Tutorial exercise sheet – Determinants and inverses

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1. Calcuate the determinants of the following matrices.

   (a) $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$; (b) $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$; (c) $\mathbf{C} = \begin{bmatrix} 3 & -6 \\ -9 & 12 \end{bmatrix}$;

2. For the matrix $\mathbf{A} = \begin{bmatrix} a & b & c \\ -1 & 1 & 3 \\ 2 & 7 & 9 \end{bmatrix}$, find:

   (a) the minor of a in $\mathbf{A}$; (b) the cofactor of a in $\mathbf{A}$; (c) the minor of b in $\mathbf{A}$; (d) the cofactor of b in $\mathbf{A}$; (e) the minor of c in $\mathbf{A}$; (f) the cofactor of c in $\mathbf{A}$; (g) the minor of b in $\mathbf{A}^\mathsf{T}$; (h) the cofactor of b in $\mathbf{A}^\mathsf{T}$; (i) the cofactor of c in $\mathbf{A}^\mathsf{T}$.

3. Calculate the determinants of the following matrices.

   (a) $\mathbf{A} = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$; (b) $\mathbf{B} = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$; (c) $\mathbf{C} = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 5 & 7 \\ 6 & 3 & 8 \end{bmatrix}$;

   (d) $\mathbf{D} = \begin{bmatrix} 2 & 7 & 6 \\ 4 & -1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$; (e) $\mathbf{E} = \begin{bmatrix} 3 & -4 & -3 \\ 2 & 7 & -31 \\ 5 & -9 & 2 \end{bmatrix}$; (f) $\mathbf{F} = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$;

   (g) $\mathbf{G} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -3 & 9 \\ 1 & -3 & 9 \end{bmatrix}$; (h) $\mathbf{H} = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$; (j) $\mathbf{J} = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$.

4. Find the values of t for which $\begin{vmatrix} t & 1 & -2 \\ 2 & t+1 & 1 \\ -1 & -4 & t-2 \end{vmatrix} = 17$.

5. In each case, determine whether the statement is true or false, and justify your answer.

   (a) The value of the determinant is independent of which row or column the cofactor expansion takes place along.

   (b) The minor M_{ij} of $\mathbf{A}$ is the same as the cofactor A_{ij} if and only if $i + j$ is even.

   (c) The determinant of a matrix whose elements are all positive must be positive.

   (d) If a matrix has a row or column of zeros, then its determinant is zero.

   (e) If each element in a matrix doubles, its determinant also doubles.

   (f) For a square matrix $\mathbf{A}$, $\det(\mathbf{A}) = \det(\mathbf{A}^\mathsf{T})$.

6. What is the maximum number of zeros a 3×3 matrix can have without having a zero determinant?

7.
$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}.$$

(a) Compute $3\mathbf{A} - \mathbf{B}$.

(b) What is $\mathbf{A}^2$?

(c) Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

(d) Show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(e) The matrix $\mathbf{A}$ satisfies $\mathbf{A}^2 - 5\mathbf{A} + \mathbf{I} = \mathbf{0}$. Use this to find a formula for $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$, and show that the matrix $\mathbf{A}^{-1}$ obtained equals the matrix $\mathbf{A}^{-1}$ in part (c).

8. (a) Find the matrix $\mathbf{X}$, given that: $\begin{bmatrix} 1 & 3 \\ 5 & 5 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 13 & 23 \\ 25 & 45 \end{bmatrix}$.

(b) Find the matrix $\mathbf{Y}$, given that: $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \mathbf{Y} = \begin{bmatrix} 17 & 8 \\ 14 & 6 \end{bmatrix}$.

(c) Find the matrix $\mathbf{Z}$, given that: $\begin{bmatrix} -4 & 2 \\ 8 & -2 \end{bmatrix} \mathbf{Z} = \begin{bmatrix} 0 & 7 \\ 1 & -6 \end{bmatrix}$.

9. Find the inverse of
$$\mathbf{Y} = \begin{bmatrix} -1 & 0 & 2 \\ -10 & -1 & 0 \\ 3 & 8 & 2 \end{bmatrix}.$$

10.
$$\text{Consider } \mathbf{K} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 3 & 1 & 2 \end{bmatrix}. \text{ Given that } \mathbf{K}^{-1} = -\frac{1}{a} \begin{bmatrix} b & -3 & -4 \\ -10 & -1 & c \\ d & 5 & -4 \end{bmatrix},$$

determine the values of a, b, c and d.

11.
$$\text{Consider } \mathbf{R} = \begin{bmatrix} 0 & 8 & 6 \\ 1 & 0 & -4 \\ 2 & 3 & 1 \end{bmatrix}. \text{ Given that } \mathbf{R}^{-1} = \frac{1}{a} \begin{bmatrix} b & -5 & 16 \\ \frac{9}{2} & 6 & c \\ d & -8 & 4 \end{bmatrix},$$

determine the values of a, b, c and d.