About proof

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What is proof?

- An explanation of why a statement is true.
- A logical statement showing that a given conclusion is guaranteed from the starting point.
- Starting with a set of basic axioms (and possibly other results proved from those axioms).
- Uses deductive reasoning (logical certainty), not inductive reasoning (reasonable expectation).

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- Corollary: a true statement that follows simply from a theorem (or similar).
- Conjecture: a statement believed to be true, but not proven.

Statements

- We did a lot of work earlier in the module on propositions and logical connectives.
- A proof is often made of a series of these.
- Some theorems contain "if", and some contain "if and only if".
  - "if p, then q": $p \implies q$;
  - "p only if q": $p \iff q$.

Exploring a theorem

- Explore examples. The aim is to test the theorem and see how it behaves in different circumstances.
  - Find trivial examples.
  - Find extreme examples.
  - Find non-examples.

Creating examples and counterexamples

- One great way to explore a new theorem or conjecture is to try out some examples.
- Euler: "Some facts can be seen more clearly by example than by proof."
- Examples can also be used to disprove statements.
- An example which shows a statement to be false is called a counterexample.

Contraposition

- $p \implies q$ is not equivalent to $\neg p \implies \neg q$.
- e.g., “if $x = 2$, then $\sqrt{x}$ is irrational” is true,
  but “if $x \neq 2$, then $\sqrt{x}$ is not irrational” would mean $\sqrt{2}$ is the only irrational square root.

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"if $x = 2$, then $\sqrt{x}$ is irrational"	$\iff$	"if $\sqrt{x}$ is rational, then $x \neq 2$"

Examples

- Mathematics is often taught in the following way: "This is how the product rule for differentiation works, here are some examples, now you do some exercises like the example I've shown you".
- This is a 'worked example', which have their place for learning techniques but not for developing true understanding.
- For higher level mathematics, we are interested to explore problems that require you to think and apply what you have learned in situations you have not previously seen. (Sound familiar?)

Proof as problem solving

- Mathematicians solve problems – proof is the guarantee that our solutions are correct.
- The problem solving advice applies when trying to prove a conjecture or theorem.
- Remember that the first stage was exploring the problem and forming a plan.
  1. **Plan**. Understand what has been said. What would a proof look like? Try some cases, come up with some examples. Draw a picture. Have you seen a similar theorem before? Separate into parts, or try some special cases. Can you make a more general or more specific version and prove that?