Nim – how to find which pile to change

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How to win at Multi-pile Nim

If the Nim sum is not zero, the next player can win. First, they remove some sticks to make the Nim sum zero. Then their opponent is forced to make the Nim sum non-zero again. They return it to zero, and continue doing this until they take the last stick.

Let's see this in action with a simple game:

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We convert the heap sizes to binary and apply $\oplus$.

3	0	1	1
2	0	1	0
4	1	0	0
	1	0	1

The Nim sum of this game is not zero, so it is an $\mathcal{N}$ position. To make the Nim sum zero, we can change the size of the heap of four sticks to one. Doing so, we obtain

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3	0	1	1
2	0	1	0
1	0	0	1
	0	0	0

If you consider this game, you may notice that whoever plays on this game cannot win. Whatever your opponent does, you can continue to return the Nim sum to zero until you take the last stick.

Nim theorem

The routine outlined above can be formalised.

Theorem. *A game of* ***Multi-pile Nim*** *with heaps* $*x_1, *x_2, \ldots, *x_n$ *is a* $\mathcal{P}$ *position if and only if the Nim sum of the sizes of the heaps is zero. That is,*

$$x_1 \oplus x_2 \oplus \ldots \oplus x_n = 0.$$

Otherwise, it is an $\mathcal{N}$ *position.*

Proof. We aim to show that,

- if the current Nim sum is zero, the game is in a $\mathcal{P}$ position (the next player cannot win, assuming perfect play);

- if the current Nim sum is not zero, the game is in an $\mathcal{N}$ position (the next player can force a win).

Making a move in this game means reducing the size of one of the heaps. Say the move changes $*x_k$ to $*x'_k$, where $x'_k \in \{0, 1, \ldots, x_{k-1}\}$. Note that $x_k \neq x'_k$, so $x_k \oplus x'_k \neq 0$.

Call the Nim sum of our game in its current position $X = x_1 \oplus \ldots \oplus x_k \oplus \ldots \oplus x_n$, and the Nim sum after the move has been made $X' = x_1 \oplus \ldots \oplus x'_k \oplus \ldots \oplus x_n$.

Taking $X \oplus X'$, we can pair each heap before and after the move.

$$\begin{aligned} X \oplus X' &= (x_1 \oplus \ldots \oplus x_k \oplus \ldots \oplus x_n) \oplus (x_1 \oplus \ldots \oplus x'_k \oplus \ldots \oplus x_n) \\ &= (x_1 \oplus x_1) \oplus \ldots \oplus (x_k \oplus x'_k) \oplus \ldots \oplus (x_n \oplus x_n). \end{aligned}$$

Since $a \oplus a = 0$ and most of the heaps have not changed size, this reduces to give

$$X \oplus X' = x_k \oplus x'_k.$$

The Nim sum of the whole game before and after a move is the same as the Nim sum of the affected heap before and after the move. We can rearrange this to find the Nim sum of the game after the move is made, X'.

$$\begin{aligned} X \oplus X' &= x_k \oplus x'_k \\ X \oplus X \oplus X' &= X \oplus (x_k \oplus x'_k) \\ X' &= X \oplus (x_k \oplus x'_k). \end{aligned}$$

So the Nim sum after the move can be found from the Nim sum of the change made and the Nim sum of the whole game before the move is made.

We can immediately say that if $X = 0$, then $X' \neq 0$ since $x_k \oplus x'_k \neq 0$. So if the Nim sum starts at zero, it cannot return to zero after a single move. Since the end game with zero sticks remaining in any pile has Nim sum zero, this is enough to show that if you make the Nim sum zero, you can prevent your opponent from winning – they make the position non-zero, then you make it zero, and repeat until you take the last stick. Thus any position with $X = 0$ is a $\mathcal{P}$ position.

Now consider the case where $X \neq 0$.

Consider X in binary. If $X \neq 0$, then there must be at least one non-zero bit in X. Say the most significant non-zero bit occurs in the dth place.

Now there must exist a heap with a 1 in the dth place also. Choose this as x_k, the heap we will change with our move. Then choose to change this to $x'_k = x_k \oplus X$. This means

$$\begin{aligned} X' &= X \oplus (x_k \oplus x'_k) \\ &= X \oplus (x_k \oplus x_k \oplus X) \\ &= (X \oplus X) \oplus (x_k \oplus x_k) \\ &= 0. \end{aligned}$$

So if the current Nim sum is not zero, it is always possible to make it zero with one move by careful choice of which heap to change and what to change it to, and we have seen that the player playing from a zero position cannot win. This means if $X \neq 0$ the game is in a $\mathcal{N}$ position. □

How to win at Multi-pile Nim – revisited

This proof is constructive, in that it tells you what to do. In section , there was a point where we chose to change the heap of four sticks to a heap of one stick. How did we know this was the right move to make? We did this because the Nim sum of the game was five and $4 \oplus 5 = 1$.

To play **Multi-pile Nim** calculate X, the Nim sum of the current position. Then

- if $X = 0$, you have lost[1];

[1] In a real game, you could play *give them enough rope* and see if your opponent makes a mistake.

- if $X \neq 0$:
  - identify a heap x_k which has a 1 in the same position as the most significant 1 in X;
  - remove sticks to change this heap's size to $x_k \oplus X$.

Example

Consider this Nim game:

Convert the heap sizes to binary and apply $\oplus$.

9	1	0	0	1
2	0	0	1	0
14	1	1	1	0
	0	1	0	1

So the Nim sum is $9 \oplus 2 \oplus 14 = 5$. Since $5 \neq 0$, the game is in a $\mathcal{N}$ position and we can win. Notice that 5 (0101) has its most significant non-zero bit in the 4s place. The heap $*14$ also has a 1 in this place. So we choose to alter $*14$. To find how many to change it to, we calculate $14 \oplus 5$.

14	1	1	1	0
5	0	1	0	1
	1	0	1	1

So $14 \oplus 5 = 11$ and a winning move changes $*14$ to $*11$ by removing three sticks.
To check this, we calculate the Nim sum after the move.

9	1	0	0	1
2	0	0	1	0
11	1	0	1	1
	0	0	0	0

The Nim sum is zero, therefore we hand the game to our opponent in a $\mathcal{P}$ position.