Counting up to symmetry

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1 Symmetries as permutations

Recall that $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ represents a function that maps

$$\begin{aligned} \sigma(1) &= 2 \\ \sigma(2) &= 4 \\ \sigma(3) &= 3 \\ \sigma(4) &= 1. \end{aligned}$$

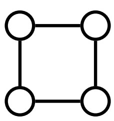
We can write this as a product of disjoint cycles.
In general:

- $(\dots xy \dots)$ means x shifts to y;
- $(x \dots y)$ means y shifts to x.

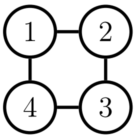
For example

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} &= (124)(3); \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} &= (14)(23); \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} &= (1234); \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} &= (1)(2)(3)(4). \end{aligned}$$

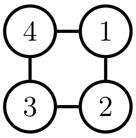
We can write the symmetries of a square as permutations. Consider the square



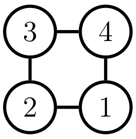
First we consider the rotations.

e: 

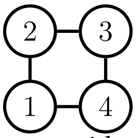
$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix} = (1)(2)(3)(4).$$

r: 

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix} = (1234).$$

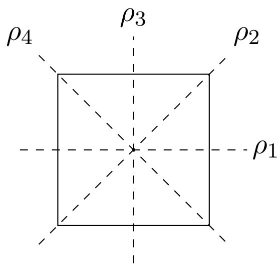
r^2: 

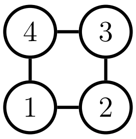
$$\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} = (13)(24).$$

r^3: 

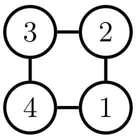
$$\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix} = (1234).$$

Next we consider the reflections, labelled as follows.

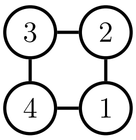


ρ_1: 

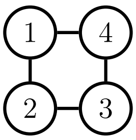
$$\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} = (14)(23).$$

ρ_2: 

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 1 & 4\end{pmatrix} = (13)(2)(4).$$

ρ_3: 

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix} = (12)(34).$$

ρ_3: 

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 3 & 2\end{pmatrix} = (1)(24)(3).$$

2 Cycle notation

Let x_i represent a cycle of length i.

Then, for example, we can represent

- (1) as x_1;
- (12) as x_2;

- (1234) as x_4;
- (13)(24) as x_2^2;
- (1)(2)(3)(4) as x_1^4;
- (1)(2)(34) as $x_1^2 x_2$.

We can represent the symmetries of a square in cycle notation.

e: x_1^4
r: x_4
r^2: x_2^2
r^3: x_4
ρ_1: x_2^2
ρ_2: $x_1^2 x_2$
ρ_3: x_2^2
ρ_4: $x_1^2 x_2$

In a colouring problem, each member of a cycle must be coloured the same way, since the elements are mapped onto each other by the symmetry in question.

For example, (13) means $1 \to 3$ and $3 \to 1$, so 1 and 3 must be the same colour.

So if there are m colours, each x_i represents a set of elements that can be coloured m different ways.

3 Burnside's Theorem

Theorem. *For G a group of permutations of $\{1, 2, \ldots, n\}$, the cycle index of G is*

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} (\textit{cycle representation of } \pi).$$

Then the number of m-colourings of configurations acted upon by G is $P(m, m, \ldots, m)$.

3.1 Example

For colouring the vertices of a square, we have

$$\begin{aligned} P(x_1, x_2, x_3, x_4) &= \frac{1}{8}\left(x_1^4 + x_4 + x_2^2 + x_4 + x_2^2 + x_1^2 x_2 + x_2^2 + x_1^2 x_2\right) \\ &= \frac{1}{8}\left(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2 x_2\right). \end{aligned}$$

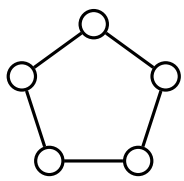
We can 2-colour this in

$$P(2,2,2,2) = \frac{1}{8}(2^4 + 2 \times 2 + 3 \times 2^2 + 2 \times 2^2 \times 2) = 6$$

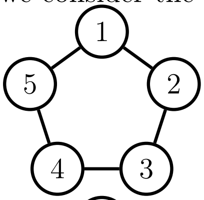
ways.

4 Example

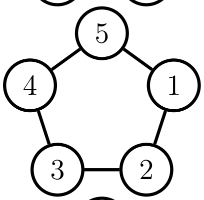
How many ways are there to colour the vertices of a regular pentagon with up to 5 colours?



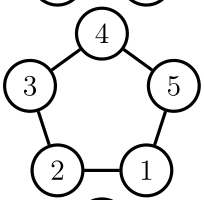
First we consider the five rotations.

e 

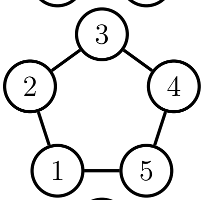
$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 3 & 4 & 5\end{pmatrix} = (1)(2)(3)(4)(5) \qquad x_1^5$$

r 

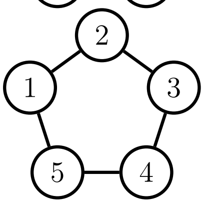
$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 1 & 2 & 3 & 4\end{pmatrix} = (15432) \qquad x_5$$

r^2 

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 5 & 1 & 2 & 3\end{pmatrix} = (14253) \qquad x_5$$

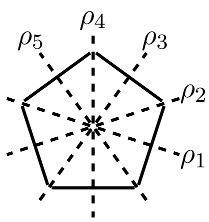
r^3 

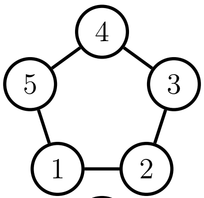
$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 1 & 2\end{pmatrix} = (13524) \qquad x_5$$

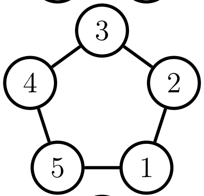
r^4 

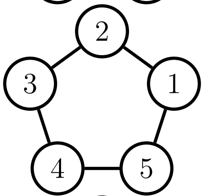
$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 4 & 5 & 1\end{pmatrix} = (12345) \qquad x_5$$

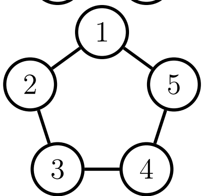
Next we consider the following five reflections.

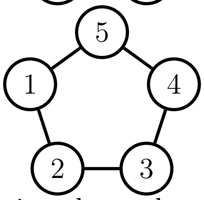


ρ_1  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix} = (14)(23)(5) \quad x_1 x_2^2$

ρ_2  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = (13)(2)(45) \quad x_1 x_2^2$

ρ_3  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix} = (12)(35)(4) \quad x_1 x_2^2$

ρ_4  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix} = (1)(25)(34) \quad x_1 x_2^2$

ρ_5  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (15)(24)(3) \quad x_1 x_2^2$

Collecting the cycle representations together, we have

$$P(x_1, x_2, x_3, x_4, x_5) = \frac{1}{10}\left(x_1^5 + 4x_5 + 5x_1x_2^2\right).$$

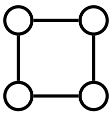
Thus we can 5-colour the shape in

$$P(5,5,5,5,5) = \frac{1}{10}\left(5^5 + 4 \times 5 + 5 \times 5 \times 5^2\right) = 377$$

ways.

5 Pólya enumeration

For colouring the vertices of a square like this



we obtained the function

$$P(x_1, x_2, x_3, x_4) = \frac{1}{8}\left(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2\right).$$

Elements in x_1 (e.g. (1)) can be coloured either

- blue: $1 + b$;
- green: $1 + g$.

So together these can be coloured $(1+b)(1+g) = 1+b+g+bg$ ways.

There is one element in x_1 and we require that it has a colour, therefore its colours can be expressed as $b+g$.

For x_2 (e.g. (12)), we have

- blue: $1+b+b^2$;
- green: $1+g+g^2$.

This gives $(1+b+b^2)(1+g+g^2) = 1+g+g^2+b+bg+bg^2+b^2+b^2g+b^2g^2$.

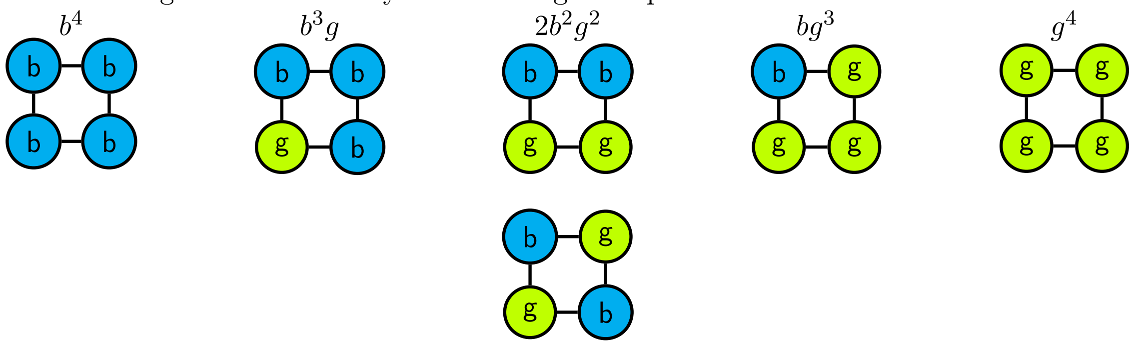
Those terms which represent two elements coloured with the same colour are: b^2+g^2.

In general, x_i can be coloured in b^i+g^i ways.

Now consider

$$\begin{aligned} P(b+g, b^2+g^2, b^3+g^3, b^4+g^4) &= \frac{1}{8}\left((b+g)^4 + 2(b^4+g^4)^2 + 3(b^2+g^2)^2 + 2(b+g)^2(b^2+g^2)\right) \\ &= b^4 + b^3g + 2b^2g^2 + bg^3 + g^4. \end{aligned}$$

These terms generate the 6 ways of colouring the square.



5.1 Example

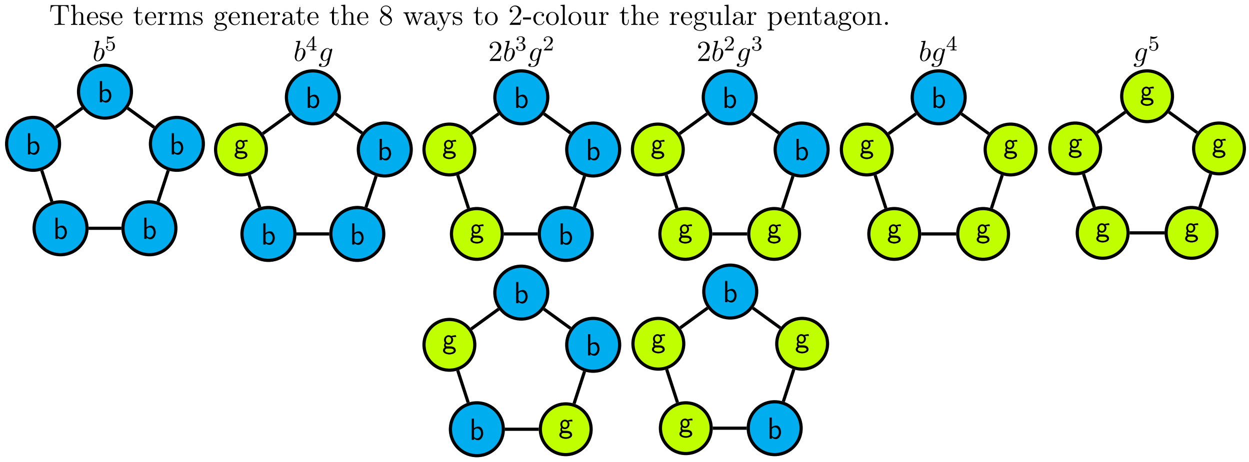
For colouring the vertices of the regular pentagon we found

$$P(x_1, x_2, x_3, x_4, x_5) = \frac{1}{10}\left(x_1^5 + 4x_5 + 5x_1x_2^2\right).$$

Then

$$\begin{aligned} P(b+g, \ldots, b^5+g^5) &= \frac{1}{10}\left((b+g)^5 + 4(b^5+g^5) + 5(b+g)(b^2+g^2)^2\right) \\ &= b^5 + b^4g + 2b^3g^2 + 2b^2g^3 + bg^4 + g^5. \end{aligned}$$

These terms generate the 8 ways to 2-colour the regular pentagon.



6 Colour Match puzzle

The problem of how many different tiles are used in the Colour Match puzzle can be answered as part of the wider question of four-colouring the vertices of a square.

Here we only have the identity and the rotations, since the tiles cannot be flipped over (the coloured dots don't show through to the back of the tiles). This means

$$P(x_1, x_2, x_3, x_4) = \frac{1}{4}\left(x_1^4 + 2x_4 + x_2^2\right).$$

Elements of a cycle x_i of length i can be coloured the same in $b^i + g^i + r^i + y^i$ ways. To generate the ways of colouring, we use

$$\begin{aligned}
&P(b+g+r+y, r^2+b^2+g^2+y^2, r^3+b^3+g^3+y^3, r^4+b^4+g^4+y^4)\\
=&\frac{1}{4}\left((r+b+g+y)^4 + 2(r^4+b^4+g^4+y^4) + (r^2+b^2+g^2+y^2)^2\right)\\
=&b^4 + b^3g + b^3r + b^3y + 2b^2g^2 + 3b^2gr + 3b^2gy + 2b^2r^2 + 3b^2ry + 2b^2y^2 + bg^3 + 3bg^2r\\
&+ 3bg^2y + 3bgr^2 + 6bgry + 3bgy^2 + br^3 + 3br^2y + 3bry^2 + by^3 + g^4 + g^3r + g^3y + 2g^2r^2\\
&+ 3g^2ry + 2g^2y^2 + gr^3 + 3gr^2y + 3gry^2 + gy^3 + r^4 + r^3y + 2r^2y^2 + ry^3 + y^4.
\end{aligned}$$

The relevant term is $6bgry$, the coefficient of which tells us there are six ways of colouring the tiles with each colour used for exactly one vertex.