Inclusion-exclusion and generators and enumerators

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Exercise

1. You have three cards labelled **A**, **B** and **C**. How many ways are there of rearranging these three cards into different orders?
2. You have a bag of three balls labelled with letters **A**, **B** and **C**. How many different ways there are of pulling two balls out of the bag, not worrying about the order?
3. You have three cards labelled **A**, **B** and **C**. You shuffle them and deal them face up onto a table, saying "A, B, C" as you place each card. What is the probability that you say the name of at least one card as you deal it?
4. You have three cards, two blue and one green. How many ways can you arrange them into sequences of length 1–3?

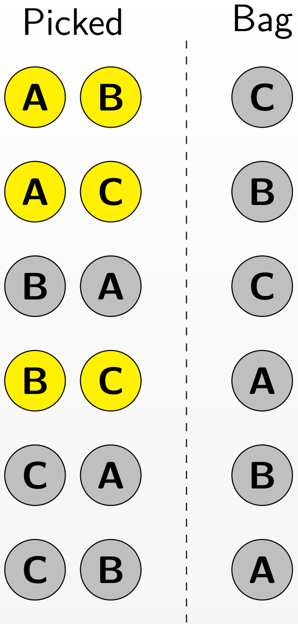
Exercise – answer 1

- There are $3! = 6$ arrangements of three cards.

1. A B C
2. A C B
3. B A C
4. B C A
5. C A B
6. C B A

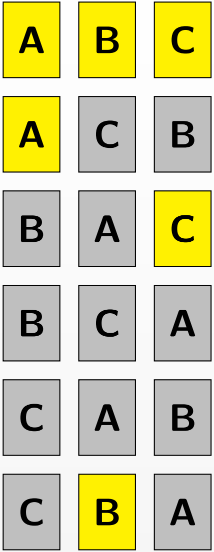
Exercise – answer 2

- There are $3! = 6$ arrangements of three balls.
- Since we don't care what order we draw these from the bag, there are $\binom{3}{2} = 3$ ways (three are reorderings).



Exercise – answer 3

- There are $3! = 6$ arrangements of three cards (as we saw previously).
- In two of these, we have no card in its right place.
- The probability that you never say a card when you place that card is $\frac{2}{6} = \frac{1}{3}$.
- The probability that you say at least one card when you place it is $1 - \frac{1}{3} = \frac{2}{3}$.



Derangements

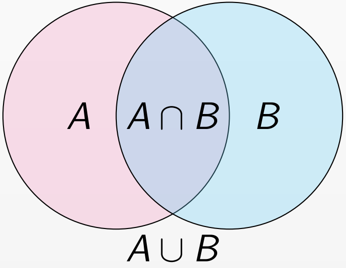
- Derangements are the permutations where no item is in its proper place.
- De Moivre argued that if A is the set of permutations of $\{a, b, c\}$ in which a is in its correct place, and if B and C are similarly defined, then the number of derangements is

$$3! - |A \cup B \cup C|$$

Inclusion-exclusion

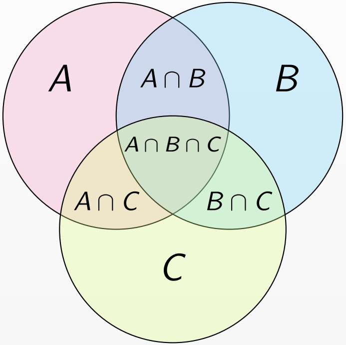
- If $|A|$ and $|B|$ are the numbers of elements in sets A and B, respectively, then how many elements are in $A \cup B$?
- If we add $|A| + |B|$ we have overcounted — we have *included* the ones in both sets twice.
- We must therefore *exclude* these duplicates from our count.

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



- If we do the same thing with sets A, B and C, we include all the sets and exclude their overlaps, now we have included the intersection $A \cap B \cap C$ three times and excluded it three times, so we must include once more to ensure it is in the final count.

$$
\begin{aligned}
|A \cup B \cup C| = & |A| + |B| + |C| \\
& - |A \cap B| - |A \cap C| - |B \cap C| \\
& + |A \cap B \cap C|.
\end{aligned}
$$



$A \cup B \cup C$

- This generalises and is called the *inclusion-exclusion principle*.

Derangements

- So the number of derangements of $\{a, b, c\}$ is

$$
\begin{aligned}
&3! - |A \cup B \cup C| \\
=&3! - (|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|) \\
=&3! - ((|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + (|A \cap B \cap C|)) \\
=&3! - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - (|A \cap B \cap C|).
\end{aligned}
$$

Derangements

- We can choose one item to put in its correct place as $\binom{3}{1}$, then rearrange the other two elements in $2!$ ways, for a total of $\binom{3}{1}2!$ with one item in the correct place.

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- We can choose two items to put in their correct places as $\binom{3}{2}$, then rearrange the other one element in 1! ways, for a total of $\binom{3}{2}1!$ with two items in the correct place.

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- We can choose two items to put in their correct places as $\binom{3}{2}$, then rearrange the other one element in $1!$ ways, for a total of $\binom{3}{2}1!$ with two items in the correct place.
- We can choose three items to put in their correct places as $\binom{3}{3}$, then rearrange the other zero elements in $0!$ ways, for a total of $\binom{3}{3}0!$ with three items in the correct place.

Derangements

- We can choose one item to put in its correct place as $\binom{3}{1}$, then rearrange the other two elements in 2! ways, for a total of $\binom{3}{1}2!$ with one item in the correct place.
- We can choose two items to put in their correct places as $\binom{3}{2}$, then rearrange the other one element in 1! ways, for a total of $\binom{3}{2}1!$ with two items in the correct place.
- We can choose three items to put in their correct places as $\binom{3}{3}$, then rearrange the other zero elements in 0! ways, for a total of $\binom{3}{3}0!$ with three items in the correct place.

$$3! - \binom{3}{1}2! + \binom{3}{2}1! - \binom{3}{3}0! = 2.$$

Derangements

- The number of derangements of six elements is

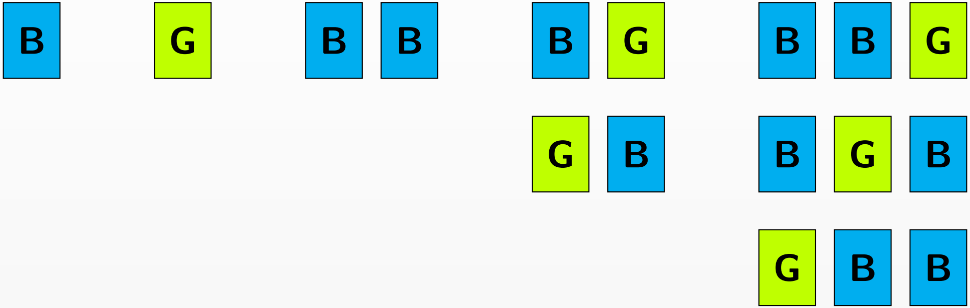
$$
\begin{aligned}
&6! - |A \cup B \cup C \cup D \cup E \cup F| \\
&= 6! - \binom{6}{1}5! + \binom{6}{2}4! - \binom{6}{3}3! + \binom{6}{4}2! - \binom{6}{5}1! + \binom{6}{6}0! \\
&= 265.
\end{aligned}
$$

Derangements

- The probability of a derangement of three objects is
$$\frac{2}{3!} = \frac{2}{6} = \frac{1}{3}.$$
- The probability of a derangement of six objects is
$$\frac{265}{6!} = \frac{265}{720} \approx 0.368.$$
- Note $\frac{1}{e} \approx 0.3679$.
- In fact, the number of derangements of n objects is always the integer nearest to $\frac{n!}{e}$, e.g. $\frac{3!}{e} \approx 2.21$, $\frac{6!}{e} \approx 264.87$.

Exercise – answer 4

- This is fairly straightforward to enumerate by hand:



- So there are 8.
- Is there a better way to work this out? What if it is a more complicated problem?

Exercises

1. Expand $(x+a)(x+b)(x+c)$.
2. Consider a set $A = \{a, b, c\}$. What are the possible subsets of A?

Enumeration

- $(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc.$
- Subsets of $\{a, b, c\}$:
  - $\{\}$;
  - $\{a\}$;
  - $\{b\}$;
  - $\{c\}$;
  - $\{a, b\}$;
  - $\{a, c\}$;
  - $\{b, c\}$;
  - $\{a, b, c\}$.

Enumeration

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  - $\{b, c\}$;
  - $\{a, b, c\}$.
- Notice how the expansion has enumerated the subsets?
- Subsets of a set are combinations without repetition.

Enumeration

- It is a shame that the coefficient of x didn't match the subset size, e.g. the coefficient of x^2 gave us subsets of size 1: $\{a\}, \{b\}, \{c\}$.
- We can align these with an alternative expression:

$$(1+ax)(1+bx)(1+cx) = 1+(a+b+c)x+(ab+ac+bc)x^2+abcx^3.$$

- Notice how now the subsets of size n are given by the x^n term.

Enumeration

- If we just want to count the subsets of each size, we don't need to include a, b, c (enumerator):

$$(1+x)(1+x)(1+x) = 1 + 3x + 3x^2 + x^3.$$

  (N.B. this is how the coefficients of the expansion of $(1+x)^n$ give the nth row of Pascal's triangle.)

- If we just want to list the subsets, we don't need to include x (generator):

$$(1+a)(1+b)(1+c) = 1 + a + b + c + ab + ac + bc + abc.$$

Arranging cards

- Our problem was:
  - You have three cards, two blue and one green. How many ways can you arrange them into sequences of length 1–3?
- We can represent:
  - no cards: 1;
  - one green card: g;
  - one blue card: b;
  - two blue cards: b^2.
- We make two functions, each representing a selection of mutually exclusive possibilities:
  - $1 + g$ to represent choosing either no green card or one green card;
  - $1 + b + b^2$ to represent choosing zero, one or two blue cards.

Generating functions

- Now we simply multiply the two functions together.

$$(1+g)(1+b+b^2) = 1+b+g+b^2+gb+gb^2.$$

- Each term of the expansion represents a selection from the set of three cards.

Generating functions

- Now we simply multiply the two functions together.

$$(1+g)(1+b+b^2) = 1+b+g+b^2+gb+gb^2$$

- Each term of the expansion represents a selection from the set of three cards.
  - b represents $\boxed{\mathbf{B}}$;
  - g represents $\boxed{\mathbf{G}}$;
  - b^2 represents $\boxed{\mathbf{B}}\ \boxed{\mathbf{B}}$.

Generating functions

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$$(1+g)(1+b+b^2) = 1+b+g+b^2+gb+gb^2$$

- Each term of the expansion represents a selection from the set of three cards.
  - gb represents **G** **B**.
  - There are $2! = 2$ ways to arrange these two cards, so really gb represents **G** **B** and **B** **G**.
  - We didn't have this rearrangement problem with b^2 because the two blue cards are identical.

Generating functions

- Now we simply multiply the two functions together.

$$(1+g)(1+b+b^2) = 1+b+g+b^2+gb+gb^2$$

- Each term of the expansion represents a selection from the set of three cards.
  - The term gb^2 represents G B B
    - There are 3! ways to arrange three cards, but 2! of them are the same, because the two blue cards are identical.
    - So we can arrange gb^2 in $\frac{3!}{2!} = 3$ ways.

Generating functions

- Now we simply multiply the two functions together.

$$(1+g)(1+b+b^2) = 1 + b + g + b^2 + gb + gb^2$$

- Each term of the expansion represents a selection from the set of three cards.
  - The first term in our expansion is 1, which represents choosing none of the cards. Whether this is a valid sequence depends on the context and the rules you are following.

Generating functions

- Note that the number of each card was given by its power. For example, zero blue cards is represented by $b^0 = 1$, one blue card by $b^1 = b$, two blue cards by b^2.
- If I wish to count combinations of cards that definitely include at least n cards, I do not include powers $< n$ in the function.
- For example, to count sets of blue and green cards that include at least one blue card, I would use

$$(1 + g)(b + b^2) = b + gb + b^2 + gb^2.$$

Robot caterpillar

Robot caterpillar

- A harder example, just to show the power of this technique.

An unbelievable claim

Enter combinatorics

- It isn't hard to establish an upper bound somewhat less than ∞.
- If all eight segments were different, the number of ways of arranging these would yield $8!$ different caterpillars.

Including shorter caterpillars

- If all eight segments were different, for sequences using $k \leq 8$ positions:
    - we choose k segments from the 8 possible segments $\left(\text{which is } \binom{8}{k}\right)$

Including shorter caterpillars

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- so the number of possible sequences would be

$$\binom{8}{k} k!$$

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  - and there are $k!$ ways to arrange these,
- so the number of possible sequences would be

$$\binom{8}{k} k! = \frac{8!}{(8-k)!}.$$

Notice that for $k = 8$, this reduces to $8!$, as you might expect.

Putting it together

- The number of possible sequences using any number from 1 to 8 segments would be the sum of this arrangement for all possible lengths, i.e.

$$\sum_{k=1}^{8} \frac{8!}{(8-k)!} = 109600.$$

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Putting it together

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$$\sum_{k=1}^{8} \frac{8!}{(8-k)!} = 109600.$$
- N.B. $109{,}600 < \infty$.
- For this problem, though, it's definitely too large.

Complicating factors

- The caterpillar has three identical segments which instruct it to move forwards, two to turn left, two to turn right and one to stop and play music.
- For the avoidance of doubt, some other features and restrictions:
  - there are eight positions that can hold segments, the head is always present;
  - any number of segments from 1–8 may be used, and not 0;
  - the order of segments matters, because it is a sequence of commands;
  - segments are connected (by USB), so there can be no gaps in a sequence – if a position is unfilled, the caterpillar ends at the preceding segment.

Notation

- Forwards: F;
- Left: L;
- Right: R;
- Music: M.

Notation

- Forwards: F;
- Left: L;
- Right: R;
- Music: M.
- Denote a sequence by a string from left to right (the head is to the left of the string).
- e.g. $FFFM$, $FFMF$, $FLRFLRFM$, LRL, F, and so on.

Over-counting

- Previously, we were over-counting.
- For example, we counted *FF* and *FF* as different caterpillars, even though they are functionally the same.

Generating functions

- For the three F segments, we can use the function $1 + f + f^2 + f^3$ to represent these.
- For the two R segments, use $1 + r + r^2$.
- For the two L segments, use $1 + l + l^2$.
- For the one M use $1 + m$.

Generating functions

- To find out how many different ways there are of selecting from these segments, expand

$$(1 + f + f^2 + f^3)(1 + r + r^2)(1 + l + l^2)(1 + m).$$

Generating functions

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$$(1+f+f^2+f^3)(1+r+r^2)(1+l+l^2)(1+m).$$

- The expansion has 72 terms.
- One is formed by multiplying the four 1s together to get 1, which represents selecting no segments. This isn't a valid caterpillar.
- So there are 71 different ways of forming caterpillars up to length 8.

Example caterpillar combination

- One of the terms of the expansion is f^3rl^2m.
- This represents choosing F, F, F, R, L, L and M for a caterpillar of length 7.

- But some of these are duplicates; how many depends which segments are involved.

Have a go

- How many ways can you form different[1] caterpillars from the following terms of the 71-term expansion?
  - m;
  - r^2;
  - f^3l;
  - f^3rl^2m.

[1] By ‘different’, I mean if I went out of the room and you changed one into the other, I would be able to identify the change when I came back into the room.

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m:

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m: 1;

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m: 1;
  - r^2:

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m: 1;
  - r^2: $\frac{2!}{2!} = 1$;

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m: 1;
  - r^2: $\frac{2!}{2!} = 1$;
  - $f^3 l$:

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m: 1;
  - r^2: $\frac{2!}{2!} = 1$;
  - f^3l: $\frac{4!}{3!} = 4$;

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m: 1;
  - r^2: $\frac{2!}{2!} = 1$;
  - f^3l: $\frac{4!}{3!} = 4$;
  - f^3rl^2m:

Have a go – answers

- How many ways can you form caterpillars from the following terms of the 71-term expansion?
  - m: 1;
  - r^2: $\frac{2!}{2!} = 1$;
  - $f^3 l$: $\frac{4!}{3!} = 4$;
  - $f^3 r l^2 m$: $\frac{7!}{3!2!} = 420$

All 71 combinations

- Doing this calculation for all 71 combinations and summing yields 5,023 different caterpillars.
- Not endless, but certainly enough to keep us busy for a while!