Proof exploration

Peter Rowlett

Investigate the following theorems. What are they saying? Can you rewrite them in your own words? Try out some special cases. What might a proof look like?

1. Suppose that $m, n \in \mathbb{N}$. Then mn is even if and only if m and n are even.

2. An integer n is divisible by 9 if and only if the sum of its digits is equivalent to $0 \pmod 9$.

3. If A and B are finite sets, then $|A \times B| = |A||B|$.

4. A train goes 500 miles along a straight track, without stopping, completing the trip with an average speed of exactly 50 miles per hour. It travels, however, at different speeds along the way. There must be a segment of 50 miles that the train traverses in precisely one hour.

5. Let $n \in \mathbb{N}$ with $n \geq 3$. For n distinct points on a circle connect consecutive points by a straight line. The sum of the interior angles of the resulting shape is $(n-2) \times 180°$.

6. If $n \in \mathbb{N}$ and $n \geq 7$, then
$$\frac{n}{n^2 - 8n + 12} \geq \frac{1}{n}.$$

7. Let A be a finite set. Let S be the set of all subsets of A. Then $|S| = 2^{|A|}$.
(Note: The set S is called the *power set* of A.)

8. For sets A, B and C we have

   (a) $A - (B \cup C) = (A - B) \cap (A - C)$;
   (b) $A - (B \cap C) = (A - B) \cup (A - C)$;
   (c) $A \neq B$ if and only if $(A - B) \cup (A - C)$;
   (d) $A \cup B \subseteq C$ if and only if $A \subseteq C$ and $B \subseteq C$.

   What happens in the extreme case(s) where some (or all) sets are empty?