Labelling and colouring

Peter Rowlett

Sheffield Hallam University

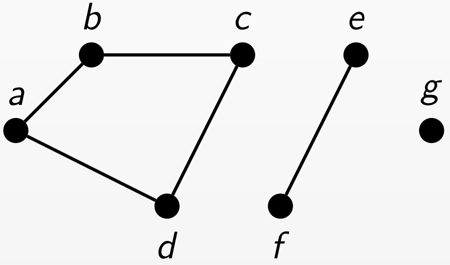
p.rowlett@shu.ac.uk

More definitions

Reachability

- A vertex b is *reachable* from vertex a if there is a path from a to b.
- Any vertex is always reachable from itself by a path of length 0.
- It is not necessary for a vertex in a graph to have any edges connecting it to other vertices, or for all parts of a graph to be reachable from all others.

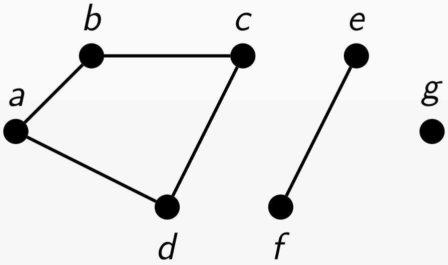
- For example, here
  - a is only reachable from b, c and d;
  - e is only reachable from f;
  - g is not reachable from any other vertex.



Components

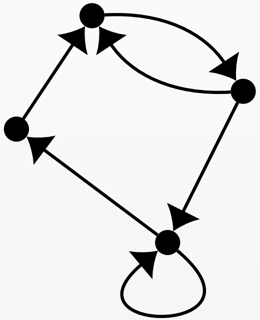
- A subgraph of a graph $G = (V, E)$ is a subset of the vertices and the edges which connect them.
- A component of a graph is a connected subgraph that is not part of any larger connected subgraph.

- For example, this graph has three components.



Directed graph (digraph)

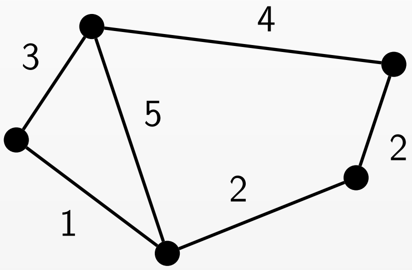
- A *directed graph* (or *digraph*) is like the graphs we have seen except that the edges have a direction, i.e. the set E is made of *ordered* pairs of vertices $\{(a, b) \mid a, b \in V\}$.
- For example,



Labelling

Edge-labelling

- We have seen graphs with the node labelled, for example with letters, which is a *vertex-labelling*.
- We can also label the edges, an *edge-labelling*.
- If the edges are labelled with positive numbers, we can call this a *weighted graph*.



Labelling

- Consider a graph $G = (V, E)$.
- Suppose we have a set, L, of *labels*.
- A *vertex labelling* of a graph is a function $V \to L$.
- An *edge labelling* of a graph is a function $E \to L$.

Magic squares

8	1	6
3	5	7
4	9	2

Magic squares

$$\begin{array}{c|c|c|l}
 & & & = 15 \\
8 & 1 & 6 & = 15 \\ \hline
3 & 5 & 7 & = 15 \\ \hline
4 & 9 & 2 & = 15 \\
\parallel & \parallel & \parallel & = 15 \\
15 & 15 & 15 &
\end{array}$$

Magic squares

- A *magic square* is a square grid of numbers such that all columns, rows and diagonals sum to the same amount, which is called the *magic constant*.
- A *semi-magic square* has this property for rows and columns, but not necessarily diagonals.

Lo Shu magic square

- This is a famous magic square, called the Lo Shu magic square.

8	1	6
3	5	7
4	9	2

What is this magic square doing in the graph theory section?

8	1	6
3	5	7
4	9	2

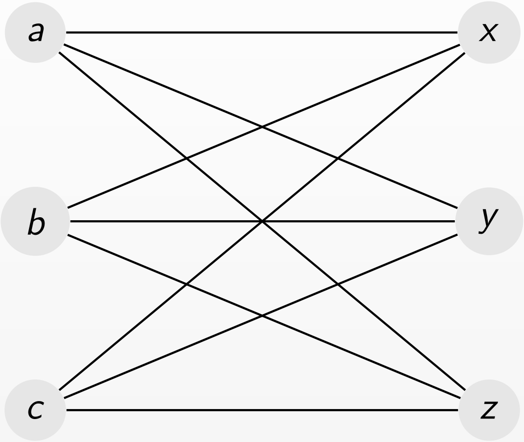
Label each row and column with a letter

8	1	6	a
3	5	7	b
4	9	2	c
x	y	z	

Now draw a graph

- A complete bipartite graph.

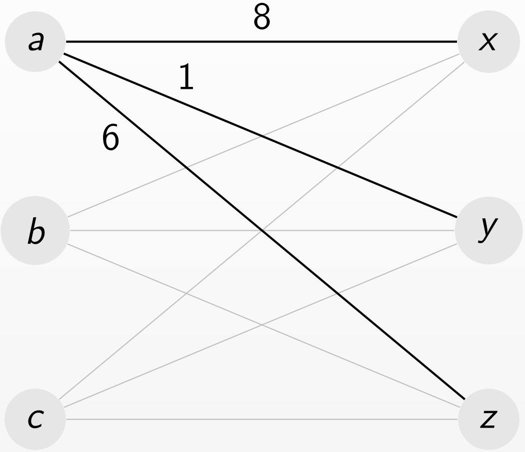
8	1	6	a
3	5	7	b
4	9	2	c
x	y	z	



Label the graph

- Label edges with the value where that row and column intersect.

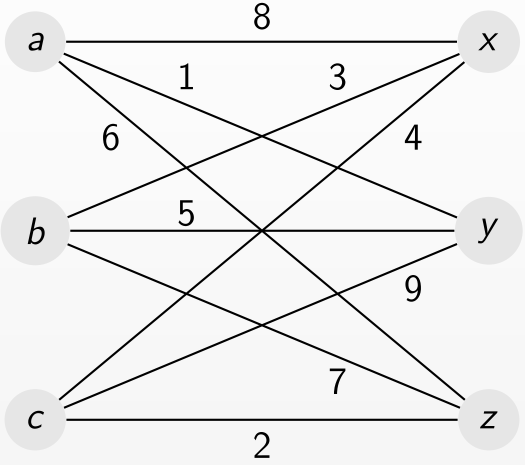
8	1	6	a
3	5	7	b
4	9	2	c
x	y	z	



Label the graph

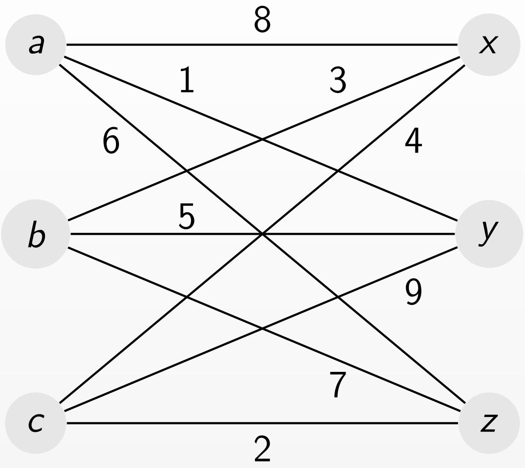
- Label edges with the value where that row and column intersect.

8	1	6	a
3	5	7	b
4	9	2	c
x	y	z	



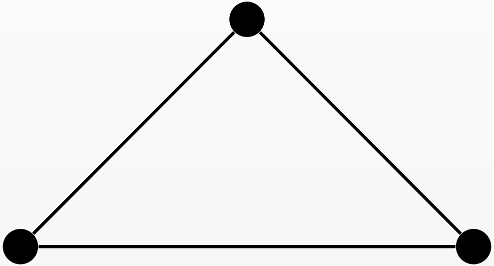
Vertex-magic graph

- At each vertex, if you sum the values of the labels of its edges, you get the same number, 15.
- This shouldn't be a surprise, because this is the same property as the magic square.
- Note we didn't use the diagonals, so we only require a semi-magic square.



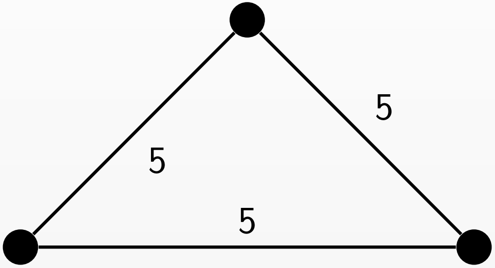
Vertex-magic labelling of other graphs

- Label this graph so it has magic constant 10.



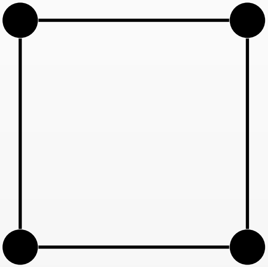
Vertex-magic labelling of other graphs

- Label this graph so it has magic constant 10.



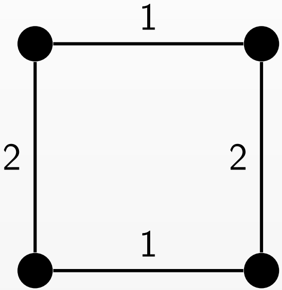
Vertex-magic labelling of other graphs

- Label this graph so it has magic constant 3.



Vertex-magic labelling of other graphs

- Label this graph so it has magic constant 3.



Magic vertex labelling

- We have seen a *vertex-magic labelling* is an edge labelling with integers such that the sum of the labels at any vertex is the same constant.
- An *edge-magic labelling* is a vertex labelling with integers such that the sum of the labels connected by any edge is constant.
- A *total-magic labelling* is one that has labels on both edges and vertices so that the sum of the labels on any vertex and its connected edges sums to the same value.
- A *Zero-sum magic graph* is a graph where the edges can be labelled such that the vertex-magic constant is 0.

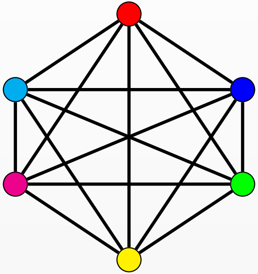
Colouring

Graph colouring

- Let C be a set of labels called *colours*.
- A *colouring* of a graph is a vertex labelling $V \to C$, such that no two adjacent vertices have the same colour.
- A graph is *k-colourable* if we can use k colours to give a colouring.
- A graph G that is k-colourable, but not $(k-1)$-colourable, is called *k-chromatic* and we say that it has *chromatic number* $\chi(G) = k$.
- The colours needn't be actual colours, but they can be. What's important is that labels are assigned to vertices so that no adjacent vertices have the same label.

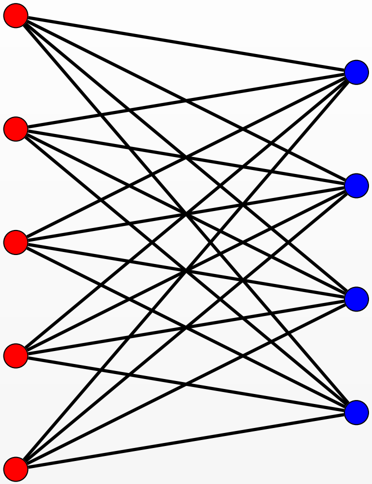
Complete graphs

- Complete graphs need as many colours as they have vertices. So $\chi(K_n) = n$.



Bipartite graphs

- A **bipartite graph** G is 2-colourable, i.e., $\chi(G) = 2$.
- e.g. the graph $K_{5,4}$.



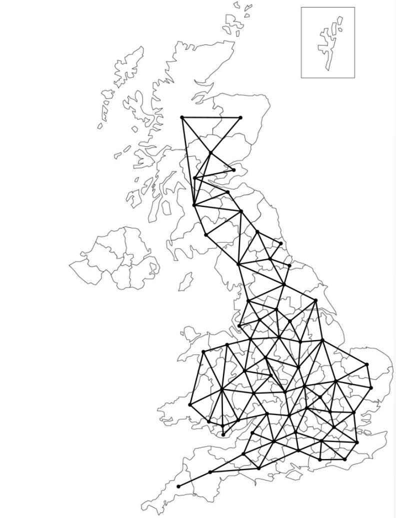
Four Colour Theorem

- No more than four colours are required to colour the regions of any map so that no two adjacent regions have the same colour.

Maps

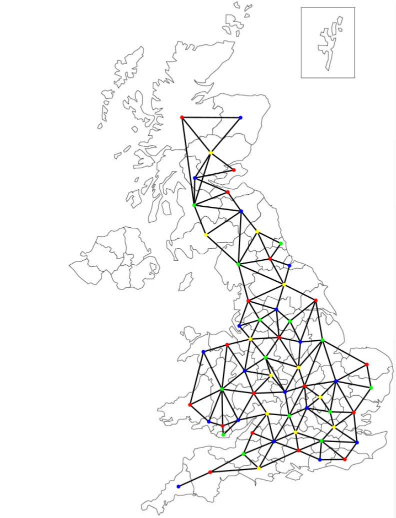
Credit: Alex Corner.

Maps



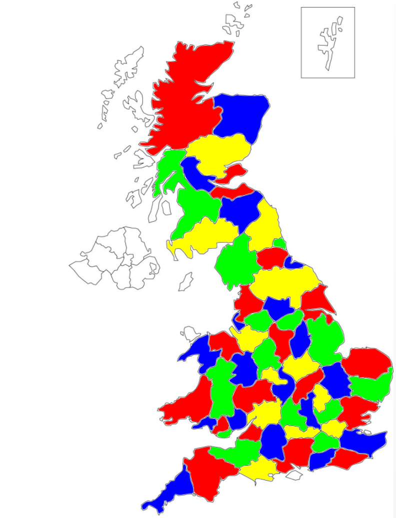
Credit: Alex Corner.

Maps



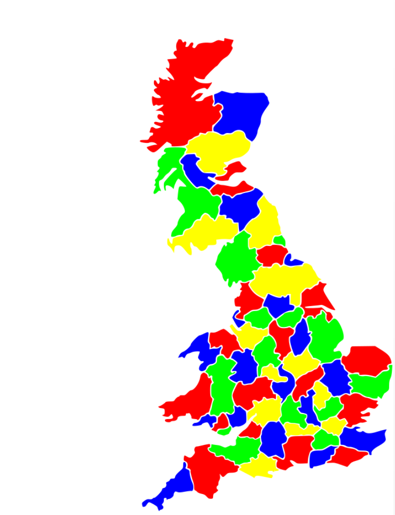
Credit: Alex Corner.

Maps



Credit: Alex Corner.

Maps



Credit: Alex Corner.

Four Colour Theorem

- Earlier we saw graphs that require > 4 colours, for example K_n for $n > 4$.
- The Four Colour Theorem only applies to loopless planar graphs.