Worksheet 1: Impartial games

1. **Nim** Find all the winning moves, if any, in the following games.

   (a)

   (b)

   (c)

   (d)

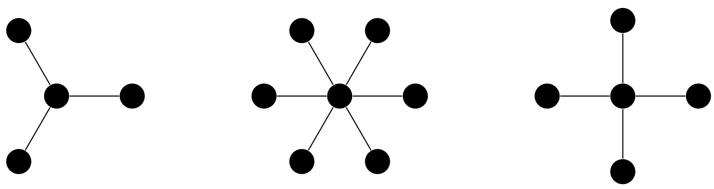
   (There are 32 sticks in the fourth pile.)

   (e)

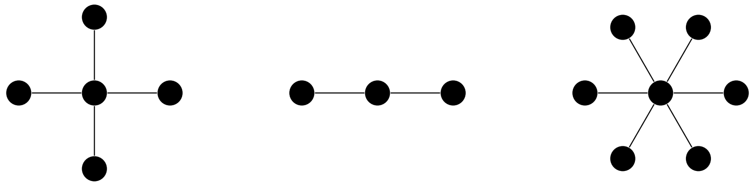
   (Ten in the third and twelve in the fourth.)

2. **Cutthroat Stars** Find all the winning moves, if any, in the following games.

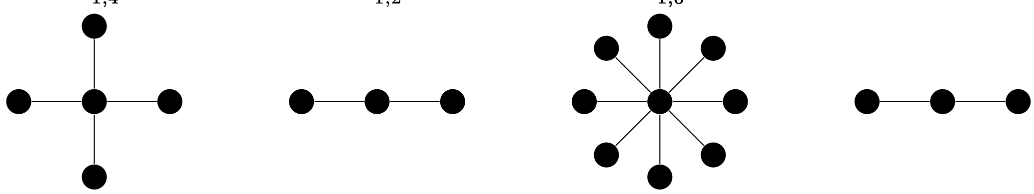
   $K_{1,3}$ $K_{1,6}$ $K_{1,4}$

   (a)

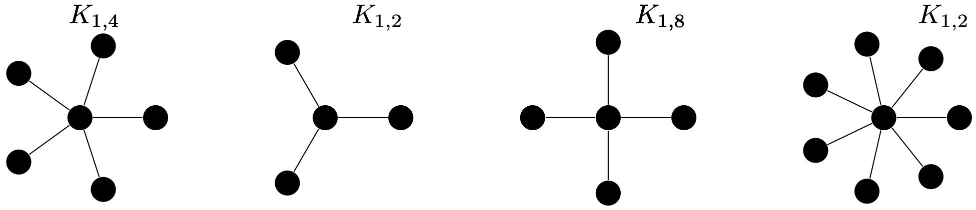
   $K_{1,4}$ $K_{1,2}$ $K_{1,6}$

   (b)

   $K_{1,4}$ $K_{1,2}$ $K_{1,8}$ $K_{1,2}$

   (c)

   $K_{1,5}$ $K_{1,3}$ $K_{1,4}$ $K_{1,7}$

   (d)

3. **Another game**

   Consider this game, in which two players take turns.

   Start with a non-negative integer n.

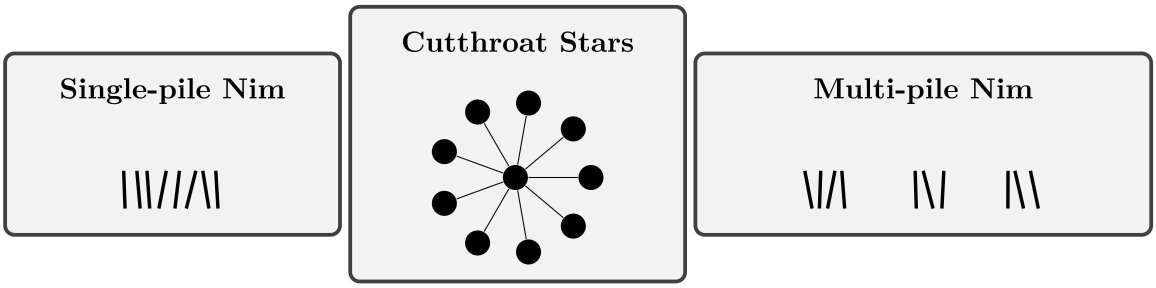
   Each turn, a player must change n by calculating one of $\lfloor \frac{n}{2} \rfloor$, $\lfloor \frac{n}{3} \rfloor$ or $\lfloor \frac{n}{6} \rfloor$.

   The winner is the player who makes $n = 0$.

   Find the equivalent Nim heaps for values of n from 0 to 12 and determine whether this is a $\mathcal{N}$-position or a $\mathcal{P}$-position.

   N.B. The brackets $\lfloor \ldots \rfloor$ are the 'floor' function. The floor of a real number x, written $\lfloor x \rfloor$, is the largest integer which is less than or equal to x. For example $\lfloor 5 \rfloor = 5$, $\lfloor 10.2 \rfloor = 10$ and $\lfloor 3.6 \rfloor = 3$.

4. **Multiple games** You and one other player are playing three impartial games simultaneously, such that the player to make the last move in the last game wins. It is your turn. Justifying your answer using a Nim sum argument, what is the optimal move assuming perfect play?



   Single-pile Nim: There is one heap of sticks. Players take turn to remove either 1, 2 or 3 sticks from a pile.

   Cutthroat Stars: There is one star graph. Players take turns to remove one vertex from the graph. When a vertex is removed, all the incident edges and any isolated vertices are also removed.

   Multi-pile Nim: There are three heaps of sticks. Players take turns to remove any positive number of sticks from precisely one heap.