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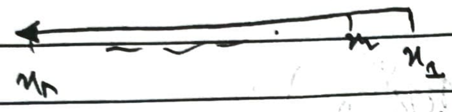
Question Three

$$m\ddot{x}_i(t+\tau) = F_{B_x}(t) = A\left(\frac{\dot{x}_{i-1}(t) - \dot{x}_i(t)}{\rho}\right) \to (3.1)$$

given in the question, A = constant.



at equilibrium: $\rho = \dfrac{1}{x_{i-1} - x_i}$ (if the order $x_{i-1} > x_i$ is maintained)

inserting ρ into (3.1), we get

$$m\ddot{x}_i(t+\tau) = A\left(\dot{x}_{i-1}(t) - \dot{x}_i(t)\right)\left(x_{i-1}(t) - x_i(t)\right)$$

$\downarrow A/m = \lambda$ (λ - is a positive priorly unknown proportionality constant)

$$\cancel{\ddot{x}_i(t+\tau) = \lambda \frac{d}{dt}}$$

$$\ddot{x}_i(t+\tau) = \dot{v}_i(t+\tau) = \lambda \frac{d}{dt}\left[\frac{(x_{i-1}(t) - x_i(t))^2}{2}\right]$$

On ~~d~~ Integrating we get:

$$\boxed{v_i(t+\tau) = \lambda\left[\frac{(x_{i-1}(t) - x_i(t))^2}{2}\right] + \alpha_i} \to (3.2)$$

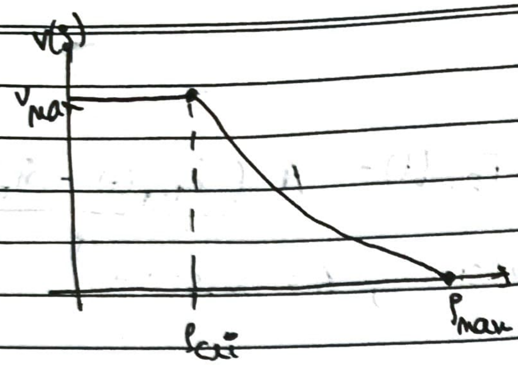
we know that $\rho = \dfrac{1}{x_{i-1} - x_i}$; putting this in (3.2):

$$\cancel{v_i(t+\tau) = \frac{\lambda}{2\rho^2} + \alpha_i}$$

$$\boxed{V(\rho) = \frac{\lambda}{2\rho^2} + \alpha_i} \to (3.3)$$

~~This relation is only for $\rho > \rho_{cri}$~~

$\because$ at $\rho < \rho_{cri}$ $V = V_{max}$ a constant



ρ_{cri} is the ~~p~~ density after which velocity of all the cars starts falling.

we know that $V(\rho_{max}) = 0$

$$V(\rho_{max}) = \frac{\lambda}{2\rho_{max}^2} + \alpha_i = 0 \qquad \text{from } (3.3)$$

$$\boxed{\alpha_i = \frac{-\lambda}{2\rho_{max}^2}}$$

$$\therefore \boxed{V(\rho) = \frac{\lambda}{2}\left(\frac{1}{\rho^2} - \frac{1}{\rho_{max}^2}\right)} \rightarrow (3.4)$$

~~for $\rho \le \rho_{cri}$ we know~~

Assuming that for $\rho \le \rho_{cri}$ all cars have the same velocity $V = V_{max}$, we apply the condition of continuity at $\rho = \rho$ ~~max~~ critical.

$$V(\rho_{cri}) = V_{max}$$

$$\therefore \frac{\lambda}{2}\left(\frac{1}{\rho_{cri}^2} - \frac{1}{\rho_{max}^2}\right) = V_{max}$$

$$\cancel{\lambda = \frac{2V_{max}\rho_{cri}^2\rho_{max}^2}{\rho_{max}\rho_{max} - \rho_{cri}^2}}$$

$$\boxed{\lambda = \frac{2V_{max}\rho_{cri}^2\rho_{max}^2}{\rho_{max}^2 - \rho_{cri}^2}} \rightarrow (3.8)$$

also, $\rho_{max} = \frac{1}{L}$ since $v(\rho_{max}) = 0$ and cars have been stopped, there will be one car followed by another without any gap.

inserting λ from 3.5 to $v(\rho)$ in 3.4 ;

$$V(\rho) = \frac{v_{max}\,\rho_{cri}^2\,\rho_{max}^2}{\rho_{max}^2 - \rho_{cri}^2}\left(\frac{1}{\rho^2} - \frac{1}{\rho_{max}^2}\right)$$

$$\cancel{V(\rho) = \frac{v_{max}\,\rho_{cri}^2}{\rho^2}\,\frac{(\rho_{max}^2 - \rho^2)}{(\rho_{max}^2 - \rho_{cri}^2)}}$$

$$\boxed{V(\rho) = \frac{v_{max}\,\rho_{cri}^2}{\rho^2}\left[\frac{\rho_{max}^2 - \rho^2}{\rho_{max}^2 - \rho_{cri}^2}\right]} \rightarrow 3.6$$

Now, we check for maximum flux using the density.

$$j(\rho) = \rho\, v(\rho)$$

$$j(\rho) = \begin{cases} \rho\, v_{max} & \rho \le \rho_{cri} \\ \frac{v_{max}\,\rho_{cri}^2}{\rho}\left[\frac{\rho_{max}^2 - \rho^2}{\rho_{max}^2 - \rho_{cri}^2}\right] & \rho > \rho_{cri} \end{cases}$$

~~if $\rho \le \rho_{cri}$~~

differentiating $j(\rho)$ for $\rho > \rho_{cri}$ and equating it to zero :

we get $$-\frac{v_{max}\,\rho_{cri}^2}{(\rho_{max}^2 - \rho_{cri}^2)}\left(\frac{\rho_{max}^2}{\rho_{opt}^2} + 1\right) = 0$$

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$$\rho_{opt}^2 = -\rho_{max}^2$$

$\rho_{opt} = i\,\rho_{max}$ which is not possible since ρ_{opt} is not real.

$\therefore$ given the boundary condition ρ_{cri} and the fact that $V(\rho)$ falls when ρ closer ρ_{cri}, flux $(\rho V(\rho))$ will be maximum at ρ_{cri}.

So
$$\boxed{\begin{aligned} V(\rho_{opt}) &= V(\rho_{crit}) = V_{max} \\ j(\rho_{opt}) &= j(\rho_{crit}) = \rho_{crit} V_{max} \end{aligned}} \rightarrow (3.8)$$

Now we can find the difference b/w the displacements caused by perturbed propog and the unperturbed ~~con~~ case.

$y_i(t)$ = displacement of ith car at eqm ($x_N > \dots x_{..} > x_2$ is maintained)

~~displacement $y_i(t)$ can be~~ is ~~given~~

$$(3.9) \leftarrow \boxed{y_i(t) = Vt - (i-1)(d+L)}$$

d = distance b/w car
L = length of car

$\dot{y}_i(t) = V$

At $\rho = \rho_{opt}$, $V = V_{max}$ from (3.7)
and since at eq^m we consider ρ_{opt}

$$\boxed{\dot{y}_i(t) = V_{max}} \qquad \boxed{y_i(t) = V_{max}\, t}$$

from 3.2 :

$$v_i(t+\tau) = \lambda\left[\frac{(x_{i-1}(t)^2 - x_i(t))^2}{2}\right] + \alpha_i$$

inserting values ~~from~~ of λ, α_i, we get :

$$\cancel{v_i(t+\tau) = \frac{v_{max}\rho_{ci}^2}{\rho^2}\left[x_{i-1}(t)^2 - x_i(t)^2\right.}$$

$$\boxed{v_i(t+\tau) = \frac{v_{max}\,\rho_{ci}^2\,\rho_{max}^2}{\rho_{max}^2 - \rho_{ci}^2}\left[(x_{i-1}(t) - x_i(t))^2 - \frac{1}{\rho_{max}^2}\right]} \quad (3.8)$$

Now we consider the case when perturbation happens. The ~~x~~ instance when perturbation occurs is $t = t_0 = 0$

at $t < 0$, we consider eqm conditions when each car is moving at same speed.

At $t=0$, the lead driver applies the brakes for a short time t, then accelerates back to velocity v_{max}

The lead car

$$v_1(t) = \begin{cases} v_{max} & t<0 \\ v_{max}(1-b(t)) & t \in [0, t_1) \\ v_{max} & t > t_1 \end{cases}$$

we integrate this to get displacement of lead car in perturbed condition

$$x_1(t) = \begin{cases} v_{max}t & t<0 \\ v_{max}(t - B(t)) & t \in [0, t_1) \end{cases}$$

perturbed position of lead car.

$$B(t) = \begin{cases} 0 & t<0 \\ \int_0^t b(s)\,ds & \text{for } t \in [0, t_1) \end{cases}$$

now, we will find the difference $z_s(t)$ which is :

$z_s(t)$ = true position of the car − unperturbed pos^n of the car
$= x_s(t) - y_s(t)$

$y_s(t) = v_{max} t$ since it is in eq^n.

$$\boxed{z_s(t) = \begin{cases} v_{max} t - v_{max} t = 0 & \text{for } t < 0 \\ -v_{max} B(t) & \text{for } t \in [0, t_s) \end{cases}}$$

IMPACT ON THE FOLLOWING CARS

before $t=0$, each of the trailing car will have velocity $= v_{max}$.

$z_i(t) = x_i(t) - y_i(t)$ for $i \geq 2$.

$$\boxed{z_i(t) = \begin{cases} 0 & t \leq 0 \\ x_i(t) - v_{max} t + (i-1)(d+L) & t \in [0, t_s] \end{cases} \quad \boxed{i \geq 2}}$$

(3.10)

this is from (3.9)
$y_i(t) = v_{max} t - (i-1)(d+L)$

~~taking the case when perturbation occurs~~ :

$$\cancel{z_i(t) = \begin{cases} 0 & t \leq 0 \\ x_i(t) - y_i(t) + (i-1) \times \frac{1}{\rho_{cri}} & t \in (0, t_s] \end{cases}} \Rightarrow \cancel{(3.10)}$$

$\cancel{\text{because } d+L = \frac{1}{\rho_{cri}}}$

we now differentiate $z_i(t)$, to later use it in our differential delay equation.

$$\dot{z}_i(t) = \begin{cases} 0 & t < 0 \\ \dot{x}_i(t) - V_{max} & t \in (0, t_a) \end{cases}$$

for $t > 0$

$$\dot{z}_i(t) = \dot{x}_i(t) + V_{max}$$

$$\dot{z}_i(t+\tau) = \dot{x}_i(t+\tau) - V_{max}$$

using (3.8):

$$\dot{z}_i(t+\tau) = \frac{V_{max}\,\rho_{ci}^2\,\rho_{max}^2}{\rho_{max}^2 - \rho_{ci}^2}\left[\left(x_{i-1}(t) - x_i(t)\right)^2 - \frac{1}{\rho_{max}^2}\right] - V_{max}$$

$$\therefore \boxed{V_{max} + \frac{d z_i}{dt}(t+\tau) = kV_{max}\left[\left(x_{i-1}(t) - x_i(t)\right)^2 - \frac{1}{\rho_{max}^2}\right] \quad \text{where } k = \frac{\rho_{ci}^2\,\rho_{max}^2}{\rho_{max}^2 - \rho_{ci}^2}}$$

(3.10)

Using (3.10), we have: for $t > 0$

$$z_{i-1}(t) = x_{i-1}(t) - V_{max}t + \frac{(i-2)}{\rho_{ci}}$$

and

$$z_i(t) = x_i(t) - V_{max}t + \frac{(i-1)}{\rho_{ci}}$$

$$z_{i-1}(t) - z_i(t) = x_{i-1}(t) - x_i(t) - \frac{1}{\rho_{ci}}$$

$$\therefore \boxed{x_{i-1}(t) - x_i(t) = z_{i-1}(t) - z_i(t) + \frac{1}{\rho_{ci}}}$$

putting this in (3.11), we get:

$$\cancel{V_{max} + \frac{d}{dt}\left(z_i(t+\tau)\right) = KV_{max}\left[\left(z_{i-1}(t) - z_i(t)\right)^2\right.}$$

$$\boxed{V_{max} + \frac{d}{dt}\left(z_i(t+\tau)\right) = KV_{max}\left[\left(z_{i-1}(t) - z_i(t) + \frac{1}{\rho_{ci}}\right)^2 - \frac{1}{\rho_{max}^2}\right]}$$

THIS IS THE DELAYED DIFFERENTIAL EQUATION.

for $2 \leq i \leq N$ with initial condition

$z_i(t) = 0$ for $t < 0$ and $1 \leq i \leq N$

and $z_1(t) = \begin{cases} 0 & t \leq 0 \\ -v_{max}\, B(t) & t \in (0, t_1] \end{cases}$