Matrix games

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Coin Poker

Game Theory

- We have mostly looked at *combinatorial game theory*, which is the study of simple, deterministic games.
- Today we will spend a little time looking at *classic game theory* or *economic games*.

Today

- The structure of today is to play and analyse some simple games.
- The conclusion of the analysis will probably not surprise you if you have played the game a bit.
- But the point is to see a method of analysis that can be applied to more complicated games.

Play a game: Coin Poker

Play in pairs. Play proceeds as follows

1. Both players put one token in the pot and each toss a coin,* but do not share the outcome.

*Or somehow generate 50/50 outcomes, for example put "toss a coin" into a search engine.

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    - fold, ending the game and giving the pot to player 2; or,
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2. Player 1 moves. They either:
   - fold, ending the game and giving the pot to player 2; or,
   - bet 2 more tokens.
3. Player 2 moves. They either:
   - fold, ending the game and giving the pot to player 1; or,
   - bet 2 more tokens.

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   - bet 2 more tokens.
3. Player 2 moves. They either:
   - fold, ending the game and giving the pot to player 1; or,
   - bet 2 more tokens.
4. Both coin tosses are revealed.
   - If both players have the same coin toss, the pot is split between them;
   - otherwise, the player who tossed heads wins the entire pot.

*Or somehow generate 50/50 outcomes, for example put “toss a coin” into a search engine.

Classic game theory

History

- The origins of modern game theory are in a paper *On the Theory of Games of Strategy* by John von Neumann in 1928.
- Famous developments were made in analysis of two-person zero-sum games in economics by von Neumann and others in the 1940s and 1950s.
- Later applications were found starting in the 1970s in biology.

Some terminology

- Typically, we might consider *simultaneous* games (players play at the same time) without *perfect information* (players don't know everything about the game in play).
- Games might be:
  - *Zero-sum*: choices by players can neither increase nor decrease the available resources; one player's win is the other's loss. (Opposite: *Non-zero-sum*.)
  - *Symmetric*: payoffs depend on the competing strategies, not who is playing them. (Opposite: *Asymmetric*.)

More terminology

- A game typically specifies:
  - the *players* of the game;
  - the *actions* or *strategies* available to each player at each stage; and,
  - the *payoffs* for each outcome (positive or negative values), which are what a player gains or loses from this outcome.
- Often the payoffs for two players are represented as a pair (a, b), where a is the payoff for player 1 and b is the payoff for player 2.

Payoffs

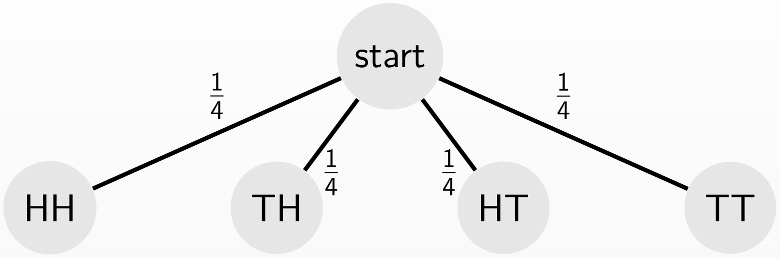
Coin Poker

- Coin Poker:
  - involves two players, 1 and 2;
  - strategies are related to whether a player chooses to bet or fold.
- Coin Poker is:
  - not exactly simultaneous, but players do not have perfect information;
  - zero-sum (either Player 1 takes the pot, or Player 2 does, or the pot is split between them);
  - asymmetric (because Player 1 moves before Player 2).
- Let's think about payoffs.

Outcomes and payoffs

- We wish to think about strategies – ways each player can approach playing the game.
- First, we need to think about the outcomes that occur when players make particular choices.

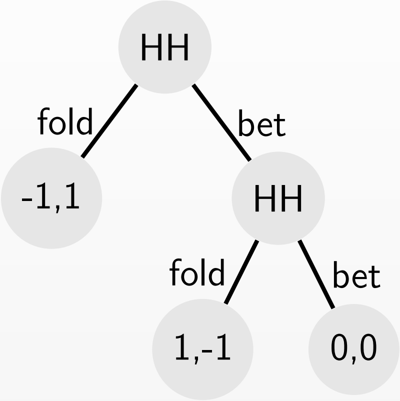
Coin Poker game tree – first move



Coin Poker game tree – HH

Player 1:

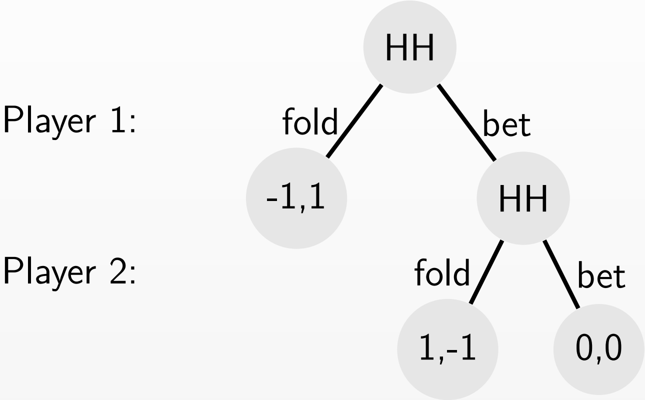
Player 2:



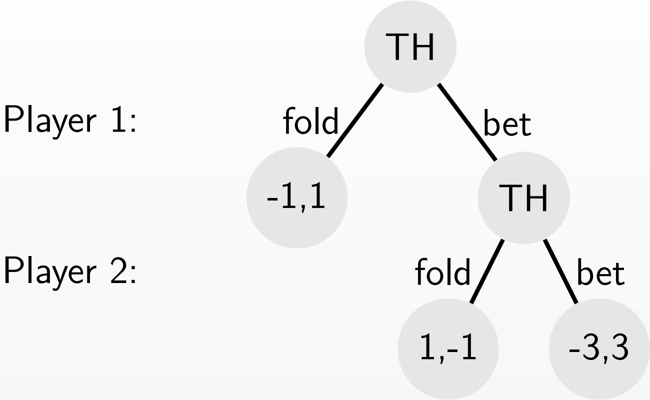
Coin Poker game tree – HH

Exercise:

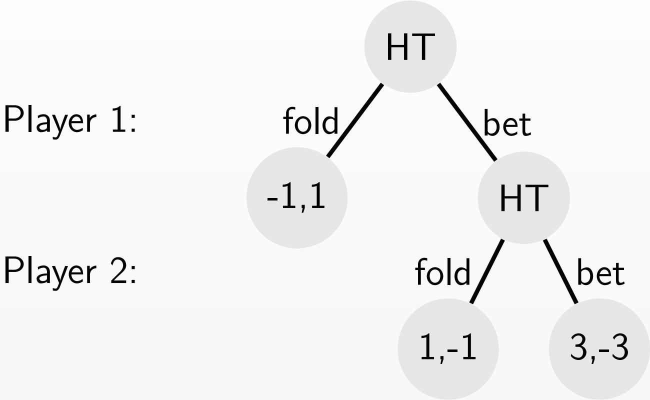
- Make the rest of the game tree for the TH, HT and TT situations.



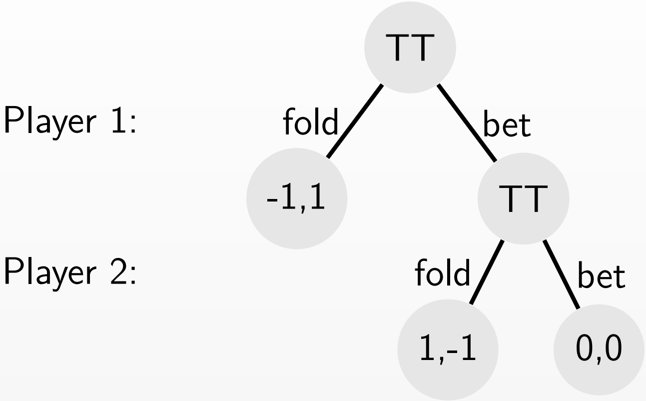
Coin Poker game tree – TH



Coin Poker game tree – HT



Coin Poker game tree – TT



Expectation

Strategies

- Say the players choose their strategies ahead of the game.
- Assuming rational play, let's consider four strategies for each player:
  - always bet;
  - never bet;
  - only H: bet if and only if you've thrown heads;
  - only T: bet if and only if you've thrown tails.

Expected outcomes

- Suppose a strategy leads to i possible outcomes x_i.
- Suppose outcome x_i occurs with probability p_i.
- Then the player's *expected outcome* is given by

$$\sum_i p_i x_i.$$

Example

- You give me £1 and choose a card at random from a standard deck of playing cards.
- I give you £10 if and only if you draw the Queen of Hearts.

Example

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- I give you £10 if and only if you draw the Queen of Hearts.
- So you expect to be 'up' £9 in one time out of 52, and 'down' £1 the rest of the time.

Example

- You give me £1 and choose a card at random from a standard deck of playing cards.
- I give you £10 if and only if you draw the Queen of Hearts.
- So you expect to be ‘up’ £9 in one time out of 52, and ‘down’ £1 the rest of the time.
- Your expected outcome is therefore

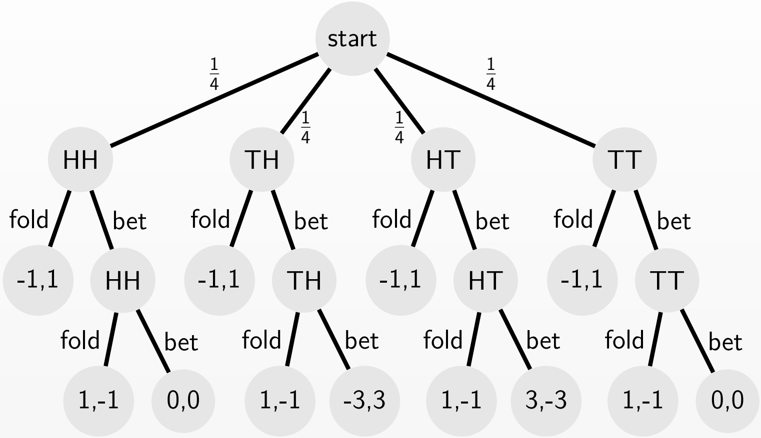
$$9 \times \frac{1}{52} + -1 \times \frac{51}{52} \approx -0.81$$

i.e. on average you expect to lose 81p playing this game.

Back to Coin Poker

- Let's consider the outcome of the situation where the players play these strategies:
  - Player 1 'always bet';
  - Player 2 'only H'.
- The outcomes for each player of these strategies depends how the coin tosses turned out.
- We can examine the possible outcomes to calculate the expected outcome.

e.g. Player 1 'always bet', player 2 'only H'



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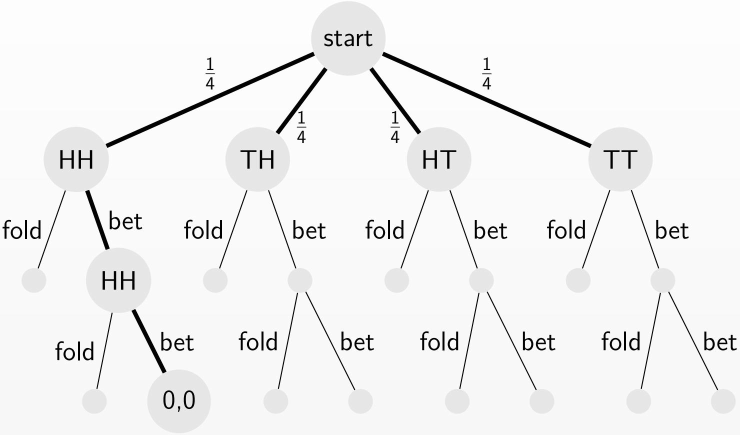
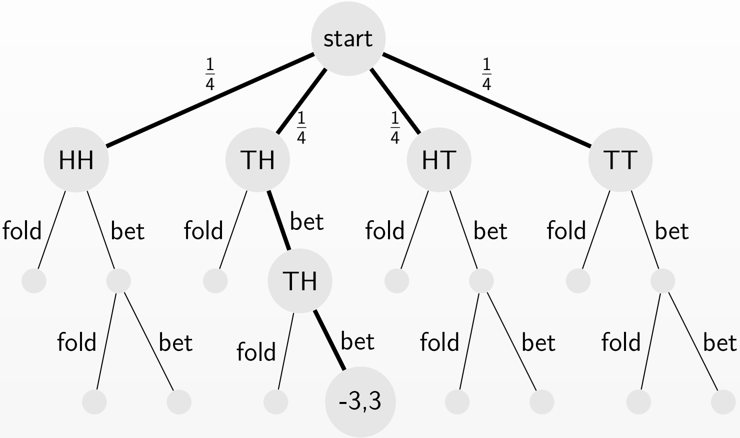


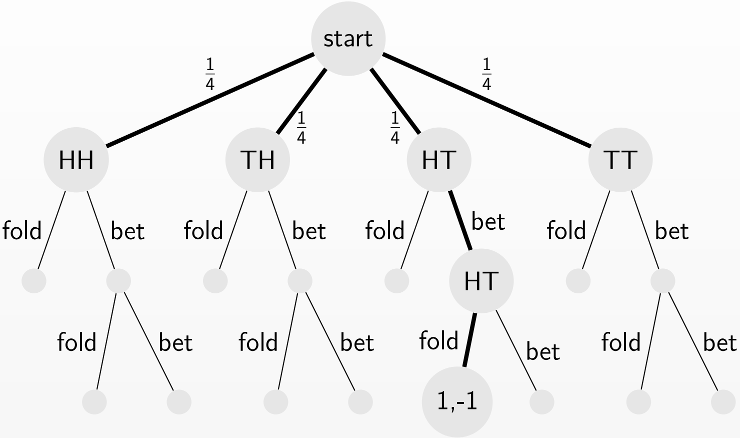
- So if the players throw HH, the outcome will be (0,0).

e.g. Player 1 ‘always bet’, player 2 ‘only H’



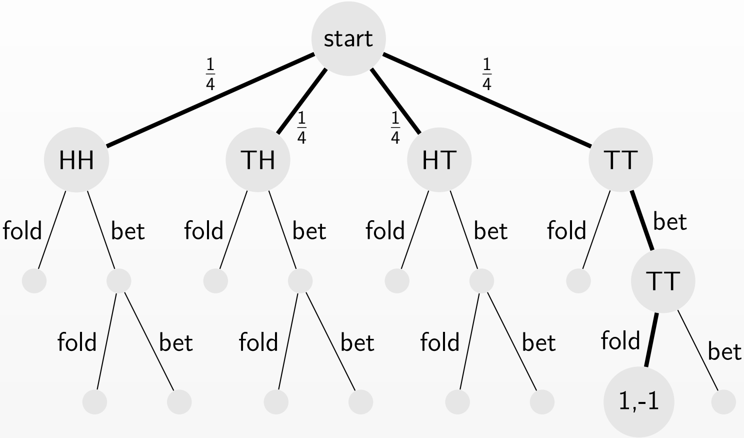
- So if the players throw TH, the outcome will be (-3,3).

e.g. Player 1 'always bet', player 2 'only H'



- So if the players throw HT, the outcome will be (1,-1).

e.g. Player 1 ‘always bet’, player 2 ‘only H’



- So if the players throw TT, the outcome will be (1,-1).

e.g. Player 1 ‘always bet’, player 2 ‘only H’

throw	probability	outcome
HH	$\frac{1}{4}$	(0,0)
TH	$\frac{1}{4}$	(-3,3)
HT	$\frac{1}{4}$	(1,-1)
TT	$\frac{1}{4}$	(1,-1)

- Putting this together, the expected outcomes from these strategies are:

$$\text{Player 1: } \frac{1}{4} \times 0 + \frac{1}{4} \times -3 + \frac{1}{4} \times 1 + \frac{1}{4} \times 1 = -\frac{1}{4};$$

e.g. Player 1 'always bet', player 2 'only H'

throw	probability	outcome
HH	$\frac{1}{4}$	(0,0)
TH	$\frac{1}{4}$	(-3,3)
HT	$\frac{1}{4}$	(1,-1)
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- Putting this together, the expected outcomes from these strategies are:

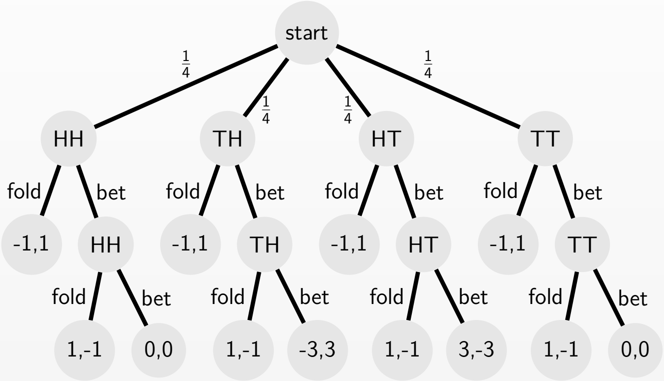
$$\text{Player 1: } \frac{1}{4} \times 0 + \frac{1}{4} \times -3 + \frac{1}{4} \times 1 + \frac{1}{4} \times 1 = -\frac{1}{4};$$
$$\text{Player 2: } \frac{1}{4} \times 0 + \frac{1}{4} \times 3 + \frac{1}{4} \times -1 + \frac{1}{4} \times -1 = \frac{1}{4}.$$

e.g. Player 1 ‘always bet’, player 2 ‘only H’

throw	probability	outcome
HH	$\frac{1}{4}$	(0,0)
TH	$\frac{1}{4}$	(-3,3)
HT	$\frac{1}{4}$	(1,-1)
TT	$\frac{1}{4}$	(1,-1)

- i.e., the expected outcome from this pair of strategies can be expressed as the payoff $\left(-\frac{1}{4}, \frac{1}{4}\right)$.

- Exercise[†]: Work out the payoffs for the following:
  1. Player 1 'always bet', Player 2 'always bet';
  2. Player 1 'always bet', Player 2 'never bet';
  3. Player 1 'always bet', Player 2 'only T';
  4. Player 1 'only H', Player 2 'only T'.



[†]If you finish: how many of the 16 can you work out?

Answers

1. Player 1 ‘always bet’, Player 2 ‘always bet’: $(0, 0)$;
2. Player 1 ‘always bet’, Player 2 ‘never bet’: $(1, -1)$;
3. Player 1 ‘always bet’, Player 2 ‘only T’: $\left(\frac{5}{4}, -\frac{5}{4}\right)$;
4. Player 1 ‘only H’, Player 2 ‘only T’: $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

Payoff matrices

Payoff matrix

- A game can be represented in a *payoff matrix*.
- Here is a two-player game where each player has two strategies and
  - a, b, c and d are the payoffs for Player 1 in the four scenarios.
  - w x, y and z are the payoffs for Player 2, similarly.

		Player 2	
		Strategy 1	Strategy 2
Player 1	Strategy 1	(a, w)	(b, x)
	Strategy 2	(c, y)	(d, z)

Our Coin Poker strategies in a payoff matrix

Player 2

		Always bet	Never bet	Only H	Only T
Player 1	Always bet	$(0,0)$	$(1,-1)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{5}{4},-\frac{5}{4}\right)$
	Never bet	$(-1,1)$	$(-1,1)$	$(-1,1)$	$(-1,1)$
	Only H	$\left(\frac{1}{4},-\frac{1}{4}\right)$	$(0,0)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{2},-\frac{1}{2}\right)$
	Only T	$\left(-\frac{5}{4},\frac{5}{4}\right)$	$(0,0)$	$(-1,1)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$

Dominance

Dominance

- We say that player 1's strategy of row i *dominates* their strategy of row k if every entry in row i is greater than or equal to the corresponding entry in row k.
- i.e. in any circumstances, they would be better playing strategy i than strategy k.
- And similarly for player 2 with columns.

Dominance: example

- For example, consider this game

		Player 2 D	Player 2 E
Player 1	A	$(2,-2)$	$(1,-1)$
Player 1	B	$(3,-3)$	$(0,0)$
Player 1	C	$(-1,1)$	$(0,0)$

- For every option in strategy C, Player 1 would be better off using strategy A.
- So we say Player 1's strategy A *dominates* strategy C.

Dominance: example

- Consequently, we can safely remove strategy C as it will never be used. This gives a simpler game to analyse.

		Player 2	
		D	E
Player 1	A	$(2, -2)$	$(1, -1)$
	B	$(3, -3)$	$(0, 0)$

- Looking at the columns, we see that Player 2 is always better off playing strategy E.
- So we say that strategy E *dominates* strategy D, and we remove strategy D.

Dominance: example

- We are now left with only one possible outcome for this game,

		Player 2
		E
Player 1	A	$(1, -1)$
	B	$(0, 0)$

- Now, Player 1 will play strategy A and the outcome of this game will be $(1, -1)$.

Dominance

- This process is called *iterative removal of dominated strategies* and can help clarify a game by removing outcomes that would not occur.

Back to our Coin Poker strategies...

- Exercise: Apply iterative removal of dominated strategies to this game.

		Player 2			
		Always bet	Never bet	Only H	Only T
Player 1	Always bet	$(0,0)$	$(1,-1)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{5}{4},-\frac{5}{4}\right)$
	Never bet	$(-1,1)$	$(-1,1)$	$(-1,1)$	$(-1,1)$
	Only H	$\left(\frac{1}{4},-\frac{1}{4}\right)$	$(0,0)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{2},-\frac{1}{2}\right)$
	Only T	$\left(-\frac{5}{4},\frac{5}{4}\right)$	$(0,0)$	$(-1,1)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$

Back to our Coin Poker strategies...

		Player 2			
		Always bet	Never bet	Only H	Only T
Player 1	Always bet	$(0,0)$	$(1,-1)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{5}{4},-\frac{5}{4}\right)$
	Never bet	$(-1,1)$	$(-1,1)$	$(-1,1)$	$(-1,1)$
	Only H	$\left(\frac{1}{4},-\frac{1}{4}\right)$	$(0,0)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{2},-\frac{1}{2}\right)$
	Only T	$\left(-\frac{5}{4},\frac{5}{4}\right)$	$(0,0)$	$(-1,1)$	$\left(-\frac{1}{4},\frac{1}{4}\right)$

- Notice that Player 1's strategy 'Only T' is dominated by their 'Always bet'.

Back to our Coin Poker strategies. . .

- So we remove Player 1's strategy 'Only T'.

Player 2

Player 1		Always bet	Never bet	Only H	Only T
	Always bet	$(0, 0)$	$(1, -1)$	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{5}{4}, -\frac{5}{4}\right)$
	Never bet	$(-1, 1)$	$(-1, 1)$	$(-1, 1)$	$(-1, 1)$
	Only H	$\left(\frac{1}{4}, -\frac{1}{4}\right)$	$(0, 0)$	$\left(-\frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{2}, -\frac{1}{2}\right)$

- In this reduced game, notice that Player 2's 'Only H' dominates all other strategies.

Back to our Coin Poker strategies...

- So we remove all Player 2's strategies except 'Only H'.

		Player 2
		Only H
Player 1	Always bet	$\left(-\frac{1}{4}, \frac{1}{4}\right)$
	Never bet	$(-1, 1)$
	Only H	$\left(-\frac{1}{4}, \frac{1}{4}\right)$

- Since Player 1 prefers $-\frac{1}{4}$ to -1, their strategy 'Never bet' is dominated by the other two.

Back to our Coin Poker strategies. . .

- We remove Player 1's 'Never bet' strategy.

		Player 2
		Only H
Player 1	Always bet	$\left(-\frac{1}{4}, \frac{1}{4}\right)$
	Only H	$\left(-\frac{1}{4}, \frac{1}{4}\right)$

- We conclude that optimal play leads Player 1 to choose either 'Always bet' or 'Only H' and Player 2 to choose 'Only H', with the expected outcome $\left(-\frac{1}{4}, \frac{1}{4}\right)$.

Coin Poker conclusions

- This is not a good game to be the first player!
- In fact, it isn't a particularly interesting game.
  - Player 2 bets only when they have a head (no choice).
  - Player 1 has a choice to either always bet or bet only when they throw a head, but the outcome doesn't change (the illusion of choice).
- Real poker has:
  - a much more complicated probability mechanic than a simple coin toss;
  - the ability to 'raise', allowing a more iterative betting dynamic – crucially involving bluffing.

Odds and Evens

Odds and Evens

- Play in pairs.
- Both players at the same time show either one finger or two.
- If the total number of fingers shown is even, Player 2 gives that number of points to Player 1.
- If the total number of fingers shown is odd, Player 1 gives that number of points to Player 2.

Equilibria

Nash equilibrium

- A useful concept is the *Nash equilibrium*: a pair of strategies, each of which is the best response to the other.
- This is the state when no player can gain an advantage by changing their action while the others keep theirs.
- In some games we see *pure* Nash equilibria – pairs of single (pure) strategies that form an equilibrium.

Example: driving game

- Here is a payoff matrix for which side of the road to drive on.
- Two players are driving in opposite directions towards each other on the same road.

		Player 2	
		L	R
Player 1	L	$(1, 1)$	$(-1, -1)$
	R	$(-1, -1)$	$(1, 1)$

Example: driving game

- One way to find equilibria is:
  - for each row, highlight the column player's best response;

		Player 2	
		L	R
Player 1	L	$(1, \boxed{1})$	$(-1, -1)$
	R	$(-1, -1)$	$(1, 1)$

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Example: driving game

- One way to find equilibria is:
  - for each row, highlight the column player's best response;
  - for each column, highlight the row player's best response;

Player 2

		L	R
Player 1	L	$(\boxed{1}, \boxed{1})$	$(-1, -1)$
	R	$(-1, -1)$	$(1, \boxed{1})$

Example: driving game

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  - for each column, highlight the row player's best response;

		Player 2	
		L	R
Player 1	L	$(\boxed{1}, \boxed{1})$	$(-1, -1)$
	R	$(-1, -1)$	$(\boxed{1}, \boxed{1})$

Example: driving game

- One way to find equilibria is:
  - for each row, highlight the column player's best response;
  - for each column, highlight the row player's best response;
  - entries with both values highlighted represent equilibria.

		Player 2	
		L	R
Player 1	L	$(1, 1)$	$(-1, -1)$
	R	$(-1, -1)$	$(1, 1)$

Example: driving game

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  - for each row, highlight the column player's best response;
  - for each column, highlight the row player's best response;
  - entries with both values highlighted represent equilibria.

		Player 2	
		L	R
Player 1	L	$(1, 1)$	$(-1, -1)$
	R	$(-1, -1)$	$(1, 1)$

- So this game has two equilibria, equally good for either player.
- (In this game, which happens in practice requires coordination.)

Exercise: Volunteering game

- Two friends need to decide who should wash up.
- Each has two options:
  - volunteer (V) to wash up;
  - stay silent (S) and hope the other person does it.
- Both want the washing up done, so the outcome where neither person volunteers is bad.
- But both would rather the other person did it.
- If both volunteer, they flip a coin to decide who will do the washing up.

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Player 2

		S	V
Player 1	S	$(-10, -10)$	$(2, -2)$
	V	$(-2, 2)$	$(-1, -1)$

Exercise: Volunteering game

- Look at the rows and columns and work out the equilibria options in this game.

		Player 2	
		S	V
Player 1	S	$(-10,-10)$	$(2,-2)$
	V	$(-2,2)$	$(-1,-1)$

Answer: Volunteering game

		Player 2	
		S	V
Player 1	S	$(-10, -10)$	$(2, -2)$
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Answer: Volunteering game

		Player 2	
		S	V
Player 1	S	$(-10, -10)$	$(2, \boxed{-2})$
	V	$(-2, \boxed{2})$	$(-1, -1)$

Answer: Volunteering game

		Player 2	
		S	V
Player 1	S	$(-10, -10)$	$(2, \boxed{-2})$
	V	$(\boxed{-2}, \boxed{2})$	$(-1, -1)$

Answer: Volunteering game

		Player 2	
		S	V
Player 1	S	$(-10,-10)$	(2, -2)
	V	(-2, 2)	$(-1,-1)$

Answer: Volunteering game

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Answer: Volunteering game

		Player 2	
		S	V
Player 1	S	$(-10, -10)$	$(2, -2)$
	V	$(-2, 2)$	$(-1, -1)$

- So we have two equilibria.
- But neither player really wants to *always* volunteer.
- Which strategy should each player play?

Mixed strategies

Mixed strategies

- In a situation without a pure Nash equilibrium, it is possible to obtain a Nash equilibrium using *mixed strategies*, in which each player plays different strategies with some probability.
- We are looking for a combination of strategies for Player 1 so that Player 2 gets the same payoff no matter what they do, and vice versa.

Example: Volunteering game

		Player 2	
		S	V
Player 1	S	$(-10, -10)$	$(2, -2)$
	V	$(-2, 2)$	$(-1, -1)$

- We can obtain two matrices, one giving the payoffs for Player 1 (**R** for rows) and one for Player 2 (**C** for columns).

$$\mathbf{R} = \begin{bmatrix} -10 & 2 \\ -2 & -1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -10 & -2 \\ 2 & -1 \end{bmatrix}$$

Example: Volunteering game

- Say:
  - Player 1 plays S with probability p (and so plays V with probability $1 - p$);
  - Player 2 plays S with probability q (and so plays V with probability $1 - q$).

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- So let

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Example: Volunteering game

- Say:
  - Player 1 plays S with probability p (and so plays V with probability $1-p$);
  - Player 2 plays S with probability q (and so plays V with probability $1-q$).
- So let

$$\mathbf{p} = \begin{bmatrix} p & 1-p \end{bmatrix} \quad \& \quad \mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$$

- Then
  - the expected outcome for Player 1 is $\mathbf{pRq}$;
  - the expected outcome for Player 2 is $\mathbf{pCq}$;

Example: Volunteering game

- For an equilibrium, we are looking for a situation where:
  1. Player 1 chooses values for $\mathbf{p}$ such that Player 2 gets the same payoff no matter what they choose for $\mathbf{q}$.
  2. Player 2 chooses values for $\mathbf{q}$ such that Player 1 gets the same payoff no matter what they choose for $\mathbf{p}$.

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  2. Player 2 chooses values for $\mathbf{q}$ such that Player 1 gets the same payoff no matter what they choose for $\mathbf{p}$.
- For 1: the expected outcome for Player 2 is $\mathbf{pCq}$, so we are looking for the values of the entries in $\mathbf{pC}$ to be the same.
- For 2: similarly, for Player 1's choice of $\mathbf{p}$ to make no difference, we are looking for the values of the entries in $\mathbf{Rq}$ to be the same.

Example: Volunteering game

- Considering Player 1's choices to restrict the options for Player 2:

$$\mathbf{pC} = \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -10 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -12p+2 & -p-1 \end{bmatrix}$$

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- Player 2 has the same expected outcome for either strategy if:

$$-12p+2 = -p-1 \implies p = \frac{3}{11}$$

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- Player 2 has the same expected outcome for either strategy if:

$$-12p + 2 = -p - 1 \implies p = \frac{3}{11}$$

- This suggests Player 1 should stay silent $\frac{3}{11}$ of the time and volunteer $\frac{8}{11}$ of the time.

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- Similarly for Player 2's choices:

$$\mathbf{Rq} = \begin{bmatrix} -10 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} -12q+2 \\ -q-1 \end{bmatrix}$$

- Player 1 has the same expected outcome for either strategy if:

$$-12q+2 = -q-1 \implies q = \frac{3}{11}$$

- This suggests Player 2 should stay silent $\frac{3}{11}$ of the time and volunteer $\frac{8}{11}$ of the time.

Example: Volunteering game

- The expected outcome for Player 1 (**R**) in this equilibrium is:

$$\begin{bmatrix} \frac{3}{11} & \frac{8}{11} \end{bmatrix} \begin{bmatrix} -10 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{11} \\ \frac{8}{11} \end{bmatrix} = -\frac{14}{11}$$

- The expected outcome for Player 2 (**C**) in this equilibrium is:

$$\begin{bmatrix} \frac{3}{11} & \frac{8}{11} \end{bmatrix} \begin{bmatrix} -10 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{11} \\ \frac{8}{11} \end{bmatrix} = -\frac{14}{11}$$

Back to Odds and Evens

- If the total number of fingers shown is even, Player 1 takes that number of points from Player 2.
- If the total number of fingers shown is odd, Player 2 takes that number of points from Player 1.

		Player 2	
		1	2
Player 1	1	$(2, -2)$	$(-3, 3)$
	2	$(-3, 3)$	$(4, -4)$

Back to Odds and Evens

- Say Player 1 shows one finger with probability p and two fingers with probability $1-p$.
- Say Player 2 shows one finger with probability q and two fingers with probability $1-q$.
- What values of p and q create a Nash equilibrium?
- What are the expected outcomes for each player?

		Player 2: 1	Player 2: 2
Player 1	1	$(2,-2)$	$(-3,3)$
	2	$(-3,3)$	$(4,-4)$

Answer: Odds and Evens

- Start by obtaining two matrices, one giving the payoffs for Player 1 (**R** for rows) and one for Player 2 (**C** for columns).

$$\mathbf{R} = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

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- Start by obtaining two matrices, one giving the payoffs for Player 1 (**R** for rows) and one for Player 2 (**C** for columns).

$$\mathbf{R} = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

- Then we require the entries of **pC** to be the same and the entries of **Rq** to be the same.

Answer: Odds and Evens

- Consider Player 1's choices:

$$\mathbf{pC} = \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3-5p & 7p-4 \end{bmatrix}$$

- Player 2 has the same expected outcome for either strategy if:

$$3 - 5p = 7p - 4 \implies p = \frac{7}{12}$$

- This suggests Player 1 should show one finger $\frac{7}{12}$ of the time and two fingers $\frac{5}{12}$ of the time.

Answer: Odds and Evens

- Consider Player 2's strategy $\begin{bmatrix} q \\ 1-q \end{bmatrix}$.

$$\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} 5q-3 \\ 4-7q \end{bmatrix}$$

- Player 1 has the same expected outcome for either strategy if:

$$5q - 3 = 4 - 7q \implies q = \frac{7}{12}$$

- This suggests Player 2 should show one finger $\frac{7}{12}$ of the time and two fingers $\frac{5}{12}$ of the time.

Answer: Odds and Evens

- Expected outcome for Player 1 (**R**) is:

$$\begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} \frac{7}{12} \\ \frac{5}{12} \end{bmatrix} = -\frac{1}{12}$$

- Expected outcome for Player 2 (**C**) is:

$$\begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} \frac{7}{12} \\ \frac{5}{12} \end{bmatrix} = \frac{1}{12}$$

Answer: Odds and Evens

- This analysis suggests both players should show one finger $\frac{7}{12}$ of the time and two fingers $\frac{5}{12}$ of the time.
- Even though this represents an equilibrium (the best response of each player), the game works in favour of Player 2 on average they take $\frac{1}{12}$ points from Player 1.