Implication

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My friend Katie

- If someone is called Katherine, their name can be shortened to Katie.
- My friend is called Katie.
- Is her name Katherine?

Implies

- The connective $\implies$ is called "implies".
- A proposition $p \implies q$ can be read "p implies q" or "if p, then q".

Special meaning

- In standard language, if we say "A happening implies B will happen", we are making a causal link.
- i.e. we are saying that A causes B.
- In logic we separate this into two parts:
  - that the two things happen together;
  - that one thing causes the other.
- When we say $p \implies q$, we are dealing with the first of these.
- This can make $p \implies q$ a bit strange to deal with.

Shapes

- Say
  - p: The shape has four sides.
  - q: The shape is a square.
- Are the following true?
  1. $p \implies q$;
  2. $q \implies p$.
  3. $\neg p \implies \neg q$;
  4. $\neg q \implies \neg p$.

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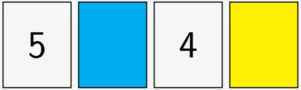
  3. $\neg p \implies \neg q$;
  4. $\neg q \implies \neg p$.
- $q \implies p$ is true, but $p \implies q$ is false.
- $\neg p \implies \neg q$ is true, but $\neg q \implies \neg p$ is false.

If and only if

- If both $p \implies q$ and $q \implies p$ are true, then we write $p \iff q$ and say "p if and only if q".

Wason selection task

A set of cards each have a number on one side and a colour on the other. You are shown four of these cards on a table. Which card or cards must you turn over in order to test that if a card shows an even number on one side, then its opposite face is blue?



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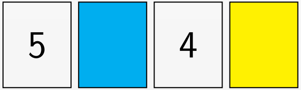

- This question contains these two propositions:
  - p: a card has an even number on one side;
  - q: a card has a blue side.
- The puzzle is asking us to test the implication $p \implies q$.

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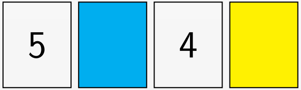

- This question contains these two propositions:
  - p: a card has an even number on one side;
  - q: a card has a blue side.
- The puzzle is asking us to test the implication $p \implies q$.
- Here is what we know:

Card	p	q
5	false	?
4	true	?
blue	?	true
yellow	?	false

Wason selection task

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- p: a card has an even number on one side;
- q: a card has a blue side.

- We are testing $p \implies q$, so the other side of the '5' card doesn't matter.
- When p is true, is q also true? So turn the '4' card.

Wason selection task

Card	p	q
5	false	?
4	true	?
blue	?	true
yellow	?	false

- p: a card has an even number on one side;
- q: a card has a blue side.
- Similarly, we don't need to know what is on the other side of the blue card.
- We do need to check the yellow card. If this is even, it would mean p is true and q is false, which would contradict $p \implies q$.

$p \implies q$

- $p \implies q$ is true unless p is true and q is false.
- The truth table is as follows. The '$p \implies q$' column refers to the truth of the implication, not the truth of either p or q.

p	q	$p \implies q$
true	true	true
true	false	false
false	true	true
false	false	true

- If p is false, this tells us nothing to refute the idea that $p \implies q$, so we say the implication is true.

Example

- Consider the following statements:
  - p: "$5 = 7$."
  - q: "I will give you £100."
- $p \implies q$ means if $5 = 7$, then I will give you £100.
- Since $5 \neq 7$, this tells us nothing to refute the idea that $p \implies q$, so we say the *implication* "$p \implies q$" is true.

Raven paradox

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- Hypothesis: “all ravens are black”.

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- Hypothesis: "all ravens are black".
- We can express this as two propositions:
  - p: "Something is a raven";
  - q: "Something is black".
- We are testing $p \implies q$.

Raven paradox

- The fact that p being false does nothing to refute $p \implies q$ is used in the Raven paradox.
- Hypothesis: “all ravens are black”.
- We can express this as two propositions:
  - p: “Something is a raven”;
  - q: “Something is black”.
- We are testing $p \implies q$.
- Suppose we see a yellow banana. p is false and q is false.
- This provides an example of $\neg p \implies \neg q$.
- Since this observation did not refute our hypothesis, this provides evidence our hypothesis is correct.

If q is always true

- It is also worth nothing that if q is always true, then $p \implies q$ is true regardless of p. In fact, q being true tells us nothing about the truth value of p.
- For example, consider these statements:
  - p: "Alice likes carrots."
  - q: "$1 + 1 = 2$."
- Since q is always true, we can say both
  - If Alice likes carrots, then $1 + 1 = 2$;
  - If Alice doesn't like carrots, then $1 + 1 = 2$.
- i.e. $p \implies q$ and $\neg p \implies q$.

Reminder: Smullyan's island

- Raymond Smullyan sets a series of puzzles on an island in which there are two types of inhabitants:
  - 'knights', who always tell the truth;
  - 'knaves', who always lie.
- Everyone on the island is either a knight or a knave.

Example

- Here are some propositions about an inhabitant of Smullyan's island:
  - p: "Alice says 'one plus one equals three'."
  - q: "Alice is a knave."
- Since knights always tell the truth and knaves always lie, $p \implies q$.

Example

- The only situation where $p \implies q$ is considered false is if p is true and q is false. Consider these statements:
  - p: "Alice is a knave."
  - q: "Alice says she is a knave."
- If Alice were a knight ($\neg p$), she would tell the truth and not claim to be a knave ($\neg q$).
- If Alice were a knave (p), she would lie and claim to not be a knave ($\neg q$).
- Since q is always false, if Alice is a knave, then p is true and $p \implies q$ is false.

$p \implies q$

- It may be helpful then to understand $p \implies q$ as meaning one of these equivalent statements:
  - "it is not the case that p is true and q is not true": $\neg(p \wedge \neg q)$.
  - "either q is true or p is not true": $\neg p \vee q$.

A puzzle from last week

You are a visitor to the island and meet three of the inhabitants: Alice, Bob, and Carol. You ask Alice "How many knights are among you?" Alice answers, but mumbles so you don't hear her answer. You ask Bob, "What did Alice say?" Bob replies "Alice said there is one knight among us." At this point Carol said "Bob is lying."

What types are Bob and Carol?

- We can set up a series of propositions and use these to make arguments about this situation.
    - a: "Alice is a knight."
    - b: "Bob is a knight."
    - c: "Carol is a knight."
    - p: "Alice said exactly one of the three is knight."
    - q: "Bob said 'Alice said there is one knight among us'."
    - r: "Carol said 'Bob is lying'."

- a: "Alice is a knight."
- b: "Bob is a knight."
- c: "Carol is a knight."
- p: "Alice said exactly one of the three is knight."
- q: "Bob said 'Alice said there is one knight among us'."
- r: "Carol said 'Bob is lying'."

- Since we know r is true, Carol and Bob must be of different types.
  - $r \wedge c \implies \neg b$;
  - $r \wedge \neg c \implies b$.
- This means that either b is true or c is true, but not both, so there is one knight among Bob and Carol.

- a: "Alice is a knight."
- b: "Bob is a knight."
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- p: "Alice said exactly one of the three is knight."
- q: "Bob said 'Alice said there is one knight among us'."
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- If Alice did not say that exactly one of the three is a knight and Bob lied that she did, then Bob must be a knave, so $q \wedge \neg p \implies \neg b$.
- If Alice were a knight, there would be two knights among the trio and Alice could not lie by saying there was only one, so $a \implies \neg p$.
- If Alice were a knave, then it is true that there is exactly one knight amongst them and Alice could not make this true claim, so $\neg a \implies \neg p$.

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- b: "Bob is a knight."
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- We know q is true because we heard Bob say what is claimed. Since p is always false and we have established $q \wedge \neg p \implies \neg b$, it follows that Bob is a knave and Carol is a knight.