Markov chains

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Markov chains are used to analyse systems which has a number of different *states* and *transitions* occur between these with some probability in discrete time intervals.

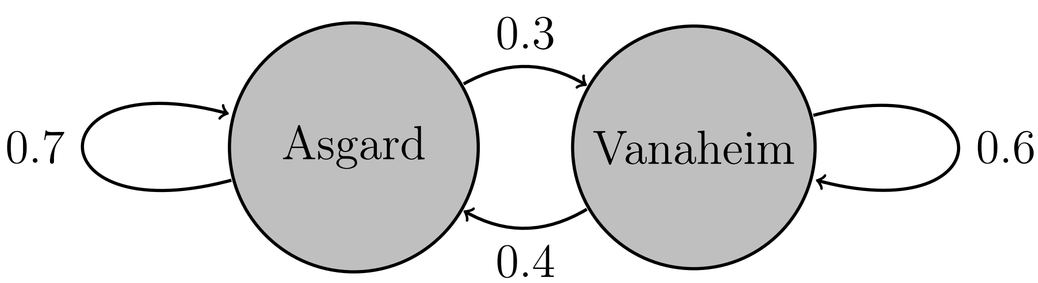
1 Representation

1.1 State-transition diagram

A Markov Chain can be represented as a labelled, directed graph where the vertices are the states and the edges indicate the transitions and are labelled with the transition probabilities.

For example, here is a simple state-transition diagram for a customer loyalty scenario.

- If someone buys Asgard brand breakfast cereal this month, the probability that they will buy the same brand next month is 0.7, otherwise they will transition to buy Vanaheim brand breakfast cereal.
- If they buy Vanaheim this month, the probability that they will buy the same brand next month is 0.6, otherwise they will transition to buy Asgard.



Note that the labels on the outgoing edges of each node sum to 1. This is because these are probabilities.

1.2 Matrix

A Markov Chain can be represented as a matrix. Here we represent the current state by the columns and the next state by the rows, so that the probability of a transition to state i from state j is represented by the element on the ith row, jth column of the matrix.

For example, the transition probabilities for the breakfast cereal scenario above can be expressed in the following table.

		Now	
		Asgard	Vanaheim
Next	Asgard	0.7	0.4
	Vanaheim	0.3	0.6

Note that entries in each column sum to 1. This is because these are all the probabilities of transitioning *from* one state.

Representing this table as a matrix $\mathbf{P}$, we obtain

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$$

We can represent the current system using a 2×1 column vector $\mathbf{s}$, with the first entry representing Asgard and the second entry representing Vanaheim. For example, if we know someone bought Vanaheim this month, we can represent this as

$$\mathbf{s} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Such a vector could be used to represent the probabilities of being in either state, or of the numbers or proportion of a population being in either state.

We can use our matrix of probabilities $\mathbf{P}$ and our current state vector $\mathbf{s}$ to predict future behaviour of the system by examining $\mathbf{Ps}$.

Confusing note: Because we have the current state as columns and the next state as rows, we use a column vector and multiply $\mathbf{Ps}$ to get the next time-step. We might have chosen to represent the transition matrix using rows as 'now' and columns as 'next' (i.e. the transpose of the matrix we have used), in which case we would use a row vector and multiply $\mathbf{sP}$ by this to get the next time-step. You may see either convention used so it is good to be aware of the difference - there is no good multiplying $\mathbf{Ps}$ if your matrix $\mathbf{P}$ is formed the other way around!

2 Next time-step

We could consider a scenario where we know that 100 people bought Asgard this month and 90 people bought Vanaheim, and use our transition matrix to predict next month's numbers. Here we form a column vector with the numbers for each cereal

$$\mathbf{s} = \begin{bmatrix} 100 \\ 90 \end{bmatrix},$$

and multiply as before.

$$\mathbf{Ps} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 100 \\ 90 \end{bmatrix} \approx \begin{bmatrix} 106 \\ 84 \end{bmatrix}.$$

Here we find that we predict 106 people to buy Asgard next month and 84 to buy Vanaheim.

Note that the two numbers sum to the same number, 190, as in the original column vector. This is a useful check to make sure nothing has gone wrong. In a simple model such as this, we have assumed that everyone buys cereal each month and that no new customers enter the system.

3 Next few time steps

As the number of Asgard purchases has increased in one time step, we might wonder whether we can make predictions about this system over subsequent time steps.

Let's call the initial state vector $\mathbf{s}_0$, then we can talk about $\mathbf{s}_i$ after i time steps. So the vector $\begin{bmatrix} 106 \\ 84 \end{bmatrix}$ above is $\mathbf{s}_1$.

We can make a prediction about $\mathbf{s}_2$ by multiplying $\mathbf{s}_1$ by the matrix of transition probabilities again.

$$\begin{aligned} \mathbf{s}_2 &= \mathbf{P}\mathbf{s}_1 \\ &= \mathbf{P}\left(\mathbf{P}\mathbf{s}_0\right) \\ &= \mathbf{P}^2\mathbf{s}_0. \end{aligned}$$

This extends, so we see that we can make a prediction about time step i by finding

$$\mathbf{s}_i = \mathbf{P}^i\mathbf{s}_0.$$

3.1 Example

Say we are interested in the number of people buying cereal each month over the next year. Our time step is one month, so we are interested in finding

$$\mathbf{s}_i = \mathbf{P}^i\mathbf{s}_0$$

for $i \in 2, \ldots, 12$.

i	$\mathbf{s}_i = \mathbf{P}^i\mathbf{s}_0$	i	$\mathbf{s}_i = \mathbf{P}^i\mathbf{s}_0$
2	$\begin{bmatrix} 107.8 \\ 82.2 \end{bmatrix}$	8	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$
3	$\begin{bmatrix} 108.3 \\ 81.7 \end{bmatrix}$	9	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$
4	$\begin{bmatrix} 108.5 \\ 81.5 \end{bmatrix}$	10	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$
5	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$	11	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$
6	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$	12	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$
7	$\begin{bmatrix} 108.6 \\ 81.4 \end{bmatrix}$		

4 Long-term behaviour

When we considered the next few time steps, you may have noticed that the behaviour of our cereal customers settled down to one decimal place after only a few time steps. Other Markov chains may not settle down so quickly, but in general we can look for a long-term behaviour using eigenvectors.

The situation where the behaviour has stabilised corresponds to

$$\mathbf{P}\mathbf{s}_n = \mathbf{s}_n$$

for some n. In fact, this is exactly the situation for an eigenvalue $\lambda_1 = 1$.

We can find the eigenvalues of $\mathbf{P}$ using

$$\begin{aligned}\begin{vmatrix}0.7-\lambda & 0.4\\ 0.3 & 0.6-\lambda\end{vmatrix} &= (0.7-\lambda)(0.6-\lambda) - 0.4\times 0.3\\ &= \lambda^2 - 1.3\lambda + 0.3\end{aligned}$$

Now $\lambda^2 - 1.3\lambda + 0.3 = 0$ when $\lambda = 1$ or 0.3.

We want an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$, so let $\mathbf{v} = \begin{bmatrix}x\\y\end{bmatrix}$ and consider

$$\begin{aligned}\begin{bmatrix}0.7-\lambda_1 & 0.4\\ 0.3 & 0.6-\lambda_1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}0.7-1 & 0.4\\ 0.3 & 0.6-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\\ &= \begin{bmatrix}-0.3 & 0.4\\ 0.3 & -0.4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\end{aligned}$$

From row 1:

$$\begin{aligned}-0.3x + 0.4y &= 0\\ 0.3x &= 0.4y\\ 3x &= 4y\end{aligned}$$

So say $\mathbf{v} = \begin{bmatrix}4\\3\end{bmatrix}$.

However, this cannot be $\mathbf{s}_n$. Remember that the state vectors $\mathbf{s}_i$ either track the proportion of people in each state or the number of people in each state (depending how we set up our model).

Here we used $\mathbf{s}_0 = \begin{bmatrix}100\\90\end{bmatrix}$ to represent the cereal-buying habits of 190 people. This means our $\mathbf{s}_n$ must have entries that sum to 190 also.

So what we need for $\mathbf{s}_n$ is a vector with entries in the ratio 4 : 3 that sums to 190, i.e.:

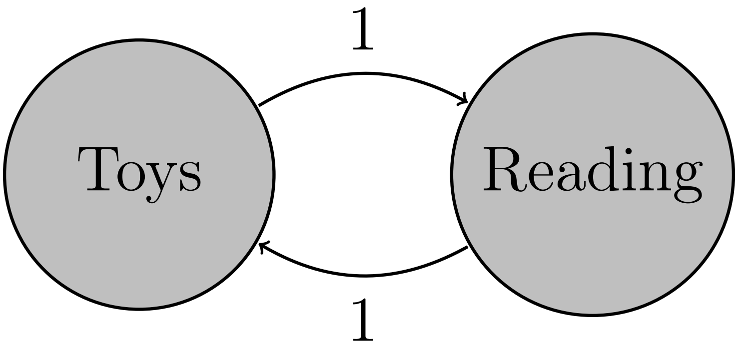
$$\mathbf{s}_n = 190 \times \begin{bmatrix}\frac{4}{7}\\ \frac{3}{7}\end{bmatrix} \approx \begin{bmatrix}108.6\\ 81.4\end{bmatrix}.$$

Note that if you come across a system that has been running for some time, you might expect this (as a modelling assumption) to be already operating in a stable long-term behaviour.

5 A warning

It is important to remember the modelling context in which you are working. Remember that any matrix with columns that sum to 1 has an eigenvalue 1, but it is important to consider what aspect of your model this relates to.

For example, consider this scenario: My son makes a new decision about what to do every ten minutes and he is very fickle. If he is playing with his toys, he will want to switch to reading. If he is reading, he will want to switch to playing with his toys.



We represent this using a transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This says that whatever the state this time, the other option will happen next time.

Say he is currently playing with his toys. So let $\mathbf{s}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$\begin{aligned}
\mathbf{s}_1 = \mathbf{P}\mathbf{s}_0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\mathbf{s}_2 = \mathbf{P}\mathbf{s}_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathbf{s}_3 = \mathbf{P}\mathbf{s}_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&\vdots
\end{aligned}$$

Hopefully it's clear that this system will simply oscillate between two states forever (or until bedtime), so there is no single stable state that, once reached, will not be moved away from.

We can, however, find an eigenvector which ought to correspond to the long-term behaviour of this system. Considering the eigenvalue $\lambda = 1$ we can find a general eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$ using

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + y \\ x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this we see $x = y$ and an eigenvector with entries in the ratio $1 : 1$ which sum to 1 is:

$$\mathbf{s}_n = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

The model is set up for my son to be in one state or the other, and so this vector cannot be a future state of the system.

This eigenvector does make sense if we think of the state vector $\mathbf{s}_i$ as telling us the *probability* of my son being in either state at time step i. Then this eigenvector $\mathbf{s}_n$ is telling us that the long-term behaviour of the system is that we expect him to spend half of his time playing with his toys and half his time reading. What this is not telling us is what he is doing in time step n.