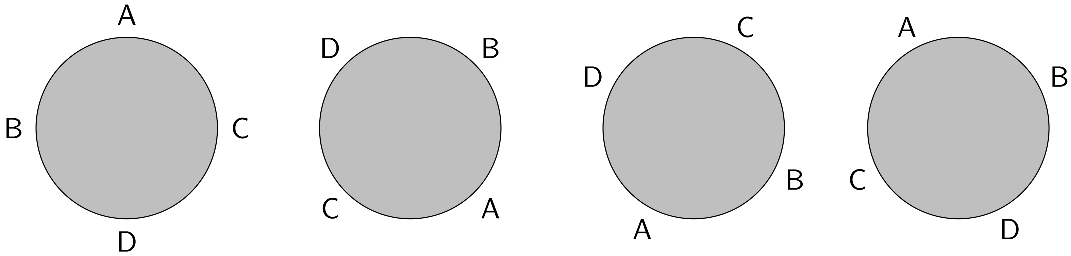
# Worksheet 5: Combinations and permutations

1. (a) If $A, B, C, D$ are people sat around a circular table, in how many different ways can they be seated? Note that a rotation around the table is not considered a different seating arrangement, so here (a) and (b) are considered identical, whereas (b), (c) and (d) are all distinct.




(a) (b) (c) (d)

   (b) How many songs are there with six different notes? (Mersenne, 1636)

   (c) The Arabic alphabet contains 28 letters.

      i. How many words of two letters can be formed?
      ii. How many words of three letters can be formed?
      iii. How many words of four letters can be formed?
      iv. How many words of five letters can be formed?

   (These problems were investigated by al-Khalīl in the 8th century.)

2. Ibn Munim (d.1228) investigated colours of silk thread in Marrakech. He asked in how many ways can threads in up to 10 colours be combined to form a silk tassel.

   For example, consider tassels of 3 colours.

   - Choosing from three colours, there is one way to form such a tassel (this is the '1' in the table below).
   - If we choose from four colours, the fourth colour can be combined with two of the first three colours in 3 ways (again, shown in the table).
   - With five colours, the fifth colour could be combined with two of the first four, and so on.

   In this way he built up the following table. Complete it.

| Using this colour and earlier: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| For a tassel of 10 colours | – | – | – | – | – | – | – | – | – | |
| For a tassel of 9 colours | – | – | – | – | – | – | – | – | | |
| For a tassel of 8 colours | – | – | – | – | – | – | – | | | |
| For a tassel of 7 colours | – | – | – | – | – | – | | | | |
| For a tassel of 6 colours | – | – | – | – | – | | | | | |
| For a tassel of 5 colours | – | – | – | – | | | | | | |
| For a tassel of 4 colours | – | – | – | | | | | | | |
| For a tassel of 3 colours | – | – | 1 | 3 | | | | | | |
| For a tassel of 2 colours | – | | | | | | | | | |
| For a tassel of 1 colours | | | | | | | | | | |

3. For the permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 4 & 6 & 2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 4 & 6 & 2 \end{pmatrix}$$

   evaluate (a) $\sigma_2\sigma_1$; (b) $\sigma_1\sigma_2$; (c) $\sigma_1^{-1}$; (d) $\sigma_2^{-1}$; (e) the order of $\sigma_3$.

4. With three dice, if we do not care about the order that the numbers are thrown:
   (a) How many ways are there to throw these so each die shows a different number?
   (b) How many ways are there to throw these so two dice show the same number and the third is different?
   (c) Thus, how many essentially different throws are there of three dice?
   (d) What are the possible throws of two dice so that their sum is the number on the third die?

5. (a) Chen Houyao (1648–1722) asked: how many ways can we draw a hand of nine playing cards from a deck of thirty distinct cards?
   (b) Recall that

$$P(A) = \frac{\text{Number of ways } A \text{ could happen}}{\text{Total number of possible outcomes}}.$$

      i. From a standard pack of 52 playing cards, what is the probability of drawing at random any Queen?
      ii. There are five balls in a bag. Four are red and one is blue. A ball is drawn at random. What is the probability of drawing a red ball?
      iii. There are five balls in a bag. Four are red and one is blue. A ball is drawn at random. What is the probability of drawing a blue ball?
      iv. What is the probability that my birthday is today?
      v. What is the probability that it is not my birthday today?

6. (a) For a simple lottery, say that players must select six different numbers from 1–49. Six of these numbers will be drawn in the lottery and players win if they match all six balls.
      i. Calculate the probability of winning the jackpot in this simple lottery using appropriate binomial coefficients.
      ii. Give a general formula for lotteries with different numbers of balls.
   (b) Suppose now that we allow prizes for matching smaller numbers of balls. For example, suppose we offer a smaller prize for matching 5 balls from 6.
      i. Calculate the probability of matching 5 balls from 6 using binomial coefficients.
      ii. Give a general formula for matching a number of balls that is smaller than the number drawn.

7. A mancala board has a series of holes into which small stones are placed. The small mancala board pictured has six holes. In how many ways can we distribute 10 indistinguishable stones among these holes?