Partizan games

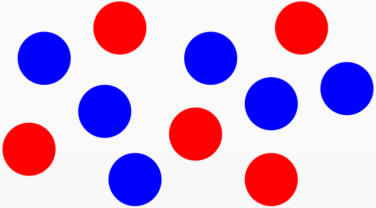
Peter Rowlett

Sheffield Hallam University

p.rowlett@shu.ac.uk

A simple partizan game

- **Pick up your colour**: Playing on a pile of blue and red counters, Left picks up any number of blue counters and Right picks up any number of red counters. The first player who cannot move loses.



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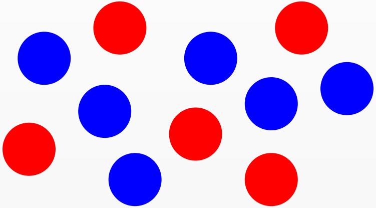
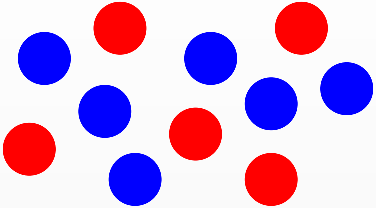


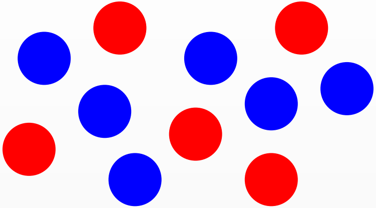
- Note: **L**eft plays b**L**ue and **R**ight plays **R**ed.

A simple partizan game



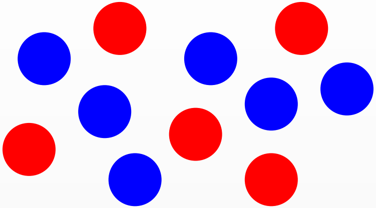
- This is different to the games we have been playing so far because it is a *partizan* game: Left and Right have their own counters which only they can move, and different goals.

A simple partizan game



- It is a simple game because, e.g. in this position it is a win for Left going either first or second because Left has more counters.

Pick up your colour



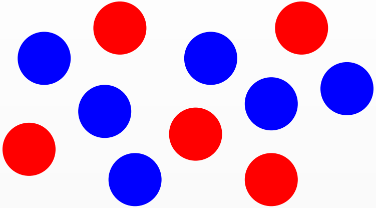
- Let's think about the game in terms of the number of counters advantage Left has over Right.
- So we add $+1$ for each blue counter and -1 for each red counter.
- The game position shown is

$$6 + (-5) = 1.$$

Notation for partizan games

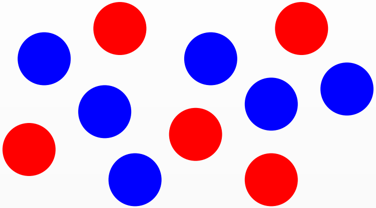
- Let
$$G = \{\underbrace{a_1, a_2, a_3, \ldots}_{L \text{ positions}} \mid \underbrace{b_1, b_2, b_3, \ldots}_{R \text{ positions}}\}$$
be a game where L represents the positions Left can move to and R represents the positions Right can move to.

Pick up your colour



- Left must remove between one and six blue counters, so they can move the game to any position in $\{0, -1, -2, -3, -4, -5\}$.
- Right must remove between one and five red counters, so they can move the game to any position in $\{2, 3, 4, 5, 6\}$.

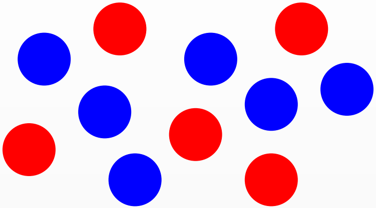
Pick up your colour



- So we can write this game position we called 1 as

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

Pick up your colour



- So we can write this game position we called 1 as

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

- Notice that a move by either player makes their own position worse.

Conway process

- A strange way of inventing numbers

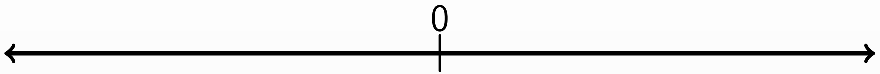
Conway process

- A strange way of inventing numbers
- On day 0 the number 0 is 'invented'.
- Then on day n there are 2^n new numbers 'invented'.
- If on a day we have numbers $a_1 < a_2 < \ldots < a_k$ then the next day we create:
    - The largest integer smaller than a_1;
    - The smallest integer larger than a_k;
    - for every pair a_i, a_{i+1} with $i \in \{1, \ldots, k-1\}$:

$$\frac{a_i + a_{i+1}}{2}.$$

Conway process: early stages

Day 0:



Conway process: early stages

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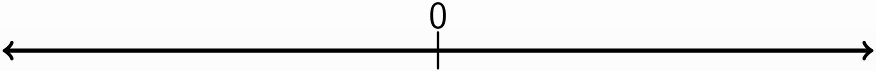
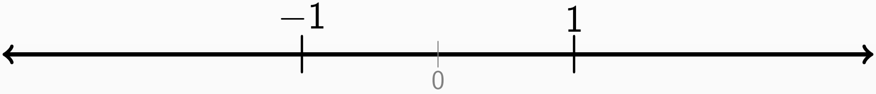


Day 1:



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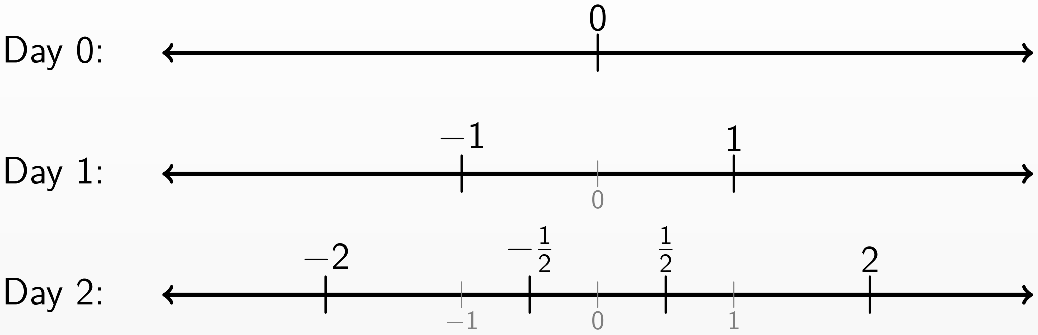
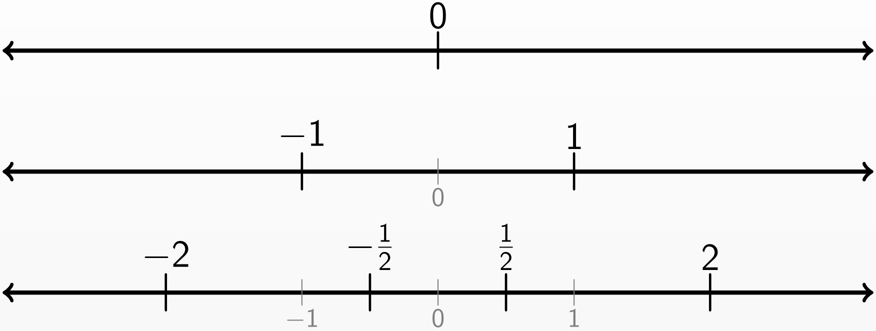


Conway process: early stages

Day 0:

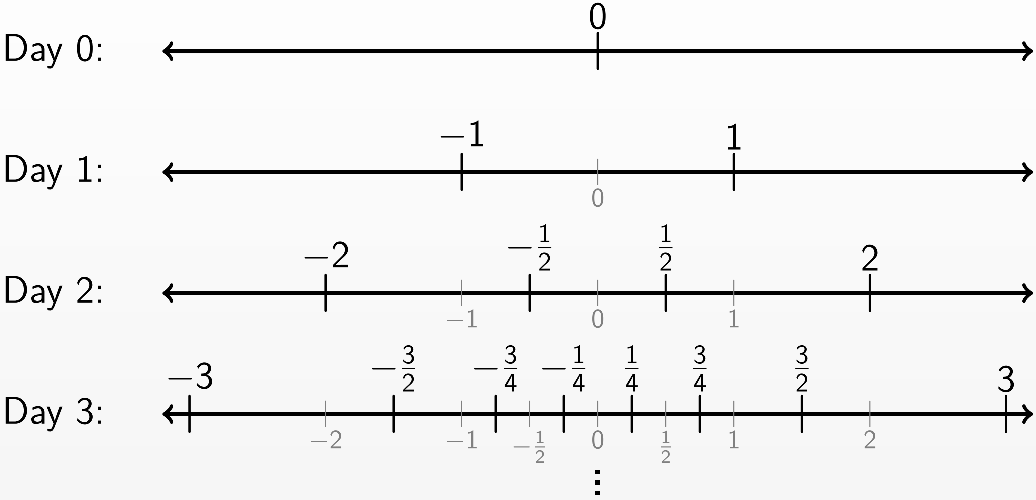
Day 1:

Day 2:



What numbers are invented on day 3?

Conway process: early stages



Consider a game

- Consider a simple game G in which
  - Left can change in one move to any of $\alpha > a_1 > a_2 > \cdots > a_n$;
  - Right can change in one move to any of $\beta < b_1 < b_2 < \cdots < b_n$.
- We can write this

$$G = \{\alpha, a_1, a_2, \ldots, a_n \mid \beta, b_1, b_2, \ldots, b_n\}.$$

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$$G = \{\alpha, a_1, a_2, \ldots, a_n \mid \beta, b_1, b_2, \ldots, b_n\}.$$

- Then there are three options:
  1. $\{\alpha, a_1, a_2, \ldots, a_n\} = \{\beta, b_1, b_2, \ldots, b_n\}$:
     - Then we don't need to distinguish between Left and Right and can just write $G = \{\alpha, a_1, a_2, \ldots, a_n\}$.
     - Actually, we've seen games likes this and labelled them using the $*$ notation.
     - They are equivalent to Nim heaps.

Consider a game

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$$G = \{\alpha, a_1, a_2, \ldots, a_n \mid \beta, b_1, b_2, \ldots, b_n\}.$$

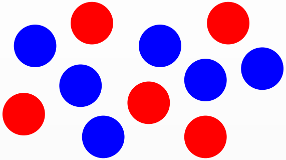
- Then there are three options:

  2. $\alpha < \beta$:
     - G is a number;
     - It's the 'simplest' number between α and β;
     - That is, the first to be 'invented' by the Conway process.
     - We're going to consider games like this today.

Numbers that are games

- In a lot of partizan games a move by either Left or Right makes their own position worse.
- So for a game position $\{a \mid b\}$ with $a < b$:
  - a move by Left to position c will have $c < a$, so we now have position $\{c \mid b\}$ with $c < b$;
  - a move by Right to position d will have $d > b$, so we now have position $\{a \mid d\}$ with $a < d$.

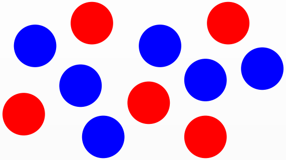
Back to Pick up your colour



- We called this game

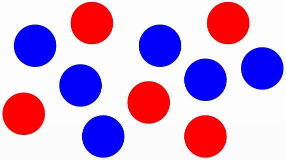
$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

Back to Pick up your colour



- We called this game
$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$
- Note that $0 < 2$.

Back to Pick up your colour

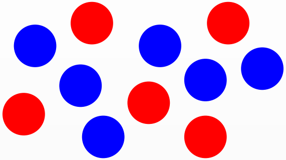


- We called this game

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

- Note that $0 < 2$.
- This game position is the first number between 0 and 2 to be invented by the Conway process.

Back to Pick up your colour



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Day 1: 

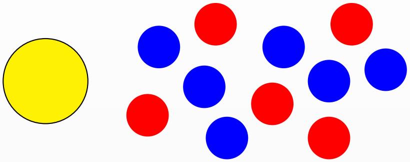
The third option

- Consider a simple game G in which
  - Left can change in one move to any of $\alpha > a_1 > a_2 > \cdots > a_n$;
  - Right can change in one move to any of $\beta < b_1 < b_2 < \cdots < b_n$.
- We can write this

$$G = \{\alpha, a_1, a_2, \ldots, a_n \mid \beta, b_1, b_2, \ldots, b_n\}.$$

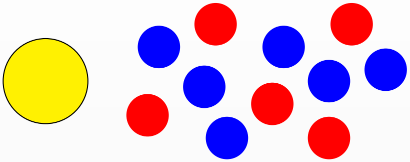
- Then there are three options:
  3. $\alpha > \beta$: This is called a 'hot game' or a 'switch'.

Pick up your colour with a twist



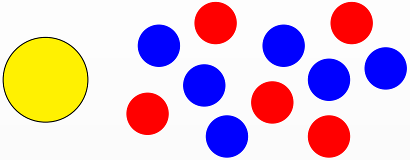
- This is the same game except if you take the yellow token it places five counters of your colour on the board.

Pick up your colour with a twist



- The options for each player are:
  - Left can remove between one and six blue counters, moving to one of $\{0, -1, -2, -3, -4, -5\}$, or take the yellow token, adding five blue counters and so moving the game to 6;
  - Right can remove between one and five red counters, moving to one of $\{2, 3, 4, 5, 6\}$, or take the yellow token, adding five red counters and so moving the game to -4.

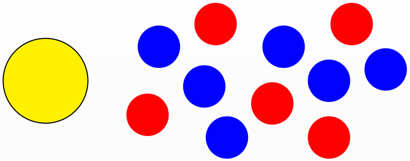
Pick up your colour with a twist



- We can write this as

$$G = \{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

Pick up your colour with a twist

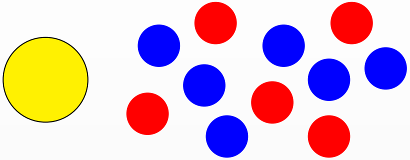


- We can write this as

$$G = \{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

- Note that $6 > -4$.

Pick up your colour with a twist

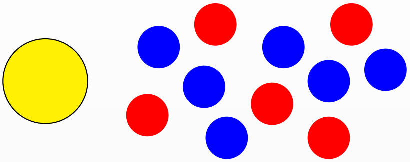


- We can write this as

$$G = \{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

- Note that $6 > -4$.
- This is 'hot': everyone wants to play it as quickly as possible.
- We call other positions 'cold': each player makes their own position worse by playing.

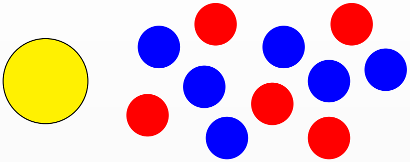
Pick up your colour with a twist



- Consider the yellow token alone.
- This is not $*5$ because it does not move the game to the same position regardless of who takes it.
- It is worth $+5$ to Left and -5 to Right. We can write this

$$\pm 5 = \{5 \mid -5\}$$

Pick up your colour with a twist



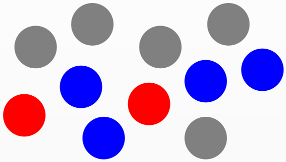
- Another way of scoring this game, then, is:
- The blue and red dots were 1.
- So the blue and red dots with the yellow token are

$$1 \pm 5\{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

There are more exotic games

- Same game except in their turn players can:
  - pick up counters that are their colour or grey; and,
  - change counters of their opponent's colour to grey.

Pick up or change colour:



There are more exotic games

- Here, Left can move the game to 0 by removing the counter.
- Right can move the game to $*1$ by changing the blue counter to grey.

Pick up or change colour:

There are more exotic games

- This game position is therefore

$$G = \{\ 0 \mid *1\ \}.$$

Pick up or change colour:

There are more exotic games

- This game position is therefore

$$G = \{\ 0 \mid *1\ \}.$$

- This is called $\uparrow$ ('up').

Pick up or change colour:

There are more exotic games

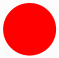
- This game position is therefore
$$G = \{\ 0 \mid *1\ \}.$$
- $\uparrow$ is infinitesimal but positive ($\uparrow > 0$).

Pick up or change colour:

There are more exotic games

- $\{ \ *1 \mid 0 \ \} = \ \downarrow < 0$ is similarly defined.

Pick up or change colour:



Back to partizan games that are numbers

- In a game

$$G = \{\alpha, a_1, a_2, \ldots, a_n \mid \beta, b_1, b_2, \ldots, b_n\},$$

  say that $\alpha > a_1 > a_2 > a_3 > \ldots$ and $\beta < b_1 < b_2 < b_3 < \ldots$, i.e. α is the largest option for Left and β is the smallest option for Right.
- Then G is the first number between α and β to be 'invented' by the Conway process.

Back to partizan games that are numbers

- In a game

$$G = \{\alpha, a_1, a_2, \ldots, a_n \mid \beta, b_1, b_2, \ldots, b_n\},$$

  say that $\alpha > a_1 > a_2 > a_3 > \ldots$ and $\beta < b_1 < b_2 < b_3 < \ldots$, i.e. α is the largest option for Left and β is the smallest option for Right.
- Then G is the first number between α and β to be 'invented' by the Conway process.
- If there are only options for Left — $\{\alpha, a_1, a_2, a_3, \ldots \mid \}$ — the game position is the earliest number greater than α.
- If there are only options for Right — $\{ \mid \beta, b_1, b_2, b_3, \ldots\}$ — the game position is the earliest number less than β.

Hackenbush

Hackenbush is played on a graph with black and grey edges. **L**eft removes b**L**ack edges and **R**ight removes g**R**ey edges. After a removal, any part of the graph not connected to the ground floats away, out of the game. The last player to make a move wins.