Boolean algebra

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Truth tables

- Remember we can prove two propositions are equivalent by showing they have the same truth table.
- For example, the truth table practice worksheet contained these:

p	q	$\neg(p \wedge q)$
T	T	F
T	F	T
F	T	T
F	F	T

p	q	$\neg p \vee \neg q$
T	T	F
T	F	T
F	T	T
F	F	T

- Therefore we can say $\neg(p \wedge q) \iff \neg p \vee \neg q$.
- This is one of De Morgan's Laws, which we will meet shortly.

Boolean algebra

- Uses the logical connectives we have been working with, such as $\wedge$, $\vee$, and $\neg$.
- 'True' is represented using 1.
- 'False' is represented using 0.
- For example, if p is the proposition "some pigs can fly" we would say $p = 0$.
- Boolean algebra follows some algebra laws you will be familiar with, and some extras that will be less familiar.

Commutative law

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$q \wedge p$
1	1	1
1	0	0
0	1	0
0	0	0

Commutative law

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$q \wedge p$
1	1	1
1	0	0
0	1	0
0	0	0

- It doesn't matter which way around we consider p and q with $\wedge$ or $\vee$.

$$p \vee q = q \vee p \qquad \& \qquad p \wedge q = q \wedge p$$

Commutative law

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$q \wedge p$
1	1	1
1	0	0
0	1	0
0	0	0

- It doesn't matter which way around we consider p and q with $\wedge$ or $\vee$.

$$p \vee q = q \vee p \qquad \& \qquad p \wedge q = q \wedge p$$

- Note that ordinary multiplication and addition have commutativity too:

$$a + b = b + a \qquad \& \qquad a \times b = b \times a$$

Associative law

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
1	1	1	1	1	1	1
1	1	0	1	1	1	1
1	0	1	1	1	1	1
1	0	0	1	1	0	1
0	1	1	1	1	1	1
0	1	0	1	1	1	1
0	0	1	0	1	1	1
0	0	0	0	0	0	0

Associative law

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
1	1	1	1	1	1	1
1	1	0	1	1	1	1
1	0	1	1	1	1	1
1	0	0	1	1	0	1
0	1	1	1	1	1	1
0	1	0	1	1	1	1
0	0	1	0	1	1	1
0	0	0	0	0	0	0

- Therefore $(p \vee q) \vee r = p \vee (q \vee r)$.

Associative law

- This works for $\wedge$ as well (you could check by writing out the truth tables), so we have:

$$(p \vee q) \vee r = p \vee (q \vee r) \qquad \& \qquad (p \wedge q) \wedge r = p \wedge (q \wedge r)$$

Associative law

- This works for $\wedge$ as well (you could check by writing out the truth tables), so we have:

$$(p \vee q) \vee r = p \vee (q \vee r) \qquad \& \qquad (p \wedge q) \wedge r = p \wedge (q \wedge r)$$

- Again multiplication and addition have associativity too:

$$(a + b) + c = a + (b + c) \qquad \& \qquad (a \times b) \times c = a \times (b \times c)$$

Identity

- The identity leaves what it is acting upon unchanged.

p	0	$p \vee 0$
1	0	1
0	0	0

p	1	$p \wedge 1$
1	1	1
0	1	0

Identity

- The identity leaves what it is acting upon unchanged.

p	0	$p \vee 0$
1	0	1
0	0	0

p	1	$p \wedge 1$
1	1	1
0	1	0

- In Boolean algebra, we have

$$p \vee 0 = p \qquad \& \qquad p \wedge 1 = p$$

Identity

- The identity leaves what it is acting upon unchanged.

p	0	$p \vee 0$
1	0	1
0	0	0

p	1	$p \wedge 1$
1	1	1
0	1	0

- In Boolean algebra, we have

$$p \vee 0 = p \qquad \& \qquad p \wedge 1 = p$$

- Once again, we have parallels in addition and multiplication:

$$a + 0 = a \qquad \& \qquad a \times 1 = a$$

Distributive law

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	0	0
0	1	1	1	0
0	1	0	1	0
0	0	1	1	0
0	0	0	0	0

$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
1	1	1
1	0	1
0	1	1
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0

Distributive law

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
1	1	1	1	1
1	1	0	1	1
1	0	1	1	1
1	0	0	0	0
0	1	1	1	0
0	1	0	1	0
0	0	1	1	0
0	0	0	0	0

$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
1	1	1
1	0	1
0	1	1
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0

- Therefore $(p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Distributive law

- This works for $\wedge$ as well (you could check by writing out the truth tables), so we have:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \qquad \& \qquad p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

Distributive law

- This works for $\wedge$ as well (you could check by writing out the truth tables), so we have:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \qquad \& \qquad p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

- This is similar to the rule that multiplication distributes over addition:

$$a \times (b + c) = (a \times b) + (a \times c)$$

Some less familiar ones

Idempotent law

- If I add a number to itself, I get double the number.
- But if I tell you p is true and p is true, I'm not saying p is doubly true.

p	$p \wedge p$
1	1
0	0

p	$p \vee p$
1	1
0	0

Idempotent law

- If I add a number to itself, I get double the number.
- But if I tell you p is true and p is true, I'm not saying p is doubly true.

p	$p \wedge p$
1	1
0	0

p	$p \vee p$
1	1
0	0

- So we have

$$p \wedge p = p \qquad \& \qquad p \vee p = p$$

Negation & Double negation

1	0	$\neg 0$	$\neg 1$
1	0	1	0

p	$\neg p$	$\neg\neg p$
1	0	1
0	1	0

Negation & Double negation

1	0	$\neg 0$	$\neg 1$
1	0	1	0

p	$\neg p$	$\neg\neg p$
1	0	1
0	1	0

- So

$$\neg 1 = 0, \qquad \neg 0 = 1 \qquad \& \qquad \neg\neg p = p$$

Tautology

- A tautology is a statement that is always true.
- We can see that $p \vee \neg p$ is a tautology.

p	$\neg p$	$p \vee \neg p$
1	0	1
0	1	1

Contradiction

- A contradiction is a statement that is never true.
- We can see that $p \wedge \neg p$ is a contradiction.

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

Annihilation law

p	0	$p \wedge 0$
1	0	0
0	0	0

p	1	$p \vee 1$
1	1	1
0	1	1

Annihilation law

p	0	$p \wedge 0$
1	0	0
0	0	0

p	1	$p \vee 1$
1	1	1
0	1	1

- So

$$p \wedge 0 = 0 \qquad \& \qquad p \vee 1 = 1$$

Absorption law

p	q	$p \vee q$	$p \wedge (p \vee q)$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	0	0

p	q	$p \wedge q$	$p \vee (p \wedge q)$
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	0

Absorption law

p	q	$p \vee q$	$p \wedge (p \vee q)$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	0	0

p	q	$p \wedge q$	$p \vee (p \wedge q)$
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	0

- So

$$p \wedge (p \vee q) = p \qquad \& \qquad p \vee (p \wedge q) = p$$

De Morgan's Laws

- The only ones named after a person, British mathematician Augustus De Morgan (1806–1871).

$$\neg(p \wedge q) = \neg p \vee \neg q$$
$$\neg(p \vee q) = \neg p \wedge \neg q$$