MA5710

MMI

9/9/2021

Assignment 02

"Traffic Dynamics - Macroscopic Models"

Preamble (as discussed in the class)

I $\Rightarrow$

$$\boxed{\begin{array}{l} \rho_t + j(\rho)_x = 0, \quad x \in \mathbb{R}, t > 0 \\ \rho(x,0) \equiv \rho_0(x), \quad x \in \mathbb{R} \end{array}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} \dfrac{\partial \rho}{\partial t} + j'(\rho)\dfrac{\partial \rho}{\partial x} = 0 \\ \rho(x,0) = \rho_0(x) \end{array}}$$

II $\Rightarrow$ Characteristics Base Curve Equation

$$\boxed{x'(t) = j'(\rho_0(x_0)), \quad x(0) = x_0}$$

$$\Uparrow$$

III $\Rightarrow$

$$\boxed{\rho(x,t) = \rho_0(x_0)}$$

$$\Downarrow$$

$$\boxed{\rho(x,t) = \rho_0\left(x(t) - j'(\rho_0(x_0))\,t\right)}$$

IV $\Rightarrow$ Green Light Problem

$$j(\rho) = \begin{cases} \rho(1-\rho), & 0 \le \rho \le 1 \\ 0, & \rho > 1 \end{cases}$$

$(\rho_{max} = 1)$

$$\rho_0(x) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases}$$

V $\Rightarrow$ Rarefaction fan – Rarefaction waves

$\Downarrow$

$$\rho(x,t) = \begin{cases} 0, & x > t \\ \frac{1}{2}\left(1 - \frac{x}{t}\right), & -t < x \le t \\ 1, & x \le -t \end{cases}$$

Question 1

Draw the plot using MATLAB/PYTHON for the above solution for $x \in [-4, 4]$, $t \in [0, 3]$.

Question 2

Show that if

$$\rho(x,t) = f\left(\frac{x - x_0}{t}\right)$$

satisfies the conservation law

$\rho_t + j'(\rho)\rho_x = 0$ with $\rho_0(x) = \begin{cases} \rho_\ell, & x \leq x_0 \\ \rho_r, & x > x_0 \end{cases}$

then f must be the right inverse of j'. That is $j'(f(y)) = y$ for all $y \in [j'(\rho_\ell), j'(\rho_r)]$.

Question 3

Suppose j' is continuous and monotone on its domain. Show that f takes the values ρ_ℓ and ρ_r on the two edges of the cone between the characteristic base curves $x_\ell := x_0 + j'(\rho_\ell)t$ and $x_r := x_0 + j'(\rho_r)t$

VI ⇒

Intersecting characteristics – Shock waves

$$j(\rho) = 4\rho(2-\rho)$$

$$\rho_0(x) = \begin{cases} 1, & x \le 1 \\ \frac{1}{2}, & 1 < x \le 3 \\ 3/2, & x > 3 \end{cases}$$

Question 4

Construct the rarefaction wave solution emerging from $x = 1$ and show that

$$\rho(x,t) = 1 - \frac{x-1}{8t}$$

Question 5

Construct the shock solution for $t > 2$ and show that

$$\rho(x,t) = \begin{cases} 1, & x \le 5-2t \\ 3/2, & x > 5-2t \end{cases}$$

Question 6

Plot, using MATLAB/PYTHON, the complete solution of the above problem (as discussed in the class) for $x \in [-4, 4]$, and $t \in [0, 3]$

Question 7

Consider $j(\rho) = 2\rho - \rho^2$ with initial

Condition $\rho_0(x) = \begin{cases} 1/2, & x \leq 0 \\ 5/4, & x > 0 \end{cases}$

Show that this leads to a shock wave propagating in the traffic flow and determine the shock speed.

VII ⇒ $\boxed{\text{Numerical Schemes}}$

h – space step size

k – time step size

1. Upwind Method:

If $j'(f) > 0$

$$f_i^{n+1} = f_i^n - \frac{k}{h}\left(j(f_i^n) - j(f_{i-1}^n)\right)$$

If $j'(f) < 0$

$$f_i^{n+1} = f_i^n - \frac{k}{h}\left(j(f_{i+1}^n) - j(f_i^n)\right)$$

2. Lax – Friedrichs Scheme:

$$f_i^{n+1} = \frac{1}{2}\left(f_{i+1}^n + f_{i-1}^n\right) - \frac{k}{2h}\left(j(f_{i+1}^n) - j(f_{i-1}^n)\right)$$

3. Richtmyer Two-Step Lax-Wendroff Scheme:

$$f_i^* = \frac{1}{2}\left(f_i^n + f_{i+1}^n\right) - \frac{k}{h}\left(j(f_{i+1}^n) - j(f_i^n)\right)$$

$$f_i^{n+1} = f_i^n - \frac{k}{h}\left(j(f_i^*) - j(f_{i-1}^*)\right)$$

4. Mac-Cormack Scheme:

$$
\begin{cases}
\rho_i^{*} = \rho_i^{n} - \dfrac{k}{h}\left(j(\rho_{i+1}^{n}) - j(\rho_i^{n})\right) \\[2ex]
\rho_i^{n+1} = \dfrac{1}{2}\left(\rho_i^{n} + \rho_i^{*}\right) - \dfrac{k}{h}\left(j(\rho_i^{*}) - j(\rho_{i-1}^{*})\right)
\end{cases}
$$

5. Gudonov Method:

if $j'(\rho) \geq 0$ and $j'(\rho_{i+1}^{n}) \geq 0$

$$\rho_i^{*} = \rho_i^{n}$$

if $j'(\rho) < 0$ and $j'(\rho_{i+1}^{n}) < 0$

$$\rho_i^{*} = \rho_{i+1}^{n}$$

if $j'(\rho) \geq 0$ and $j'(\rho_{i+1}^{n}) < 0$

$$
\rho_i^{*} = \begin{cases} \rho_i^{n} & \text{if } s \geq 0 \\ \rho_{i+1}^{n} & \text{if } s < 0 \end{cases}
$$

where s is speed of the shock

if $j'(\rho) < 0$ and $j'(\rho_{i+1}^{n}) \geq 0$

ρ_i^{*} is the unique soln. of $j'(\rho_i^{*}) = 0$.

$$
\left\{ \rho_i^{n+1} = \rho_i^{n} - \frac{k}{h}\left(j(\rho_i^{*}) - j(\rho_{i-1}^{*})\right) \right.
$$

Question 8

Implement all the above mentioned 5 methods for (i) Green Light Problem and
(ii) Intersecting characteristics Problem
as given earlier.

Which method performs better? Compare the Solution with the explicit solution (as given in the class)

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