Impartial games

Peter Rowlett

Sheffield Hallam University

p.rowlett@shu.ac.uk

Combinatorial games

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.
- Some games that are not combinatorial games: Poker,

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.
- Some games that are not combinatorial games: Poker, Bridge,

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.
- Some games that are not combinatorial games: Poker, Bridge, Monopoly,

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.
- Some games that are not combinatorial games: Poker, Bridge, Monopoly, Pong,

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.
- Some games that are not combinatorial games: Poker, Bridge, Monopoly, Pong, Paper-Scissor-Stone,

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.
- Some games that are not combinatorial games: Poker, Bridge, Monopoly, Pong, Paper-Scissor-Stone, international diplomacy,

Combinatorial games

- Combinatorial game theory generally deals with turn-based games of no chance with perfect information.
- Some examples: Noughts and Crosses, Chess, Draughts, Go, Dots and Boxes, Hex, Nim.
- Some games that are not combinatorial games: Poker, Bridge, Monopoly, Pong, Paper-Scissor-Stone, international diplomacy, Darwinian evolution.

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;
  - it is *deterministic*, i.e. there is no chance: games that do not involve dice, spinners, card deals and other generators of random occurrences;

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;
  - it is *deterministic*, i.e. there is no chance: games that do not involve dice, spinners, card deals and other generators of random occurrences;
  - players have *perfect information*: each player is aware of all the details of the game position (or game state) and the set of available moves at all times (i.e. no hidden cards, etc.);

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;
  - it is *deterministic*, i.e. there is no chance: games that do not involve dice, spinners, card deals and other generators of random occurrences;
  - players have *perfect information*: each player is aware of all the details of the game position (or game state) and the set of available moves at all times (i.e. no hidden cards, etc.);
  - it is *finite*: there are a finite number of available positions;

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;
  - it is *deterministic*, i.e. there is no chance: games that do not involve dice, spinners, card deals and other generators of random occurrences;
  - players have *perfect information*: each player is aware of all the details of the game position (or game state) and the set of available moves at all times (i.e. no hidden cards, etc.);
  - it is *finite*: there are a finite number of available positions;
  - it *ends*: games end in a finite sequence of moves;

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;
  - it is *deterministic*, i.e. there is no chance: games that do not involve dice, spinners, card deals and other generators of random occurrences;
  - players have *perfect information*: each player is aware of all the details of the game position (or game state) and the set of available moves at all times (i.e. no hidden cards, etc.);
  - it is *finite*: there are a finite number of available positions;
  - it *ends*: games end in a finite sequence of moves;
  - there is a *winner*:

Some terminology

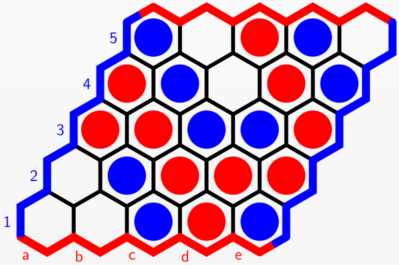
- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;
  - it is *deterministic*, i.e. there is no chance: games that do not involve dice, spinners, card deals and other generators of random occurrences;
  - players have *perfect information*: each player is aware of all the details of the game position (or game state) and the set of available moves at all times (i.e. no hidden cards, etc.);
  - it is *finite*: there are a finite number of available positions;
  - it *ends*: games end in a finite sequence of moves;
  - there is a *winner*:
    - in *normal play*, the last player to move wins;

Some terminology

- A *combinatorial game* will generally meet some or all of the following conditions:
  - it is *two-player*: two players play the game;
  - it is *turn-based*: the two players alternate turns;
  - it is *deterministic*, i.e. there is no chance: games that do not involve dice, spinners, card deals and other generators of random occurrences;
  - players have *perfect information*: each player is aware of all the details of the game position (or game state) and the set of available moves at all times (i.e. no hidden cards, etc.);
  - it is *finite*: there are a finite number of available positions;
  - it *ends*: games end in a finite sequence of moves;
  - there is a *winner*:
    - in *normal play*, the last player to move wins;
    - in *misére play*, the last player to move loses.

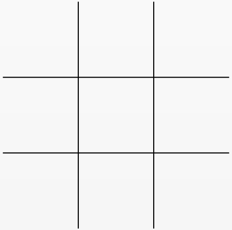
Combinatorial games

- Reminder: a *two-player*, *turn-based*, *deterministic*, *finite* game of *perfect information* that *ends* with a *winner*.
- For example
  - Hex meets all these conditions.



Combinatorial games

- Reminder: a *two-player*, *turn-based*, *deterministic*, *finite* game of *perfect information* that *ends* with a *winner*.
- For example
  - Noughts and Crosses is a *two-player*, *turn-based*, *deterministic*, *finite* game of *perfect information* that *ends*, but may end in a draw.



Some terminology

- The two players of a combinatorial game are traditionally called *Left* (or L) and *Right* (or R); if it matters which player goes first they are sometimes labelled A and B (and perhaps given appropriate names, such as Alice and Bob);
- An *impartial game* is one where the options for L and R are the same, and a *partizan game* is one which is not impartial.

Some terminology

- *Perfect play* is the behaviour for a player that leads to the best possible outcome for that player regardless of the response of the other player (which may be the absolute best outcome if this is known, or the best expected outcome);
- A *solved game* is a game for which the outcome can be correctly predicted from any position, assuming both players play perfectly.
- *Symmetry strategy*: Whenever your opponent does something on one part of the board, you mimic this move in another part. A strategy that maintains a simple symmetry is called *Tweedledum-Tweedledee**.

*Named after characters you may know from Lewis Carroll's 1871 novel *Through the Looking-Glass, and What Alice Found There*, but that isn't important.

Single-pile Nim

Game: She Loves Me She Loves Me Not

- Play in pairs.
- Play takes place on a single daisy.
- Choose a number of petals.
- Players alternately remove exactly one petal from the daisy.
- The last player to remove a petal wins.

Question: what is a winning strategy for this game?

Terminology: Parity

- All that matters to work out who will win **She Loves Me She Loves Me Not** is the parity of the starting position.
- (A number's parity is whether the number is odd or even.)
- Parity is often an important concept in analysing combinatorial games.

Terminology: $\mathcal{N}$- and $\mathcal{P}$-positions

- A game in $\mathcal{N}$-position is one where the next player to play can force a win. A game that starts in $\mathcal{N}$-position is a win for the player who goes first (assuming they play perfectly).
- A game in $\mathcal{P}$-position is one where the previous player who played (or the second to play) can force a win.

Terminology: $\mathcal{N}$- and $\mathcal{P}$-positions

- A game in $\mathcal{N}$-position is one where the next player to play can force a win. A game that starts in $\mathcal{N}$-position is a win for the player who goes first (assuming they play perfectly).
- A game in $\mathcal{P}$-position is one where the previous player who played (or the second to play) can force a win.
- A game of **She Loves Me She Loves Me Not** that starts with an odd number of petals is in $\mathcal{N}$-position, and so is a winner for player 1.
- A game that starts with an even number of petals is in $\mathcal{P}$-position, and so is a win for player 2

Game: Single-pile Nim

- Last week we played Single-pile Nim. Rules:
  - Play in pairs.
  - Place a pile of sticks[†] in front of you.
  - Players take turns to take 1, 2 or 3 sticks from the pile.
  - The player who takes the last match wins.

[†]or stones or counters or, indeed, matches.

How to win at Single-pile Nim

- Single-pile Nim is in $\mathcal{P}$-position if the number of sticks is congruent to $0 \pmod 4$, and it is in $\mathcal{N}$-position if the number of sticks is not congruent to $0 \pmod 4$.

How to win at Single-pile Nim

- Single-pile Nim is in $\mathcal{P}$-position if the number of sticks is congruent to $0 \pmod 4$, and it is in $\mathcal{N}$-position if the number of sticks is not congruent to $0 \pmod 4$.
- So a winning move is to make the number of sticks congruent to $0 \pmod 4$ at the end of your turns.

How to win at Single-pile Nim

- Single-pile Nim is in $\mathcal{P}$-position if the number of sticks is congruent to $0 \pmod 4$, and it is in $\mathcal{N}$-position if the number of sticks is not congruent to $0 \pmod 4$.
- So a winning move is to make the number of sticks congruent to $0 \pmod 4$ at the end of your turns.
- Put another way:
  - Make the number of sticks at the end of your turn be $4k$ for some integer k.
  - When your opponent takes x sticks, you take $4 - x$.

How to win at Single-pile Nim

- Single-pile Nim is in $\mathcal{P}$-position if the number of sticks is congruent to $0 \pmod 4$, and it is in $\mathcal{N}$-position if the number of sticks is not congruent to $0 \pmod 4$.
- So a winning move is to make the number of sticks congruent to $0 \pmod 4$ at the end of your turns.
- Put another way:
  - Make the number of sticks at the end of your turn be $4k$ for some integer k.
  - When your opponent takes x sticks, you take $4 - x$.
- Note this is not always possible – if the number of sticks is already congruent to $0 \pmod 4$ at the start of your turn, the game is in $\mathcal{P}$-position already and you have lost (assuming perfect play).

Multi-pile Nim

Game: Multi-pile Nim

- Last week we played Multi-pile Nim (a different game that looks similar because they both use sticks[‡]).
- Rules:
  - Play in pairs, take turns.
  - Start with several heaps[§] of sticks.
  - A move is to remove a positive number of sticks from one of the piles.
  - The player to take the last stick wins.

An example 3-pile Nim game:

[‡] or stones or counters or, indeed, matches.

[§] or piles

Concept: Nim sum

- Notation: write $*x$ to denote a Nim heap with x sticks.
- We can add two Nim heaps $*a$ and $*b$ by saying

$$*a + *b = *(a \oplus b).$$

The number $a \oplus b$ is called the Nim sum of $*a$ and $*b$.

Concept: Nim sum

- The Nim sum of k non-negative integers $x_1, x_2, \ldots, x_k$, written $x_1 \oplus x_2 \oplus \ldots \oplus x_k$, is obtained by adding the numbers in binary without carrying (an operation called exclusive-or or XOR and denoted $\oplus$).
- N.B., the multiplication table for single-digit $\oplus$ is:

$\oplus$	**0**	**1**
0	0	1
1	1	0

Concept: Nim sum

- For example, to compute the Nim sum of $5 \oplus 4 \oplus 13$:

$$\begin{array}{c|cccc} 5 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 \\ 13 & 1 & 1 & 0 & 1 \\ \hline & 1 & 1 & 0 & 0 \end{array}$$

- For each column of bits (binary digits), if the column has an odd number of 1s, the column sum is 1; otherwise, the column sum is 0.
- Here, 1100 is binary for 12, so we say $5 \oplus 4 \oplus 13 = 12$.

Concept: Nim sum

- Nim sum has the following properties:
  - *commutativity*: $a \oplus b = b \oplus a$;
  - *associativity*: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
  - $a \oplus a = 0$;
  - $a \oplus b \oplus c = 0$ if and only if $a \oplus b = c$.

How to win at Multi-pile Nim

- A game of Nim with heaps $*x_1, *x_2, \ldots, *x_k$ is a $\mathcal{P}$-position if and only if its Nim sum
$$x_1 \oplus x_2 \oplus \ldots \oplus x_k = 0.$$
- Some observations that follow from this theorem and its proof:
  - If the Nim sum is 0, this is a $\mathcal{P}$-position. The previous player can win by making the Nim sum 0 on every turn that follows.
  - If the Nim sum is not 0, this is an $\mathcal{N}$-position. The next player can win by making the Nim sum 0 on every turn that follows.

How to win at Multi-pile Nim

- A game of Nim with heaps $*x_1, *x_2, \ldots, *x_k$ is a $\mathcal{P}$-position if and only if its Nim sum
$$x_1 \oplus x_2 \oplus \ldots \oplus x_k = 0.$$
- Some special cases:
  - If there is only a single heap left, the winning move is to take everything. This works because it leaves no sticks, a Nim sum of 0.
  - If there are two heaps, then the winning move, if there is one, is to make the two heaps the same size, then play Tweedledum-Tweedledee (note that this works because $a \oplus a = 0$).

Nim-like games and mex

An important distinction

- Important: we must remember the difference between

Multi-pile Nim with one heap

- We can remove any number of sticks from any heap, just there happens to be only one heap.
- For example:

  |\\///\|

  (This game is in the $\mathcal{N}$ position, we can win by taking the whole pile.)

Single-pile Nim

- A totally different game where each turn we can take 1, 2 or 3 sticks.
- For example:

  |\\///\|

  (This game is in the $\mathcal{P}$ position, we cannot win because the size of the pile is congruent to 0 (mod 4).)

Something to notice

- Have you noticed that each Nim heap is like its own little game of *Multi-pile Nim with one heap*?
- You can win any of the individual heaps by taking all its sticks, the question is can you be the player to win the *last* heap by taking *its* last stick?

Nim sum

- We saw a way to add Nim heaps together using Nim sum.
- For example, we can say

$$*5 + *3 + *9 + *2 = *(5 \oplus 3 \oplus 9 \oplus 2) = *13.$$

Nim heaps

- We can define Nim heaps by thinking about what we can change them to.
- For a Nim heap with zero sticks, there are no moves that can be made.

Nim heaps

- We can define Nim heaps by thinking about what we can change them to.
- For a Nim heap with zero sticks, there are no moves that can be made.
- The set of available moves is therefore the empty set $\{\}$.

Nim heaps

- We can define Nim heaps by thinking about what we can change them to.
- For a Nim heap with zero sticks, there are no moves that can be made.
- The set of available moves is therefore the empty set $\{\}$.
- So we say $*0 = \{\}$.

Nim heaps

- Now consider a Nim heap $*1$:

- We can change this to $*0$ only. Let's write $*1$ in terms of a set of available moves:

$$*1 = \{*0\}.$$

Nim heaps

- Now consider $*2$:

  ||

- We can take one stick, changing this to $*1$, or take both sticks, changing this to $*0$.
- So

$$*2 = \{*0, *1\}.$$

Nim heaps

- Similarly, we can define

$$*3 = \{*0, *1, *2\};$$

Nim heaps

- Similarly, we can define

$$*3 = \{*0, *1, *2\};$$

$$*4 = \{*0, *1, *2, *3\};$$

Nim heaps

- Similarly, we can define

$$*3 = \{*0, *1, *2\};$$

$$*4 = \{*0, *1, *2, *3\};$$

$$\vdots$$

- For a general Nim heap with n sticks, we can define

$$*n = \{*0, *1, *2, \ldots, *(n-1)\}.$$

Nim heaps

- Similarly, we can define

$$*3 = \{*0, *1, *2\};$$

$$*4 = \{*0, *1, *2, *3\};$$

$$\vdots$$

- For a general Nim heap with n sticks, we can define

$$*n = \{*0, *1, *2, \dots, *(n-1)\}.$$

- We could say that $*n$ is the Nim heap for which:
  - we can move to all positions $*k$ for $0 \leq k < n$;
  - but we cannot move to position $*n$.

Mex

- The ***m**inimum **ex**cluded value* (mex) of a set of non-negative integers is the smallest non-negative integer not included in the set.
- For example, in the set $\{0, 1, 2, 3, 5, 7, 8\}$ the mex is 4.

Mex

- The ***m**inimum **ex**cluded value* (mex) of a set of non-negative integers is the smallest non-negative integer not included in the set.
- For example, in the set $\{0, 1, 2, 3, 5, 7, 8\}$ the mex is 4.
- In terms of our Nim heap, notice that for a Nim heap $*n$ the number n is the mex of the sizes of the positions we can move to in one turn.

$$*n = \{*0, *1, *2, \ldots, *(n-1)\}.$$

A strategy for Nim-like games

Two games

- Say we play two games simultaneously. Each turn, players make one move in one game. The winner is the player who moves last overall.

Game A

- The first player chooses any value from $\{0, 1, 2, 3, 4, 6, 7, 8, 9, \ldots\}$.
- Then players take turns to reduce the integer to any value ≥ 0.

Multi-pile Nim with one heap of 5 sticks.

|/\|/

Two games

- Say we play two games simultaneously. Each turn, players make one move in one game. The winner is the player who moves last overall.

Game A

- The first player chooses any value from $\{0, 1, 2, 3, 4, 6, 7, 8, 9, \ldots\}$.
- Then players take turns to reduce the integer to any value ≥ 0.

Multi-pile Nim with one heap of 5 sticks.

I/\I/

- But there's a twist! You must go first and play in Game A.

Two games

Game A

Multi-pile Nim

$$\{0, 1, 2, 3, 4, 6, 7, 8, 9, \ldots\}$$

I/\I/

- If you choose to move Game A to a position > 5, then I can reduce it to 5 in the next move and then play Tweedledum-Tweedledee until I win.

Two games

Game A | *Multi-pile Nim*

$$\{0, 1, 2, 3, 4, 6, 7, 8, 9, \ldots\}$$

I/\I/

- If you choose to move Game A to a position > 5, then I can reduce it to 5 in the next move and then play Tweedledum-Tweedledee until I win.
- If you choose to move Game A to a position < 5, then I can reduce the Nim heap to match in the next move and then play Tweedledum-Tweedledee until I win.

Two games

Game A

$$\{0, 1, 2, 3, 4, 6, 7, 8, 9, \ldots\}$$

Multi-pile Nim

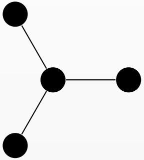
I/\I/

- The number 5 has some significance to Game A.
- Note that 5 is the mex of both games.
- In some sense we can call the opening position in game A equivalent to the Nim heap $*5$.

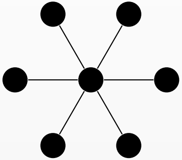
Cutthroat Stars

Star graphs

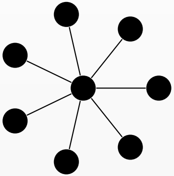
- A star graph $K_{1,n}$ is a graph with a single central node, n radial nodes, and n edges connecting the central node to each radial node.
- Some examples:



$K_{1,3}$



$K_{1,6}$

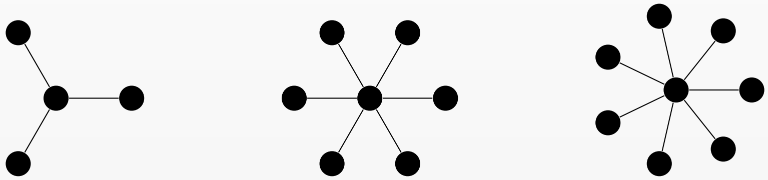


$K_{1,7}$

Cutthroat Stars

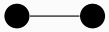
Played on one or more star graphs. Players take turns to remove one vertex from one of the graphs on the board. When a vertex is removed, all the incident edges and any isolated vertices are also removed. The first player who cannot make a legal move loses.

Here is a sample starting position.



Cutthroat Stars on $K_{1,1}$

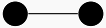
- When there is no graph left, there are no available moves and we are in the position $*0 = \{\}$.



- For $K_{1,1}$, removing either node removes the whole graph, so the set of available moves is
$$\{*0\}.$$
- The mex is 1 and this position is equivalent to $*1 = \{*0\}$.

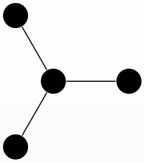
Cutthroat Stars on $K_{1,2}$

- Removing the central node deletes the whole graph, which is $*0$.
- Removing either radial node moves to $K_{1,1}$, which is $*1$.
- The set of available moves is

$\{*0,$  $\}.$

- This is equivalent to $*2 = \{*0, *1\}$.

Cutthroat Stars on $K_{1,3}$

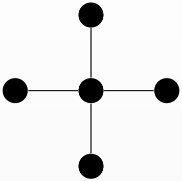


- Removing the central node deletes the whole graph, which is $*0$.
- Removing any radial node moves to $K_{1,2}$, which is $*2$.
- The set of available moves is

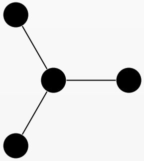
$$\{*0,\quad$$  $$\}.$$

- This is equivalent to $*1 = \{*0, *2\}$.

Cutthroat Stars on $K_{1,4}$



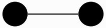
- The set of available moves is

$\{*0,$  $\}.$

- This is equivalent to $*2 = \{*0, *1\}$.

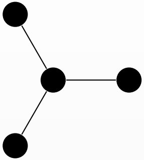
Cutthroat Stars mex values

- In fact, the mex values oscillate.

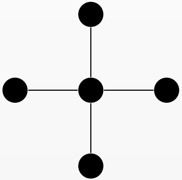


$K_{1,1}$ $(*1)$

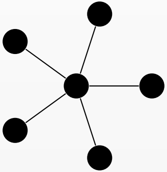
$K_{1,2}$ $(*2)$



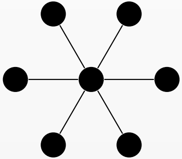
$K_{1,3}$ $(*1)$



$K_{1,4}$ $(*2)$



$K_{1,5}$ $(*1)$



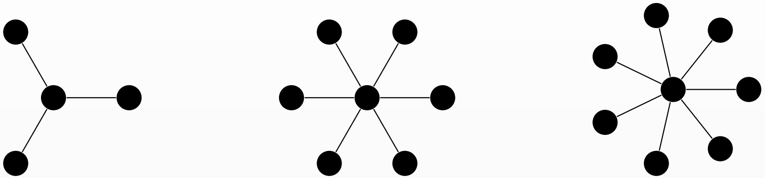
$K_{1,6}$ $(*2)$

$\vdots$

Notice

1. This isn't straightforwardly Nim, because mex values don't simply decrease as we remove nodes.
2. In fact, removing a radial node from some graphs increases the mex value, making it equivalent to adding sticks to the heap.
3. The graph $K_{1,n}, n > 0$ is equivalent to $*2$ if n is even and equivalent to $*1$ if n is odd.

Our example game



- These are equivalent to $*1$, $*2$ and $*1$.
- So this game is equivalent to a Multi-pile Nim game:

| /\ \

Our example game

| /\   \

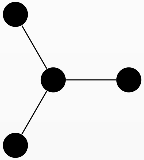
- We can solve this using Nim sum.

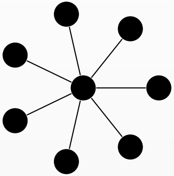
$$
\begin{array}{c|cc}
*1 & 0 & 1 \\
*2 & 1 & 0 \\
*1 & 0 & 1 \\
\hline
 & 1 & 0
\end{array}
$$

- i.e. $1 \oplus 2 \oplus 1 = 2$.
- The relevant heap is $*2$, which we must change to $2 \oplus 2 = 0$.
- The winning move is to completely remove the middle heap, then play Tweedledum-Tweedledee.

Our example game

- In terms of Cutthroat Stars, the equivalent move is to take the central node from the middle graph, moving it from $*2$ to $*0$.





- Now, if your opponent takes the central node of one graph, you take the central node of the other and win.
- If they take a radial node of $K_{1,3}$, turning it into $K_{1,2}$, you take a radial node from $K_{1,7}$, turning it into $K_{1,6}$.

Single-pile Nim

Reminder: Single-pile Nim

- Players take turns to take 1, 2 or 3 sticks from a pile.
- The player who takes the last stick wins.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with no sticks

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with no sticks

- We cannot play.
- This is therefore equivalent to $*0 = \{\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with one stick

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with one stick

- We must remove one stick to get to $*0$.
- This is therefore equivalent to $*1 = \{*0\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 2 sticks

||

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 2 sticks

||

- We can remove one or two sticks.
- The available moves are

$$\{*0, |\}$$

- This is therefore equivalent to $*2 = \{*0, *1\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 3 sticks

\|\|\\

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 3 sticks

|||

- We can remove one, two or three sticks.
- The available moves are

$$\{*0, \,|\, , \,||\, \}$$

- This is therefore equivalent to $*3 = \{*0, *1, *2\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 4 sticks

\|\|\\/

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 4 sticks

- We can remove one, two or three sticks.
- The available moves are

$$\{ | , || , ||| \}$$

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 4 sticks

|||/

- We can remove one, two or three sticks.
- The available moves are

{| , || , |||}

- These are equivalent to

$$\{*1, *2, *3\}.$$

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 4 sticks

|||/

- We can remove one, two or three sticks.
- The available moves are

$$\{ | \, , \, || \, , \, ||| \}$$

- These are equivalent to

$$\{*1, *2, *3\}.$$

- This position is therefore equivalent to $*0 = \{*1, *2, *3\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 5 sticks

|/\|/

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 5 sticks

|/\|/

- We can remove one, two or three sticks.
- The available moves are

$$\{\ ||, |||\backslash, |||/\ \}$$

- This is therefore equivalent to $*1 = \{*2, *3, *0\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 6 sticks

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 6 sticks

\|\//|

- We can remove one, two or three sticks.
- The available moves are

{ \|\, \|\/ , |/\|/ }

- This is therefore equivalent to $*2 = \{*3, *0, *1\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 7 sticks

||\//|\

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 7 sticks

\|\//|\

- We can remove one, two or three sticks.
- The available moves are

  { \|\/ , |/\|/ , \|\//| }

- This is therefore equivalent to $*3 = \{*0, *1, *2\}$.

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 8 sticks

Single-pile Nim is equivalent to Multi-pile Nim

Single-pile Nim with 8 sticks

|\\///\|

- We can remove one, two or three sticks.
- The available moves are

{|/\|/, \|\//|, \|\//|\}

- This is therefore equivalent to $*0 = \{*1, *2, *3\}$.

Single-pile Nim is equivalent to Multi-pile Nim

- This pattern continues, so for integer $k \geq 0$:
  - $4k$ sticks is equivalent to $*0$;
  - $4k+1$ sticks is equivalent to $*1$;
  - $4k+2$ sticks is equivalent to $*2$;
  - $4k+3$ sticks is equivalent to $*3$;
- i.e. divide by 4 and the remainder is the size of the equivalent Multi-pile Nim heap.

Playing different games

Playing different games

- For games that are equivalent to Nim heaps, we don't need each part to be from the same game.
- For example, we can imagine we are playing three games simultaneously and the player who makes the last move in the last game wins.

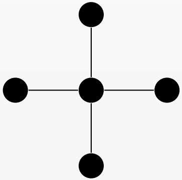
Single-pile Nim

|/\|/

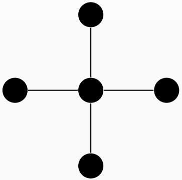
Multi-pile Nim

||

Cutthroat Stars



Playing different games

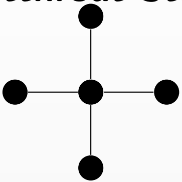
Single-pile Nim	Multi-pile Nim	Cutthroat Stars
|/\\\|/	\\\|	
$*1$	$*2$	$*2$

Playing different games

$$*1 \qquad\qquad *2 \qquad\qquad *2$$

- $1 \oplus 2 \oplus 2 = 1$.
- A winning move is therefore to move the Single-pile Nim game to $*0$ by removing one stick.

Playing different games

Single-pile Nim	**Multi-pile Nim**	**Cutthroat Stars**
|||/	||	 $K_{1,4}$
$*0$	$*2$	$*2$