Lecture 2

Independent, Mutually Exclusive Events and Tree Diagrams

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Aims

- Understand the importance of independent events.
- Understand the idea of mutually exclusive events.
- Use tree diagrams to analyse probability models.

Independent and Mutually Exclusive Events

- Independence is a very important concept in statistics.
- Sometimes misused when it is assumed without justification.

Definition 2.1 (***Independent Events***)

- Two events, A and B say, are independent if and only if,

$$\mathbf{P(A \text{ and } B) = P(A) \times P(B)}$$

- i.e., the probability that both A and B happen is the product of the individual probabilities.
- The advantage of independent events is that, to find the probability of their joint occurrence, we can simply multiply together their individual probabilities.

Example 2.1 (*Independent Events*)

Some events that are independent include:

- Outcomes on successive tosses of a coin or die. (*What happened on the previous throw does not affect what happens on subsequent throws.*)
- The sex of babies. (*The sex of each baby is determined at random, notwithstanding the sexes of previous babies.*)

Note: independence is a property that must be established or assumed before it can be used to simplify calculations.

The next example shows how it might be used in practice.

Example 2.2 (*Power plant safety*)

Suppose a power plant has two safety systems, a primary system which works with probability 0.999, and a backup system which works with probability 0.890. Assuming the two systems operate independently, what is the reliability or safety of the plant?

- It is easy to work with complementary events.
- We have,

$$P(\text{plant safe}) = 1\text{-}P(\text{plant fails}) \qquad \textit{(Complement Law)}$$

But,

$$\begin{aligned} P(\text{Plant fails}) &= P(\text{both systems fail}) \\ &= P(\text{primary fails}) \times P(\text{backup fails}) \\ &= (1-0.999)\times(1-0.890) \\ &= 0.00011 \end{aligned}$$

so that,

$$P(\text{plant safe}) = 1 - 0.00011 = \mathbf{0.99989}.$$

Important points

1. Such calculations have often been used to arrive at unrealistic figures for the safety of complex operating processes, e.g. nuclear power plants.

    For example, it's easy to check that with three backup systems, each with a reliability of 0.99, the probability of failure assuming independence is 0.000001 - a reassuringly small figure!

2. However, we can only make these calculations if we can justify the assumption of independence.

    For example, it's not unusual to find that backup systems that are not used very often can be more unreliable than supposed when actually called upon.

Mutually Exclusive Events

This concept simply describes outcomes or events that cannot occur simultaneously.

Definition 2.2 (*Mutually Exclusive Events*)

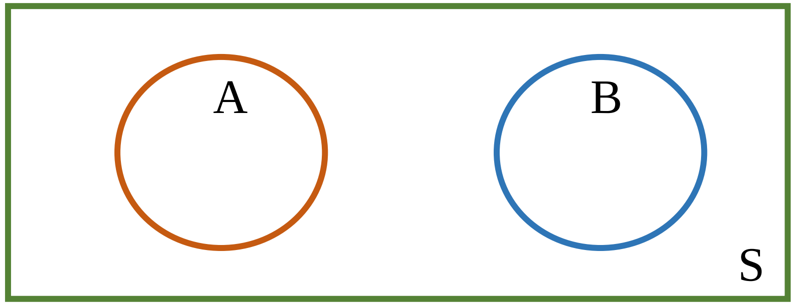
Two events, A and B say, are said to be mutually exclusive if

$$\mathbf{P(A \text{ and } B) = 0}$$

i.e. a joint probability of zero means it is impossible for them both to occur.

Mutually Exclusive Events

The idea of mutually exclusive events can be represented with a Venn diagram as follows,



The events A and B have no elementary events in common (*no overlap/intersection*).

- An example might be,

A={score even number} = {2,4,6}

and B={score odd number} = {1,3,5}

on the throw of a die, clearly mutually exclusive events!

Non-Mutually Exclusive Events

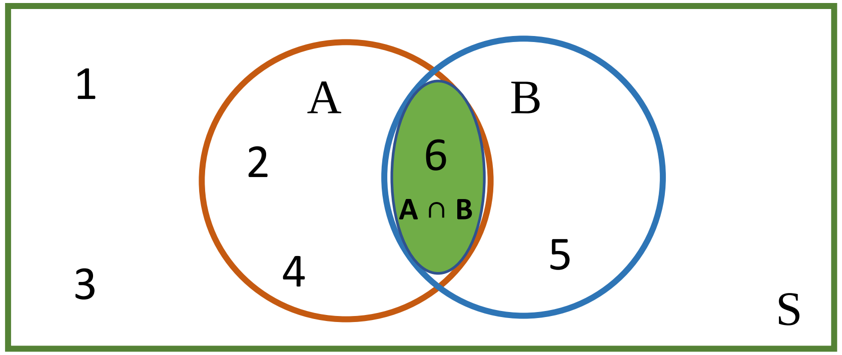
Consider the events,

A = {score even number} = {2,4,6}

and B = {score number > 4} = {5,6}

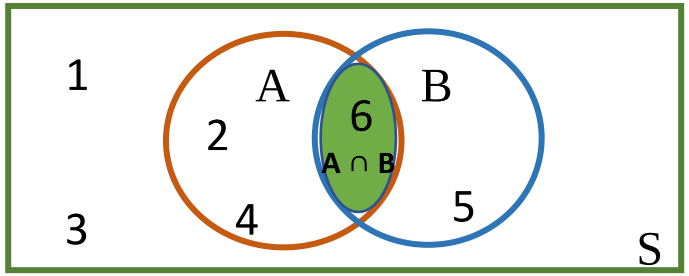
on the throw of a die.

- We can represent the outcomes using a Venn diagram as follows,



The set of events in common to both A and B is known as their **intersection** and in set terms is written as ∩.

The addition Law of probability



- Here we consider the calculation of probabilities involving an "**or**" statement, i.e.

P(A or B) (sometimes written as **P(AUB)**).

- In terms of a Venn diagram, this kind of probability is represented by the combined elements of the two sets, A and B.
- However, if we simply add up the elements of A and the elements of B, we will clearly have counted their elements of **($A \cap B$)** **twice**.
- The answer is to **subtract one copy** of the elements of $A \cap B$. This results in the following theorem,

$$\boxed{\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}}$$

Comment: *This is the simplest "inclusion/exclusion" law. Can you derive a further result for* ***$P(A \cup B \cup C)$****?*

Example 2.3

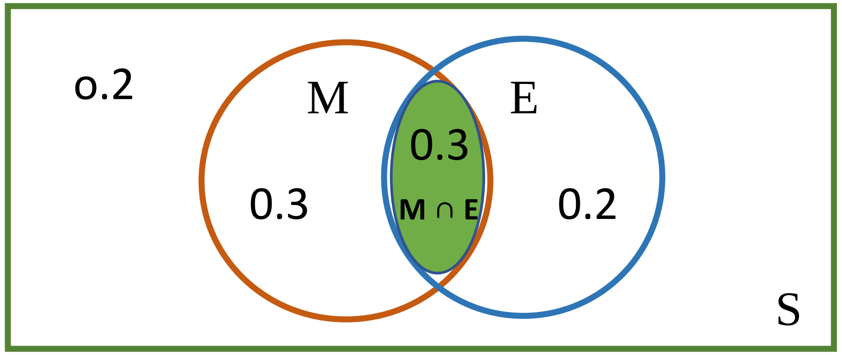
In a particular city, 60% of people read the morning paper, 50% of people read the evening paper and 30% read both. What is the probability that and individual selected at random reads the morning or the evening paper?

In an obvious notation, we have,

$$P(M) = 0.6,\ P(E) = 0.5 \text{ and } P(M \cap E) = 0.3$$

Therefore, $P(M \cup E) = 0.6 + 0.5 - 0.3 = 0.8$

This calculation can be represented by the following Venn diagram,



Example 2.4 (*Mode of travel*)

In lecture 1, we looked at example of mode of travel and where people lived. We can use this table to illustrate the **addition law** and **whether events are independent**.

- To calculate, for example, P(Car ∪ Town)

we have,

(The Proof/Demonstration is:) $P(\text{Car} \cup \text{Town}) = \dfrac{25 + 40 + 30}{100}$

Or $= \dfrac{65}{100} + \dfrac{70}{100} - \dfrac{40}{100}$

$=$ **0.95**

	Live		
	Town	Rural	**Total**
Car	40	30	**70**
Bus	25	5	**30**
Total	**65**	**35**	100

- Are the events travel by Car and live in Town independent?

We have, $P(\text{Car}) = \dfrac{70}{100}$ and $P(\text{Town}) = \dfrac{65}{100}$

P(A and B) = P(A)× P(B)

However, $P(\text{Car} \cap \text{Town}) = \dfrac{40}{100} \neq \dfrac{70}{100} \times \dfrac{65}{100}$ (*Hence not independent*)

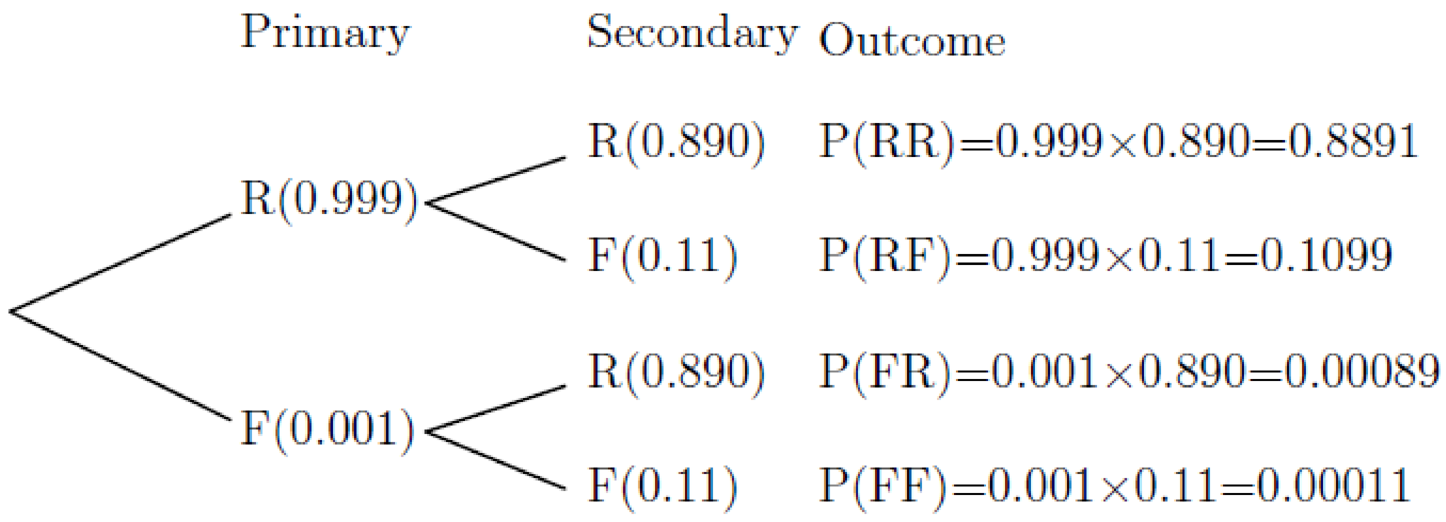
More on Tree Diagrams

- If we can assume independent events, we can analyse fairly complicated examples using tree diagrams.

- The idea is simply that at each branch of the tree, we multiply the current probability of that branch by the probability of each subsequent event occurring.

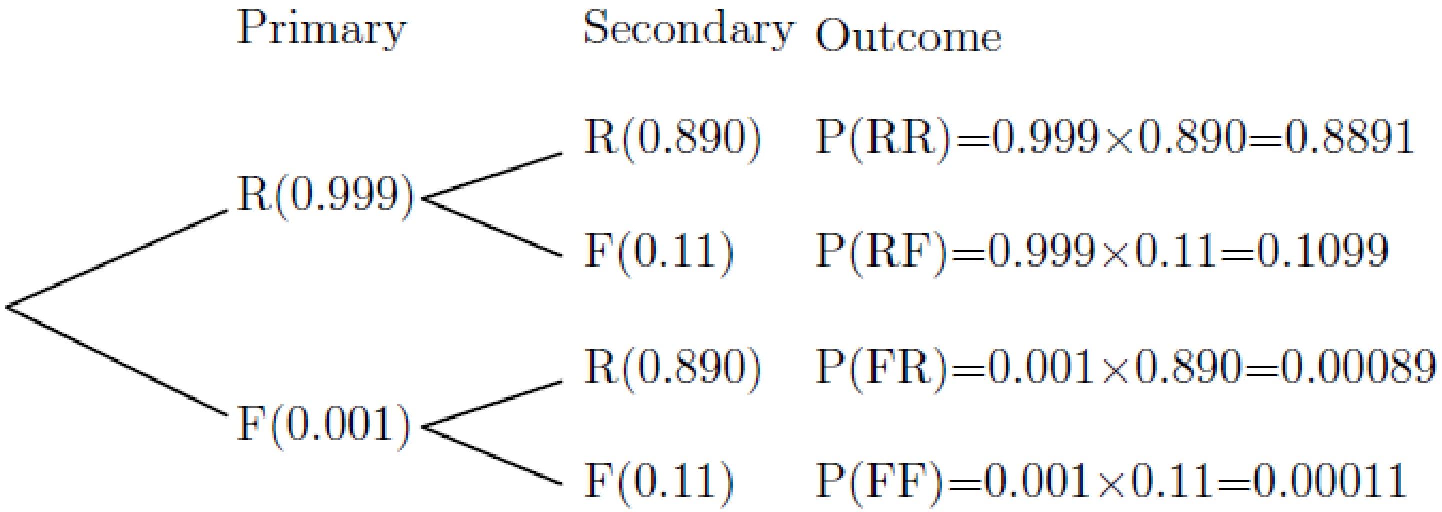
Example 2.5 (*Power Plant safety*)

Assuming independence, we can model the operation of the plant using the following tree diagram,



Note: The resulting set of probabilities should **always sum to one** as they represent the set of elementary outcomes. In this case we have,

Example 2.5 (*Power Plant safety*)



In this case we have,

P(RR) + P(RF) + P(FR) + P(FF) = 0.8891 + 0.1099 + 0.0089 + 0.00011

= 1.0, as required.

Kolmogorov's axioms (A Brief History)

The very first investigations into probability theory were made in the context of gambling games such as cards and roulette. Some of the most famous mathematicians of the 17th and 18th centuries applied their minds to the study of the odds of such games.

However, the theory of probability was not put on a properly **formal basis** until the seminal work of **Andrei kolmogorov** in Grundbegriffe der Wahscheinlichkeitsrechnung (*Foundations of the Theory of probability*) was published in 1933.

In essence, **Kolmogorov** was able to show that a maximum of **four simple axioms** was necessary and sufficient to define the way any and all probabilities must behave.

We will consider these axioms and some of their consequences in the next few slides.

Kolmogorov's Axiomatic Approach to Probability

If S has a finite number of members, Kolmogorov showed that as few as three axioms are necessary and sufficient to define the probability function P:

Axiom 1: Let A be any event defined over S. Then P(A) ≥ 0.

Axiom 2: P(S) = 1

Axiom 3: Let A and B be any two **mutually exclusive events** defined over S. Then

P(A ∪ B) = P(A) + P(B)

When S has an infinite number of members, a fourth axiom is needed:

Axiom 4: Let $A_1, A_2, \ldots$ be events defined over S. If $A_i \cap A_j = \emptyset$ for each $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The expression $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots$

Using these axioms we can prove some of the results we have already seen. Examples are on next slide.

Kolmogorov’s Axiomatic Approach to Probability

Theorem 2.1 P(A^c) = 1 − P(A)

The Proof/Demonstration is:

By **Axiom 2** and the definition of complementary events, we have

$$P(S) = 1 = P(A \cup A^c)$$

But A and A^c are mutually exclusive so, by **Axiom 3**

$$P(A \cup A^c) = P(A) + P(A^c)$$

and the result follows.

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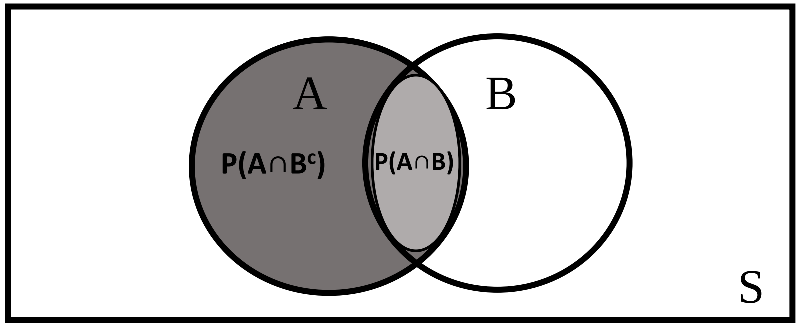
Contingency table *(Once again)*

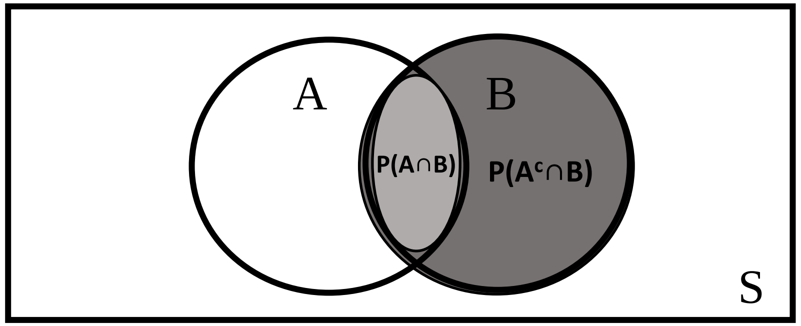
P(A) = P($A \cap B^c$) + P($A \cap B$)

P(B) = P($A^c \cap B$) + P($A \cap B$)

P(A)



P(B)



P(A) + P(A^c) = 1

P(B) + P(B^c) = 1

P(A) = 1 – P(A^c)

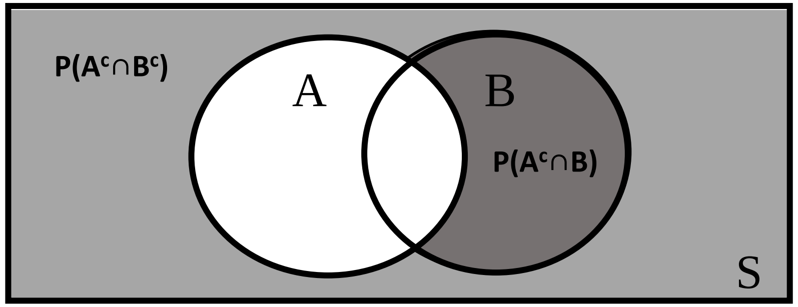
P(B) = 1 – P(B^c)

	P(A)	P(A^c)	Total
P(B)	$P(A \cap B)$	$P(A^c \cap B)$	$\boldsymbol{P(B)}$
P(B^c)	$P(A \cap B^c)$	$P(A^c \cap B^c)$	$\boldsymbol{P(B^c)}$
Total	$\boldsymbol{P(A)}$	$\boldsymbol{P(A^c)}$	**1**

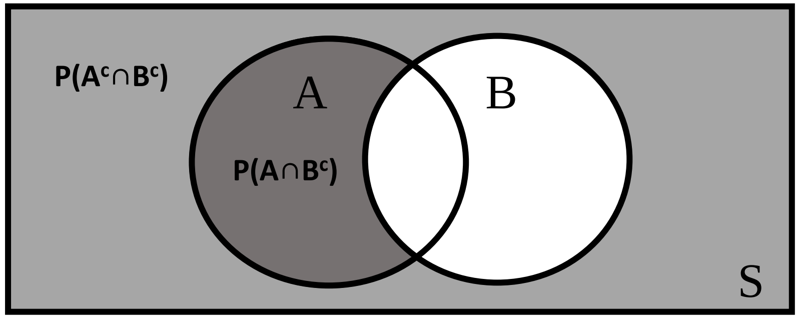
P(A^c) = $P(A^c \cap B) + P(A^c \cap B^c)$

P(B^c) = $P(A \cap B^c) + P(A^c \cap B^c)$

P(A^C)



P(B^C)



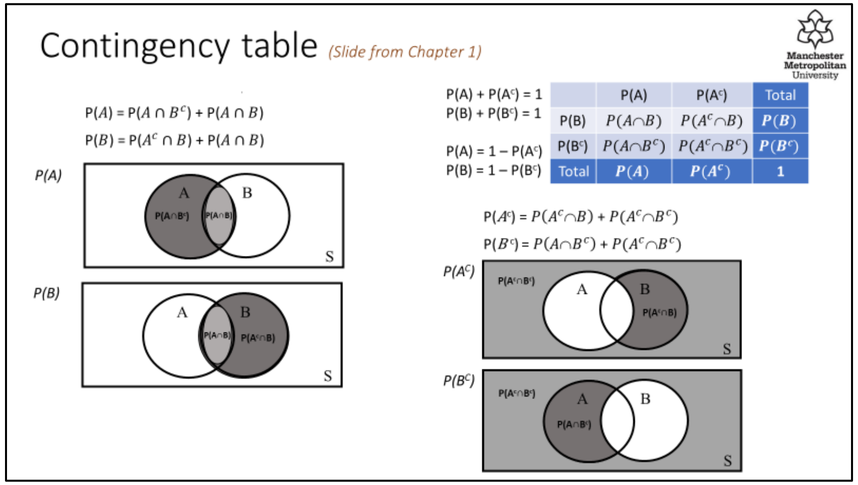
Kolmogorov’s Axiomatic Approach to Probability

Theorem 2.2 P($A \cup B$) = P(A) + P(B) - P($A \cap B$), *the addition law of probability*

The Proof/Demonstration is:

From **Axiom 3**, we can write

$$P(A) = P(A \cap B^c) + P(A \cap B)$$
$$P(B) = P(A^c \cap B) + P(A \cap B)$$



Adding these two equations gives,

$$P(A) + P(B) = \left[P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)\right] + P(A \cap B)$$

- Using a **special case of Axiom 4**, where the number of events is finite, the sum in **[ ]** is **P(A ∪ B)** - check with a Venn diagram. Therefore,

$$P(A) + P(B) = [P(A \cup B)] + P(A \cap B)$$

- Subtracting P(A ∩ B) from both sides gives the required result.