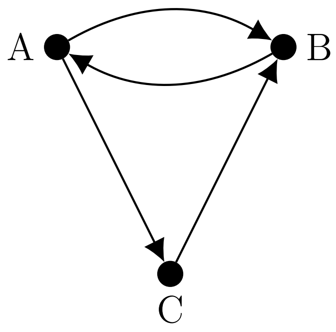
Basic PageRank algorithm

Peter Rowlett

1 Scenario



This information is represented in the following table.

		From		
		A	B	C
To	A	0	1	0
	B	$\frac{1}{2}$	0	1
	C	$\frac{1}{2}$	0	0

Now as a matrix.

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

2 Basic approach

Imagine everyone starts with $\frac{1}{3}$ of the available vote, and then iterate until the share of vote stabilises.

$$\text{step 1:} \quad \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix}$$

$$\text{step 2:} \quad \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$

$$\text{step 3:} \quad \vdots$$

We are looking for a situation where

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

i.e. where the values of $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ stabilise.

3 Enter eigenvectors

Notice that this corresponds precisely to $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ being an eigenvector for eigenvalue $\lambda = 1$.

Notes:

- Since A, B and C in this vector are proportions of the available vote, we must have $A + B + C = 1$.
- A matrix where all columns sum to 1 always has an eigenvalue $\lambda = 1$ – though it doesn't hurt to check; if you find yours doesn't have $\lambda = 1$ then maybe you've made a mistake forming the matrix?

For $\lambda = 1$, let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then we want non-zero x, y and z for

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} - \lambda \mathbf{I} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the first row: $-x+y=0$, i.e. $y=x$.
From the third row: $\frac{1}{2}x - z = 0$, i.e. $z = \frac{1}{2}x$.
The ratio of $x:y:z$ is therefore $2:2:1$ so choose

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}.$$

We choose $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ to be an eigenvector such that $A+B+C=1$ since A, B and C are shares of the available vote.

So people A and B draw with $\frac{2}{5}$ths of the available vote.