Proof methods – exercises

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1. Choose any five consecutive positive whole numbers, and multiply them together. Did you get a multiple of 120? Will you always get a multiple of 120? How would you convince someone?

2. Prove or disprove the following.

   (a) Let m be an integer. If m is odd, then m^2 is odd.

   (b) Suppose that $p \in \mathbb{Q}$ and $p^2 \in \mathbb{Z}$. Then, $p \in \mathbb{Z}$.

   (c) Let m and n be real numbers. If $n > m > 0$, then
   $$\frac{m+1}{n+1} > \frac{m}{n}.$$

   (d) Let A, B and C be sets. Then,
   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

   (e) Let A and B be sets. Then, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

3. Show that the sum of two consecutive odd numbers is a multiple of 4. What is the converse and is it true?

4. Let $f : X \to Y$. Suppose A and B are subsets of X. Show $f(A \cup B) = f(A) \cup f(B)$.

5. Prove or disprove the following.

   (a) $a + b = c \implies a^2 + b^2 = c^2$.

   (b) Let $x \in \mathbb{Z}$. Then $-x$ is negative.

6. The following is a proof that $1 = 2$. You may have reason to doubt that this is a true fact! Where is the error?

   Theorem. $1 = 2$.

   Proof. Let $a = b$, where $a, b \in \mathbb{Z}$. Then,

$$\begin{array}{rcll} ab & = & a^2 & \text{since } a = b, \\ ab - b^2 & = & a^2 - b^2 & \text{by subtracting } b^2 \text{ from both sides,} \\ b(a-b) & = & (a+b)(a-b) & \text{by factoring,} \\ b & = & a+b & \text{by dividing both sides by } a - b, \\ b & = & 2b & \text{since } a = b, \\ 1 & = & 2 & \text{by dividing by } b. \end{array}$$

   $\square$

7. Prove or disprove the following.

   (a) Suppose $n \in \mathbb{N}$. Then $n^3 - n$ is a multiple of 3.

   (b) Suppose $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.

   (c) The square of any integer is of the form $3k$ or $3k + 1$ for some $k \in \mathbb{Z}$.

   (d) Suppose $a = bc$ for $a, b, c \in \mathbb{R}$. If two of a, b or c are non-zero, then so is the third.

8. Prove or disprove the following.

   (a) There are no positive integers x and y such that $x^2 - y^2 = 1$.

   (b) The sum of a rational and an irrational number is an irrational number.

   (c) $\sqrt{3}$ is irrational.

   (d) $\sqrt{4}$ is irrational. (What happens is you try the same approach as for $\sqrt{3}$?)

   (e) There are no positive integer solutions to $x^2 + x + 1 = y^2$.

   (f) Suppose $x, y \in \mathbb{Z}$. Then $\sqrt{x^2 + y^2} \neq x + y$.