# Worksheet 7: Inclusion-exclusion and generators and enumerators

1. (a) Ten lockers are numbered 1–10 and are all closed. The first person to pass the lockers opens all lockers. The second person closes even numbered lockers. The third person changes the state of every third locker. This continues, with the $n$th person changing the state of lockers labelled with multiples of $n$. At the end of this process, how many lockers are now open?
   (b) What if there were one hundred lockers?
2. Use an inclusion-exclusion argument to answer these questions:
   (a) How many binary strings of length 10 (e.g. 1011010010, 0000000001, 0101010101, ...) either start with a '1' or end with '00'?
   (b) How many numbers from 1 to 10,000 (inclusive) are divisible by 3 or 5?
3. Give a function which counts the number of ways to distribute five tokens among three children if
   (a) there are no restrictions;
   (b) each child gets at least one token;
   (c) the oldest child gets at least two tokens.
4. A classic puzzle asks how many ways there are to make 50¢ from US coins.
   (a) Give a function which counts the number of ways to combine up to five 10¢ coins and up to two 25¢ coins.
   (b) Use this to count the number of ways to make 50¢ from 10¢ and 25¢ coins.
   (c) Give a function which could be used to count the number of ways to make 50¢ from 1¢, 5¢, 10¢ and 25¢ coins.
5. A word game is played with four letters $\{A, B, C, D\}$. These letters are written on tiles and occur in the following frequencies: $(A, 1)$, $(B, 3)$, $(C, 1)$, $(D, 4)$. Each player is given three tiles at the start of the game.
   (a) Give a function which generates the number of ways that tiles can be selected.
   (b) In how many ways can the first set of three tiles be chosen, given that we don't care about the order in which these are selected?
   (c) What sets of three tiles can be chosen in this way?
   (d) If we cared in what order the tiles were drawn from the bag, how many ways would there be of drawing three starting tiles?
6. A shop sells coloured sweets in red, purple, yellow and orange.
   (a) Give a generating function that counts the number of ways to make bags of sweets with at least six red ones and an even number of yellow ones.
   (b) How many ways can Jennie select twelve sweets so that she has at least six red ones and an even number of yellow ones?

**Note** — You may find technology useful for expanding some of the larger brackets (e.g. Python's SymPy via e.g. `expand((1+a+a**2)*(b+1))`, or WolframAlpha).