MMI - Assignment 5

Question 4

Linearize the Lotka Volterra ~~method~~ model and apply ODE solver.

a) without fishing

In rescaled model :

$$u' = u(1-v) \quad \text{— (i)}$$

$$v' = \alpha v(u-1) \qquad \because \alpha = \frac{d}{a} = \frac{3}{4} = 0.75$$

$$v' = 0.75 v(u-1) \quad \text{— (ii)}$$

eqm point $(u,v) = (1,1)$ $\quad f(x,y) = g(x,y) = 0.$

shifting the origin to the eqm point :

$$u = U + 1 \Rightarrow U = u - 1$$

$$v = V + 1 \Rightarrow V = v - 1$$

U, V are both functions of time.

$$U' = u' \quad \text{— (iii)}$$

$$V' = v' \quad \text{— (iv)}$$

using (i)-(iv) : $U' = -V(U+1) = -V - UV$

$$V' = 0.75U(V+1) = 0.75UV + 0.75U$$

now, the quadratic terms for U', V' i.e. $-UV$ and $0.75UV$ can be ignored as they are much smaller than the linear terms for small $|U|$ and $|V|$ when we consider points near the new origin i.e the eqm.

$$\therefore \boxed{\begin{aligned} U' &= -V \\ \text{and } V' &= 0.75U \end{aligned}} \quad \text{— (v)}$$

OR

$$\begin{bmatrix} U' \\ V' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0.75 & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \quad \Big\} \text{ linearized model.}$$

$$\begin{bmatrix} 0 & -1 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

$$\boxed{\bar{x}' = A\bar{x}}$$

b) with fishing

unscaled equations are modified to:

$$N' = N(a - bP - \delta)$$
$$P' = P(cN - d - \delta)$$

Upon rescaling:

$$\tau = at \qquad \frac{d\tau}{dt} = a$$

$$x(\tau) = \frac{c}{d} N(t) \Rightarrow x'(\tau) = \frac{c}{d} N'(t) \times \frac{dt}{d\tau} = \frac{c}{d \times a} N'(t)$$

$$y(\tau) = \frac{b}{a} P(t) \Rightarrow y'(\tau) = \frac{b}{a} P'(t) \times \frac{dt}{d\tau} = \frac{b}{a^2} P'(t)$$

$$\therefore x'(\tau) = \frac{c}{da}\left(\frac{d}{c}\right) x(\tau)\left(a - b\left(\frac{a}{b}\right) y(\tau) - \delta\right)$$

$$\boxed{x'(\tau) = x(\tau)\left(1 - \frac{\delta}{a} - y(\tau)\right)} \rightarrow \text{(i)}$$

and

$$y'(\tau) = \frac{b}{a^2}\left(\frac{a}{b}\right) y(\tau)\left(c\left(\frac{d}{c}\right) x(\tau) - d - \delta\right)$$

$$\boxed{y'(\tau) = \alpha\, y(\tau)\left(x(\tau) - 1 - \frac{\delta}{d}\right)} \rightarrow \text{(ii)}$$

Shifting the origin to eqm, we get:

$$X(\tau) = x(\tau) - 1 - \frac{\delta}{d}, \quad Y(\tau) = y(\tau) - 1 + \frac{\delta}{a}$$

$$X'(\tau) = x'(\tau) \quad \text{and} \quad Y'(\tau) = y'(\tau)$$

$$\therefore \boxed{X'(\tau) = -\left(X(\tau) + 1 + \frac{\delta}{d}\right) Y(\tau)}$$

$$\boxed{Y'(\tau) = \alpha\left(Y(\tau) + 1 - \frac{\delta}{a}\right) X(\tau)}$$

ignoring quadratic terms we get:

$$\boxed{\begin{bmatrix} \bar{X}' \\ \bar{Y}' \end{bmatrix} = \begin{bmatrix} 0 & -1 - \frac{\delta}{d} \\ \alpha\left(1 - \frac{\delta}{a}\right) & 0 \end{bmatrix} \cdot \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}}$$

Question 5

selection fishing : only prey population is fished at rate $\delta > 0$.

$$N' = N(a - bP - \delta)$$
$$P' = P(cN - d)$$

$\frac{dt}{d\tau} = \frac{1}{a}$

$\Downarrow$ scaling

$\tau = at$

$x = \frac{c}{d} N$

$y = \frac{b}{a} P$

$$x' = \frac{c}{d} \frac{N'}{a} = \frac{c}{d} \frac{N}{a}(a - bP - \delta) = \frac{x}{a}(a - ay - \delta)$$

$$\boxed{x' = x\left(1 - y - \frac{\delta}{a}\right)}$$

$$y' = \frac{b}{a} \frac{P'}{a} = \frac{b}{a \times a} P(cN - d) = \frac{b}{a^2} \left(\frac{a}{b}\right) y \left(c\left(\frac{d}{c}\right)x - d\right)$$

$$\boxed{y' = \alpha y(x - 1)}$$

Shifting the origin :

$X = x - 1$; $x = X + 1$

$Y = y - 1 + \frac{\delta}{a}$; $y = Y + 1 - \frac{\delta}{a}$

$$X' = -(X+1)Y$$
$$Y' = \alpha X\left(Y + 1 - \frac{\delta}{a}\right)$$

$\Downarrow$ ignoring the quadratic terms.

$$\boxed{\begin{aligned} X' &= -Y \\ Y' &= \alpha X\left(1 - \frac{\delta}{a}\right) \end{aligned}}$$

$$s' = \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ \alpha\left(1-\frac{s}{a}\right) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \alpha\left(1-\frac{s}{a}\right) & 0 \end{bmatrix} s$$

where $s = \begin{bmatrix} x \\ y \end{bmatrix}$.

$\Downarrow$

linearized model when fishing is selective only for the prey and not the predator, at rate $s>0$.

The phase portrait for the linearized and non linearized models, ~~both~~ when selective fishing occurs does not change the phase model by lot and they are similar to the non linearized / linearized ~~the~~ phase ~~pos~~ portraits when fishing is ~~done~~ not selective respectively.

Date: / /

Q6 ii) As shown in class; the linearized form of the Lotka-Volterra model (revised)

is:

$$\begin{pmatrix} N' \\ P' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{-aad}{c} & \frac{-bd}{c} \\ \frac{a}{b}(c-ad) & 0 \end{pmatrix}}_{A} \begin{pmatrix} N \\ P \end{pmatrix}$$

here the equilibrium point $= \left(\frac{d}{c}, \frac{a}{bc}(c-ad)\right)$

If eigenvalues of A be λ_1, λ_2:

$$\lambda_{1,2} = \frac{-aad}{2c} \pm \sqrt{\left(\frac{aad}{2c}\right)^2 - \frac{ad}{c}(c-ad)}$$

Now, let's take the different a values:

(I) $a_1 = 0.1$

~~$\lambda_{1,2}$ root part $= \left(\frac{0.1\times4\times3}{3}\right)^2$~~

root part $= \left(\frac{0.1\times4\times3}{3}\right)^2 - \frac{4\times3\times2}{3}(1.5-(0.1)3)$

$= 0.16 - 9 < 0$

$\therefore$ the eigenvalues will be complex and the real value of the eigenvalues is negative.

$\therefore$ The system will be stable (spiral)

(II) $k_2 = 0.01$

root part = $\left(\frac{0.01 \times 4 \times 3}{3}\right)^2 - \frac{4 \times 3 \times 2}{3}\left(1.5 - (0.01)(3)\right)$

$= 0.0016 - 11.76 << 0$

like $a_1 = 0.1$, even $a = 0.02$ has complex eigenvalues with real part < 0. $\therefore$ indicating a stable spiral.

This is also shown in the phase portraits plotted in the code file.

(III) for eigenvalues 0.001 and 0.0001:

the real part will become very small compared to the imaginary part ~~since (a^k will~~ and therefore the eigenvalues will only be complex

The ~~model~~ phase portrait will therefore become elliptical, as depicted ~~by the phase~~ in the code file and the model will be similar to the Lotka Volterra model which did not account ~~for~~ for the logistic growth law. But the system will be unstable.

Stable = structurally stable.

Final answer:

system = stable for $a = 0.1, 0.01$

system = unstable for $a = 0.001, 0.0001$

for eq^m point $\left(\frac{d}{c}, \frac{a}{bc}(c - ad)\right)$

for eq^m point $(0, 0)$: system not possible since no population will exist!

Date: / /

Question 7

(i) N_2 preys on N_1
N_3 preys on N_1, N_2
N_4 has a symbiotic relationship with N_2 and is neutral wrt the other two.

Resulting system of eqns:

$$\boxed{N_1' = a_1 N_1 \left(1 - \frac{N_1}{K}\right) - b_1 N_1 N_2 - b_2 N_1 N_3} \rightarrow \text{(i)}$$

K = carrying capacity of N_1

$$\boxed{N_2' = c_1 N_1 N_2 - b_3 N_2 N_3 - a_2 N_2 + d_1 N_2 N_4} \quad \text{(ii)}$$

~~$N_3' = c_2 N_1 N_3 + d$~~

$$\boxed{N_3' = c_2 N_1 N_3 + c_4 N_3 N_2 - a_3 N_3} \rightarrow \text{(iii)}$$

$$\boxed{N_4' = d_2 N_2 N_4 - a_4 N_4} \rightarrow \text{(iv)}$$

(ii) Equilibrium points:

$N_1', N_2', N_3', N_4' = 0$ for eqm points:

a) $\boxed{N_1, N_2, N_3, N_4 = 0}$ – (I)

b) Let $N_1 = 0$:

$\therefore$ $N_2(-b_3 N_3 - a_2 + d_1 N_4) = 0$
$N_3(c_4 N_2 - a_3) = 0$
$N_4(d_2 N_2 - a_4) = 0$

solutions: ~~$N_3 = 0$~~, $\boxed{N_1 = 0,\ N_3 = 0,\ N_4 = a_2/d_1,\ N_2 = \frac{a_4}{d_2}}$ (II)

(III) $\leftarrow$ $\boxed{N_1 = 0,\ N_4 = 0,\ N_3 = -a_2/b_3,\ N_2 = a_3/c_4}$

$\times$ not possible as $N_3 < 0$.

$$\boxed{\text{if } N_2 = \frac{a_2}{c_4} = \frac{a_4}{d_2}, \quad d_1 N_4 - b_3 N_3 = a_2} \quad \text{— (IV)}$$

only if $\frac{a_2}{c_4} = \frac{a_4}{d_2}$ and $N_1 = 0$

d) let $N_3 = 0$:

$$\therefore \quad N_4 (d_2 N_2 - a_4) = 0$$

$$N_2 (c_1 N_1 - a_2 + d_1 N_4) = 0$$

$$N_1 \left(a_1 \left(1 - \frac{N_1}{K}\right) - b_1 N_2 \right) = 0$$

$$\boxed{\text{solutions: (IV): } N_3 = 0, \; N_4 = 0, \; N_1 = \frac{a_2}{c_1}, \; N_2 = \frac{a_1}{b_1}\left(1 - \frac{N_1}{K}\right)}$$

(VI): $N_3 = 0, \; N_2 = a_4 / d_2, \; N_1 = (a_1 - b_1 N_2) \frac{K}{a_1}$

$$N_4 = \frac{1}{d_1}\left(a_2 - \frac{c_1 K}{a_1}(a_1 - b_1 N_2)\right)$$

$$\Downarrow$$

$$\text{(VI)} \quad \boxed{\begin{aligned} &N_3 = 0, \; N_2 = a_4 / d_2, \; N_1 = \left(a_1 - \frac{b_1 a_4}{d_2}\right)\frac{K}{a_1} \\ &N_4 = \frac{1}{d_1}\left(a_2 - \frac{c_1 K}{a_1}\left(a_1 - \frac{b_1 a_4}{d_2}\right)\right) \end{aligned}}$$

d) let $N_4 = 0$:

$$N_1 \left(a_1 \left(1 - \frac{N_1}{K}\right) - b_1 N_2 - b_2 N_3 \right) = 0$$

$$N_2 (c_1 N_1 - b_3 N_3 - a_2) = 0$$

$$N_3 (c_2 N_1 + c_4 N_2 - a_3) = 0$$

$$\boxed{\text{(VII): } N_4 = 0, \; N_2 = 0, \; N_1 = a_3 / c_2, \; N_3 = \frac{a_1}{b_2}\left(1 - \frac{a_3}{c_2 K}\right)}$$

$N_4 = 0$,

I have manually found the eqm points, but I have also found them in the matlab code. Along with the eigenvalues.

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Question 8

(i)

$$x' = \sigma(y - x) \quad (1)$$
$$y' = rx - y - xz \quad (2)$$
$$z' = xy - bz \quad (3)$$

$$x' = y' = z' = 0 \overset{(1)}{\Rightarrow} \boxed{x = y}$$

$$\downarrow$$

$$(2)$$

$$\downarrow$$

$$x(r - 1 - z) = 0 \quad \text{OR} \quad \boxed{x = 0}$$

$$\boxed{z = r - 1}$$

$$\xrightarrow{(3)} x^2 = b(r-1)$$

$$x = \pm\sqrt{b(r-1)}$$

$\therefore$ two solutions $= P_1 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$

$P_2 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$

$P_3 = (0, 0, 0)$: trivial solution

P_1, P_2, P_3 are eq^m points.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -x \\ 0 & x & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(ii) done in the python code.

(iii) let's take the eqm point as P_3:

$P_3 = (0, 0, 0)$;
therefore the ~~syst~~ equilibrium point is at the origin.

~~the~~

∴ linearization becomes: (after ignoring quadratic terms)

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

to find eigen values:

$$\det \begin{bmatrix} -10-\lambda & 10 & 0 \\ 28 & -1-\lambda & 0 \\ 0 & 0 & -8/3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\downarrow R_2 + \frac{1}{10+\lambda} 28$$

$$\det \begin{bmatrix} -10-\lambda & 10 & 0 \\ 0 & -1-\lambda+\frac{10\times 28}{10+\lambda} & 0 \\ 0 & 0 & -8/3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$(-\lambda - 8/3)\cancel{(10+\lambda)}\left(\frac{-10-\lambda^2-11\lambda+10\times 28}{\cancel{10+\lambda}}\right) = 0$$

$$\boxed{\lambda_1 = -8/3} \quad \lambda_{2,3} = -11/2 \pm 0.5\sqrt{81+4\times 28}$$

~~Ay~~

All three eigenvalues and negative and real when $r<1$. $\therefore$ when $\lambda<1$: the point P_1 is asymptomatically stable, there are no other real fixed points. 15/9

when $r>1$; λ_1, λ_2 remain negative
λ_3 becomes positive.
$\therefore$ ~~when $r>1$~~ there is a stable manifold through P_1, which is a 2-D surface immersed in phase space and tangent to eigenvectors belonging to λ_1, λ_2.

(ii) eq^m point = P_1 (linearization about P_3 is very comparable)

$$P_1 = \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$$

$$x' = 10(y-x)$$
$$y' = rx - y - xz$$
$$z' = xy - 8/3\, z$$

we take $X = x - \sqrt{b(r-1)}$
$Y = y - \sqrt{b(r-1)}$
$Z = z - (r-1)$

~~$\therefore X'$ becomes~~

$\therefore X', Y', Z'$ become:

$$X' = 10\left(Y + \sqrt{b(r-1)} - X - \sqrt{b(r-1)}\right)$$
$$\boxed{X' = 10(Y - X)}$$

$$Y' = r\left(X + \sqrt{b(r-1)}\right) - \left(Y + \sqrt{b(r-1)}\right) - \left(X + \sqrt{b(r-1)}\right)\left(Z + r - 1\right)$$
$$= rX - Y - XZ - \sqrt{b(r-1)}(r-1) - X(r-1) - Z\sqrt{b(r-1)}$$
$$= -Y + X + Z\sqrt{\tfrac{8}{3}(r-1)}$$

$$z' = \left(X + \sqrt{b(r-1)}\right)\left(Y + \sqrt{b(r-1)}\right) - \frac{8}{3}(Z + r - 1)$$

$$z' = \sqrt{b(r-1)}\,(X+Y) - \frac{8}{3}Z$$

this is done by ignoring multivariable terms: XY, XZ, YZ and also constants.

$$\therefore \boxed{\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & \sqrt{8/3(r-1)} \\ \sqrt{8/3(r-1)} & \sqrt{8/3(r-1)} & -8/3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}$$

using an online eigenvalue calculator, eigenvalues are found and ~~they are~~ the polynomial of eigenvalues is:

$$p(\lambda) = 3\lambda^3 + 41\lambda^2 + 8(r+10)\lambda + 160(r-1)$$

(1) For $1 < r < r_1 \approx 1.3456$
3 real negative roots and $\therefore P_2$ is a locally attractive node. Two eigenvalues will be equal if $r = r_1$

(2) $r_1 < r < r_2 \approx 24.737$, one real negative eigenvalue and pair of complex eigenvalues $\lambda_2 = \bar{\lambda}_3$ with $\mathrm{Re}\,\lambda_2 < 0$. phase diagram near P_2 is a stable spiral.

(3) For $r > r_2$: real eigenvalue $\lambda_1 < 0$ is ~~regain~~ retained, but for complex conjugate pair, we have $\mathrm{Re}\,\lambda_2 > 0$

Date: / /

$\therefore P_2$ is not a stable eq^n.

The same ~~conclusing~~ conclusions can be made of P_3.