Worksheet 1: Impartial games – answers

1. (a) Nim sum is 1. Remove 1 stick from any pile.
   (b) Nim sum is 3. Remove 1 stick from the 2-heap, or remove 3 sticks from the 3-heap or 7-heap.
   (c) Nim sum is 13. Remove 5 sticks from the 9-heap.
   (d) Nim sum is 46. Remove 18 sticks from the 32-heap.
   (e) Nim sum is 0. No winning move possible.

2. **Cutthroat Stars** Find all the winning moves, if any, in the following games.
   (a) $K_{1,3}$, $K_{1,6}$ and $K_{1,4}$ are equivalent to $*1$, $*2$ and $*2$, respectively. The game is therefore in a $\mathcal{N}$-position. Remove the central node of $K_{1,3}$.
   (b) $K_{1,4}$, $K_{1,2}$ and $K_{1,6}$ are equivalent to $*2$, $*2$ and $*2$, respectively. The game is therefore in a $\mathcal{N}$-position. Remove the central node of any graph.
   (c) $K_{1,4}$, $K_{1,2}$, $K_{1,8}$ and $K_{1,2}$ are equivalent to $*2$, $*2$, $*2$ and $*2$, respectively. The game is therefore in a $\mathcal{P}$-position and there is no winning move.
   (d) $K_{1,5}$, $K_{1,3}$, $K_{1,4}$ and $K_{1,7}$ are equivalent to $*1$, $*1$, $*2$ and $*1$, respectively. The game is therefore in a $\mathcal{N}$-position. Remove a radial node from either $K_{1,4}$ or $K_{1,5}$.

3.

n	Set of available options	mex	Nim equivalent	Type
$n = 0$	$\{\}$	0	$*0$	$\mathcal{P}$
$n = 1$	$\left\{\lfloor\frac{1}{2}\rfloor, \lfloor\frac{1}{3}\rfloor, \lfloor\frac{1}{6}\rfloor\right\} = \{0, 0, 0\}$	1	$*1$	$\mathcal{N}$
$n = 2$	$\left\{\lfloor\frac{2}{2}\rfloor, \lfloor\frac{2}{3}\rfloor, \lfloor\frac{2}{6}\rfloor\right\} = \{1, 0, 0\}$	2	$*2$	$\mathcal{N}$
$n = 3$	$\left\{\lfloor\frac{3}{2}\rfloor, \lfloor\frac{3}{3}\rfloor, \lfloor\frac{3}{6}\rfloor\right\} = \{1, 1, 0\}$	2	$*2$	$\mathcal{N}$
$n = 4$	$\left\{\lfloor\frac{4}{2}\rfloor, \lfloor\frac{4}{3}\rfloor, \lfloor\frac{4}{6}\rfloor\right\} = \{2, 1, 0\}$	3	$*3$	$\mathcal{N}$
$n = 5$	$\left\{\lfloor\frac{5}{2}\rfloor, \lfloor\frac{5}{3}\rfloor, \lfloor\frac{5}{6}\rfloor\right\} = \{2, 1, 0\}$	3	$*3$	$\mathcal{N}$

- For $6 \leq n \leq 11$, note that $\lfloor\frac{n}{2}\rfloor \geq 3$, $\lfloor\frac{n}{3}\rfloor \geq 2$ and $\lfloor\frac{n}{6}\rfloor \geq 1$, so the set of available options $\{\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{3}\rfloor, \lfloor\frac{n}{6}\rfloor\}$ cannot contain 0. Therefore values of n for $6 \leq n \leq 11$ have mex 0 and are equivalent to $*0$, a $\mathcal{P}$-position.
- For $n = 12$, the available options are $\left\{\lfloor\frac{12}{2}\rfloor, \lfloor\frac{12}{3}\rfloor, \lfloor\frac{12}{6}\rfloor\right\} = \{6, 4, 3\}$. Using the values above, this is equivalent to $*0, *3, *2$, so the mex is 1 and $n = 12$ is equivalent to $*1$, a $\mathcal{N}$-position.

4. Single-pile Nim is equivalent to $*0$. Cutthroat Stars is equivalent to $*1$. Multi-pile Nim is equivalent to $*4$. Overall Nim sum is $0 \oplus 1 \oplus 100 = 101 = *5$. Play in Multi-pile Nim by removing 3 sticks from the 4-heap.