

Patterns that break

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- ▶ A number's prime factorisation is when we write the number as a product of its prime factors.
- ▶ For example,
 - ▶ $6 = 2 \times 3$;
 - ▶ $18 = 2 \times 3 \times 3$.
- ▶ Notice 6 has an even number of factors and 18 has an odd number of factors.
- ▶ Let $E(n)$ be the size of the set

$$\{k \in \mathbb{Z}^+ \mid 2 < k \leq n \wedge k \text{ has an even number of factors}\}$$

- ▶ Let $O(n)$ be the size of the set

$$\{k \in \mathbb{Z}^+ \mid 2 < k \leq n \wedge k \text{ has an odd number of factors}\}$$

n	No. factors	$E(n)$	$O(n)$
1	0	1	0
2	1	1	1
3	1	1	2
4	2	2	2
5	1	2	3
6	2	3	3
7	1	3	4
8	3	3	5
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- ▶ In 1962, a counterexample was found.
- ▶ The smallest counterexample is $n = 906, 150, 257$.

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 - ▶ There are no $a, b, c \in \mathbb{Z}^+$ which satisfy

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = 4$$

until

$a = 154476802108746166441951315019919837485664325669565431700026634898253202035277999$

$b = 36875131794129999827197811565225474825492979968971970996283137471637224634055579$

$c = 4373612677928697257861252602371390152816537558161613618621437993378423467772036$

► What pattern starts 1, 1, 2, 3, 5, 8, 13 ...?