Matrices

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1 Matrices

A *matrix* is a rectangular array of elements which we enclose with square brackets (note that some sources you may read use rounded brackets). The numbers in a matrix can be positive, negative, zero, fractions, decimals and so on. For example, the following are matrices:

$$\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & \sqrt{17} \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ \pi & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To refer to a matrix later, we might label it, usually with a capital letter. For example:

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 2 & 3 & \sqrt{17} \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ \pi & 0 \end{bmatrix}.$$

Since matrices all have different sizes, we refer to a matrix by its number of *rows* and number of *columns*, in that order. Above, matrix $\mathbf{A}$ has two rows and two columns, so we call this a 2×2 matrix (we say "two by two matrix"). Similarly, $\mathbf{B}$ is 3×3, $\mathbf{C}$ is 1×3 and $\mathbf{H}$ is 3×2.

A general $m \times n$ matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Where a_{ij} represents the element in the ith row and jth column.

Some terminology

A *square matrix* is one that has the same number of rows as columns. For example:

$$\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 2 & \pi & 7 & 19 \\ -1 & 0 & -1 & 0 & 0 & 5 \\ 3 & -1 & 0 & 56 & 2 & -3 \\ 3 & -1 & 0 & 57 & 2 & -3 \\ 0 & 0 & 0 & 9 & -9 & -1 \\ 1 & -1 & 1 & 5 & -3 & \sqrt{2} \end{bmatrix}.$$

A *diagonal matrix* is a square matrix that has zeros everywhere except possibly on the *leading diagonal*, which runs from top left to bottom right. For example:

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

An *identity matrix* is a diagonal matrix with all the diagonal elements equal to 1. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The letter $\mathbf{I}$ is usually used to refer to the identity matrix.

A *zero matrix* is a matrix for which every element is zero. For example:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}.$$

2 Matrix addition and subtraction

We can only add and subtract matrices that are the same size, that is, those that have the same number of rows and columns.

When two matrices are compatible in this way we can add (or subtract) them by adding (or subtracting) the elements in corresponding positions.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a+r & b+s & c+t \\ d+u & e+v & f+w \\ g+x & h+y & i+z \end{bmatrix}$$

3 Scalar multiplication

We can always add a matrix to itself. For example,

$$\mathbf{A} + \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & -2 \end{bmatrix}$$

We can write $\mathbf{A} + \mathbf{A}$ as $2\mathbf{A}$. Notice that all the elements of $\mathbf{A}$ have doubled in $2\mathbf{A}$. This illustrates how to multiple a matrix by a number (a *scalar*), which we call *scalar multiplication*.

To multiply a matrix by a scalar, multiply each element in the matrix by that number. For example:

$$3 \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

$$k \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

4 Matrix multiplication

Consider the matrix multiplication $\mathbf{AB}$. The product of two matrices can only be found if the number of columns in the first matrix, $\mathbf{A}$, is the same as the row of rows in the second, $\mathbf{B}$.

So if $\mathbf{A}$ is a $p \times q$ matrix and $\mathbf{B}$ is a $r \times s$ matrix, we can only multiply them together if $q = r$. If this is the case and we multiply them together, the answer matrix would be a $p \times s$ matrix.

For example, here we multiple a 3×2 matrix by a 2×3 matrix. The result will be a 3×3 matrix.

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

To multiply two matrices, $\mathbf{AB}$, where possible, first move along the first row of $\mathbf{A}$ and multiply each element by a corresponding element moving down the first column of $\mathbf{B}$. Then multiple the first row of $\mathbf{A}$ by the second column of $\mathbf{B}$, then the third, and so on. When you have used every column in $\mathbf{B}$, move onto the second row of $\mathbf{A}$ and multiply this by the first column of $\mathbf{B}$, then the second, and so on. Eventually you will find that you fill every element you expect to find in $\mathbf{AB}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix};$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj + bm + cp & ak + bn + cq & al + bo + cr \\ dj + em + fp & dk + en + fq & dl + eo + fr \\ gj + hm + ip & gk + hn + iq & gl + ho + ir \end{bmatrix}.$$

For example:

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 6 & 1 & -3 \end{bmatrix}.$$

Consider

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 10 \end{bmatrix}.$$

Here we have a 2×2 matrix multiplied by a 2×3 matrix, so the result is a 2×3 matrix.

Now consider

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

Here we have a 2×3 matrix multiplied by a 2×2 matrix, which is not possible because the dimensions are not compatible.

Note that just because $\mathbf{AB}$ is possible, does not mean that $\mathbf{BA}$ is possible. We say that matrix multiplication is not commutative.

Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

Notice that multiplication by an identity matrix leaves a matrix unchanged, i.e. $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.

5 Transpose of a matrix

The transpose of a matrix $\mathbf{A}$, denoted $\mathbf{A}^\mathsf{T}$, is obtained by exchanging the rows and columns of the matrix $\mathbf{A}$. For example,

$$\begin{bmatrix} 1 & 0 & 9 \\ -1 & 0 & 2 \\ 3 & 1 & -6 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 1 \\ 9 & 2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 3 & 1 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$