Some graph theory proofs

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1 Vertex degree

Theorem 1.1. *Every simple graph with $n > 1$ vertices has two vertices of the same degree.*

To prove this, we use an important mathematical result called the pigeonhole principle. Since we are going to prove this on the way to proving the theorem above, here it is labelled a lemma.

Lemma 1.2. *If we have $n \in \mathbb{N}$ elements and wish to distribute these among $m \in \mathbb{N}$ sets with $n > m$, then one of the m sets must contain more than one element.*

This is sometimes demonstrated with an example where you have, say, 10 letters and 9 pigeonholes. If you distribute the 10 letters into the 9 pigeonholes, there must be one pigeonhole that contains more than one letter. A more formal formulation would be that if you have sets A, B and $|A| > |B|$ then there is no injective function from $A \to B$. We'll stick with putting elements in sets as above.

Proof. We will prove this by induction on the number of sets.

Base case: Consider 2 elements being distributed in 1 set. We thus have a set containing more than one element.

Inductive step: Assume that if we have j elements to distribute in $k-1$ sets, with $j > k-1$, then one of the $k-1$ sets contains more than one element.

Now consider k sets and $l > k$ elements. There are three cases for what is in the first set:

1. More than one element. Then one of the sets (the first) contains more than one element.
2. One element. Now there are $l-1 > k-1$ elements to fit in the remaining $k-1$ sets. By the inductive hypothesis, one of the $k-1$ sets contains more than one element.
3. Zero elements. Now there are $l > k-1$ elements to fit in the remaining $k-1$ sets. Again, by the inductive hypothesis, one of these contains more than one element.

In each case, we see that there is a set containing more than one element, as required. $\square$

Now we are ready to address Theorem 1.1.

Proof. Let G be an arbitrary simple graph with $n > 1$ vertices. Either

1. G contains no connected component. Then every vertex has degree 0, and therefore we have at least two vertices of the same degree.
2. G contains at least one connected component (which may or may be the whole graph) with n edges. Then any given vertex v can connect to at most $n-1$ vertices. If we assign vertices to sets according to their degree (so all the vertices of degree 1 go in one set, all the vertices of degree 2 go in another, and so on). Now we have $n-1$ sets into which we wish to assign n vertices. By the pigeonhole principle, there must be a set with at least two vertices assigned, and hence at least two vertices with the same degree.

Either way, we have two vertices of the same degree, as required. $\square$