Proof methods – exercises

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1. Prove or disprove the following.

   (a) Let m be an integer. If m is odd, then m^2 is odd.

   (b) Suppose that $p \in \mathbb{Q}$ and $p^2 \in \mathbb{Z}$. Then, $p \in \mathbb{Z}$.

   (c) Let m and n be real numbers. If $n > m > 0$, then
   $$\frac{m+1}{n+1} > \frac{m}{n}.$$

   (d) Let A, B and C be sets. Then,
   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

   (e) Let A and B be sets. Then, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

2. Show that the sum of two consecutive odd numbers is a multiple of 4. What is the converse and is it true?

3. Let $f : X \to Y$. Suppose A and B are subsets of X. Show $f(A \cup B) = f(A) \cup f(B)$.

4. Prove or disprove the following.

   (a) $a + b = c \implies a^2 + b^2 = c^2$.

   (b) Let $x \in \mathbb{Z}$. Then $-x$ is negative.

5. The following is a proof that $1 = 2$. You may have reason to doubt that this is a true fact! Where is the error?

   Theorem. $1 = 2$.

   Proof. Let $a = b$, where $a, b \in \mathbb{Z}$. Then,
   $$\begin{array}{rcll} ab &=& a^2 & \text{since } a = b, \\ ab - b^2 &=& a^2 - b^2 & \text{by subtracting } b^2 \text{ from both sides,} \\ b(a-b) &=& (a+b)(a-b) & \text{by factoring,} \\ b &=& a+b & \text{by dividing both sides by } a - b, \\ b &=& 2b & \text{since } a = b, \\ 1 &=& 2 & \text{by dividing by } b. \end{array}$$
   $\square$

6. Prove or disprove the following.

   (a) Suppose $n \in \mathbb{N}$. Then $n^3 - n$ is a multiple of 3.
   (b) Suppose $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.
   (c) The square of any integer is of the form $3k$ or $3k + 1$ for some $k \in \mathbb{Z}$.
   (d) Suppose $a = bc$ for $a, b, c \in \mathbb{R}$. If two of a, b or c are non-zero, then so is the third.

7. Prove or disprove the following.

   (a) There are no positive integers x and y such that $x^2 - y^2 = 1$.
   (b) The sum of a rational and an irrational number is an irrational number.
   (c) $\sqrt{3}$ is irrational.
   (d) $\sqrt{4}$ is irrational. (What happens is you try the same approach as for $\sqrt{3}$?)
   (e) There are no positive integer solutions to $x^2 + x + 1 = y^2$.
   (f) Suppose $x, y \in \mathbb{Z}$. Then $\sqrt{x^2 + y^2} \neq x + y$.