Quantifiers

Peter Rowlett

1 For all

Often in maths we use the construction “for all”, for example we might write “$x^2 \geq 0$ for all $x \in \mathbb{R}$”.

The symbol $\forall$ is a shorthand for this, so we could equivalently write $\forall x \in \mathbb{R}\ (x^2 \geq 0)$. We tend to write the $\forall$ at the start, and the statement being acted on is placed in brackets for clarity. The symbol $\forall$ is an upside-down ‘A’.

To show $\forall x\ (p)$, imagine someone gives you an x and you have to show that p is true for that x. You have to be able to do this no matter what x they give you.

2 There exists

Another term we often use is “there exists”, for example we might write “there exists $x \in \mathbb{Z}^+$ such that $x^2 = 9$”.

The symbol $\exists$ is a shorthand for this, so we could equivalently write $\exists x \in \mathbb{Z}^+\ (x^2 = 9)$.

To show $\exists x\ (p)$, imagine you get to choose an x so that p is true for that x. You can choose any x.

3 Negation

To negate a statement p is to find an equivalent statement which means $\neg p$. If p is “All cats are grey”, then $\neg p$ is not “All cats are not grey”, so negation is not as simple as putting ‘not’ in the right place.

In fact, the negation of “All cats are grey” is “Some cats are not grey”. So $\exists$ negates $\forall$ (and vice versa).

The following hold

- $\neg(\forall x\ (p)) \iff \exists x\ (\neg p)$;
- $\neg(\exists x\ (p)) \iff \forall x\ (\neg p)$.

To negate a sentence with quantifiers on p, change every $\forall$ to $\exists$, change every $\exists$ to $\forall$, and replace p with its negation.