Worksheet 7: Inclusion-exclusion and generators and enumerators – solutions

1. (a) 7;

   (b) 68.

   (The lockers that are open at the end of this process are those labelled with square numbers.)

2. (a) There are 2^9 strings that are 1 followed by nine other bits. There are 2^8 strings that are eight bits followed by 00. There are 2^7 strings that are 1 followed by seven bits then 00. The total is $2^9 + 2^8 - 2^7 = 640$.

   (b) There are 3333 numbers that are divisible by 3, 2000 that are divisible by 5, and 666 that are divisible by 15. The total is $3333 + 2000 - 666 = 4667$.

3. (a) $(1 + x + x^2 + x^3 + x^4 + x^5)^3$;

   (b) $(x + x^2 + x^3 + x^4 + x^5)^3$;

   (c) $(x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4 + x^5)^2$.

4. (a) Let d represent the 10¢ (dime) and q represent the 25¢ (quarter). Then $(1 + d + d^2 + d^3 + d^4 + d^5)(1 + q + q^2)$.

   (b) $(1 + d + d^2 + d^3 + d^4 + d^5)(1 + q + q^2) = d^5q^2 + d^5q + d^5 + d^4q^2 + d^4q + d^4 + d^3q^2 + d^3q + d^3 + d^2q^2 + d^2q + d^2 + dq^2 + dq + d + q^2 + q + 1$. The two of these that total 50¢ are q^2 and d^5.

   (c) Let c represent the 1¢ (cent) and n represent the 5¢ (nikel). Then the number of ways is represented by

   $$(1 + c + c^2 + \cdots + c^{50})(1 + n + n^2 + \cdots + n^{10})(1 + d + d^2 + d^3 + d^4 + d^5)(1 + q + q^2).$$

5. (a) Given that a corresponds to tile A, and so on:
   $(1 + a)(1 + b + b^2 + b^3)(1 + c)(1 + d + d^2 + d^3 + d^4)$.

   (b) There are 12 order-3 terms in the expansion: ab^2, abc, abd, acd, ad^2, b^3, b^2c, b^2d, bcd, bd^2, cd^2, d^3.

   (c) These correspond to ABB, ABC, ABD, ACD, ADD, BBB, BBC, BBD, BCD, BDD, CDD, DDD.

   (d) The order-3 terms in the expansion can be rearranged as follows.

selection	arrangements
ab^2	$\frac{3!}{2!} = 3$
abc	$3! = 6$
abd	$3! = 6$
acd	$3! = 6$
ad^2	$\frac{3!}{2!} = 3$
b^3	1
b^2c	$\frac{3!}{2!} = 3$
b^2d	$\frac{3!}{2!} = 3$
bcd	$3! = 6$
bd^2	$\frac{3!}{2!} = 3$
cd^2	$\frac{3!}{2!} = 3$
d^3	1

The sum of the last column is 44, so there are 44 different ways of drawing three tiles from the set.

6. (a) $(x^6 + x^7 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x + x^2 + x^3 + \dots)^2$.

   (b) 50.