Permutations

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1 Definition and notation

Definition. If A is a set, a *permutation* is a bijection $\sigma\colon A \to A$.

We will often consider permutations on a set $\mathbb{Z}(n) = \{x \in \mathbb{Z} \mid 1 \leq x \leq n\}$.

Example

Consider a permutation $\sigma\colon \mathbb{Z}(4) \to \mathbb{Z}(4)$ which maps the elements $\{1, 2, 3, 4\}$ onto $\{a, b, c, d\}$.

- a, b, c and d must be distinct, because σ is injective.
- $\{a, b, c, d\}$ must contain all the elements of $\mathbb{Z}(4)$, because σ is surjective.

Say σ is the permutation that maps $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 4$ and $\sigma(4) = 1$. We can represent this permutation using two-line notation as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

In this notation, the numbers are written out on the top line, with the second line showing where they are mapped to.

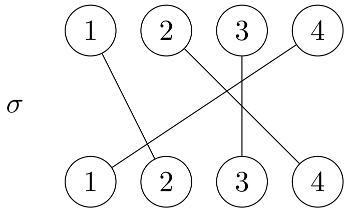
There is also a cycle notation where the permutation is written as a product of *disjoint cycles*. In this representation, we have

$$\sigma = (1\,2\,4)(3).$$

It is common to omit 1-cycles, so this could also be represented

$$\sigma = (1\,2\,4).$$

We can also draw the permutation as a diagram.



2 Composing Permutations

If σ, $\tau\colon \mathbb{Z}(n) \to \mathbb{Z}(n)$ are permutations, then the composite function $\sigma \circ \tau\colon \mathbb{Z}(n) \to \mathbb{Z}(n)$ is also a permutation, since the composite of bijections is also a bijection. This notation tells us that we perform τ first, followed by σ, often just written as $\sigma\tau$ instead of $\sigma \circ \tau$.

2.1 Example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1\,2\,4)$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4)$$

In the two-line notation, we can write out the first permutation to be performed as normal, and then what the second permutation does on the line below. Then we just read off the resulting permutation by ignoring the middle row.

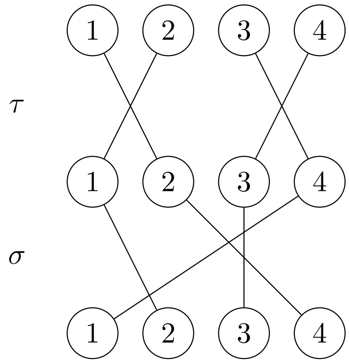
$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

In the one-line notation, we write the cycles out in order, before then following each element through the cycles from right to left.

$$\sigma\tau = (1\,2\,4)(1\,2)(3\,4) = (1\,4\,3)(2) = (1\,4\,3)$$

In this example, we can think about what happens to each of the numbers from 1 to 4. For 1, the right-most cycle $(3\,4)$ doesn't move it, the next cycle $(1\,2)$ swaps it with 2, and then the left-most cycle $(1\,2\,4)$ sends that 2 to 4. So 1 ultimately ends up at 4. Similarly, 2 is swapped with 1 by $(1\,2)$ before being sent back to 2 by the cycle $(1\,2\,4)$. The number 3 is sent to 4 by $(3\,4)$, is not affected by $(1\,2)$, but is then sent to 1 by the cycle $(1\,2\,4)$. Finally, 4 is sent to 3, then left alone, then sent to 1 by the left-most cycle.

Using diagrams, we can think of this as stacking the diagram for τ on top of the diagram for σ.



Note that for $\tau\sigma$ we have

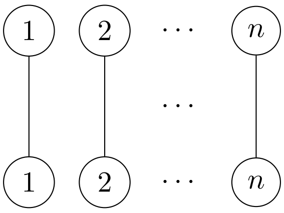
$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \neq \sigma\tau.$$

In general, composition of permutations is not commutative.

3 Identity Permutation

There is a very simple permutation which doesn't relabel anything at all. This is called the identity permutation, defined by $e_n \colon \mathbb{Z}(n) \to \mathbb{Z}(n)$ where $e_n(x) = x$. In the two-line and one-line notation, this is just

$$e_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & n \end{pmatrix} = (1)(2)\dots(n-1)(n).$$



If we compose an identity permutation with any other permutation σ, then this leaves σ unchanged:

$$\sigma e_n = \sigma = e_n \sigma.$$

4 Inverse Permutations

Every permutation σ has an inverse σ^{-1}, which we think of as 'un-shuffling' the numbers we started with.

Example

Using the permutation

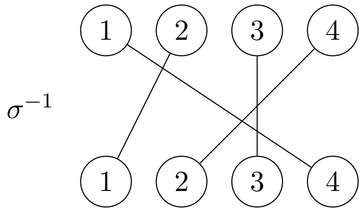
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1\,2\,4)$$

then the inverse is easy to find. In two-line notation, we simply read the permutation from bottom to top, rather than top to bottom. We can write the bottom row on the top row, and vice-versa, before reordering.

$$\sigma^{-1} = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

In the one-line notation, we can simply write the cycle backwards and then rearrange to have the lowest number at the front if we desire.

$$\sigma^{-1} = (4\,2\,1) = (1\,4\,2)$$



Inverse permutations have the property that

$$\sigma\sigma^{-1} = e_n = \sigma^{-1}\sigma.$$

5 Inverses of Composites

Taking inverses of composites is a little trickier, but not by much. If we take the permutations σ and τ from before, then we can show that $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (1\,2\,4)$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4)$$

We know that $\sigma\tau = (1\,4\,3)$, so we should find that $(\sigma\tau)^{-1} = (1\,3\,4)$.

$$\begin{aligned}\tau^{-1}\sigma^{-1} &= (3\,4)^{-1}(1\,2)^{-1}(1\,2\,4)^{-1}\\ &= (3\,4)(1\,2)(1\,4\,2)\\ &= (1\,3\,4)\\ &= (\sigma\tau)^{-1}.\end{aligned}$$

6 Symmetric Groups

Definition. The set of all permutations of $\mathbb{Z}(n)$ forms a group, under composition of permutations. This group is called the **symmetric group** of degree n. We use the notation S_n to denote this group.

Consider S_4. How many elements does it have? i.e. how many permutations are there of four elements?

There are:

- 4 choices for the first number;
- 3 remaining choices for the second number;
- 2 remaining choices for the third number;
- 1 remaining choices for the fourth number.

Therefore there are $4 \times 3 \times 2 \times 1 = 5! = 24$ elements in S_4.

In general, there are $n!$ permutations of $\mathbb{Z}(n)$, so $n!$ elements in S_n, where

$$n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1.$$

Thus, $1! = 1$. What about $0!$?

Consider this way of defining the factorial:

$$\begin{aligned}(n+1)! &= (n+1)n!\\ n! &= \frac{(n+1)!}{(n+1)}.\end{aligned}$$

So

$$0! = \frac{(0+1)!}{(0+1)} = \frac{1!}{1} = \frac{1}{1} = 1.$$

Another way to think about this: $n!$ is the number of ways of arranging n objects, and there's only one way to arrange 0 objects.

7 Choosing objects without repetition

Say we have a pile of ten numbered balls and we want to choose three of them. Choosing the first, there are 10 possibilities for which ball that could be. Then we come to choose the second. Well, now there are only 9 balls remaining, so there are 9 ways we could choose the second one. Similarly, there are 8 ways to choose the third. In total, then, there are $10 \times 9 \times 8$ ways to choose three balls from ten. To denote this using factorials, say:

$$10 \times 9 \times 8 = \frac{10 \times 9 \times 8 \times 7 \times \ldots \times 1}{7 \times \ldots \times 1} = \frac{10!}{(10-3)!} = 720.$$

In general, we can pick r objects from n without repetition in

$$\frac{n!}{(n-r)!}$$

ways.

In fact, the story doesn't end there. Within our 720 different ways of choosing three balls from ten, there will be lots that are just rearrangements of each other. For example (assuming the balls are numbered 1–10), we might draw ball number 7 first, then ball number 3, then ball number 9. What if we had chosen ball 3, then 7, then 9? Depending on the circumstances we are dealing with, we might consider this different or we might think these two are the same outcome really.

In fact, we know that the number of ways of arranging 3 balls is $3! = 6$. So our 720 selections could be reduced by a factor of 6 if we didn't care about the order in which they were chosen.

That would give

$$\frac{10!}{(10-3)!} \times \frac{1}{3!} = \frac{10!}{3!(10-3)!} = \frac{720}{6} = 120.$$

In general, the number of ways of choosing r objects from n without repetition when we don't care about the order of the r objects is

$$\frac{n!}{r!(n-r)!}.$$

This is the binomial coefficient and can be written using the notation $\binom{n}{r}$.

$\binom{n}{r}$ gives the entries on the nth row of Pascal's triangle, for $r = 0, 1, \ldots, n$.