Maps

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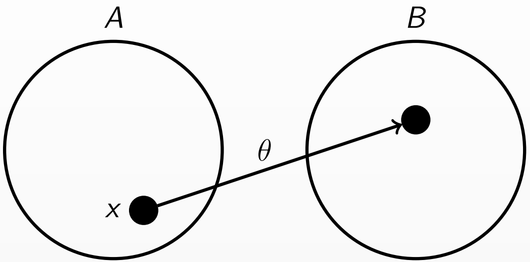
Maps

- For two sets A and B, we define a map θ from A to B where there is some rule that assigns to each element of A a corresponding element of B.
- We write $\theta : A \to B$.
- For $x \in A$, we write $\theta(x) \in B$.
- A is called the *domain* of θ.
- B is called the *range* of θ.



Example

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  - if $x \in P$, then $\phi(x)$ is the height of x in millimetres measured to the nearest millimetre.
- Note that θ and ϕ are not equal. For two maps to be equal, we require three conditions to all be true:
  1. θ and ϕ have the same domain;
  2. θ and ϕ have the same range;
  3. for every $x \in A$, $\theta(x) = \phi(x)$.

Functions

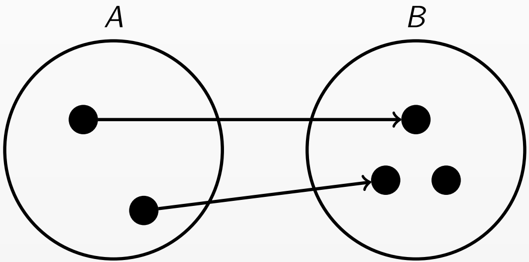
- Functions provide examples of maps between sets.
- For example, let $f : \mathbb{R} \to \{y \in \mathbb{R} \mid y \geq 0\}$ be defined so that

$$f(x) = x^2 \quad \text{for all} \quad x \in \mathbb{R}.$$

Injective, surjective, bijective maps

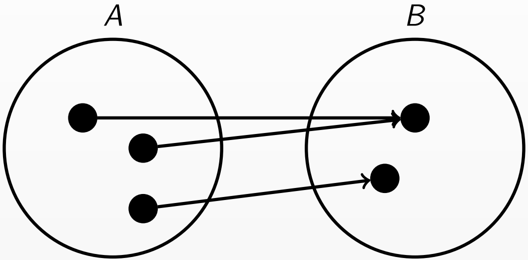
Injective (one-to-one)

- A map $\theta : A \to B$ is *injective* if whenever x and y are distinct elements of A, then $\theta(x)$ and $\theta(y)$ are distinct elements of B.
- We can say
$$\theta(x) = \theta(y) \implies x = y.$$



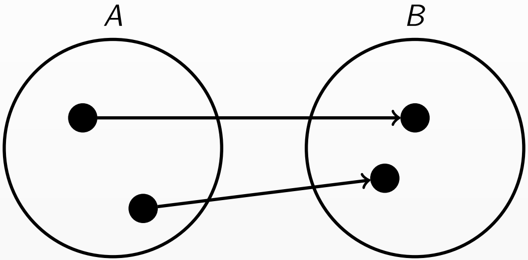
Surjective

- A map $\theta : A \to B$ is *surjective* if for each element $y \in B$ there is at least one element of $x \in A$ such that $\theta(x) = y$.



Bijective

- A map $\theta : A \to B$ is *bijective* if it is both injective and surjective.



Injective, surjective and bijective

- The definition of a map $\theta : A \to B$ says every element $x \in A$ is assigned a corresponding $\theta(x) \in B$.
- A surjective map is one where there are no elements in B left out – they all have an element in A mapped onto them.
- An injective map is one where each element A is assigned a different element in B.
- A bijective map is one where every element in B has a unique element in A associated with it.

Inverse map

- With a bijective map $\theta : A \to B$, we can define its inverse, $\theta^{-1} : B \to A$ using the rule:
  - if $y \in B$, let $\theta^{-1}(y) = x$, where x is the unique element in A such that $\theta(x) = y$.
- The inverse map is only defined when a map is bijective.
- θ^{-1} is itself bijective.
- $(\theta^{-1})^{-1} = \theta$.

Using maps to compare sets

Counting

- We say a finite set A has *cardinality* n (i.e. $|A| = n$) if we can count the elements of A and get the 'answer' n.
- Say $A = \{a_1, a_2, \ldots a_n\}$ and let
$$\mathbb{Z}(n) = \{x \in \mathbb{Z} \mid 1 \leq x \leq n\}.$$
- We can think of this as a bijective map $\theta : \mathbb{Z}(n) \to A$ so that
$$\theta(1) = a_1, \quad \theta(2) = a_2, \quad \ldots \quad \theta(n) = a_n.$$

Counting

- Another way to represent this is by arranging the elements of $\mathbb{Z}(n)$ and the elements of A in one-to-one correspondance, which would demonstrate that $|A| = n$.
- For example, let C be the set of colours of the rainbow. Then

C	red	orange	yellow	green	blue	indigo	violet
$\mathbb{Z}(7)$	1	2	3	4	5	6	7

we see that $|C| = 7$.

Counting

- We can do this counting in whatever order seems sensible, for example the following is an equally good map from $\mathbb{Z}(7)$ to C which demonstrates that $|C| = 7$.

C	orange	indigo	violet	green	yellow	red	blue
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Counting

- This method of counting by producing a bijective map between sets does not just apply to finite sets.
- For example, we can show that the set of positive integers $\mathbb{Z}^+$ is the same cardinality as the natural numbers (excluding zero) $\mathbb{N}$ by producing a map as follows

$$\begin{array}{c|ccccccc} \mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \hline \mathbb{Z}^+ & 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{array}$$

- We can carry on this map forever, so there are same number of elements in each set.

Counting

- We can also produce a map between $\mathbb{N}$ and the positive even numbers.
- For example, I could simply say $\theta(x) = 2x$, so θ maps a number n onto the nth even number.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \hline 2 & 4 & 6 & 8 & 10 & 12 & \dots \end{array}$$

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- So there are the same number of natural numbers as there are integers.
- It may be surprising that a set which feels like twice as big as $\mathbb{N}$ can be the same size $\mathbb{N}$, but infinity is weird.

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- We say $\mathbb{R}$ is *uncountable*, and there are *more* real numbers than integers.
- In fact, there are more real numbers between 0 and 1 than there are integers. etc.