- The oldest written mathematics contains two kinds of content:
  1. Methods for serious, important, useful problems, like
     - What is the volume of a cylindrical granary of diameter 9 and height 10?
     - If 10 *hekat* of fat are issued to last a year, how much can be used up in a day?
     - In a fair trade, how many loaves made from 45 *hekat* of grain should be exchanged for 100 loaves made from 10 *hekat* of grain?

Image: © Trustees of the British Museum.

- The oldest written mathematics contains two kinds of content:
  2. But also playful diversions, like
     - Think of a number. Add two-thirds of the number to itself, take away a third of the result, and give the answer.
     - Find a number such that when two-thirds of it is added and then one third of the sum, a third of the total is equal to 10.
     - In each of 7 houses, there are 7 cats. Each cat catches 7 mice. Each mouse would eat 7 ears of corn, each ear of corn would, if sown, produce 7 *hekat* of grain. How many houses, cats, mice, ears and hekats in total?

Image: © Trustees of the British Museum.

What does this tell us?

- For as long as we have evidence of mathematics, it has both practical applications and a playful, curious side.
- Plato tells us the Ancient Egyptians taught maths to their children through play, and recommends this.
- Throughout history, we have evidence of mathematicians playing with puzzles and curious oddities, and trying out ideas around the maths they are using.

What does this tell us?

- Mathematics is often presented as a neat, finished product.
- But it has another side, creative, messy, playful, and experimental. Pólya called this "mathematics in the making".
- The neat presentation hides this method of discovery, but it was always there – and it involved mistakes, mis-steps, and dead-ends.

What does this tell us?

- This means you have only been shown the products of doing mathematics rather than being shown the process by which it was created.
- This module includes some of the products, but really it is about thinking, trying to teach you to solve problems and make mathematics.
- So ask questions, play around, and be curious!

What happens when you encounter an unseen problem?

If you can solve it

- Gosh, well done!
- Remember, we are interested in your problem-solving abilities. The fact you can solve this problem is lovely, but may just be a fluke or coincidence.
- Can you do it on different unseen problems, or did you get lucky?
- It's good if you are able to communicate your result clearly and show us you solved this via a sensible problem-solving method.
- Perhaps try a different problem. Better to practice when you can't just see the solution.

If you can't solve it

- Good! Here's where it gets interesting.
- There are problem-solving approaches you can try to get to a solution.
- For this module:
  - it is very important that you try a sensible problem-solving approach which has a good chance at working towards a solution;
  - the aim is to learn about problem solving, not necessarily to solve a particular problem;
  - this is especially true if your solution relies on you getting a hint or sneaking a look at the answer;
  - acknowledge that this is hard, because the emotions in problem solving are all around getting a solution.

Advice for problem solving

1. Plan

- Advice:
  - Stop and really read the problem.
  - Ask yourself: What am I being asked to do? What information have I been given? What information is missing? What would a solution look like?

1. Plan

- Advice:
  - Draw a picture. Introduce suitable notation for the information you have been given and information you are being asked for.

1. Plan

- Advice:
  - Have you seen a similar problem before? How did you solve that one?

1. Plan

- Advice:
  - Separate the problem into smaller parts and examine them separately.

1. Plan

- Advice:
  - Can you write the problem in a different way? Try it. Is what you have written actually the same problem? If not, what is different? If it is, can you solve this version of the problem?
  - Can you solve a related problem? Can you remove part of the restriction and solve a more general problem? Or, can you come up with a specific example and solve that? Do your solutions help you plan to approach the main problem?

2. Carry out your plan

- When your plan is ready, put it into action.
- Advice:
  - Check each step. Is each step correct?
  - You should expect to be stuck quite a lot of the time. Recognise that you are stuck and accept it. Calmly review where you are and try to get unstuck.
  - It is (usually!) okay to wait and mull over the problem for a while.
  - If you are sure your plan cannot work, you may need to return to the *Plan* stage.

3. Review

- First, if you found a solution, check it is correct.
  - Can you check your solution is correct?
  - Does it answer the original problem?
  - Can you get the same solution from a different method?
  - Can you work from your solution and get back to the original problem and, doing so, is the problem you get to the same as the original problem you tried to solve?

3. Review

- Second, reflect on what happened.
  - Remember that the point of solving problems is not just to get marks in assessments. The purpose here is to think about what has happened and see what you can learn.

3. Review

- Second, reflect on what happened.
  - Think about the process you took and particularly any dead ends you went down.
    - What went wrong?
    - Could you have avoided the dead ends, or were they a necessary part of solving the problem?
    - What do you wish you had know when you first attempted the problem?

3. Review

- Second, reflect on what happened.
  - What can you do now that you couldn't before?
    - Can you use the method, or the result, for some other problem?
    - Can you write down a new problem that you are now able to solve?
    - Is the problem you have solved part of a wider family of problems?
    - Can your method be adapted to solve more of them?

What does this mean for an unseen problem?

Attempting a solution

- Make sure you have understood the question and explored this understanding:
  - Rewrite the question in your own words.
  - State the crucial features of the question.
  - Draw a diagram showing the question.
  - Write an explanation of what the question is asking you to do and why this is difficult. Where does the difficulty lay?

Attempting a solution

- Try variants:
  - Try to state and solve a simpler version of the problem.
    - Remove part of the restrictions.
    - Add more restrictions.
    - Try it for a smaller number.
  - Make a conjecture and test it.

If all else fails and you can't find a solution

- Make it clear what you have tried and why you are sure that it doesn't work.
- Critique the problem – what is making it hard to solve?

If you do find a solution

- Check your answer is correct.
  - You should be mostly in a position to be able to check the answer against the original puzzle, or at least to verify your method.
  - Perhaps having solved it, you can see a different method that would work? If you get the same answer via different methods, this is good evidence you may be correct.
- Reflect on what happened.
- Play around with the puzzle.
  - Can you rephrase the puzzle into a different context?
  - Can you make a variant of the puzzle?

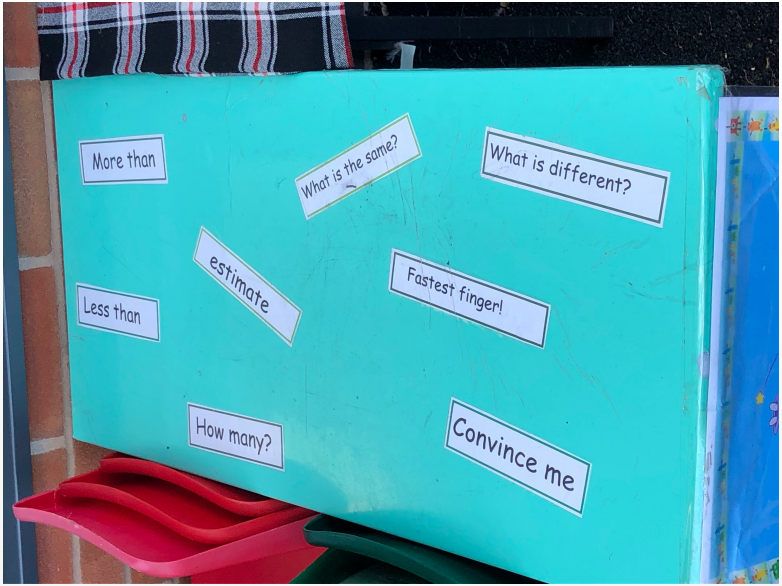
Golden rule for this module

- Introducing his section of hints and solutions, Pólya says "The reader who has earnestly tried to solve the problem has the best chance to profit by the hint and the solution."
- **Please please please**, if you have already seen a problem and know how to solve it, allow others the chance to profit from trying it themselves, rather than simply telling them the solution. Even if you do decide to tell them, try giving hints and asking questions to try to lead them to a solution, rather than giving them the solution wholesale. You and they will benefit from the practice.

No spoilers!

- It is really tempting to get invested in the problem and want to know the solution.
- If you tell someone how to answer a question, you are robbing them of the opportunity to practice and this is a serious business indeed. (This is hard, because there are positive emotions for some people associated with knowing more than others and showing this.)
- If you spoil a problem for yourself by looking at someone's work or searching online for how to answer it, more fool you.

Mathematical thinking



Mathematical thinking