Worksheet 6: Recurrence relations

1. (a) 6.
   (b) 1 is AABBC. 2 is ABCBC. 3 is AABBA.

2. (a) 5.
   (b) 8.
   (c) This generates the Fibonacci sequence.

3. Let S_n represent the number of valid secret codes of length n. By experiment, we find $S_1 = 3$ and $S_2 = 8$. At each new stage, we can add A, B or C to all the codes found at the previous stage, except those which start B and to which we are adding A. The codes that start B in the $n-1$ stage are formed by adding B to the front of all those that we had at the $n-2$ stage. Therefore $S_n = 3S_{n-1} - S_{n-2}$.

   Either through iterating the recurrence relation, or by calculating the general solution

   $$S_n = \left(\frac{1}{2} + \frac{3}{2\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} - \frac{3}{2\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n$$

   we find $S_8 = 2584$.

4. (a) $a_1 = 5, a_n = 5a_{n-1}$;
   (b) $a_1 = 3, a_n = a_{n-1}$;
   (c) $a_1 = 5, a_n = a_{n-1} + 5$;
   (d) $a_0 = 5, a_n = a_{n-1} + 4$;
   (e) $a_1 = 1, a_n = a_{n-1} + 2n - 1$.

5. Characteristic equation: $r - 2 = 0$, so $r = 2$. General solution: $T_n = \alpha(2^n) + \beta$. Since $T_1 = 1$ and $T_2 = 3$, we have

   $$1 = 2\alpha + \beta;$$
   $$3 = 4\alpha + \beta.$$

   which gives $(\alpha, \beta) = (1, -1)$ for a particular solution: $T_n = 2^n - 1$, as required.

6. (a) $S_n = \frac{8}{2^n} = \frac{1}{2^{n-3}}$;
   (b) $S_n = \frac{5}{6}(3^n) - \frac{1}{2}(-1)^n$;
   (c) $S_n = \frac{4}{7}(6^n) + \frac{3}{7}(-1)^n$;
   (d) $S_n = \left(\frac{7}{9} - \frac{1}{9}n\right)3^n$.

7. (a) $S_n = \frac{4}{2^n} - 2(5^n)$;
   (b) $S_n = 3\sin\left(\frac{\pi}{2}n\right)$;
   (c) $S_n = \frac{3}{5}(4^n) - \frac{1}{10}(-1)^n - \frac{1}{2}$;
   (d) $S_n = 5n + 1$.