Linear systems

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1 Introduction

A general linear system of n equations in n unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}$$

can be written as a matrix in the form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Write this as $\mathbf{Ax} = \mathbf{b}$.

$\mathbf{A}$ is called the coefficient matrix.

Example

Write the following linear system in matrix notation.

$$\begin{aligned}
x + 2y &= 5 \\
3x - 5y &= 14
\end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

2 Inverse matrix method

For a system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

we use the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$ to get

$$\begin{aligned}
\mathbf{A}^{-1}\mathbf{A}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{I}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}
\end{aligned}$$

Example

Solve the following linear system.

$$\begin{aligned}
2x - y + 4z &= 12 \\
x + y + 2z &= 3 \\
-3x - z &= -10
\end{aligned}$$

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & 2 \\ -3 & 0 & -1 \end{bmatrix}$$

Taking the determinant down the middle column we get

$$\det(\mathbf{A}) = -(-1)\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} + 0 = 15 \neq 0.$$

Then

$$\operatorname{cof}(\mathbf{A}) = \begin{bmatrix} -1 & -5 & 3 \\ -1 & 10 & 3 \\ -6 & 0 & 3 \end{bmatrix},$$

so

$$\operatorname{Adj}(\mathbf{A}) = \begin{bmatrix} -1 & -1 & -6 \\ -5 & 10 & 0 \\ 3 & 3 & 3 \end{bmatrix}.$$

Now

$$\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{\det(\mathbf{A})}\operatorname{Adj}(\mathbf{A}) \\
&= \frac{1}{15}\begin{bmatrix} -1 & -1 & -6 \\ -5 & 10 & 0 \\ 3 & 3 & 3 \end{bmatrix}.
\end{aligned}$$

Therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 12 \\ 3 \\ -10 \end{bmatrix}$$
$$= \frac{1}{15} \begin{bmatrix} -1 & -1 & -6 \\ -5 & 10 & 0 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ 3 \\ -10 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

The answer is that $x = 3$, $y = -2$ and $z = 1$.

3 Cramer's Rule

A *Cramer system* is any system of n linear equations in n unknowns if and only if the matrix formed by the coefficients is non-singular.

Cramer's Rule (also known as Method of Determinants) makes use of determinants to solve such a non-singular square system.

For a system

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix}$

let $\mathbf{A}_i$ be the matrix obtained by replacing the entries of the ith column of $\mathbf{A}$ by the answer vector.

Then

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}.$$

Note that for $\mathbf{Ax} = \mathbf{b}$, if $\det(\mathbf{A}) = 0$ then Cramer's Rule will not work.