papergrid

Date: / /

MA5710 : Assignment II

Devansh Sanghvi : BE19B002

Q2) $\rho(x,t) = f\left(\frac{x-x_0}{t}\right)$

satisfies conservation law :

$$\frac{\partial \rho}{\partial t} + j'(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad \rho_0(x) = \begin{cases} \rho_l, & x \le x_0 \\ \rho_r, & x > x_0 \end{cases}$$

$$\boxed{\frac{\partial f}{\partial t}\left(\frac{x-x_0}{t}\right) + j'\left(f\left(\frac{x-x_0}{t}\right)\right)\frac{\partial f}{\partial x}\left(\frac{x-x_0}{t}\right) = 0}$$

$\rightarrow \frac{\partial f}{\partial t}\left(\frac{x-x_0}{t}\right) = -f'\left(\frac{x-x_0}{t}\right)\left(\frac{x-x_0}{t^2}\right)$ using chain rule

and $\frac{\partial f}{\partial x}\left(\frac{x-x_0}{t}\right) = f'\left(\frac{x-x_0}{t}\right)\frac{1}{t}$

$$\therefore -f'\left(\frac{x-x_0}{t}\right)\left(\frac{x-x_0}{t^2}\right) + j'\left(f\left(\frac{x-x_0}{t}\right)\right)\frac{1}{t}f'\left(\frac{x-x_0}{t}\right) = 0$$

$$\therefore \boxed{j'\left(f\left(\frac{x-x_0}{t}\right)\right) = \frac{x-x_0}{t}} \rightarrow (i)$$

we know that $x'(t) = j'(\rho)$

which gives us : $x - x_0 = j'(\rho)t$

$$\therefore \boxed{\frac{x-x_0}{t} = j'(\rho)}$$

where $\rho_0(x) = \begin{cases} \rho_L & x \le x_0 \\ \rho_R & x > x_0 \end{cases}$

$\therefore$ setting $\frac{x-x_0}{t} = y$ in (i) gives : $j'(f(y)) = y$

$\forall y \in [j'(\rho_L) - j'(\rho_R)]$

papergrid

Date: / /

Q3) we know that f is the inverse of j', since j' is ~~continuous~~ continuous and and monotonous in its domain, ~~and~~ there exists a unique function g such that :

$$j'(g(y)) = y \quad \text{(i)} \quad \text{and} \quad g'(j'(\rho)) = \rho \quad \text{(ii)}$$

and $j'(f(y)) = y$ $\therefore$ $f(y)$ is the inverse of $j'(\rho)$.

On the left edge :

$$x_L = x_0 + j'(\rho_L)t$$

$$\therefore \quad j'(\rho_L) = \frac{x_L - x_0}{t}$$

$$f(j'(\rho_L)) = f\left(\frac{x_L - x_0}{t}\right)$$

using (ii) we get : $\rho_L = f\left(\frac{x_L - x_0}{t}\right)$

on the right edge

$$x_R = x_0 + j'(\rho_R)t$$

$$j'(\rho_R) = \frac{x_R - x_0}{t}$$

$$f(j'(\rho_R)) = f\left(\frac{x_R - x_0}{t}\right)$$

$$\boxed{\rho_R = f\left(\frac{x_R - x_0}{t}\right)}$$

This proves that f takes the values ρ_L and ρ_R on the two edges of core b/w the characteristic base curves.

papergrid

Date: / /

Q3) Interesting characteristics (shock waves)

$j(\rho) = 4\rho(2-\rho)$

$\boxed{j'(\rho) = 8 - 8\rho}$

$$\rho_0(x) = \begin{cases} 1 & x \le 1 \\ 1/2 & 1 < x \le 3 \\ 3/2 & x > 3 \end{cases}$$

$$j'(\rho) = \begin{cases} 0 & x \le 1 \\ 4 & 1 < x_0 \le 3 \\ -4 & x_0 > 3 \end{cases}$$

Q4) Rarefaction wave solution from $x = 1$ and show that $\rho(x,t) = 1 - \frac{x-1}{8t}$

Using smooth approx condition, we replace $\rho_0 \to \rho_0^t$

$$\rho_0^t(x_0) = \begin{cases} 1 & ; x_0 \le 1-t \\ \frac{3}{4} - \frac{x_0 - 1}{4t} & ; 1-t < x_0 \le 1+t \\ 1/2 & ; 1+t < x_0 \le 3 \\ 3/2 & ; x_0 > 3 \end{cases}$$

$$j'(\rho_0^t(x_0)) = \begin{cases} 0 & ; x_0 \le 1-t \\ 2\left(1 + \left(\frac{x_0 - 1}{t}\right)\right) & ; 1-t \le x_0 \le 1+t \\ 4 & ; 1+t < x_0 \le 3 \\ -4 & ; x_0 > 3 \end{cases}$$

where $0 < t < 2$

for $1-t \le x_0 \le 1+t$:

$$x(t) = x_0 + 2\left(1 + \left(\frac{x_0 - 1}{t}\right)\right)t \qquad x(t) = x$$

$$x_0 = x - 2(t + x_0 - 1)$$

$$3x_0 = x - 2t + 2$$

Date: / /

Q4) using smooth approximation conditions ; we can convert $f_0 \to f_0^{\epsilon}$:

$$f_0^{\epsilon}(x_0) = \begin{cases} 1 & ; x_0 \le 1-\epsilon \\ \frac{3}{4} - \left(\frac{x_0 - 1}{4\epsilon}\right) & ; 1-\epsilon < x_0 < 1+\epsilon \\ 1/2 & ; 1+\epsilon < x_0 < 3 \\ 3/2 & ; x_0 > 3 \end{cases}$$

$$j'(f_0^{\epsilon}(x_0)) = \begin{cases} 0 & ; x_0 \le 1-\epsilon \\ 2\left(1+\left(\frac{x_0-1}{\epsilon}\right)\right) & ; 1-\epsilon \le x_0 \le 1+\epsilon \\ 4 & ; 1+\epsilon < x_0 < 3 \\ -4 & ; x_0 > 3 \end{cases}$$

where $0 < \epsilon < 2$

for $1-\epsilon < x_0 < 1+\epsilon$:

$$x(t) = x_0 + 2\left(1 + \left(\frac{x_0 - 1}{\epsilon}\right)\right)t \qquad x(t) = x$$

$$x_0 = x - 2(\epsilon + x_0 - 1)\frac{t}{\epsilon}$$

$$x_0\left(1 + \frac{2t}{\epsilon}\right) = x - \frac{2t}{\epsilon}(\epsilon - 1)$$

$$\boxed{x_0 = \frac{\epsilon x - 2t(\epsilon - 1)}{2t + \epsilon}}$$

$$f^{\epsilon}(x(t), t) = f^{\epsilon}\left(x_0 + 2\left(1 + \frac{x_0 - 1}{\epsilon}\right)t, t\right)$$

$$= f_0^{\epsilon}(x_0)$$

$$= \frac{3}{4} - \frac{x_0 - 1}{4\epsilon}$$

$$= \frac{3}{4} - \frac{1}{4\epsilon}\left(\frac{\epsilon x - 2t(\epsilon - 1)}{2t + \epsilon} - 1\right)$$

$$= \frac{3}{4} - \frac{1}{4\epsilon}\left(\frac{\epsilon x - 2t\epsilon - \epsilon}{2t + \epsilon}\right)$$

papergrid

Date: / /

$$= \frac{3}{4} - \frac{1}{4}\left(\frac{x-2t-1}{2t+\varepsilon}\right)$$

Applying the limit $\varepsilon \to 0$, we get :

$$\lim_{\varepsilon\to 0} \rho^{\varepsilon}(x(t), t) = \rho(x, t) \quad (\text{for } x_0 = 1) \quad \boxed{x(t) = x}$$

$$\rho(x,t) = \frac{3}{4} - \frac{1}{4}\frac{(x-2t-1)}{(2t)}$$

$$\rho(x,t) = \frac{1}{4}\left(3 - \frac{x-1}{2t} + 1\right)$$

$$\boxed{\rho(x,t) = 1 - \frac{x-1}{8t}}$$

rarefaction waves emerging from $x_0 = 1$.

Q5) Shock solution for $t > 2$:

$x = x_0 + j'(\rho)t$

Refraction baselines and baselines for $x_0 > 3$ will start intersecting at :

$\boxed{x = x_0 - 4t}$ $\boxed{x = 1 + 4t}$

~~The shock~~ The shock will be stationary until t_s.

$x = 3$ in the above eqn, we get :

$$3 = 1 + 4t_s$$

$$\boxed{t_s = 1/2}$$

After $t_s = 1/2$, the shock waves move by a trajectory $\sigma(t)$. Shock speed after $t_s = 1/2$ will be :

$$\frac{d\sigma}{dt} = \frac{j_2 - j_1}{\rho_2 - \rho_1} = \frac{j(3/2) - j\left(1 - \frac{\sigma(t)-1}{8t}\right)}{\frac{3}{2} - \left(1 - \frac{\sigma(t)-1}{8t}\right)}$$

papergrid

Date: / /

where we have inserted the explicit form of the rarefaction fan density:

$$\rho(x,t) = 1 - \frac{x-1}{8t}$$

$$\cancel{\frac{d\sigma}{dt}} = j(3/2) = 3$$

$$j\left(1 - \frac{\sigma(t)-1}{8t}\right) = 4\left(1 - \frac{\sigma(t)-1}{8t}\right)\left(2 - \left(1 - \frac{\sigma(t)-1}{8t}\right)\right)$$

$$\therefore \frac{d\sigma}{dt} = \frac{3 - 4\left(1 - \frac{\sigma(t)-1}{8t}\right)\left(2 - \left(1 - \frac{\sigma(t)-1}{8t}\right)\right)}{\frac{1}{2} + \frac{\sigma(t)-1}{8t}}$$

$$\cancel{\frac{d\sigma}{dt} = \frac{3 - \frac{1}{2t}\left(8t - \sigma(t) + 1\right)\left(\frac{8t + \sigma(t) - 1}{8t}\right)}{\frac{8t + 4\sigma(t) - 4}{2t}}}$$

$$\cancel{\frac{d\sigma}{dt} = \frac{1}{8t}\left(\frac{6t - (8t)^2 + (\sigma(t)-1)^2}{4t + \sigma(t) - 1}\right)}$$

$$\frac{d\sigma}{dt} = \frac{3 - 4\left(1 - \frac{\sigma(t)-1}{8t}\right)\left(1 + \frac{\sigma(t)-1}{8t}\right)}{\frac{1}{2} + \frac{\sigma(t)-1}{8t}}$$

$$\frac{d\sigma}{dt} = \frac{3 - 4(1) + \cancel{4}\left(\frac{\sigma(t)-1}{4t}\right)^2}{\frac{1}{2} + \frac{\sigma(t)-1}{8t}}$$

$$= \frac{-1 + \left(\frac{\sigma(t)-1}{4t}\right)^2}{\frac{1}{2} + \frac{\sigma(t)-1}{8t}}$$

$$= \frac{-1 + \left(\frac{\sigma(t)-1}{4t}\right)^2}{\frac{1}{2}\left(1 + \frac{\sigma(t)-1}{4t}\right)}$$

papergrid

Date: / /

$$\boxed{\frac{d\sigma}{dt} = -2\left(1 - \frac{\sigma(t)-1}{4t}\right)}$$

$$\frac{d\sigma}{dt} - \frac{\sigma}{2t} = -2\left(1+\frac{1}{4t}\right)$$

Bernoulli DE (Linear), we use standard procedures

$P(t) = -\frac{1}{2t}$ $\quad Q(t) = -2\left(1+\frac{1}{4t}\right)$

$$IF = e^{\int P(t)\,dt}$$

$$IF = e^{\int -\frac{1}{2t}\,dt}$$

$$IF = \frac{1}{\sqrt{t}}$$

$$IF \times \sigma = \int \frac{1}{\sqrt{t}}(-2)\left(1+\frac{1}{4t}\right)dt$$

$$\frac{\sigma}{\sqrt{t}} = -4\sqrt{t} + \frac{1}{\sqrt{t}} + c\sqrt{t}$$

$$\sigma = 1 - 4t + c\sqrt{t}$$

$\sigma(1/2) = 3$

$3 = 1 - 2 + c$ $\quad \boxed{c = 4\sqrt{2}}$

~~$\boxed{c=4}$~~

~~$\sigma = 5 - 4t$~~

$$\sigma = 1 + 4\left(\sqrt{2t} - t\right)$$

However, trajectory is only valid till shock waves reach $x=1$ that means the end of rarefaction waves.

$$\sigma(t) = 1$$

$$1 + 4\left(\sqrt{2t} - t\right) = 1$$

$$\boxed{t = 2}$$

papergrid

Date: / /

Thus after ~~t=2~~ t=2, shock wave follows a different trajectory.

$\rho_l = 1$ and $\rho_r = 3/2$

~~Intersect~~ (This is the intersection b/w $x_0 \leq 1$ and $x_0 > 3$ base lines)

Using the Rankine-Hugoniot condition:

$$\frac{dx}{dt} = \frac{j(3/2) - j(1)}{3/2 - 1}$$

$$= \frac{3 - 4}{1/2}$$

$$\frac{dx}{dt} = -2$$

$\because x = x_0 + \frac{dx}{dt} t$

$\boxed{x = x_0 - 2t}$ → shock wave trajectory.

At $t = 2$, $x = 1$

$1 = x_0 - 4$

$\boxed{x_0 = 5}$

$\boxed{x = 5 - 2t}$ → shock wave trajectory.

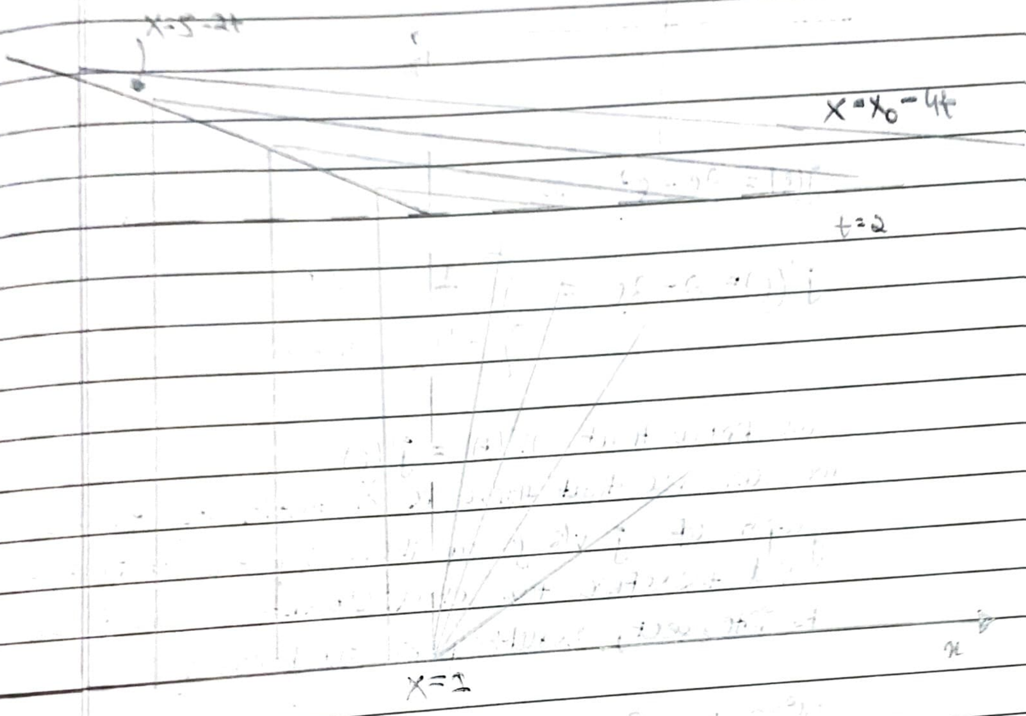
Using this, we get density to the right of the shock wave to be $3/2$ and density to the left ~~to~~ to be 1.

$$\rho(x,t) = \begin{cases} 1 & x \leq 5-2t \\ 3/2 & x > 5-2t \end{cases}$$

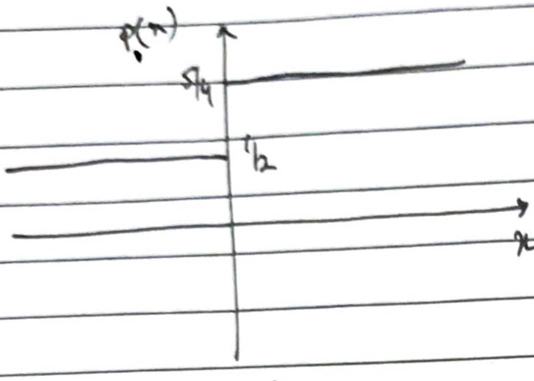
papergrid

Date: / /

Shock solution for $t > 2$



Q7) $$\rho_0(x) = \begin{cases} 1/2 & x \le 0 \\ 5/4 & x > 0 \end{cases}$$



$$j(\rho) = 2\rho - \rho^2$$

$$j'(\rho) = 2 - 2\rho = \begin{cases} 1 & x \le 0 \\ -1/2 & x > 0 \end{cases}$$

we know that $x'(t) = j'(\rho)$
we can see that there is discontinuity in the graph of j v/s ρ as shown by the derivative and therefore the characteristics are forced to intersect, resulting in shockwaves.

Using the Rankine-Hugoniot condition:

$$\frac{dx}{dt} = \frac{j_R - j_L}{\rho_R - \rho_L} = \frac{j(5/4) - j(1/2)}{5/4 - 1/2}$$

$$= \frac{2(5/4) - (5/4)^2 - 2(1/2) + (1/2)^2}{3/4}$$

$$= \frac{\frac{15}{16} - \frac{3}{4}}{3/4}$$

$$\boxed{s = \frac{dx}{dt} = 1/4}$$ this is the shock speed for the traffic flow.

Date: / /

Q8) The results we have got resonate with the results and conclusions discussed in class.

- Method (i) and (iv) do not yield the correct results. Since $j(\rho) = \rho(1-\rho)$ is 0 for both $\rho=0$ and $\rho=1$, the first iteration of our time loop from $j(\rho) = 0$.
  and this implies that:
  $$\boxed{\rho_i^{n=2} = \rho_i^{n=1}} \rightarrow \text{for methods 1 and 4}$$

  This is the case for next iteration as well, giving us the result:
  $$\rho_i^{n+1} = \rho_i^{n}$$

  Method 2 : rarefaction wave is not correctly represented.

- Method 3 and 5 : yield the result closest to actual $\rho(x,t)$.
  Methods 3 and 4 have regions that blow up. This is near the shock wave, where the partial differential equation loses its meaning.

- I couldn't write the code for Q5, but as mentioned by sir in class, Method 5 gives the closest results to the actual function. This is because it accounts for the shock wave. Values for shock waves are within range.