Vectors

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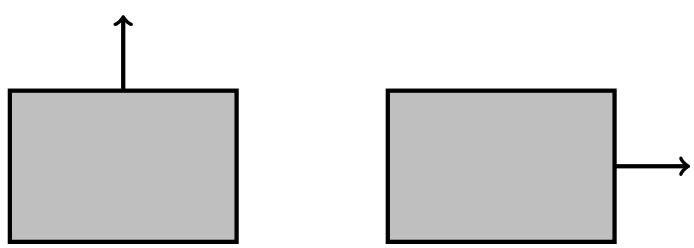
1 Position vector

A position in the (x, y)-plane with coordinates (a, b) can be represented as a vector, which is a one-column matrix $\begin{bmatrix} a \\ b \end{bmatrix}$. This does not give any more information, but it can be useful to manipulate the information in this form.

2 Scalar and vector quantities

Some quantities, such as mass or time, are scalars. We represent these by a single number which is its *magnitude*, e.g. 26kg or 39 seconds. Other quantities we might want to model have both a magnitude and a *direction*. For example, if we talk about a force of 10N applied vertically upwards or a force of 10N applied horizontally to the right, these would produce quite different effects.

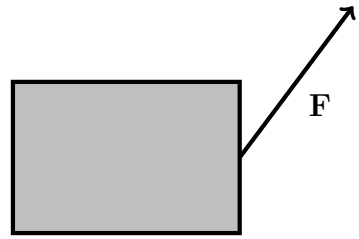


We represent quantities which have these two components magnitude and direction using a *vector*. We can represent a vector using a one-column matrix.

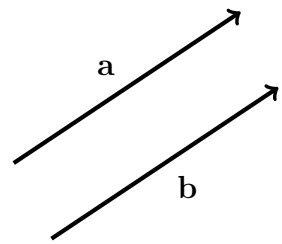
Some examples of scalar quantities are distance, area, mass, temperature, price, work, and energy. Some examples of vector quantities are displacement, force, velocity, acceleration, and momentum.

3 Representation

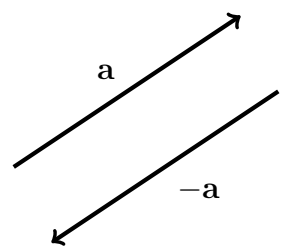
Vectors can be represented by drawing a line. For example, here the vector $\mathbf{F}$ represents a force acting on a block.



Two vectors are said to be equal if they have the same magnitude and the same direction. For example, here $\mathbf{a}$ and $\mathbf{b}$ are equal, even though they don't coincide. Unless a vector represents a position in space, it can be translated without changing the vector itself.



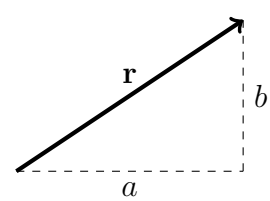
For a vector $\mathbf{a}$, the negative vector $-\mathbf{a}$ acts with the same magnitude in the opposite direction.



4 Magnitude

If $\mathbf{r}$ is a vector quantity, then its magnitude is $|\mathbf{r}|$.

If $\mathbf{r} = \begin{bmatrix} a \\ b \end{bmatrix}$, then we can draw this vector

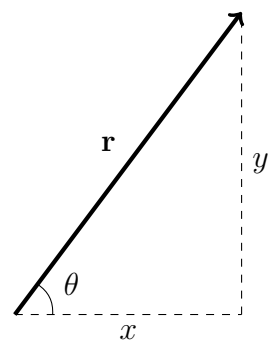


By the Pythagorean theorem,

$$|\mathbf{r}| = \sqrt{a^2 + b^2}.$$

If we know a vector $\mathbf{r}$ acts at an angle θ and has some magnitude $|\mathbf{r}|$, we can draw this as below.

There are equivalent results in higher dimensions. For example, a 3D vector $\mathbf{r} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has magnitude $|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$.



Then we can determine the two components of this force, one horizontal and one vertical, using

$$x = |\mathbf{r}| \cos(\theta);$$
$$y = |\mathbf{r}| \sin(\theta).$$

Therefore, we can write

$$\mathbf{r} = \begin{bmatrix} |\mathbf{r}| \cos(\theta) \\ |\mathbf{r}| \sin(\theta) \end{bmatrix}$$

5 Unit vectors

If we have a vector $\mathbf{r}$, we can form a vector of magnitude 1 acting in the direction of $\mathbf{r}$. This is called a unit vector and is written $\hat{\mathbf{r}}$. We can scale a vector to a unit vector by dividing by its magnitude, so

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}.$$

By convention, the unit vector on the x-axis is denoted $\mathbf{i}$. Similarly, the unit vector on the y-axis is denoted $\mathbf{j}$ and the unit vector on the z-axis (in 3D geometry) is denoted $\mathbf{k}$.

In a 2D system, $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So a vector can be represented as a combination of these quantities

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\mathbf{i} + b\mathbf{j}.$$

In a 3D system, $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So a vector can be represented as a combination of these quantities

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

6 Manipulation of vectors

Since vectors are matrices, we can add them together and apply transformations just as we have seen with vectors.

For example, if an object starts at the origin, moves along a vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then moves along a vector $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$, then we can locate its final position by matrix addition

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

7 Dot product

The *dot product*, also known as the *scalar product*, is defined for vectors $\mathbf{a}$ and $\mathbf{b}$ as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

where θ is the angle between the vectors.

The dot product is a scalar quantity – it has magnitude only.

Some properties of the dot product:

- The dot product is commutative, that is

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

- The dot product is distributive over addition, that is

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

- For a unit vector $\hat{\mathbf{b}}$, then $|\hat{\mathbf{b}}| = 1$ and

$$\mathbf{a} \cdot \hat{\mathbf{b}} = |\mathbf{a}| \cos(\theta),$$

  which is equal to the component of $\mathbf{a}$ in the direction of $\hat{\mathbf{b}}$.

- Since $\cos(0) = 1$, the dot product of a vector with itself is

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}| \cos(0) = |\mathbf{a}|^2 \qquad \text{or} \qquad |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

- If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then $\theta = 90°$, then $\cos(\theta) = 0$ and $\mathbf{a} \cdot \mathbf{b} = 0$. Indeed, if $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors and $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are perpendicular, so $\mathbf{i} \cdot \mathbf{j} = 0$. Further, since $|\mathbf{i}| = 1$, it follows that $\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}|^2 = 1$. We can use these to develop a formula for finding the dot product of two vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

That is,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) \\ &= a_1\mathbf{i} \cdot (b_1\mathbf{i} + b_2\mathbf{j}) + a_2\mathbf{j}(b_1\mathbf{i} + b_2\mathbf{j}) \\ &= a_1b_1 + a_2b_2. \end{aligned}$$

7.1 Example: angle between two vectors

Find the angle between two vectors $\mathbf{a} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

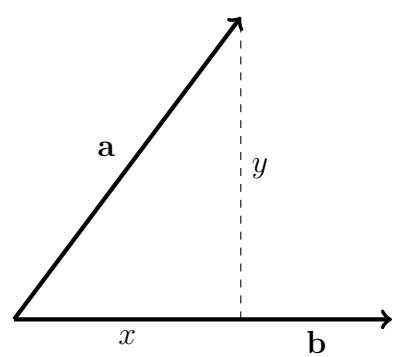
First, we can find $|\mathbf{a}| = \sqrt{7^2 + 8^2} = \sqrt{113}$ and $|\mathbf{b}| = \sqrt{5^2 + 2^2} = \sqrt{29}$.

Now note that $\mathbf{a} \cdot \mathbf{b} = 7 \times 5 + 8 \times -2 = 19$.

Therefore

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{19}{\sqrt{113}\sqrt{29}} \approx 70.6^\circ.$$

7.2 Example: component of a vector in the direction of another



Recall that for a vector $\mathbf{a} = \begin{bmatrix} x \\ y \end{bmatrix}$, we had $x = |\mathbf{a}| \cos(\theta)$.

If we take the dot product of $\mathbf{a}$ with a unit vector in the $\mathbf{b}$ direction, $\hat{\mathbf{b}}$, then we get

$$\begin{aligned} \mathbf{a} \cdot \hat{\mathbf{b}} &= |\mathbf{a}||\hat{\mathbf{b}}| \cos(\theta) \\ &= |\mathbf{a}| \cos(\theta) \quad \text{since } |\hat{\mathbf{b}}| = 1. \end{aligned}$$

So the component of the vector $\mathbf{a}$ in the direction of vector $\mathbf{b}$ is given by $\mathbf{a} \cdot \hat{\mathbf{b}}$.

8 Cross product

The *cross product* or *vector product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin(\theta)\hat{\mathbf{n}},$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

The result of the vector product is a vector at right angles to both $\mathbf{a}$ and $\mathbf{b}$.

Some properties of the cross product:

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$;
2. $\mathbf{a} \times \mathbf{a} = 0$;
3. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$;
4. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$;
5. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$.

Note the following results about the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = 0 & \mathbf{j} \times \mathbf{j} = 0 & \mathbf{k} \times \mathbf{k} = 0 \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Taking

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we get

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \end{aligned}$$

The quantities like $a_2b_3 - a_3b_2$ might look like determinants. In fact, we can define the cross product as a determinant.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$