Lecture 5 Exercises

5.1 A recursion formula. The geometric distribution has $P(X = x) = \pi(1 - \pi)^{x-1}$. Show that,

$$P(X = x) = (1 - \pi)P(X = x - 1)$$

5.2 Let X be the value observed from rolling an 8-sided die

a) What is the probability distribution of X.

b) Draw a graph of the probability distribution.

c) Find the mean and variance of X.

d) Find the expected value of

(i) $3X + 5$

(ii) $\ln(X)$

5.3 An urn contains two yellow balls and three red balls. Three balls are drawn at random from the urn without replacement.

a) Draw a tree diagram to represent the sample space for this experiment and find the probabilities of each outcome.

b) Let the random variable X denote the number of red balls drawn.

(i) Write down the probability distribution of X

(ii) Find the mean and variance of X.

5.4 A game consists of tossing a coin until the first head appears. The score recorded is the number of tosses required.

a) If the random variable Y is the number of tosses, what is the distribution of Y?

b) Write down the first 6 values of the probability distribution and draw a rough sketch.

c) Find the mean and variance of Y

5.5 Two fair dice are rolled and the total score observed.

a) Write down the probability distribution of the total score.

b) Find the mean and variance of the total score.

5.6 Two fair dice are rolled and the *maximum* score observed.

a) Write down the probability distribution of the maximum score.

b) Find the mean and variance of the maximum score.

5.7 A fair coin is tossed three times. Let the r.v. X be the number of heads in the tosses minus the number of tails.

a) Find the probability distribution of X.

b) Find the mean and variance of X.

5.8 The game of simple *Chuck-a-luck* is played by a single player against the house. The game is conducted as follows:

The player chooses any number between 1-6 inclusive and places a bet of £1. The banker thens rolls 2 fair dice. If the player's number occurs 1 or 2 times, he wins £1 or £2, respectively. If the player's number does not appear on any of the dice, he loses his £1 stake. Let the random variable X denote the player's winnings on the game.

a) Find the probability mass function of X.

b) Find $\mathrm{E}[X]$.

5.9 The random variable X has the following probability mass function:

x	1	2	3	4	5
$P(X = x)$	7c	5c	4c	3c	c

a) Find c to make this a valid mass function.

b) Find $\mathrm{E}[X]$ and $\mathrm{var}(X)$

5.10 A discrete random variable Y has mass function:

y	2	3	5	7	11
$P(Y = y)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	a	b

and $\mathrm{E}[Y] = \frac{14}{3}$

a) Find a and b.

b) Find $\mathrm{var}(Y)$.

5.11 A fair six-sided die has '1' on one face; '2' on two of its faces and '3' on the remaining three faces.

a) Let the random variable Y denote the score on a single roll of the die.

(i) Tabulate the mass function of Y.

(ii) Find the mean and variance of Y

b) Let the random variable X denote the total score on two rolls of the die.

(i) Tabulate the mass function of X.

(ii) Find the mean and variance of X

5.12 An urn contains n (> 1) balls numbered 1 to n from which two balls are drawn simultaneously. Find

a) the probability distribution of X, the larger of the two numbers drawn.

b) the expected value of X

5.13 a) A and B play a game that involves each rolling a fair die simultaneously.

(i) Let the random variable X be the absolute difference in their scores. Tabulate the probability mass function of X

(ii) Find the mean and variance of X.

b) If the value of X is 1 or 2, A wins; if it is 3, 4 or 5, B wins and if it is zero, they roll again. Find the probability that A wins on the

(i) first go;

(ii) 2nd go;

(iii) rth go.

c) Find the probability that A wins.

5.14 A discrete random variable has the following mass function,

$$f(y;\pi) = \begin{cases} \pi & y = 1 \\ (1-\pi) & y = 0 \end{cases}$$

where $0 < \pi < 1$. This is known as the Bernouilli distribution. Find $\mathrm{E}[Y]$ and $\mathrm{var}(Y)$.

5.15 Markov's inequality. Suppose that X is a positive valued random variable with mean, $\mu > 0$, i.e. $P(X = x) = 0$ for any $x < 0$. Show that, for any $a > 0$,

$$P(X \geq a) \leq \frac{\mu}{a}$$

Hint: Use the same technique as the proof of Chebychev's theorem.

5.16 Some applications of Markov's inequality

a) Scores on a test have a mean mark of 65. Find an upper bound for the probability that a student will score 80 or more.

b) The time taken to be served in a fast food restaurant has a mean of 1 minute. Find an upper bound for the probability that a customer waits for more than 3 minutes.

c) Bags of sugar sold by a supermarket have a mean weight of 1kg. Due to strict quality control procedures, no bags weigh less than 0.95kg. Find an upper bound for the probability that a bag weighs more than 1.1kg.

d) The weight of laboratory mice are random with mean 175g. No animals weigh less than 166g. Find an upper bound for the probability that a randomly selected mouse weighs more than 190g.

5.17 The number of customers using a fast food restaurant in one hour is a random variable with mean 90 and standard deviation 12. Use Chebychev's inequality to find a bound for the probability that between 60 and 120 customers will be served.

5.18 The salt content of a packet of crisps is a random variable with mean 5g and standard deviation 0.25g. According to Chebychev's theorem, between what values must be the salt content of,

(i) at least 95% of packets;

(ii) at least 99% of packets?