Worksheet 6: Recurrence relations

1. (a) We can denote poetry rhyme schemes with a sequence of letters, e.g. AABBA. Each represents a line, such that lines with the same letters rhyme with each other. AABBA is the rhyming scheme of a Limerick.

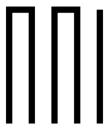
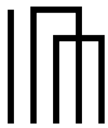
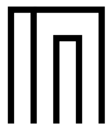
   A two-line poem has two rhyme schemes. It either rhymes or it doesn't, so AA or AB.

   How many rhyming schemes are there for a three-line poem?

   (b) The following diagrams represent different rhyme schemes for 5-line poems. Moving from left to right, each stick represents a line from the poem. Sticks are joined with a horizontal bar if they rhyme.

   Can you work out which diagram matches which of the following rhyme schemes?

   i. AABBC
   ii. AABBA
   iii. ABCBC

   1.  2.  3. 

   These diagrams are from *The Tale of Genji*, an 11th century novel by Japanese writer Murasaki Shikibu. Actually she wasn't writing about poetry, but about an incense game played in the Japanese court. Players were given five incense sticks and challenged to match the scents. But the maths is the same as for poetry rhyming schemes!

2. In 12th century India, Sanskrit poems were written using words with short and long syllables, which can be thought of as one- and two-syllable words, and poets considered the number of possible ways to write such poems.

   A line of a poem three syllables long can be made in three different ways:

   - one-syllable word, one-syllable word, one-syllable word (1-1-1)
   - one-syllable word, two-syllable word (1-2)
   - two-syllable word, one-syllable word (2-1)

   (a) If I want a line with four syllables, how many different ways can I combine one- and two-syllable words to make it?

   (b) How about a line with five syllables?

   (c) Is there a pattern in the number of ways to make lines of each length?

3. A spy agency makes codes that are 8 characters long using only the letters `A`, `B` and `C`. How many codes are there that do not contain "`AB`" (in that order)? For example, the code below has "`AB`" in the middle, so shouldn't be counted.

   AAAABCAC

4. Give a recursive definition for the following integer sequences $a_1, a_2, \ldots$ where for all $k \in \mathbb{Z}^+$ we have

   (a) $a_n = 5^n$;

   (b) $a_n = 3$;

   (c) $a_n = 5n$;

   (d) $a_n = 4n + 5$;

   (e) $a_n = n^2$.

5. Verify by forming and solving a characteristic equation, the result that a Tower of Hanoi puzzle with n discs can be solved in $T_n = 2^n - 1$ moves, given that the recurrence relation is $T_n = 2T_{n-1} + 1$.

6. Solve the following recurrence relations:

   (a) $S_1 = 4, \quad S_n = \frac{1}{2}S_{n-1}$;

   (b) $S_1 = 3, \quad S_2 = 7, \quad S_n = 2S_{n-1} + 3S_{n-2}$;

   (c) $S_0 = 1, \quad S_1 = 3, \quad S_n = 5S_{n-1} + 6S_{n-2}$;

   (d) $S_1 = 2, \quad S_2 = 5, \quad S_n = 6S_{n-1} - 9S_{n-2}$.

7. Solve the following recurrence relations:

   (a) $S_0 = 2, \quad S_1 = -8, \quad S_n = \frac{11}{2}S_{n-1} - \frac{5}{2}S_{n-2}$;

   (b) $S_0 = 0, \quad S_1 = 3, \quad S_n = -S_{n-2}$;

   (c) $S_0 = 0, \quad S_1 = 2, \quad S_2 = 9, \quad S_n = 3S_{n-1} + 4S_{n-2} + 3$;

   (d) $S_0 = 1, \quad S_1 = 6, \quad S_n = S_{n-1} + 5$.