

Proof methods – exercises

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1. In logic, we talk about *necessary* conditions and *sufficient* conditions. If p is a necessary condition for q it does not mean that p on its own is enough to guarantee q . Rather, it means p will have to be true if there is to be any question of q being true – we need p for q . It means $q \implies p$.

We also talk about *sufficient* conditions. If p is a sufficient condition for q it means that p being true is enough to say q is true, though it is possible q is true without p being true. It means $p \implies q$.

If p is necessary and sufficient for q it means $p \iff q$.

Which of the following conditions is *necessary* for $n \in \mathbb{N}$ to be divisible by 6? Which conditions are *sufficient* for n to be divisible by 6?

- (a) n is divisible by 3;
 - (b) n is divisible by 9;
 - (c) n is divisible by 12;
 - (d) $n = 24$;
 - (e) n^2 is divisible by 3;
 - (f) n is even and divisible by 3.
2. Choose any five consecutive positive whole numbers, and multiply them together. Did you get a multiple of 120? Will you always get a multiple of 120? How would you convince someone?
 3. Prove or disprove the following.
 - (a) Let m be an integer. If m is odd, then m^2 is odd.
 - (b) Suppose that $p \in \mathbb{Q}$ and $p^2 \in \mathbb{Z}$. Then, $p \in \mathbb{Z}$.
 - (c) Let m and n be real numbers. If $n > m > 0$, then

$$\frac{m+1}{n+1} > \frac{m}{n}.$$

- (d) Let A , B and C be sets. Then,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

- (e) Let A and B be sets. Then, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
4. Show that the sum of two consecutive odd numbers is a multiple of 4. What is the converse and is it true?

5. Let $f : X \rightarrow Y$. Suppose A and B are subsets of X . Show $f(A \cup B) = f(A) \cup f(B)$.

6. Prove or disprove the following.

(a) $a + b = c \implies a^2 + b^2 = c^2$.

(b) Let $x \in \mathbb{Z}$. Then $-x$ is negative.

7. The following is a proof that $1 = 2$. You may have reason to doubt that this is a true fact! Where is the error?

Theorem. $1 = 2$.

Proof. Let $a = b$, where $a, b \in \mathbb{Z}$. Then,

| | | | |
|------------|-----|------------------|---------------------------------------|
| ab | $=$ | a^2 | since $a = b$, |
| $ab - b^2$ | $=$ | $a^2 - b^2$ | by subtracting b^2 from both sides, |
| $b(a - b)$ | $=$ | $(a + b)(a - b)$ | by factoring, |
| b | $=$ | $a + b$ | by dividing both sides by $a - b$, |
| b | $=$ | $2b$ | since $a = b$, |
| 1 | $=$ | 2 | by dividing by b . |

□

8. Prove or disprove the following.

(a) Suppose $n \in \mathbb{N}$. Then $n^3 - n$ is a multiple of 3.

(b) Suppose $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.

(c) The square of any integer is of the form $3k$ or $3k + 1$ for some $k \in \mathbb{Z}$.

(d) Suppose $a = bc$ for $a, b, c \in \mathbb{R}$. If two of a , b or c are non-zero, then so is the third.

9. Prove or disprove the following.

(a) There are no positive integers x and y such that $x^2 - y^2 = 1$.

(b) The sum of a rational and an irrational number is an irrational number.

(c) $\sqrt{3}$ is irrational.

(d) $\sqrt{4}$ is irrational. (What happens if you try the same approach as for $\sqrt{3}$?)

(e) There are no positive integer solutions to $x^2 + x + 1 = y^2$.

(f) Suppose $x, y \in \mathbb{Z}$. Then $\sqrt{x^2 + y^2} \neq x + y$.