Recurrence patterns

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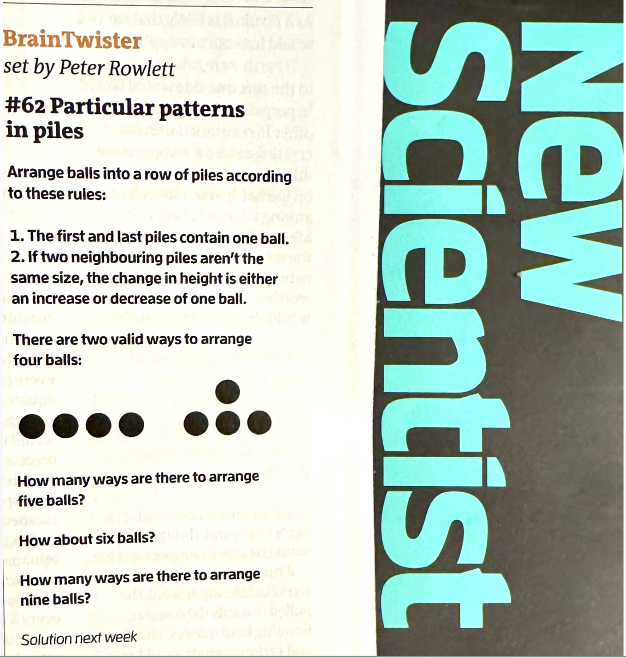
A puzzle

- *New Scientist*, 1 March 2025.



BrainTwister

set by Peter Rowlett

#62 Particular patterns in piles

Arrange balls into a row of piles according to these rules:

1. The first and last piles contain one ball.
2. If two neighbouring piles aren't the same size, the change in height is either an increase or decrease of one ball.

There are two valid ways to arrange four balls:

How many ways are there to arrange five balls?

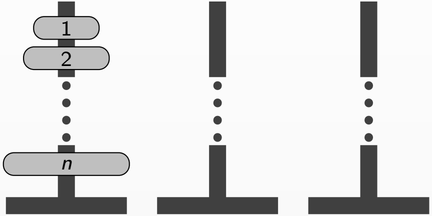
How about six balls?

How many ways are there to arrange nine balls?

Solution next week

Tower of Hanoi

- Move a stack of discs from one peg to one of the empty ones, obeying these rules:
  1. move one disc at a time which must be the upper disc on its stack;
  2. no disc may be placed on top of a disc that is smaller than it.



History

- There are various claims about the ancient origin of this puzzles.
- Including a temple (sometimes in Hanoi, or India) in which monks are playing the game with 64 discs and when they are finished the world will end.

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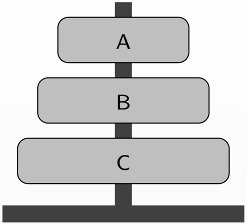
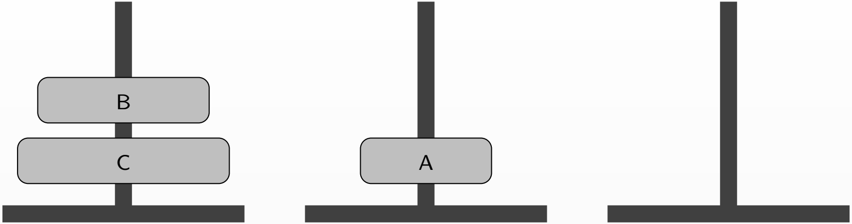
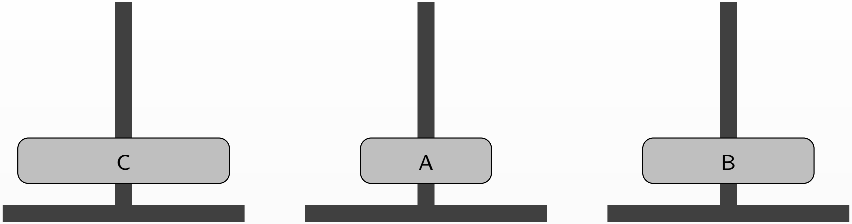
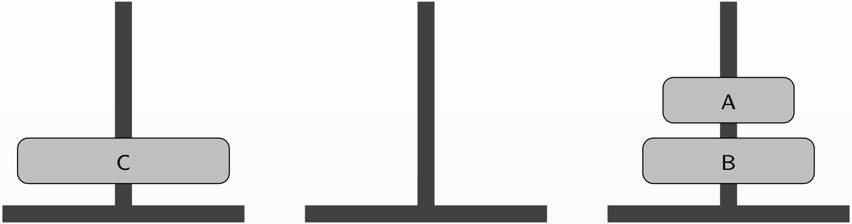
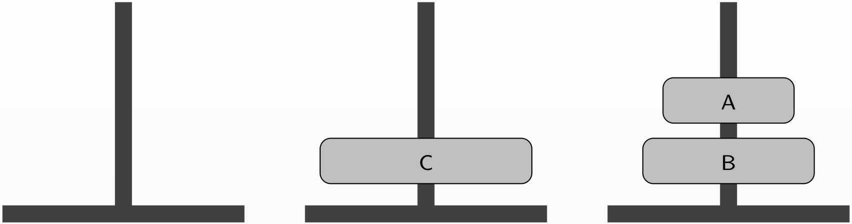
- Including a temple (sometimes in Hanoi, or India) in which monks are playing the game with 64 discs and when they are finished the world will end.
- It was invented by French mathematician Édouard Lucas in 1883.

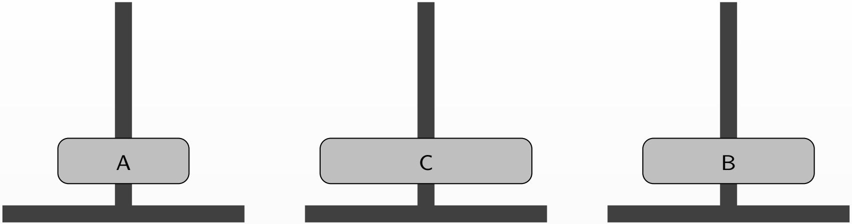

A
B
C

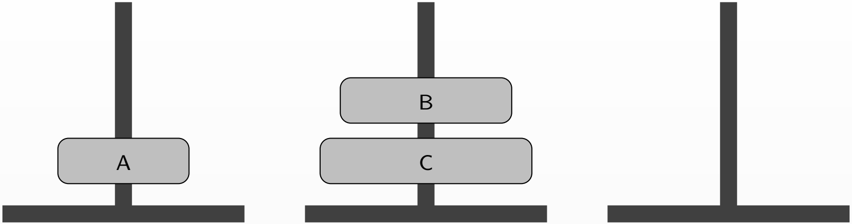

B
C
A

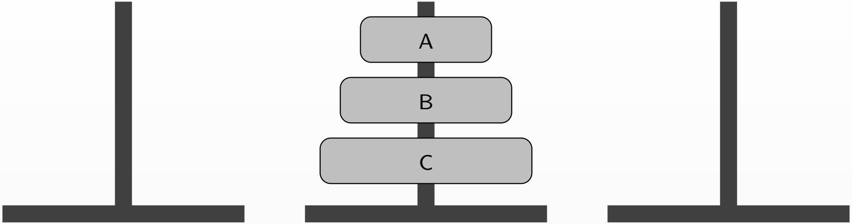

C
A
B


C
A
B


A
C
B

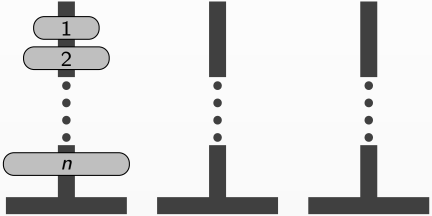

A
C
B

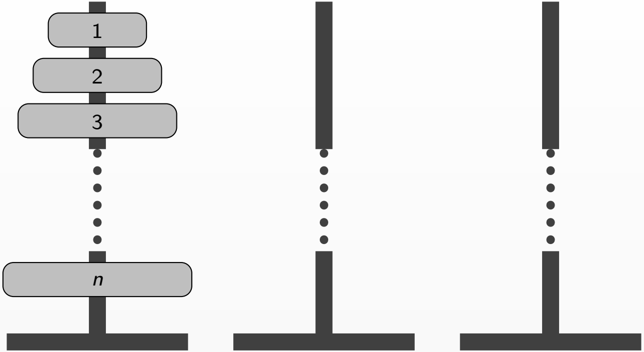

B
A
C

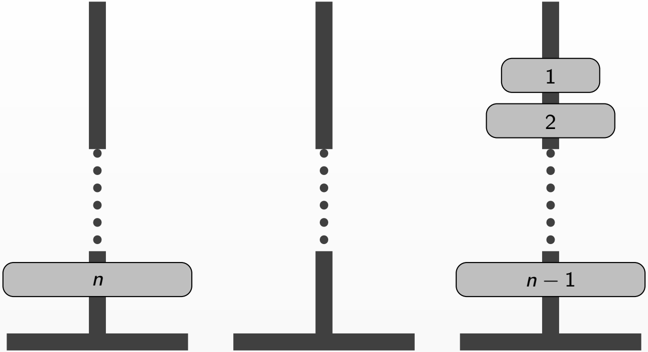

A
B
C

Tower of Hanoi

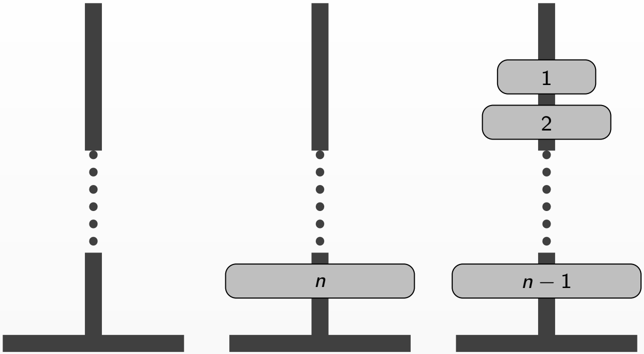
- Move a stack of discs from one peg to one of the empty ones, obeying these rules:
  1. move one disc at a time which must be the upper disc on its stack;
  2. no disc may be placed on top of a disc that is smaller than it.
- Have a go, with small numbers of discs (1, 2, 3, 4, . . .).
- What is the minimum number of moves (call this T_n for a tower with n discs)?

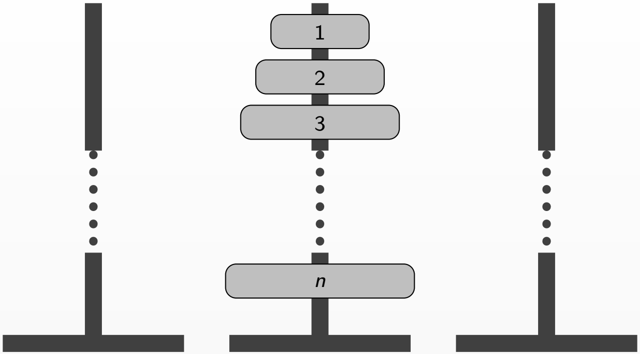



1
2
3
n



1
2
n
$n-1$


1
2
n
n − 1


1
2
3
n

Tower of Hanoi

- To move n discs, first we move $n-1$ discs, then move the nth, then move the $n-1$ on top of it.
- Moving $n-1$ discs can be done in T_{n-1} moves.
- Therefore,

$$T_n = 2T_{n-1} + 1.$$

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- Therefore,

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n	T_n
1	1
2	3
3	7
4	15
5	31
$\vdots$	$\vdots$

Tower of Hanoi

- Does this look like

$$T_n = 2^n - 1?$$

- Does this pattern continue forever?

n	2^n	T_n
1	$2^1 = 2$	1
2	$2^2 = 4$	3
3	$2^3 = 8$	7
4	$2^4 = 16$	15
5	$2^5 = 32$	31
$\vdots$	$\vdots$	$\vdots$

Can we prove $T_n = 2^n - 1$?

- We can do this by induction.
- We have $T_1 = 1$.
- Inductive hypothesis: for $1 \leq k \leq n-1$, $T_k = 2^k - 1$.
- Now consider

$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2^{n-1} - 1) + 1 \qquad \text{(inductive hypothesis with } k = n-1\text{)} \\ &= 2^n - 2 + 1 = 2^n - 1. \end{aligned}$$

$\square$