Linear systems

Peter Rowlett

1 Introduction

Many relationships are linear, and we use *linear equations* to describe such relationships, e.g. a linear equation of the variables (or 'unknowns') $x_1, x_2, \ldots x_n$ has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, a_2, \ldots a_n$ and b are constants.

A *linear system* is a collection of one or more linear equations for which we seek solutions that satisfy all the equations simultaneously.

1.1 Gaussian Elimination on equations

Example

Find the solutions of the following linear system of two equations in two unknowns:

$$x + 2y = 5 \tag{1}$$
$$3x + 5y = 14. \tag{2}$$

First multiply both sides of equation (1) by 3:

$$3x + 6y = 15. \tag{3}$$

Now take each side of (3) from the corresponding side of (2):

$$y = 1.$$

From (1) we now know

$$\begin{aligned} x + 2 &= 5 \\ x &= 3 \end{aligned}$$

So the solution is $(x, y) = (3, 1)$.

Example

Find the solutions of the following linear system of five equations in five unknowns:

$$5x - 4y + 7z + w - 3v = 3 \tag{1}$$
$$x + 10y - z + 2w + 4v = 9 \tag{2}$$
$$-3x - 5y + 8z - 2w - 3v = 0 \tag{3}$$
$$x - 7y + z - w - v = 8 \tag{4}$$
$$2x + 3y - z - 4w + 2v = 2 \tag{5}$$

Just joking! Could you do it? Would you want to?
What if it was 20 equations in 20 unknowns?
What if it was 1000 equations in 1000 unknowns?

2 Matrix notation

A general linear system of n equations in n unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}$$

can be written as a matrix in the form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Write this as $\mathbf{Ax} = \mathbf{b}$.

$\mathbf{A}$ is called the coefficient matrix.

Example

Write the following linear system in matrix notation.

$$\begin{aligned} x + 2y &= 5 \\ 3x - 5y &= 14 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

3 Inverse matrix method

For a system of equations

$$\mathbf{Ax} = \mathbf{b},$$

we use the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$ to get

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{Ix} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

Example

Solve the following linear system.

$$\begin{aligned} 2x - y + 4z &= 12 \\ x + y + 2z &= 3 \\ -3x - z &= -10 \end{aligned}$$

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & 2 \\ -3 & 0 & -1 \end{bmatrix}$$

Taking the determinant down the middle column we get

$$\det(\mathbf{A}) = -(-1)\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} + 0 = 15 \neq 0.$$

Then

$$\operatorname{cof}(\mathbf{A}) = \begin{bmatrix} -1 & -5 & 3 \\ -1 & 10 & 3 \\ -6 & 0 & 3 \end{bmatrix},$$

so

$$\operatorname{Adj}(\mathbf{A}) = \begin{bmatrix} -1 & -1 & -6 \\ -5 & 10 & 0 \\ 3 & 3 & 3 \end{bmatrix}.$$

Now

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{\det(\mathbf{A})}\operatorname{Adj}(\mathbf{A}) \\ &= \frac{1}{15}\begin{bmatrix} -1 & -1 & -6 \\ -5 & 10 & 0 \\ 3 & 3 & 3 \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \mathbf{A}^{-1} \begin{bmatrix} 12 \\ 3 \\ -10 \end{bmatrix} \\
&= \frac{1}{15} \begin{bmatrix} -1 & -1 & -6 \\ -5 & 10 & 0 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ 3 \\ -10 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.
\end{aligned}$$

The answer is that $x = 3$, $y = -2$ and $z = 1$.

4 Cramer's Rule

A *Cramer system* is any system of n linear equations in n unknowns if and only if the matrix formed by the coefficients is non-singular.

Cramer's Rule (also known as Method of Determinants) makes use of determinants to solve such a non-singular square system.

For a system

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix}$

let $\mathbf{A}_i$ be the matrix obtained by replacing the entries of the ith column of $\mathbf{A}$ by the answer vector.

Then

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}.$$

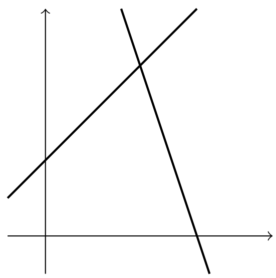
Note that for $\mathbf{Ax} = \mathbf{b}$, if $\det(\mathbf{A}) = 0$ then Cramer's Rule will not work.

5 Row operations

5.1 Number of solutions

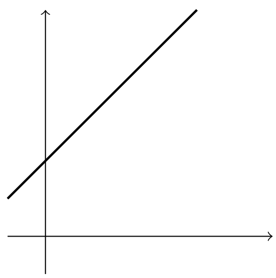
There are three possibilities for the number of solutions of a linear system:

1. Exactly one solution (consistent system).
   e.g. $3x + y = 6$ and $x - y = -1$ are both satisfied at the point $(\frac{1}{2}, \frac{3}{2})$:



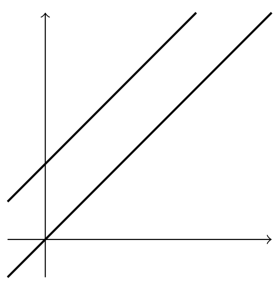
2. Infinitely many solutions (consistent system).

   e.g. $x - y = -1$ and $5x - 5y = -5$ coincide entirely, meaning they overlap completely:



3. No solutions; such a system is said to be inconsistent.

   e.g. $x - y = -1$ and $x - y = 0$ are parallel, meaning no point satisfies both lines simultaneously:



5.2 Equivalence of linear systems and matrices

Two linear systems are equivalent if and only if they have the same solution set.

The basic method for solving a system of linear equations is to replace the given system by a new equivalent system that has the same solution set but it is easier to solve.

5.3 Elementary row operations

Transforming one linear system into an equivalent system that is easier to solve uses a series of steps by applying the following three types of operations (known as *elementary row operations*):

1. Row switching: A row within the matrix can be switched with another row.

2. Row multiplication: Each element in a row can be multiplied by a non-zero constant.

3. Row addition: A row can be replaced by the sum of that row and a multiple of another row.

5.4 Augmented matrix

We can write the system $\mathbf{Ax} = \mathbf{b}$ as an *augmented matrix* as

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right].$$

This has the same meaning, it is just a compact notation we can use for row operations.

5.5 Gaussian elimination

Steps:

1. Write down the augmented matrix corresponding to the given linear system.
2. Perform a series of elementary row operations to reduce the augmented matrix to an *echelon* form. This is a matrix that satisfies these conditions:
   (a) It is *upper triangular* (entries below the main diagonal are 0);
   (b) All zero rows are at the bottom of the matrix;
   (c) The leading entry of each non-zero row after the first occurs to the right of the leading entry of the previous row.
3. Working from the bottom of the echelon matrix upwards, evaluate the unknowns using back-substitution.

Example

Find the solutions of the following linear system of three equations in three unknowns:

$$x - 2y + 3z = 3 \tag{1}$$
$$2x + y - z = 9 \tag{2}$$
$$-3x + 5y = -2 \tag{3}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 2 & 1 & -1 & 9 \\ -3 & 5 & 0 & -2 \end{array}\right]$$

We perform row operations, aiming to get the matrix into echelon form.
Let's refer to row 1 by r_1, and the others similarly.
Replace r_2 with $r_2 - 2 \times r_1$ and r_3 with $r_3 + 3r_1$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & -1 & 9 & 7 \end{array}\right]$$

Now replace r_3 with $5r_3 + r_2$ (referring to rows 2 and 3 from the transformed matrix above).

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & 0 & 38 & 38 \end{array}\right]$$

From the new row 3, we see

$$\begin{aligned} 38z &= 38 \\ z &= 1. \end{aligned}$$

Using this and row 2, we get

$$\begin{aligned} 5y - 7 &= 3 \\ 5y &= 10 \\ y &= 2. \end{aligned}$$

Finally, we can use these values in row 1:

$$\begin{aligned} x - 2 \times 2 + 3 &= 3 \\ x - 4 + 3 &= 3 \\ x - 4 &= 0 \\ x &= 4. \end{aligned}$$

The solution is $(x, y, z) = (4, 2, 1)$.

Example

Find the solutions of the following linear system of three equations in three unknowns:

$$x - 2y + 3z = 3 \tag{1}$$
$$2x + y - z = 9 \tag{2}$$
$$-3x - 4y + 5z = -8 \tag{3}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 2 & 1 & -1 & 9 \\ -3 & -4 & 5 & -8 \end{array}\right]$$

We replace r_2 by $r_2 - 2r_1$ and r_3 by $3r_1 + r_3$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & -10 & 14 & 1 \end{array}\right]$$

Now we can replace r_3 by $2r_2 + r_3$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

Since we have obtained from row 3 the contradiction $0 = 7$, we see that there are no solutions to this system. Such a system is said to be *inconsistent*.

Example

Find the solutions of the following linear system of three equations in three unknowns:

$$x - 2y + 3z = 3 \tag{1}$$
$$2x + y - z = 9 \tag{2}$$
$$-3x - 4y + 5z = -15 \tag{3}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 2 & 1 & -1 & 9 \\ -3 & -4 & 5 & -15 \end{array}\right]$$

First replace r_2 by $r_2 - 2r_1$ and r_3 by $3r_1 + r_3$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & -10 & 14 & -6 \end{array}\right]$$

Now replace r_3 by $2r_2 + r_3$:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In row 3 we have obtained $0 = 0$. This is not a contradiction, but neither does it involve z. We say there is no restriction on z in this linear system.

So let $z = t$. Then, from row 1:

$$\begin{aligned} 5y - 7t &= 3 \\ y &= \frac{1}{5}(7t + 3), \end{aligned}$$

and from row 1:

$$\begin{aligned} x - 2 \times \frac{1}{5}(7t + 3) + 3t &= 3 \\ x &= \frac{1}{5}(21 - t). \end{aligned}$$

The solution to this system is $(x, y, z) = (\frac{1}{5}(21 - t), \frac{1}{5}(7t + 3), t)$ for $t \in \mathbb{R}$.