What can go wrong? 1: Dangling nodes

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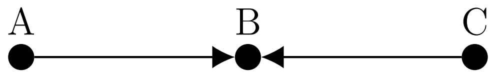
1 Scenario



This information is represented in the following table.

		From		
		A	B	C
To	A	0	0	0
	B	1	0	1
	C	0	0	0

Now as a matrix.

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2 What's the problem?

Let's find the $\lambda = 1$ eigenvalue.

$$\det\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \lambda\mathbf{I}\right) = 0$$

$$\begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0 \text{ when } \lambda = 0.$$

There is no $\lambda = 1$ eigenvalue!

Let's start with $\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and iterate:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In a sense, person B sucks up all the votes and doesn't pass any on, ultimately draining all the available votes from the system. B is called a 'dangling node', and corresponds to a column of zeros.

3 Fixing the problem

One way to deal with a dangling node is to replace the column of zeros by spreading their vote between all candidates. In a sense, they voted for everyone, rather than no one. In a voting sense, this evenly-split vote should cancel itself out, but it allows us a situation where the $\lambda = 1$ eigenvalue returns and we can produce a ranking.

Here this would be

$$\begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 1 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$$

Now we consider

$$\left(\begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 1 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} - \mathbf{I} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \frac{1}{3} & 0 \\ 1 & -\frac{2}{3} & 1 \\ 0 & \frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first row: $-x + \frac{1}{3}y = 0$, so $x = \frac{1}{3}y$.
From the third row: $\frac{1}{3} - z = 0$, so $z = \frac{1}{3}y$.
So the ratio of $x : y : z$ is $1 : 3 : 1$.

Choose $\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}$.

So person B is ranked first with $\frac{3}{5}$ths of the vote, and A and C come joint second.