

# Matrix transformations

Peter Rowlett

## 1 Introduction

In what follows, we will represent a point with coordinates  $(x, y)$  by a  $2 \times 1$  matrix

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We will consider how to use matrices to transform points on a computer screen. Transformations include scaling, translation, rotation, reflection, and shearing. Mathematically, each can be represented by a *transformation matrix*.

In general, if we transform our point using the transformation matrix

$$\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then we obtain

$$\begin{aligned} \mathbf{TX} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}. \end{aligned}$$

We can say the point  $\begin{bmatrix} x \\ y \end{bmatrix}$  has been translated to the point  $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ .

## 2 Shapes

We can consider a line as being defined by two end-points. Hence the matrix

$$\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

represents the line joining the points  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ .

Similarly, a triangle can be stored as a  $2 \times 3$  matrix, for example

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & -1 \end{bmatrix}.$$

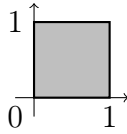
And so on.

### 3 Scaling

Consider the square

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

We can draw this.



Now if we apply the transformation

$$\mathbf{T} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

we obtain

$$\begin{aligned} \mathbf{TS} &= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 4 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

We can draw this.

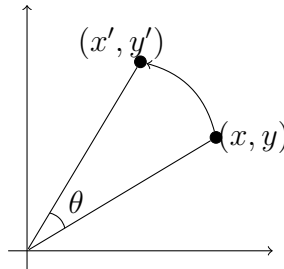


In general, scaling is done using a diagonal matrix, with the first row determining the horizontal scaling and the second the vertical.

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

### 4 Rotation

In this diagram, the point  $(x, y)$  is rotated by  $\theta$  around the origin to  $(x', y')$ .



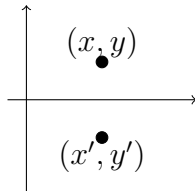
The transformation matrix that rotates a point anticlockwise around the origin by an angle  $\theta$  is given by

$$\mathbf{T} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

## 5 Reflection

### 5.1 Reflection in the $x$ -axis

In the diagram below, the point  $(x, y)$  is reflected in the  $x$ -axis to  $(x', y')$ .



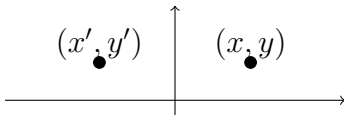
The two points have the same horizontal coordinate, so  $x' = x$ . However, the vertical coordinate has flipped, so  $y' = -y$ .

The transformation matrix that produces this effect is

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

### 5.2 Reflection in the $y$ -axis

In the diagram below, the point  $(x, y)$  is reflected in the  $y$ -axis to  $(x', y')$ .



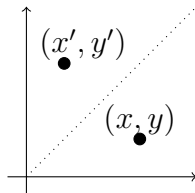
The two points have the same vertical coordinate, so  $y' = y$ . However, the horizontal coordinate has flipped, so  $x' = -x$ .

The transformation matrix that produces this effect is

$$\mathbf{T} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

### 5.3 Reflection in the line $y = x$

In the diagram below, the point  $(x, y)$  is reflected in the line  $y = x$  to  $(x', y')$ .

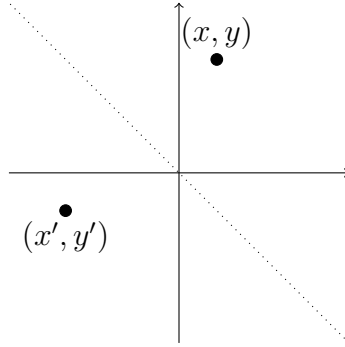


In effect, the  $x$  and  $y$  coordinates have swapped. The transformation matrix that brings this about is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

## 5.4 Reflection in the line $y = -x$

In the diagram below, the point  $(x, y)$  is reflected in the line  $y = x$  to  $(x', y')$ .

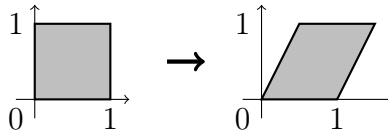


Here the  $x$  and  $y$  coordinates have swapped and also changed sign. The transformation matrix that brings this about is

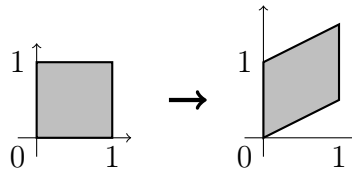
$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

## 6 Shearing

A shear is a distortion. The diagram below shows a shear in the  $x$ -direction.



And the diagram below shows a shear in the  $y$ -direction.



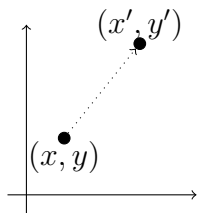
A shear is brought about by the transformation matrix

$$\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The off-diagonal terms,  $b$  and  $c$ , determine the type of shear produced, with  $b$  acting in the  $x$  direction and  $c$  in the  $y$  direction. Similarly to section 3, the element  $a$  is a scale factor in the  $x$  direction and  $d$  is a scale factor in the  $y$  direction.

## 7 Translation

A translation is a movement in a specific direction by a specific amount. For example, in the diagram below the point  $(x, y)$  is translated to  $(x', y')$ .



Consider a point at  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  which is shifted to  $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ . The translation has added 3 to the  $x$  coordinate and added 4 to the  $y$  coordinate. That is

$$4 = 1 + 3$$

$$7 = 3 + 4.$$

In general, to translate  $\begin{bmatrix} x \\ y \end{bmatrix}$  to  $\begin{bmatrix} x' \\ y' \end{bmatrix}$ , we use

$$x' = x + t_x$$

$$y' = y + t_y$$

where  $t_x$  is the translation in the  $x$  direction and  $t_y$  is the translation in the  $y$  direction.

The previous transformations all involved multiplication of coordinates by some factor, but this one is different. It requires us to add a number to the different coordinates.

To enable this to be represented by matrix multiplication, we introduce *homogeneous coordinates*. The point  $\begin{bmatrix} x \\ y \end{bmatrix}$  is represented in homogeneous coordinates by

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

The extra '1' is used to increase the order of the transformation matrix from  $2 \times 2$  to  $3 \times 3$ . We now use a transformation matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying  $\mathbf{T}$  to  $\mathbf{X}$ , we obtain

$$\mathbf{TX} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}.$$

The homogenous coordinates

$$\begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

represent the point

$$\begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}.$$

## 8 Composite transformations

Since we have transformations encoded as matrices, we can multiply these together to form composite transformations.

For example, we saw in section 5.4 that reflection in the  $y = -x$  line is obtained using

$$\mathbf{T}_{\text{ref}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and using the technique in section 4, rotation about the origin by  $70^\circ$  would be represented by

$$\mathbf{T}_{\text{rot}} = \begin{bmatrix} \cos(70^\circ) & -\sin(70^\circ) \\ \sin(70^\circ) & \cos(70^\circ) \end{bmatrix}.$$

The composite transformation is obtained by multiplying these matrices together. Remember that in matrix multiplication,  $\mathbf{AB} \neq \mathbf{BA}$ , so we must think about the order in which we want to apply the transformations.

Acting on a point  $\mathbf{X}$ , if we perform the reflection then a rotation we would calculate  $\mathbf{T}_{\text{rot}}\mathbf{T}_{\text{ref}}\mathbf{X}$ . Note that the last transformation comes first in our multiplication.

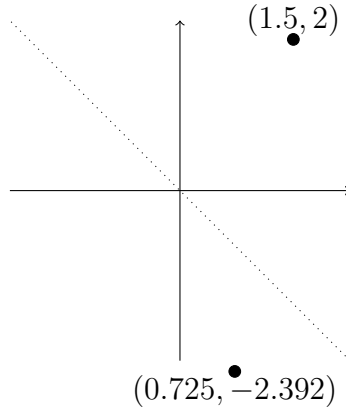
We can calculate the composite transformation matrix

$$\begin{aligned} \mathbf{T}_{\text{rot}}\mathbf{T}_{\text{ref}} &= \begin{bmatrix} \cos(70^\circ) & -\sin(70^\circ) \\ \sin(70^\circ) & \cos(70^\circ) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sin(70^\circ) & -\cos(70^\circ) \\ -\cos(70^\circ) & -\sin(70^\circ) \end{bmatrix}. \end{aligned}$$

For example, if we apply our transformation to the point  $\begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$  we obtain

$$\begin{bmatrix} \sin(70^\circ) & -\cos(70^\circ) \\ -\cos(70^\circ) & -\sin(70^\circ) \end{bmatrix} \begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.725 \\ -2.392 \end{bmatrix}.$$

This is represented on the diagram below.



If a composite transformation involves a translation, we must use homogenous coordinates for the point and all transformations.

## 8.1 Example

Determine a transformation matrix that rotates a point  $30^\circ$  anticlockwise about the point  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

The rotation matrices so far have represented rotation about the origin. To represent rotation about  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  we require three transformations:

1. translation of  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  to the origin,  $\mathbf{T}_{\text{trans}}$ ;
2. rotation of  $30^\circ$  anticlockwise,  $\mathbf{T}_{\text{rot}}$ ;
3. translation from the origin back to  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\mathbf{T}_{\text{back}}$ .

$$\mathbf{T}_{\text{trans}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{T}_{\text{rot}} = \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) & 0 \\ \sin(30^\circ) & \cos(30^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{T}_{\text{back}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The composite transformation is then

$$\mathbf{T}_{\text{back}}\mathbf{T}_{\text{rot}}\mathbf{T}_{\text{trans}} = \begin{bmatrix} 0.8660 & -0.5 & 1.1340 \\ 0.5 & 0.8660 & -0.232 \\ 0 & 0 & 1 \end{bmatrix}.$$