^{2} \times 0.85^{3}$$

$$= \mathbf{0.1382}$$

(b)

$$P(X \geq 4) = P(x=4) + P(X=5)$$

$$= {}^{5}_{4}C \times 0.15^{4}(1-0.15)^{5-4} + {}^{5}_{5}C \times 0.15^{5}(1-0.15)^{5-5}$$

$$= 5 \times 0.15^{4} \times 0.85 + 1 \times 0.15^{5}$$

$$= \mathbf{0.0022}$$

The Binomial Distribution

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Example 7.2 (*Use of Formula*)

(c) $\boldsymbol{n}$**=5;** $\boldsymbol{\pi}$**=0.15**

$$\boxed{P(X = x) = {}^{n}_{x}C\pi^{x}(1-\pi)^{n-x}, \qquad \text{for } x = 0,1,2,\ldots,n;\ and\ \ 0 < \pi < 1}$$

$$P(X \le 2) = P(x = 0) + P(X = 1) + P(X = 2)$$

$$= {}^{5}_{0}C \times 0.15^{0}(1-0.15)^{5-0} + {}^{5}_{1}C \times 0.15^{1}(1-0.15)^{5-1} + {}^{5}_{2}C \times 0.15^{2}(1-0.15)^{5-2}$$

$$= 1 \times 0.85^{5} + 5 \times 0.15 \times 0.85^{4} + 10 \times 0.15^{2} \times 0.85^{3}$$

$$= 0.4437 + 0.3915 + 0.1382$$

$$= \mathbf{0.9734}$$

The Binomial Distribution

Using Binomial tables

- Clearly, evaluating probabilities like this can be very time consuming. Because the Binomial distribution is very widely applied, it is easy to find tables of Binomial probabilities, although be very careful about the format in which they are presented.

The tables used at MMU give probabilities for selected values of n and π in the form **P(X ≤ x).**

However, any probability can be found using some simple **rules**, for example:

1. **P(X ≤ x)** - found directly from tables *{At most}*
2. **P(X ≥ x) = 1 – P(X ≤ x – 1)** - the law of complements *{At least}*
3. **P(X = x) = P(X ≤ x) – P(X ≤ x – 1)** - difference of two sets *{Exactly}*
4. **P(a ≤ X ≤ b) = P(X ≤ b) – P(X ≤ a – 1)**
   - note what happens to the end points *{Between a and b}*

The Binomial Distribution

Example 7.3 (*Ticket Machines*) – *Use of* **Table Rules and table values**

To evaluate the three probabilities for the **ticket machine example** we proceed as follows.

Firstly, identify the **column** in the Binomial table for which $\boldsymbol{n} = \mathbf{5}$ and $\boldsymbol{\pi} = \mathbf{0.15}$. The answers are found as,

- To find $P(X = 2)$, use

$$\begin{aligned} P(X = 2) &= P(X \leq 2) - P(X \leq 1) \\ &= 0.9734 - 0.8352 \\ &= \mathbf{0.1382} \end{aligned}$$

- To find $P(X \geq 4)$, use

$$\begin{aligned} P(X \geq 4) &= 1 - P(X \leq 3) \\ &= 1 - 0.9978 \\ &= \mathbf{0.0022} \end{aligned}$$

- $P(X \leq 2) = \mathbf{0.9734}$ **straight from tables** (rule 1).

1. **P(X ≤ x)** - *{At most}*
   found directly from tables
2. **P(X ≥ x) = 1 – P(X ≤ x – 1)** - *{At least}*
   the law of complements
3. **P(X = x) = P(X ≤ x) – P(X ≤ x – 1)** - *{Exactly}*
   difference of two sets
4. **P(a ≤ X ≤ b) = P(X ≤ b) – P(X ≤ a – 1) -**
   *{Between **a** and **b**}*
   - note what happens to the end points

The Binomial Distribution
Mean and Variance

It can be shown that **Mean** $(\boldsymbol{\mu})$,

$$\boxed{\textbf{Mean}(\boldsymbol{\mu}) = \textbf{E [X]} = \boldsymbol{n\ \pi}}$$

And **Variance**,

$$\boxed{\boldsymbol{var\ (X) = n\pi(1-\pi)}}$$

Note: Derivation of Mean and Variance is available on next few slides. We can discuss these in lab session (if required).

The Poisson Distribution (Chapter 8)

The Poisson Distribution

This distribution was discovered by the French mathematician S.D. Poisson in 1837. It can be **applied in a remarkable number of areas**.

Examples might include:

- the number of buses passing down Oxford Road in one minute.
- the number of passengers on each bus.
- the number of misprints in each chapter of these notes when first produced!

The Poisson Distribution

Conditions for the Poisson Distribution

The Poisson distribution is applied whenever the following conditions are met,

- the random variable of interest is the ***number of events*** *that occur in a* ***given interval***.

- **events** occur at ***random and independently*** *in a given* ***interval***.

- ***events occur uniformly*** *in a given* ***interval***, i.e. the mean **number of events** in a given interval is **proportional to the size of the interval**.

The Poisson Distribution

Example 8.1 (*Telephone Calls*)

This is the classic example of a Poisson distribution often used in textbooks. We have,

- The random variable is the **number of calls arriving** at a **switchboard** in a given **time interval**, say **one minute**.
- We assume individuals making the calls are doing so **independently** of each other and that individual calls arrive at some **random rate**.
- If we **doubled the observation period**, say to **two minutes**, we'd expect to observe **double the number of calls on average** - this is the idea of a **uniform rate**.

The Poisson Distribution

Definition 8.1 (*The Poisson probability distribution*)

The probability **mass function** for a random variable, $\boldsymbol{X}$, following a Poisson distribution with **average rate**, $\boldsymbol{\mu}$ say, is given by the **formula**,

$$\boxed{P(X = x) = \frac{\mu^x e^{-\mu}}{x!}, \qquad x = 0,1,2,3 \ldots \ldots ; \;\; \mu > 0}$$

Where,

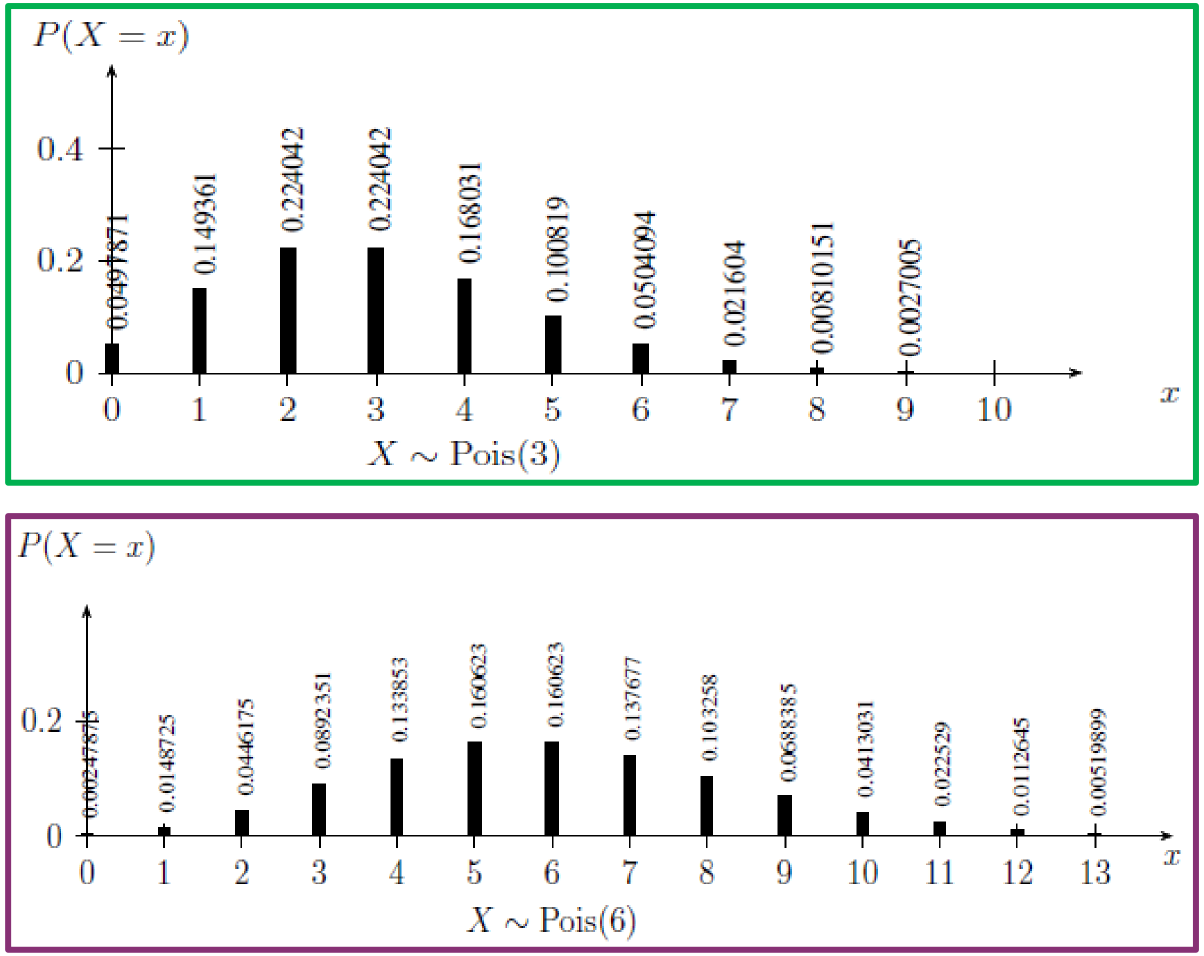
μ = average rate (mean rate)

x = number of successes observed

The Poisson Distribution

Some examples of Poisson probability distributions (**mass functions**) are shown in Figure below:

Note that the range of X **extends to +∞**, although the probabilities attached to higher values for these examples quickly become very small.



Poisson Mass Functions

The Poisson Distribution

Example 8.2 (*Telephone Calls*)

A company operates a busy switchboard. **Calls arrive** at a **mean rate of 3.5 per minute** and **leave** at a **mean rate of 4.2 per minute**. Find the probability that,

a) at least five calls **arrive in one minute**

b) exactly five calls **arrive in one minute**

C) at most 7 calls leave in one minute

d) between 4 and 9 calls inclusive leave in one minute.

Note firstly that

- The **mean rates do not need to be whole numbers**.
- Let the random variables

$$X = \{\text{number of calls } \textbf{arriving}\}$$

and

$$Y = \{\text{number of calls } \textbf{leaving}\}$$

at the switchboard.

Shorthand notation:

We can write $X \sim$ **Pois(3.5)** and $Y \sim$ **Pois(4.2)**, where the symbol $\sim$ is to be read as "**has the (probability) distribution**".

The Poisson Distribution

Example 8.2 (*Telephone Calls*)

- To use the Poisson tables, we must first identify the **mean rate** of the Poisson distribution.
- For the first two questions about $\boldsymbol{X}$ - the number of calls **arriving** we need the row labelled **3.5**.
- As with the Binomial distribution, **the tables are given in ≤ format**.

We then find,

a) $P(X \geq 5) = 1 - P(X \leq 4)$
$= 1 - 0.7254$
$= \mathbf{0.2746}$

b) $P(X = 5) = \mathrm{P}(X \leq 5) - P(X \leq 4)$
$= 0.8576 - 0.7254$
$= \mathbf{0.1322}$

The tables used at MMU give probabilities for selected values of $\boldsymbol{\mu}$ in the form **P(X ≤ x).**
However, any probability can be found using some simple **rules**, for example:

1. **P(X ≤ x)** - found directly from tables *{At most}*
2. **P(X ≥ x) = 1 − P(X ≤ x − 1)** - the law of complements *{At least}*
3. **P(X = x) = P(X ≤ x) − P(X ≤ x − 1)** - difference of two sets *{Exactly}*
4. **P(a ≤ X ≤ b) = P(X ≤ b) − P(X ≤ a − 1)**
   - note what happens to the end points *{Between **a** and **b**}*

The Poisson Distribution

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Example 8.2 (*Telephone Calls*)

- For the next two questions we need to refer to the row labelled **4.2** since, with outgoing calls, we are dealing with a Poisson distribution with this mean rate.

We then have,

c) $P(Y \leq 7) = \mathbf{0.9361}$

d) $P(4 \leq Y \leq 9) = P(Y \leq 9) - P(Y \leq 3)$

$= 0.9889 - 0.3954$

$= \mathbf{0.5935}$

1. **P(X ≤ x)** - *{At most}*
   found directly from tables
2. **P(X ≥ x) = 1 − P(X ≤ x − 1)** - *{At least}*
   the law of complements
3. **P(X = x) = P(X ≤ x) − P(X ≤ x − 1)** - *{Exactly}*
   difference of two sets
4. **P(a ≤ X ≤ b) = P(X ≤ b) − P(X ≤ a − 1)** - *{Between **a** and **b**}*
   - note what happens to the end points

The Poisson Distribution

It's worth looking at the last example **(d)** in more detail.

Consider the elements of the two sets,

$\{Y \leq 3\} \equiv \{0, 1, 2, 3\}$ and $\{Y \leq 9\} \equiv \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The **difference** of these **two sets** is then,

$$P(Y \leq 9) - P(Y \leq 3) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{0, 1, 2, 3\}$$
$$= \{4, 5, 6, 7, 8, 9\}$$
$$= P(4 \leq Y \leq 9)$$

which is the set of values for which we wish to find the probability.

The moral here is to be **careful** about **end-points** of sets - if in doubt, write down the sets in a basic form so that you can identify the appropriate values.

The Poisson Distribution

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Further properties

As mentioned in the introduction, **one** important aspect of the Poisson distribution is **uniformity**.

This means that we assume,

- events occur at a **uniform rate over the interval** of interest.
- If the size of the interval changes, then we must **change the mean rate in proportion**.

Example 8.3 (*Telephone Calls (again)*)

Find the probability that

a) at least 20 calls arrive at the exchange in a **4 minute** period

b) at most 1 call arrives in a **12 second** period.

The Poisson Distribution

Example 8.3 (*Telephone Calls (again)*)

To answer these problems we must **calculate the rate** at which calls will arrive in the **specified time period**.

- If calls arrive at the rate of 3.5 per minute, then, in a 4 minute period, we would expect calls to arrive at a rate of **4 × 3.5 = 14**.
- Similarly, in a 12 second period, we would expect calls to arrive at a rate of **3.5 ÷ 5 = 0.7** (12 secs is 1/5 of a minute).

Therefore, defining **two new variables**, we can write,

$$W = \text{number of calls in 4 minutes}$$
$$\sim \text{Pois}(14)$$
$$Z = \text{number of calls in 12 secs}$$
$$\sim \text{Pois}(0.7)$$

The Poisson Distribution

Example 8.3 (*Telephone Calls (again)*)

Once we've identified the appropriate rate, we simply use the tables as before,

a) $P(W \geq 20) = 1 - P(W \leq 19)$ $\{W \sim \text{Pois}(14)\}$

$= 1 - 0.9235$

$= \mathbf{0.0765}$

b) $P(Z \leq 1) = \mathbf{0.8442}$ $\{Z \sim \text{Pois}(0.7)\}$

The **second** useful property of the Poisson distribution is that different **Poisson variables can be added** to provide **another Poisson variable** - all we do is add together the individual rates.

The Poisson Distribution

Example 8.4 (*Telephone Calls*)

Suppose we don't differentiate between incoming and outgoing calls, e.g.

Let W = total number of calls

$= X + Y$

= Pois(3.5) + Pois(4.2)

= Pois(7.7)

and, as before, once we've identified the mean rate - in this case **7.7** calls per minute on average - we use the tables as before.

The Poisson Approximation to the Binomial Distribution

We can turn the **derivation** of the Poisson distribution using the Binomial distribution on its head to arrive at the following result.

If **π is small and n large**, then

$$\mathbf{B}(n,\ \pi) \approx \mathbf{Pois}(\mu),\ \text{where}\ \mu = n\pi$$

The **rule of thumb** generally used is that **$n\pi < 5$** for the **approximation to be valid**.

Examples:

1. Suppose n = 50 and π = 0.05, then **$n\pi$ = 2.5**, so the approximation is valid.

From Binomial tables,

$$P(X = 2) = 0.5405 - 0.2794 = \mathbf{0.2611}$$

and from the Poisson tables with mean μ = 2.5,

$$P(X = 2) = 0.5438 - 0.2783 = \mathbf{0.2565}$$

The Poisson Approximation to the Binomial Distribution

2. Suppose n = 20 and π = 0.01, then $\boldsymbol{n\pi}$ **= 0.2**, so the approximation is valid.

From Binomial tables,

$P(X = 2) = 0.9990 - 0.9831 = \mathbf{0.0159}$

and from the Poisson tables with mean μ = 0.2,

$P(X = 2) = 0.9989 - 0.9825 = \mathbf{0.0164}$

3. Suppose n = 50 and π = 0.01, then $\boldsymbol{n\pi}$ **= 0.5**, so the approximation is valid.

From Binomial tables,

$P(X = 2) = 0.9862 - 0.9106 = \mathbf{0.0756}$

and from the Poisson tables with mean μ = 0.5,

$P(X = 2) = 0.9856 - 0.9098 = \mathbf{0.0758}$

The Poisson Distribution

How to look for values if Mean (μ) is not in the table:

Example: If average rate (μ) is 1.7 and the random variable **X** follows the Poisson distribution, then find the probability of $P(X<2)$.

Solution:

Since values for μ=1.7 are not listed in the Poisson table.

≤1 Value for μ= 1.6

≤1 Value for μ= 1.8

$P(X<2) = P(X \le 1)$ = average of 0.9885 and 0.9856 {a close approximation.}

Therefore,

$$P(X<2) = P(X \le 1) = \frac{0.9885 + 0.9856}{2} = \frac{1.9741}{2} = 0.98705 = \mathbf{0.9871}$$

The Poisson Distribution
Mean and Variance

It can be shown that **Mean**,

$$\boxed{\mathbf{E\,[X] = \mu}}$$

And **Variance**,

$$\boxed{\boldsymbol{var\,(X) = \mu}}$$

Note: Derivation of Mean and Variance is available on next few slides. We can discuss these in lab session (if required).