Proof by induction.

Suppose we want to show some statement p is true for all non-negative integers $0, 1, 2, \ldots$

Let $p(n)$ be a proposition involving $n \in \mathbb{Z}$, $n \geq 0$.
(Note $p(n)$ is not a function!)

To prove a statement of the form

$$\forall n \, (p(n))$$

1. First prove $p(0)$ (base case).
2. Then prove $p(n-1) \Rightarrow p(n)$ (inductive step).

Form of final proof:

Base case: [proof of $p(0)$ here]

Inductive step: [proof of $\cancel{\forall n (p(n))}$ $p(n-1) \Rightarrow p(n)$ here].

We can prove this works using contradiction.

Theorem. Suppose

i) $p(0)$ is true.

ii) $p(n-1) \Rightarrow p(n)$ $\forall n \in \mathbb{Z}, n \geq 0$.

Then $p(n)$ is true for all $n \in \mathbb{Z}, n \geq 0$.

Proof. Suppose to the contrary that $p(n)$ is not true for all $n \in \mathbb{Z}, n \geqslant 0$.

Then there is a smallest value $j \geqslant 0$, such that $p(j)$ is false.

Now $p(j-1)$ must be true, because j is the smallest false value.

But $p(j-1) \Rightarrow p(j)$ by ii).

This is a contradiction. $\square$

Theorem. $\displaystyle\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$ for all $n \in \mathbb{N}$.

e.g. $n=2$ $\displaystyle\sum_{i=1}^{2} i = 1 + 2 = 3$

$\frac{1}{2} n(n+1) = \frac{1}{2} \times 2 \times 3 = 3$

Proof.

Base case: Let $n=1$

Then $\displaystyle\sum_{i=1}^{1} i = 1$ & $\frac{1}{2} \times 1 \times (1+1) = 1$.

Inductive step: We want to show that if the statement is true for $n=k-1$, then it must be true for $n=k$.

Assume $\displaystyle\sum_{i=1}^{k-1} i = \frac{1}{2}(k-1)(k-1+1) = \frac{1}{2} k(k-1)$.

Now consider $n=k$.

We wish to show $\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$.

We know $\sum_{i=1}^{k} i = \sum_{i=1}^{k-1} i + k$, by the definition of summation.

We assumed $\sum_{i=1}^{k-1} i = \frac{1}{2}k(k-1)$, so

$$
\begin{aligned}
\sum_{i=1}^{k} i &= \tfrac{1}{2}k(k-1) + k \\
&= k\left(\tfrac{1}{2}(k-1)+1\right) \\
&= k\left(\tfrac{1}{2}k - \tfrac{1}{2} + 1\right) \\
&= k\left(\tfrac{1}{2}k + \tfrac{1}{2}\right) \\
&= \tfrac{1}{2}k(k+1), \text{ as required.}
\end{aligned}
$$

$\square$

We showed $n=k-1 \Rightarrow n=k$

$$\sum_{i=1}^{k-1} i = \tfrac{1}{2}k(k-1) \Rightarrow \sum_{i=1}^{k} i = \tfrac{1}{2}k(k+1)$$

and we showed $n=1$ is true.

So

$n=1 \Rightarrow n=2$

$n=2 \Rightarrow n=3$

$n=3 \Rightarrow n=4 \quad \ldots$

Note we assumed $n=k-1$ was true, but did not assume $n=k$ is true.

<u>Theorem</u>. $6^n - 1$ is divisible by 5 $\forall n \in \mathbb{Z}, n \geqslant 0$.

<u>Proof</u>.

Base case: Let $n=0$. Then the statement is true since $6^0 - 1 = 0 = 5 \times 0$.

Inductive step:

Assume $6^{k-1} - 1$ is divisible by 5.

Then $$6^k - 1 = 6(6^{k-1}) - 1$$
$$= 6(5m+1) - 1 \quad \text{(inductive hypothesis)}$$

Since $6^{k-1} - 1$ is divisible by 5, we can write

$6^{k-1} - 1 = 5m$ for some $m \in \mathbb{Z}$.

So $$6(5m+1) - 1 = 30m + 6 - 1$$
$$= 30m + 5$$
$$= 5(6m+1),$$

which is divisible by 5. $\square$

Theorem. Every simple graph with $n > 1$ vertices has two vertices of the same degree.

First we will prove the pigeonhole principle.

Lemma. If we have $n \in \mathbb{N}$ elements and wish to distribute these among $m \in \mathbb{N}$ sets with $n > m$, then one of the m sets must contain more than one element.

Proof. We will prove this by induction on the number of sets.

Base case: Consider 2 elements in 1 set. This set contains more than one element.

Inductive step: Assume that if we have j elements to distribute in $k-1$ sets, with $j > k-1$, then one of the $k-1$ sets contains more than one element.

Now consider k sets and $l > k$ elements. There are 3 cases for what is in the first set:

1. More than one element. Then this is a set which contains more than one element.
2. One element. Now there are $l-1 > k-1$ elements to fit into the remaining $k-1$ sets.

By the inductive hypothesis, one of these contains more than element.

3. No elements. Now there are $l > k-1$ elements to fit into $k-1$ sets, so by the inductive hypothesis, one must contain more than one element.

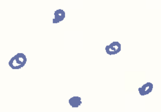
In each case, we had a set with more than one element, as required.

$\square$

Now ~~we~~ are ready to prove our theorem.

<u>Proof</u>. Let G be an arbitrary simple graph with $n > 1$ vertices.

Either:



1. G has no connected components.

Then every vertex has degree 0, so we have two vertices with the same degree.

2. G has at least one connected component.

Then any given vertex ~~has~~ can connect to at most $n-1$ other vertices.

Put all vertices of degree 1 into a set,
all vertices of degree 2 into a set,
$\vdots$
all vertices of degree $n-1$ into a set.

By the pigeonhole principle, there must be a set with at least 2 vertices, hence we have a pair of vertices with same degree.