Recurrence relations

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Definition. Let $S_0, S_1, S_2, \ldots$ be a sequence of numbers. A *linear recurrence relation with constant coefficients* is a formula which expresses S_k in terms of some (possibly all) of the terms that precede it for some $k > i$. Terms $S_0, S_1, \ldots, S_i$ are not defined by the recurrence formula, but are stated explictly (as initial conditions).

That is, we have constants $A_0, A_1, \ldots A_{n-1} \in \mathbb{R}$ and a function f such that

$$S_n = A_0 S_{n-1} + A_1 S_{n-2} + \cdots + A_{n-1} S_0 + f(n).$$

(Note that some of the A_i might be zero, and we might have $f(n) = 0$.)

1 Example

The Fibonacci Rabbits Puzzle can be stated as follows:

- A newly-born male-female pair of rabbits are placed in a field.
- After one month, newly-born rabbits are mature and begin to breed.
- One month after mating, females give birth to one male-female pair and then breed again.
- No rabbits die.
- How many rabbit pairs are there after one year?

In month 0, there are no rabbits (nothing has happened yet).
In month 1, there is 1 pair of rabbits.
In month 2, there is still 1 pair of rabbits and they mate.
In month 3, the female gives birth, so there are now 2 pairs; the initial pair mates again.
In month 4, the first pair have another pair of babies, and the first two sets of rabbits mate.

$$\vdots$$

The number of rabbits each month is made up from:

- the number that were alive last month;
- plus those that were alive two months ago give birth.

That is, we have a recurrence relation in the number after k months F_k.

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}.$$

We hope to find a solution of the form $F_n = r^n$, so substitute this into our relation.

$$\begin{aligned}
r^n &= r^{n-1} + r^{n-2} \\
r^n - r^{n-1} - r^{n-2} &= 0 \\
r^2 - r - 1 &= 0 \qquad \text{(division by } r^{n-2}\text{)} \\
r &= \frac{1 \pm \sqrt{5}}{2} \qquad \text{(by the quadratic formula).}
\end{aligned}$$

Since we have two (linearly-independent) solutions, we can form a general solution from these

$$F_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Given our initial conditions $F_0 = 0$, $F_1 = 1$, we can calculate the values of the constants α_1 and α_2.

$$\begin{aligned}
0 &= \alpha_1 + \alpha_2 \implies \alpha_2 = -\alpha_1 \\
1 &= \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) \\
&= \alpha_1 \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right) \\
&= \sqrt{5}\alpha_1 \implies \alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}.
\end{aligned}$$

So

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

2 General method

We can rearrange our recurrence relation as follows

$$C_0 S_0 + C_1 S_1 + C_2 S_2 + \cdots + C_n S_n = f(n),$$

where $C_n \in \mathbb{R}$ and $C_i = -C_n A_{n-i-1}$ for $0 \leq i < n$.

We seek a solution of the form $S_n = \alpha r^n$, where $\alpha \neq 0$ and $r \neq 0$, so we can substitute to form the *characteristic equation*

$$C_0 \alpha + C_1 \alpha r + C_2 \alpha r^2 + \cdots + C_n \alpha r^n = f(n).$$

When $f(n) = 0$ for all n, the relation is called *homogeneous*; otherwise, it is called *non-homogeneous*.

Consider the second order homogenenous recurrence relation

$$C_0 S_0 + C_1 S_1 + C_2 S_2 = C_0 \alpha + C_1 \alpha r + C_2 \alpha r^2 = 0.$$

Since $\alpha \neq 0$, this forms the quadratic equation

$$C_0 + C_1 r + C_2 r^2 = 0.$$

There are three cases for the solutions of this quadratic r_1, r_2:

1. r_1 and r_2 are distinct real numbers and we have a solution in the form $S_n = \alpha_1 r_1^n + \alpha_2 r_2^n$;

2. $r_1 = r_2$ are real and we have a solution in the form $S_n = (\alpha_1 + \alpha_2 n) r_1^n$;

3. $r_{1,2} = a\mathrm{e}^{\pm bi}$ form a complex conjugate pair and we have a solution in the form $S_n = a\left(\beta_1 \cos(bn) + \beta_2 \sin(bn)\right)$.

3 Examples

3.1 Example

Consider the recurrence relation

$$S_0 = 0, \quad S_1 = 2, \quad S_n = 2S_{n-1} + 8S_{n-2}.$$

We rearrange this to give

$$S_n - 2S_{n-1} - 8S_{n-2} = 0,$$

giving the characteristic equation

$$r^2 - 2r - 8 = 0.$$

Solving this, we find $r = 4$ or -2, so

$$S_n = \alpha_1 4^n + \alpha_2(-2)^2.$$

Since $S_0 = 0$, we have $0 = \alpha_1 + \alpha_2$, i.e. $\alpha_2 = -\alpha_1$.
Now since $S_1 = 2$,

$$\begin{aligned} 2 &= 4\alpha_1 - 2\alpha_2 \\ &= 6\alpha_1, \end{aligned}$$

i.e. $\alpha_1 = \frac{1}{3}$, $\alpha_2 = -\frac{1}{3}$.
So we have the general solution

$$S_n = \frac{1}{3}\left(4^n - (-2)^n\right).$$

3.2 Example

Consider the recurrence relation

$$S_0 = 1, \quad S_1 = 2, \quad S_n = 4(S_{n-1} - S_{n-2}).$$

We can rearrange this to give

$$S_n - 4S_{n-1} + 4S_{n-2} = 0,$$

giving the characteristic equation

$$r^2 - 4r + 4 = 0.$$

Solving this, we find $r = 2$, so

$$S_n = (\alpha_1 + \alpha_2 n)2^n.$$

Since $S_0 = 1$, we have $1 = \alpha_1$.
Now since $S_1 = 3$,

$$\begin{aligned} 3 &= (1 + \alpha_2) \times 2 \\ \alpha_2 &= \frac{3}{2} - 1 = \frac{1}{2}. \end{aligned}$$

So we have the general solution

$$S_n = \left(1 + \frac{n}{2}\right)2^n.$$

3.3 Example

Consider the recurrence relation

$$S_0 = 1, \quad S_1 = 5, \quad S_n = S_{n-1} - S_{n-2}.$$

We can rearrange this to give

$$S_n - S_{n-1} + S_{n-2} = 0,$$

giving the characteristic equation

$$r^2 - r + 1 = 0.$$

Solving this, we find $r = \frac{1 \pm \sqrt{3}}{2}$, so

$$S_n = \beta_1 \cos\left(\frac{\pi n}{3}\right) + \beta_2 \sin\left(\frac{\pi n}{3}\right).$$

Since $S_0 = 1$, we have $1 = \beta_1$.
Now since $S_1 = 5$,

$$5 = \beta_1 \cos\left(\frac{\pi}{3}\right) + \beta_2 \sin\left(\frac{\pi}{3}\right)$$

$$\beta_2 = \frac{9}{\sqrt{3}}.$$

So we have the general solution

$$S_n = \cos\left(\frac{\pi n}{3}\right) + \frac{9}{\sqrt{3}} \sin\left(\frac{\pi n}{3}\right).$$

4 Non-homogeneous recurrence relations

Just to touch on these briefly as an example, consider the following recurrence relation.

$$S_0 = 0, \quad S_1 = 1, \quad S_2 = 4, \quad S_n = 2S_{n-1} + 3S_{n-2} + 2.$$

This can be rearranged to give

$$S_n - 2S_{n-1} - 3S_{n-2} = 2,$$

which gives the characteristic equation

$$r^2 - 2r - 3 = 0.$$

Solving this, we obtain $r = -1$ or 3.

Since the right hand side is a constant, this means the solution must be of the form

$$S_n = \alpha_1(-1)^n + \alpha_2(3)^n + \beta$$

for some $\beta \in \mathbb{R}$ the particular solution.

(Note that $S_n = \alpha_1(-1)^n + \alpha_2(3)^n$ gives the solution to $S_n - 2S_{n-1} - 3S_{n-2} = 0$, so if β has the same form as this general solution, the form of the particular solution must take account of this — as βn perhaps. This is not the case here.)

Since $S_0 = 0$,$S_1 = 1$,$S - 2 = 4$, we can form the following system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 + \beta &= 0 \\ -\alpha_1 + 3\alpha_2 + \beta &= 1 \\ \alpha_1 + 9\alpha_2 + \beta &= 4 \end{aligned}$$

Solving this, we obtain $(\alpha_1, \alpha_2, \beta) = (0, \frac{1}{2}, -\frac{1}{2})$. Thus the solution is

$$S_n = \frac{1}{2}\left(3^n - 1\right).$$