Queuing

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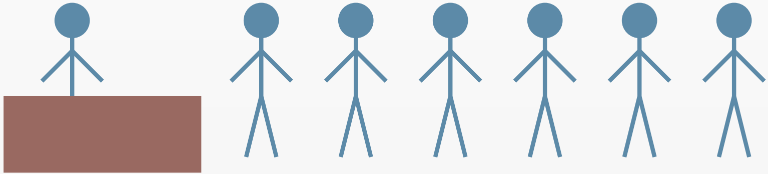
Can you give an example of a queue?

Shopping

- There are shops with a single cashier and a single queue.
- What other sorts of shop queue have you been in?
- Why are there different sorts?
- Which sort works best?

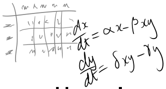


Mathematical Modelling

Scenario

A problem arises that we'd like to understand.

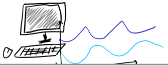
Mathematical model



A simple version of the scenario we can ask precise questions about.

Solve it

Using various mathematical methods.



Check & communicate

Does the solution make sense? Interpret and talk using the original context.

refine?

Queuing worksheet

- A simple version of a queue we can start to explore.

Answers

1. What is the average number of arrivals and departures for our simulation in **the first minute**?

$$\frac{1}{2} \times 2 + \frac{1}{2} \times 1 = 1.5.$$

Answers

2. What are the possible queue lengths after 1, 2, 3, 4 and 5 minutes?

Each round we can:

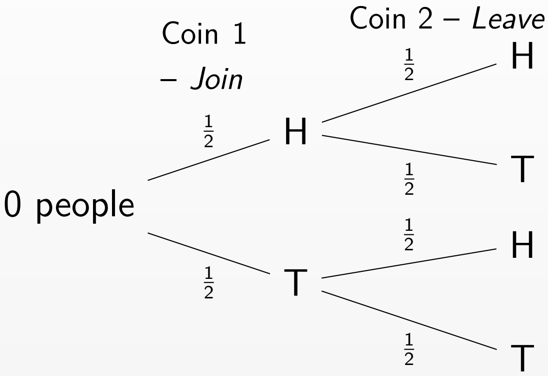
- increase by 1
  (toss a 2 and a 1);
- decrease by 1
  (toss a 1 and a 2) [if > 0];
- stay the same.

So in minute n the possible queue lengths are $0, 1, \ldots, n$.

Minutes	**Possible queue lengths**
0	0
1	0, 1
2	0, 1, 2
3	0, 1, 2, 3
4	0, 1, 2, 3, 4
5	0, 1, 2, 3, 4, 5

Answers

3. What are the probabilities of each queue length at the end of the first minute?



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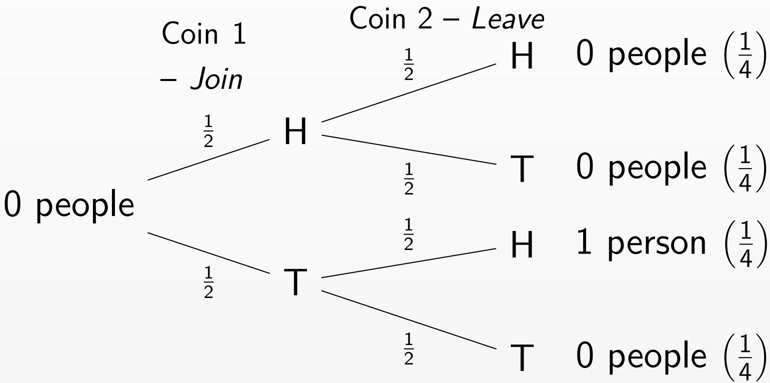
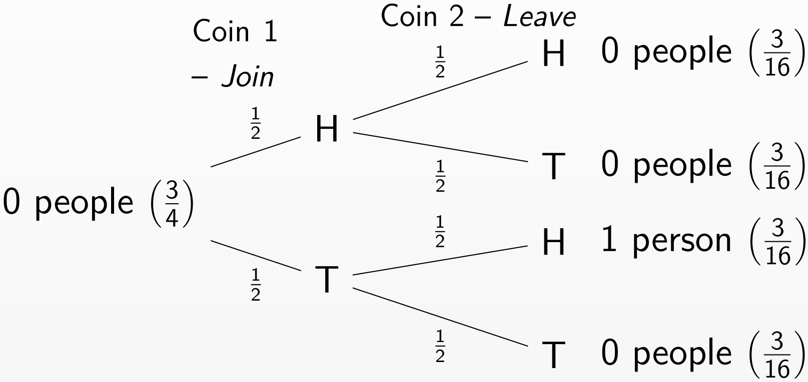


Answers

For the second minute:



Answers

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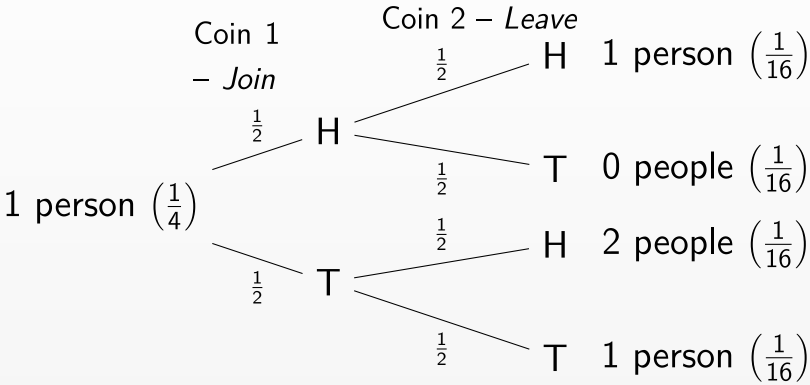

Answers

What are the probabilities of each queue length at the end of each minute?

Minutes	0	1	2	3	4	5
0	1	0	0	0	0	0
1	$\frac{3}{4}$	$\frac{1}{4}$	0	0	0	0
2	$\frac{5}{8}$	$\frac{5}{16}$	$\frac{1}{16}$	0	0	0
3	$\frac{35}{64}$	$\frac{21}{64}$	$\frac{7}{64}$	$\frac{1}{64}$	0	0
4	$\frac{63}{128}$	$\frac{21}{64}$	$\frac{9}{64}$	$\frac{9}{256}$	$\frac{1}{256}$	0
5	$\frac{231}{512}$	$\frac{165}{512}$	$\frac{165}{1024}$	$\frac{55}{1024}$	$\frac{11}{1024}$	$\frac{1}{1024}$

Answers

4. What is the average queue length at the end of each minute?

Minutes	**Length**
1	$\frac{1}{4} = 0.25$
2	$\frac{7}{16} \approx 0.44$
3	$\frac{19}{32} \approx 0.59$
4	$\frac{187}{256} \approx 0.73$
5	$\frac{437}{512} \approx 0.85$

Our simulation

- This simulation in Python a million times.
- Ending numbers in the queue:
  - Minimum: 0
  - Maximum: 5
  - Mean: 0.85
  - Median: 1
  - Mode: 0

ending number	**count**
0	451167 (45%)
1	322491 (32%)
2	160784 (16%)
3	53886 (5.4%)
4	10711 (1.1%)
5	961 (0.1%)

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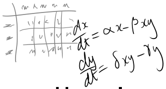

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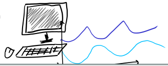

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Our simulation

- What does it do well?
- What does it do badly?
- How could it be improved?

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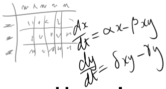

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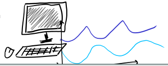

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Arrival rate

- From data about a scenario, we might be able to determine an average arrival rate per hour, say $\lambda = 6$.
- Does this mean 6 people arrive every hour?

Arrival rate

- The probability of n people joining a queue in a time period given the average number of arrivals is λ can be modelled using

$$P(n) = \frac{\lambda^n \mathrm{e}^{-\lambda}}{n!}.$$

- What is the probability of $n = 2$ arrivals within one minute if the average number of arrivals is $\lambda = 1.5$?

Answer

$$P(2) = \frac{1.5^2\,\mathrm{e}^{-1.5}}{2!} \approx 0.251.$$