Combinations and permutations

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1 Counting

Sum rule: If one item can be counted in n distinct ways, and a second can be counted in m distinct ways, and the two cannot happen simultaneously, then together there are $n + m$ ways to count these items.

For example, if I put six red balls in a bag, then I put five green balls in a bag. There are $5 + 6 = 11$ balls in the bag.

Product rule: If there are m items each of which has associated with it n items, we can count mn items in total.

For example, if I have five bags each of which contains six balls, there are $5 \times 6 = 30$ balls in total.

1.1 Example

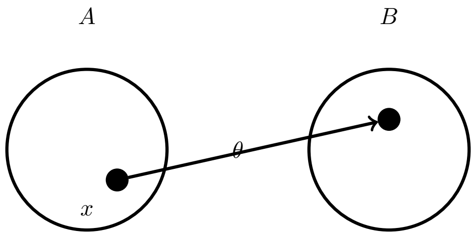
A buffet has 5 salads to choose: potato, noodle, reekeh, tabbouleh, or rice. You may take or leave each type of salad. How many different plates can you form?

Since you can either take or leave each type, there are $2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$ different plates.

One of these is the option where you take none of the salads – the empty plate. Whether this counts or not depending on the precise problem being solved.

2 Some reminders: sets, maps, functions

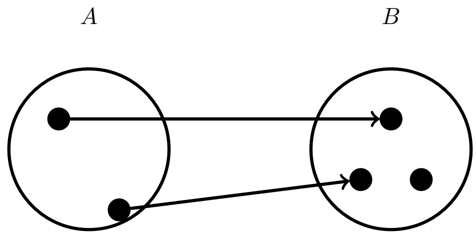
For two sets A and B, we define a map or function θ from A to B where there is some rule that assigns to each element of A a corresponding element of B.



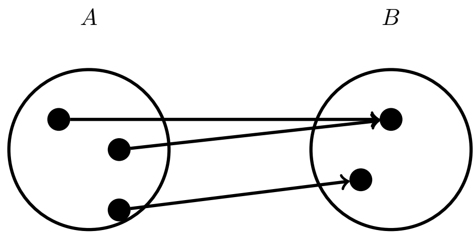
- We write $\theta : A \to B$.
- For $x \in A$, we write $\theta(x) \in B$.
- A is called the *domain* of θ.
- B is called the *range* of θ.

A map $\theta : A \to B$ is *injective* if whenever x and y are distinct elements of A, then $\theta(x)$ and $\theta(y)$ are distinct elements of B. We can say

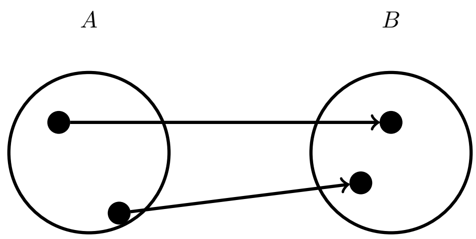
$$\theta(x) = \theta(y) \implies x = y.$$



A map $\theta : A \to B$ is *surjective* if for each element $y \in B$ there is at least one element of $x \in A$ such that $\theta(x) = y$.



A map $\theta : A \to B$ is *bijective* if it is both injective and surjective.



So:

- The definition of a map $\theta : A \to B$ says every element $x \in A$ is assigned a corresponding $\theta(x) \in B$.
- A surjective map is one where there are no elements in B left out – they all have an element in A mapped onto them.
- An injective map is one where each element A is assigned a different element in B.
- A bijective map is one where every element in B has a unique element in A associated with it.

With a bijective map $\theta : A \to B$, we can define its inverse, $\theta^{-1} : B \to A$ using the rule:

- if $y \in B$, let $\theta^{-1}(y) = x$, where x is the unique element in A such that $\theta(x) = y$.

3 Permutations

Definition. If A is a set, a *permutation* is a bijection $\sigma \colon A \to A$.

We will often consider permutations on a set $\mathbb{Z}(n) = \{x \in \mathbb{Z} \mid 1 \leq x \leq n\}$.

Example

Consider a permutation $\sigma\colon \mathbb{Z}(4) \to \mathbb{Z}(4)$ which maps $(1, 2, 3, 4)$ onto (a, b, c, d).

- a, b, c and d must be distinct, because σ is injective.
- Each of a, b, c, d be in $\mathbb{Z}(4)$, because σ is surjective.

Say σ is the permutation that maps $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 4$ and $\sigma(4) = 1$. We can represent this permutation using two-line notation as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

In this notation, the numbers are written out on the top line, with the second line showing where they are mapped to.

There is also a cycle notation where the permutation is written as a product of *disjoint cycles*. In this representation, we have

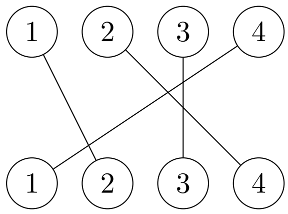
$$\sigma = (1\,2\,4)(3).$$

It is common to omit 1-cycles, so this could also be represented

$$\sigma = (1\,2\,4).$$

We can also draw the permutation as a diagram.

σ



4 Composing Permutations

If σ, $\tau\colon \mathbb{Z}(n) \to \mathbb{Z}(n)$ are permutations, then the composite function $\sigma \circ \tau\colon \mathbb{Z}(n) \to \mathbb{Z}(n)$ is also a permutation, since the composite of bijections is also a bijection. This notation tells us that we perform τ first, followed by σ, often just written as $\sigma\tau$ instead of $\sigma \circ \tau$.

4.1 Example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1\,2\,4)$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4)$$

In the two-line notation, we can write out the first permutation to be performed as normal, and then what the second permutation does on the line below. Then we just read off the resulting permutation by ignoring the middle row.

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

In the one-line notation, we write the cycles out in order, before then following each element through the cycles from right to left.

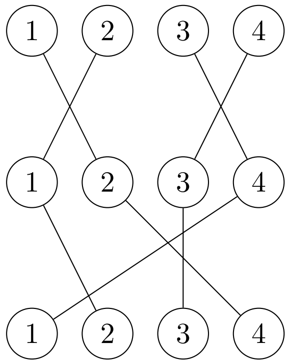
$$\sigma\tau = (1\,2\,4)(1\,2)(3\,4) = (1\,4\,3)(2) = (1\,4\,3)$$

In this example, we can think about what happens to each of the numbers from 1 to 4. For 1, the right-most cycle $(3\,4)$ doesn't move it, the next cycle $(1\,2)$ swaps it with 2, and then the left-most cycle $(1\,2\,4)$ sends that 2 to 4. So 1 ultimately ends up at 4. Similarly, 2 is swapped with 1 by $(1\,2)$ before being sent back to 2 by the cycle $(1\,2\,4)$. The number 3 is sent to 4 by $(3\,4)$, is not affected by $(1\,2)$, but is then sent to 1 by the cycle $(1\,2\,4)$. Finally, 4 is sent to 3, then left alone, then sent to 1 by the left-most cycle.

Using diagrams, we can think of this as stacking the diagram for τ on top of the diagram for σ.

τ

σ



Note that for $\tau\sigma$ we have

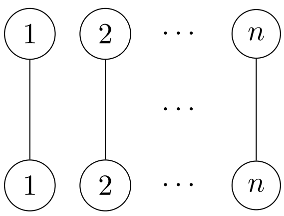
$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \neq \sigma\tau.$$

In general, composition of permutations is not commutative.

5 Identity Permutation

There is a very simple permutation which doesn't relabel anything at all. This is called the identity permutation, defined by $e_n \colon \mathbb{Z}(n) \to \mathbb{Z}(n)$ where $e_n(x) = x$. In the two-line and one-line notation, this is just

$$e_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & n \end{pmatrix} = (1)(2)\ldots(n-1)(n).$$



If we compose an identity permutation with any other permutation $\sigma \colon \mathbb{Z}(n) \to \mathbb{Z}(n)$, then this leaves σ unchanged:

$$\sigma e_n = \sigma = e_n\sigma.$$

6 Inverse Permutations

Every permutation σ has an inverse σ^{-1}, which we think of as ‘un-shuffling’ the numbers we started with.

Example

Using the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1\,2\,4)$$

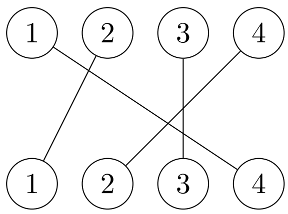
then the inverse is easy to find. In two-line notation, we simply read the permutation from bottom to top, rather than top to bottom. We can write the bottom row on the top row, and vice-versa, before reordering.

$$\sigma^{-1} = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

In the one-line notation, we can simply write the cycle backwards and then rearrange to have the lowest number at the front if we desire.

$$\sigma^{-1} = (4\,2\,1) = (1\,4\,2)$$

σ^{-1}



Inverse permutations have the property that

$$\sigma\sigma^{-1} = e_n = \sigma^{-1}\sigma.$$

7 Inverses of Composites

Taking inverses of composites is a little trickier, but not by much. If we take the permutations σ and τ from before, then we can show that $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1\,2\,4)$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4)$$

We know that $\sigma\tau = (1\,4\,3)$, so we should find that $(\sigma\tau)^{-1} = (1\,3\,4)$.

$$\begin{aligned} \tau^{-1}\sigma^{-1} &= (3\,4)^{-1}(1\,2)^{-1}(1\,2\,4)^{-1} \\ &= (3\,4)(1\,2)(1\,4\,2) \\ &= (1\,3\,4) \\ &= (\sigma\tau)^{-1}. \end{aligned}$$

8 Symmetric Groups

Recall that a group is a set and an operation such that the set under that operation has closure, associativity, an identity, and inverses.

Definition. The set of all permutations of $\mathbb{Z}(n)$ forms a group, under composition of permutations. This group is called the **symmetric group** of degree n. We use the notation S_n to denote this group.

Consider S_4. How many elements does it have? i.e. how many permutations are there of four elements?

There are:

- 4 choices for the first element;
- 3 remaining choices for the second element;
- 2 remaining choices for the third element;
- 1 remaining choices for the fourth element.

Therefore there are $4 \times 3 \times 2 \times 1 = 4! = 24$ elements in S_4.

In general, there are $n!$ permutations of $\mathbb{Z}(n)$, so $n!$ elements in S_n, where

$$n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1.$$

Thus, $1! = 1$. What about $0!$?

Consider this way of defining the factorial:

$$\begin{aligned}(n+1)! &= (n+1)n! \\ n! &= \frac{(n+1)!}{(n+1)}.\end{aligned}$$

So

$$0! = \frac{(0+1)!}{(0+1)} = \frac{1!}{1} = \frac{1}{1} = 1.$$

Another way to think about this: $n!$ is the number of ways of arranging n objects, and there's only one way to arrange 0 objects.

9 Arranging objects

An arrangement (or ordering) of a set of objects is a permutation.

Example

Consider arranging three letters A, B and C. How many ways can these be arranged? Six:

$$\begin{gathered}ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA\end{gathered}$$

Each of these is a permutation of the letters.

$$\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix} \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix} \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

10 Choosing objects

10.1 Choosing objects with repetition

Say we have an infinite pile of As, Bs and Cs and we want to make a word using four of these letters, we can do this in 3^4 ways (3 because there are three letters, multiplied 4 times because we are choosing from these three 4 times).

10.2 Choosing objects without repetition

Say we have a pile of ten numbered balls and we want to choose three of them. Choosing the first, there are 10 possibilities for which ball that could be. Then we come to choose the second. Well, now there are only 9 balls remaining, so there are 9 ways we could choose the second one. Similarly, there are 8 ways to choose the third. In total, then, there are $10 \times 9 \times 8$ ways to choose three balls from ten. To denote this using factorials, say:

$$10 \times 9 \times 8 = \frac{10 \times 9 \times 8 \times 7 \times \ldots \times 1}{7 \times \ldots \times 1} = \frac{10!}{(10-3)!} = 720.$$

In general, we can pick r objects from n without repetition in

$$\frac{n!}{(n-r)!}$$

ways.

In fact, the story doesn't end there. Within our 720 different ways of choosing three balls from ten, there will be lots that are just rearrangements of each other. For example (assuming the balls are numbered 1–10), we might draw ball number 7 first, then ball number 3, then ball number 9. What if we had chosen ball 3, then 7, then 9? Depending on the circumstances we are dealing with, we might consider this different or we might think these two are the same outcome really.

In fact, we know that the number of ways of arranging 3 balls is $3! = 6$. So our 720 selections could be reduced by a factor of 6 if we didn't care about the order in which they were chosen.

That would give

$$\frac{10!}{(10-3)!} \times \frac{1}{3!} = \frac{10!}{3!(10-3)!} = \frac{720}{6} = 120.$$

In general, the number of ways of choosing r objects from n without repetition when we don't care about the order of the r objects is

$$\frac{n!}{r!(n-r)!}.$$

This is the binomial coefficient and can be written using the notation $\binom{n}{r}$.

10.3 Counting the ones you didn't select

From a bag of 10 balls, if I choose 4 balls to draw out from the bag I can do this in $\binom{10}{4}$ ways. However, this is the same as choosing 6 balls to leave behind in the bag, which would be $\binom{10}{6}$. Thus

$$\binom{n}{r} = \binom{n}{n-r}.$$

10.4 Being careful with overcounting

A standard deck of playing cards has 52 cards in four suits of 13 cards each: hearts, clubs, diamonds and spades.

We can draw a hand of five cards from the deck in $\binom{52}{5} = 2598960$ ways.

We can draw from a standard deck five cards that do not include a club in $\binom{39}{5} = 575757$ ways.

This means we can draw five cards including at least one club in $\binom{52}{5} - \binom{39}{5} = 2023203$ ways.

Can we count this another way? For example, we could choose a club in $\binom{13}{1}$ ways, then add four other cards from the rest of the deck in $\binom{51}{4}$ ways. But

$$\binom{13}{1}\binom{51}{4} = 3248700 > 2023203.$$

What is happening? In fact, $\binom{13}{1}\binom{51}{4}$ is overcounting. For example, the first club could be the Three of Clubs, then the other cards could be the Six of Clubs and three other cards. Or the first club could be the Six of Clubs and the other cards could the Three of Clubs and the same three other cards. Our second methods counts both of these separately.

It is important to be very clear what you are counting in combinatorics problems to avoid overcounting.

11 Stars and bars

Suppose we wish to paint eight blocks with four colours, each of which may be used to paint zero or more blocks. How many ways are there to paint the blocks?

This problem can be thought of as equivalent to putting our eight blocks in four different bins, each representing one of our colours.

If we represent our blocks with stars ($*$) and separate our bins with a vertical bar $|$, we can represent ways of painting our blocks like this:

$$* \mid * * \mid * * * \mid * *$$

That would be one block with the first colour, two with the second and fourth colours, and three blocks with the third colour.

Here is another colouring:

$$* * \mid * \mid \mid * * * * *$$

Here there are two blocks with the first colour, one with the second, none with the third and five with the fourth colour.

We can also think of this notation as eleven symbols, eight of which are stars and three of which are bars. The choice of which symbols are bars determines the size of the bins. We can choose three symbols from eleven in $\binom{11}{3} = 165$ ways, so there are 165 ways to colour our blocks with four colours.

In general, we can arrange n stars amongst k bars by choosing k symbols from $n + k$ to be our bars (or equivalently n symbols to be our stars). That is, we can do this in

$$\binom{n+k}{k} = \binom{n+k}{n}$$

ways.