Determinants and inverses

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1 Determinants

The *determinant* is a number (N.B. not a matrix) associated with a square matrix. We refer to the determinant of a matrix $\mathbf{A}$ using either $\det(\mathbf{A})$ or $|\mathbf{A}|$ and write

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}, \qquad \det(\mathbf{B}) = \begin{vmatrix} -1 & 2 & 7 \\ 14 & 2 & 8 \\ 6 & -1 & 0 \end{vmatrix}.$$

1.1 2×2 determinants

The 2×2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has associated with it a determinant, calculated as follows.

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 4.$$

1.2 3×3 determinants

The process for finding a 3×3 determinant is more complicated than for a 2×2, though we will ultimately see they are the same process.

First, recall that a general 3×3 matrix can be written:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1.2.1 Minors

Each element, a_{ij}, in a square matrix A has associated with it a *minor*, M_{ij}. The minor is the value of the determinant that results from removing the row and column of the element under consideration. For example, the minor M_{23}, which is associated with element a_{23}, is found by deleting the second row and third column, and calculating the determinant of what is left.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example

Find M_{11} and M_{32}.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 5 & 2 \end{bmatrix}; \quad M_{11} = \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix}, \quad M_{32} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}.$$

1.2.2 Cofactors

Each element, a_{ij}, in a square matrix $\mathbf{A}$ has associated with it a *cofactor*, A_{ij}. This is found using the minor by calculating

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

For example, for the general matrix $\mathbf{A}$, cofactor A_{23} is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example

Find A_{11} and A_{32}.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 5 & 2 \end{bmatrix}; \quad A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix}, \quad A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}.$$

1.2.3 General rule for cofactors

For a general matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

the cofactor of a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the *minor*, the determinant of the matrix obtained by deleting the ith row and jth column of the matrix $\mathbf{A}$.

1.2.4 Calculating the determinant of a 3×3 matrix

First, choose any row or column. Move along the row or column, multiplying the value of each of the three elements in that row or column by the corresponding cofactor. Sum these three products to get the determinant.

Example

Calculate the determinant of matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 3 & -5 & 2 \end{bmatrix} = 1 \times \begin{vmatrix} -1 & 3 \\ -5 & 2 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & -1 \\ 3 & -5 \end{vmatrix} = 1 \times 13 - 2 \times -5 + 3 \times -7 = 2.$$

This works along any row or column - so choose one that looks easy!

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 5 & 2 \end{bmatrix} = -2 \times \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} = -2 \times -11 = 22.$$

A good way to check you have the right answer is to calculate the determinant along different rows or columns and see if your answers match.

1.2.5 General rule for a 3×3 determinant

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For 3×3 matrix $\mathbf{A}$, the determinant can be found on the pth row using

$$\det(\mathbf{A}) = a_{p1}A_{p1} + a_{p2}A_{p2} + a_{p3}A_{p3} = \sum_{k=1}^{3} a_{pk}A_{pk},$$

or down the qth column using

$$\det(\mathbf{A}) = a_{1q}A_{1q} + a_{2q}A_{2q} + a_{3q}A_{3q} = \sum_{k=1}^{3} a_{kq}A_{kq},$$

where $A_{ij} = (-1)^{i+j}M_{ij}$ and M_{ij} is the determinant of the matrix formed by removing the ith row and jth column from $\mathbf{A}$.

1.3 2×2 determinants follow the same process as 3×3

For a general 2×2 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Moving along the first row, first find the minors M_{11} and M_{12}.

$$M_{11} = \left|a_{22}\right| = a_{22}, \qquad M_{12} = \left|a_{21}\right| = a_{21}.$$

Now calculate the cofactors A_{11} and A_{12}.

$$A_{11} = (-1)^{1+1} M_{11} = \left|a_{22}\right| = a_{22}, \qquad A_{12} = (-1)^{1+2} M_{12} = -\left|a_{21}\right| = -a_{21}.$$

Now calculate the determinant using

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}A_{11} + a_{12}A_{12} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

1.4 Higher order determinants

Though we won't usually calculate these by hand, higher order determinants can be calculated using the same process. For example, for a general 4×4 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

we calculate the minor M_{ij} in the same way by deleting the ith row and jth column, leaving a 3×3 determinant. For example,

$$M_{34} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

Then the cofactor is calculated following the usual procedure.

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

Obviously this is more work because there are four 3×3 determinants to calculate.

Higher order determinants follow in the same way. For the $n \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

calculate the determinant along the pth row using

$$\det(\mathbf{A}) = a_{p1}A_{p1} + a_{p2}A_{p2} + \ldots + a_{pn}A_{pn} = \sum_{k=1}^{n} a_{pk}A_{pk},$$

or down the qth column using

$$\det(\mathbf{A}) = a_{1q}A_{1q} + a_{2q}A_{2q} + \ldots + a_{nq}A_{nq} = \sum_{k=1}^{n} a_{kq}A_{kq},$$

where $A_{ij} = (-1)^{i+j}M_{ij}$ and M_{ij} is the determinant of the matrix formed by removing the ith row and jth column from $\mathbf{A}$.

1.5 Properties of determinants

In general, for two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$:

- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$;
- $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$ for scalar k.

2 Inverting matrices

2.1 Matrix of cofactors

The matrix of cofactors for a matrix $\mathbf{A}$, $\mathrm{cof}(\mathbf{A})$, is the matrix where the (i, j) entry is the cofactor of a_{ij} in $\mathbf{A}$.

2.2 Adjoint matrix

The adjoint matrix of $\mathbf{A}$, $\mathrm{Adj}(\mathbf{A})$, is the transpose of the matrix of cofactors.

Example

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\mathrm{cof}(\mathbf{A}) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix},$$

so

$$\mathrm{Adj}(\mathbf{A}) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

2.3 Identity matrix

An identity matrix is a diagonal matrix with all diagonal elements $= 1$. For example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is called the identity for matrix multiplication because it leaves the matrix unchanged, i.e. for a matrix $\mathbf{A}$

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}.$$

2.4 Inverse of a matrix

For any scalar a, there is a multiplicative inverse of a, such that $ab = 1$, so that $a = b^{-1}$ and $b = a^{-1}$.

Similarly with matrices, for matrices $\mathbf{A}$ and $\mathbf{B}$, if

$$\mathbf{AB} = \mathbf{I}$$

then $\mathbf{A}$ is called the *left inverse* of $\mathbf{B}$ and $\mathbf{B}$ is called the *right inverse* of $\mathbf{A}$.

For square matrices $\mathbf{A}$ and $\mathbf{B}$, we simply say that $\mathbf{A}$ is the *inverse* of $\mathbf{B}$ and $\mathbf{B}$ is the *inverse* of $\mathbf{A}$.

A matrix that is invertible is called non-singular (or regular). A matrix that is not invertible is called singular. A matrix is invertible if and only if its determinant is zero.

We write $\mathbf{A}^{-1}$ for the inverse of $\mathbf{A}$.

2.4.1 Properties

1. $\mathbf{A}^{-1}$ is unique. It is impossible that there are two different inverse matrices for $\mathbf{A}$.
2. A square matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.

2.5 Finding an inverse matrix

Steps for finding an inverse matrix:

1. Find the determinant, $\det(\mathbf{A})$, and check that it is non-zero.
2. Form the cofactors matrix $\mathrm{cof}(\mathbf{A})$.
3. Find the adjoint matrix $\mathrm{Adj}(\mathbf{A})$.
4. Find the inverse, $\mathbf{A}^{-1}$.

If $\mathbf{A}$ is a square and non-singular matrix, then $\mathbf{A}^{-1}$ is given by:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{Adj}(\mathbf{A}).$$

Example

Find the inverse of the following matrix.

$$\mathbf{A} = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}$$

1. First find the determinant.

$$\begin{vmatrix} -4 & -2 \\ 5 & 5 \end{vmatrix} = -4 \times 5 - -2 \times 5 = -10.$$

2. Now form the matrix of cofactors.

$$\operatorname{cof}(\mathbf{A}) = \begin{bmatrix} 5 & -5 \\ 2 & -4 \end{bmatrix}.$$

3. Now the adjoint matrix

$$\operatorname{Adj}(\mathbf{A}) = \begin{bmatrix} 5 & 2 \\ -5 & -4 \end{bmatrix}.$$

4. Now find the inverse

$$\mathbf{A}^{-1} = \frac{1}{-10} \times \begin{bmatrix} 5 & 2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}.$$