Matrix transformations

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1 Introduction

In what follows, we will represent a point with coordinates (x, y) by a 2×1 matrix

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We will consider how to use matrices to transform points on a computer screen. Transformations include scaling, translation, rotation, reflection, and shearing. Mathematically, each can be represented by a *transformation matrix.*

In general, if we transform our point using the transformation matrix

$$\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then we obtain

$$\begin{aligned} \mathbf{TX} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}. \end{aligned}$$

We can say the point $\begin{bmatrix} x \\ y \end{bmatrix}$ has been translated to the point $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$.

2 Shapes

We can consider a line as being defined by two end-points. Hence the matrix

$$\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

represents the line joining the points $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Similarly, a triangle can be stored as a 2×3 matrix, for example

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & -1 \end{bmatrix}.$$

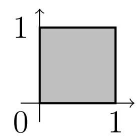
And so on.

3 Scaling

Consider the square

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

We can draw this.



Now if we apply the transformation

$$\mathbf{T} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

we obtain

$$\begin{aligned} \mathbf{TS} &= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 4 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

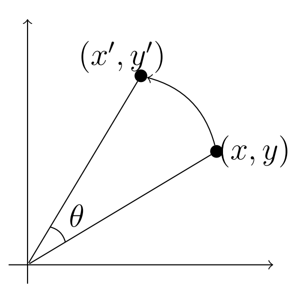
We can draw this.

In general, scaling is done using a diagonal matrix, with the first row determining the horizontal scaling and the second the vertical.

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

4 Rotation

In this diagram, the point (x, y) is rotated by θ around the origin to (x', y').



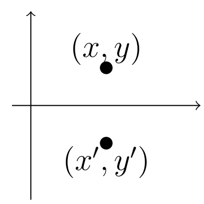
The transformation matrix that rotates a point anticlockwise around the origin by an angle θ is given by

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

5 Reflection

5.1 Reflection in the x-axis

In the diagram below, the point (x, y) is reflected in the x-axis to (x', y').



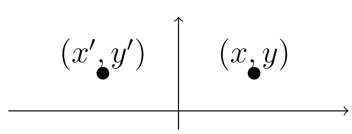
The two points have the same horizontal coordinate, so $x' = x$. However, the vertical coordinate has flipped, so $y' = -y$.

The transformation matrix that produces this effect is

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

5.2 Reflection in the y-axis

In the diagram below, the point (x, y) is reflected in the y-axis to (x', y').



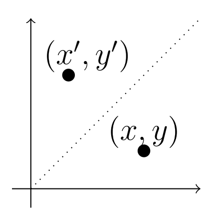
The two points have the same vertical coordinate, so $y' = y$. However, the horizontal coordinate has flipped, so $x' = -x$.

The transformation matrix that produces this effect is

$$\mathbf{T} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

5.3 Reflection in the line $y = x$

In the diagram below, the point (x, y) is reflected in the line $y = x$ to (x', y').

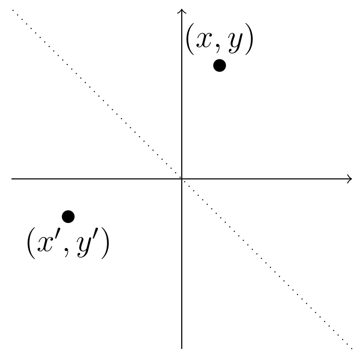


In effect, the x and y coordinates have swapped. The transformation matrix that brings this about is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

5.4 Reflection in the line $y = -x$

In the diagram below, the point (x, y) is reflected in the line $y = x$ to (x', y').

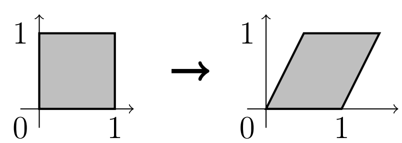


Here the x and y coordinates have swapped and also changed sign. The transformation matrix that brings this about is

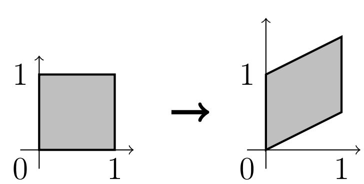
$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

6 Shearing

A shear is a distortion. The diagram below shows a shear in the x-direction.



And the diagram below shows a shear in the y-direction.



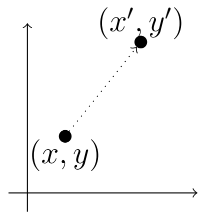
A shear is brought about by the transformation matrix

$$\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The off-diagonal terms, b and c, determine the type of shear produced, with b acting in the x direction and c in the y direction. Similarly to section 3, the element a is a scale factor in the x direction and d is a scale factor in the y direction.

7 Translation

A translation is a movement in a specific direction by a specific amount. For example, in the diagram below the point (x, y) is translated to (x', y').



Consider a point at $\begin{bmatrix}1\\3\end{bmatrix}$ which is shifted to $\begin{bmatrix}4\\7\end{bmatrix}$. The translation has added 3 to the x coordinate and added 4 to the y coordinate. That is

$$\begin{aligned}4 &= 1 + 3\\ 7 &= 4 + 4.\end{aligned}$$

In general, to translate $\begin{bmatrix}x\\y\end{bmatrix}$ to $\begin{bmatrix}x'\\y'\end{bmatrix}$, we use

$$\begin{aligned}x' &= x + t_x\\ y' &= y + t_y\end{aligned}$$

where t_x is the translation in the x direction and t_y is the translation in the y direction.

The previous transformations all involved multiplication of coordinates by some factor, but this one is different. It requires us to add a number to the different coordinates.

To enable this to be represented by matrix multiplication, we introduce *homogeneous coordinates*. The point $\begin{bmatrix}x\\y\end{bmatrix}$ is represented in homogeneous coodinates by

$$\mathbf{X} = \begin{bmatrix}x\\y\\1\end{bmatrix}.$$

The extra '1' is used to increase the order of the transformation matrix from 2×2 to 3×3. We now use a transformation matrix

$$\mathbf{T} = \begin{bmatrix}1 & 0 & t_x\\0 & 1 & t_y\\0 & 0 & 1\end{bmatrix}.$$

Applying $\mathbf{T}$ to $\mathbf{X}$, we obtain

$$\mathbf{TX} = \begin{bmatrix}1 & 0 & t_x\\0 & 1 & t_y\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix} = \begin{bmatrix}x + t_x\\y + t_y\\1\end{bmatrix}.$$

The homogenous coordinates

$$\begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

represent the point

$$\begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}.$$

8 Composite transformations

Since we have transformations encoded as matrices, we can multiply these together to form composite transformations.

For example, we saw in section 5.4 that reflection in the $y = -x$ line is obtained using

$$\mathbf{T}_{\text{ref}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and using the technique in section 4, rotation about the origin by 70° would be represented by

$$\mathbf{T}_{\text{rot}} = \begin{bmatrix} \cos(70^\circ) & -\sin(70^\circ) \\ \sin(70^\circ) & \cos(70^\circ) \end{bmatrix}.$$

The composite transformation is obtained by multiplying these matrices together. Remember that in matrix multiplication, $\mathbf{AB} \neq \mathbf{BA}$, so we must think about the order in which we want to apply the transformations.

Acting on a point $\mathbf{X}$, if we perform the reflection then a rotation we would calculate $\mathbf{T}_{\text{rot}}\mathbf{T}_{\text{ref}}\mathbf{X}$. Note that the last transformation comes first in our multiplication.

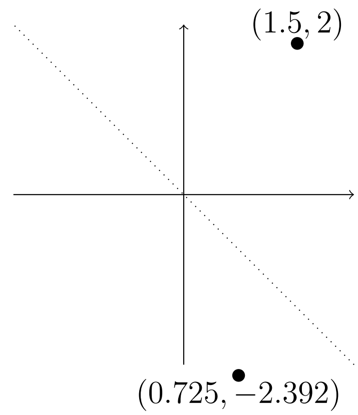
We can calculate the composite transformation matrix

$$\begin{aligned} \mathbf{T}_{\text{rot}}\mathbf{T}_{\text{ref}} &= \begin{bmatrix} \cos(70^\circ) & -\sin(70^\circ) \\ \sin(70^\circ) & \cos(70^\circ) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sin(70^\circ) & -\cos(70^\circ) \\ -\cos(70^\circ) & -\sin(70^\circ) \end{bmatrix}. \end{aligned}$$

For example, if we apply our transformation to the point $\begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$ we obtain

$$\begin{bmatrix} \sin(70^\circ) & -\cos(70^\circ) \\ -\cos(70^\circ) & -\sin(70^\circ) \end{bmatrix} \begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.725 \\ -2.392 \end{bmatrix}.$$

This is represented on the diagram below.



If a composite transformation involves a translation, we must use homogenous coordinates for the point and all transformations.

8.1 Example

Determine a transformation matrix that rotates a point 30° anticlockwise about the point $\begin{bmatrix}1\\2\end{bmatrix}$.

The rotation matrices so far have represented rotation about the origin. To represent rotation about $\begin{bmatrix}1\\2\end{bmatrix}$ we require three transformations:

1. translation of $\begin{bmatrix}1\\2\end{bmatrix}$ to the origin, $\mathbf{T}_{\text{trans}}$;
2. rotation of 30° anticlockwise, $\mathbf{T}_{\text{rot}}$;
3. translation from the origin back to $\begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{T}_{\text{back}}$.

$$\begin{aligned}\mathbf{T}_{\text{trans}} &= \begin{bmatrix}1 & 0 & -1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix};\\ \mathbf{T}_{\text{rot}} &= \begin{bmatrix}\cos(30^\circ) & -\sin(30^\circ) & 0\\ \sin(30^\circ) & \cos(30^\circ) & 0\\ 0 & 0 & 1\end{bmatrix};\\ \mathbf{T}_{\text{back}} &= \begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}.\end{aligned}$$

The composite transformation is then

$$\mathbf{T}_{\text{back}}\mathbf{T}_{\text{rot}}\mathbf{T}_{\text{trans}} = \begin{bmatrix}0.8660 & -0.5 & 1.1340\\ 0.5 & 0.8660 & -0.232\\ 0 & 0 & 1\end{bmatrix}.$$