Proof methods – exercises

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1. In logic, we talk about *necessary* conditions and *sufficient* conditions. If p is a necessary condition for q it does not mean that p on its own is enough to guarantee q. Rather, it means p will have to be true if there is to be any question of q being true – we need p for q. It means $q \implies p$.

   We also talk about *sufficient* conditions. If p is a sufficient condition for q it means that p being true is enough to say q is true, though it is possible q is true without p being true. It means $p \implies q$.

   If p is necessary and sufficient for q it means $p \iff q$.

   Which of the following conditions is *necessary* for $n \in \mathbb{N}$ to be divisible by 6? Which conditions are *sufficient* for n to be divisible by 6?

   (a) n is divisible by 3;
   (b) n is divisible by 9;
   (c) n is divisible by 12;
   (d) $n = 24$;
   (e) n^2 is divisible by 3;
   (f) n is even and divisible by 3.

2. Choose any five consecutive positive whole numbers, and multiply them together. Did you get a multiple of 120? Will you always get a multiple of 120? How would you convince someone?

3. Prove or disprove the following.

   (a) Let m be an integer. If m is odd, then m^2 is odd.
   (b) Suppose that $p \in \mathbb{Q}$ and $p^2 \in \mathbb{Z}$. Then, $p \in \mathbb{Z}$.
   (c) Let m and n be real numbers. If $n > m > 0$, then
   $$\frac{m+1}{n+1} > \frac{m}{n}.$$
   (d) Let A, B and C be sets. Then,
   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
   (e) Let A and B be sets. Then, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

4. Show that the sum of two consecutive odd numbers is a multiple of 4. What is the converse and is it true?

5. Let $f : X \to Y$. Suppose A and B are subsets of X. Show $f(A \cup B) = f(A) \cup f(B)$.

6. Prove or disprove the following.
   (a) $a + b = c \implies a^2 + b^2 = c^2$.
   (b) Let $x \in \mathbb{Z}$. Then $-x$ is negative.

7. The following is a proof that $1 = 2$. You may have reason to doubt that this is a true fact! Where is the error?

   Theorem. $1 = 2$.

   Proof. Let $a = b$, where $a, b \in \mathbb{Z}$. Then,

$$\begin{array}{rcll} ab & = & a^2 & \text{since } a = b, \\ ab - b^2 & = & a^2 - b^2 & \text{by subtracting } b^2 \text{ from both sides,} \\ b(a-b) & = & (a+b)(a-b) & \text{by factoring,} \\ b & = & a + b & \text{by dividing both sides by } a - b, \\ b & = & 2b & \text{since } a = b, \\ 1 & = & 2 & \text{by dividing by } b. \end{array}$$

$\square$

8. Prove or disprove the following.
   (a) Suppose $n \in \mathbb{N}$. Then $n^3 - n$ is a multiple of 3.
   (b) Suppose $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.
   (c) The square of any integer is of the form $3k$ or $3k + 1$ for some $k \in \mathbb{Z}$.
   (d) Suppose $a = bc$ for $a, b, c \in \mathbb{R}$. If two of a, b or c are non-zero, then so is the third.

9. Prove or disprove the following.
   (a) There are no positive integers x and y such that $x^2 - y^2 = 1$.
   (b) The sum of a rational and an irrational number is an irrational number.
   (c) $\sqrt{3}$ is irrational.
   (d) $\sqrt{4}$ is irrational. (What happens is you try the same approach as for $\sqrt{3}$?)
   (e) There are no positive integer solutions to $x^2 + x + 1 = y^2$.
   (f) Suppose $x, y \in \mathbb{Z}$. Then $\sqrt{x^2 + y^2} \neq x + y$.