Permutations – exercises

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1. For the permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 4 & 6 & 2 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}$$

evaluate (a) $\sigma_2\sigma_1$; (b) $\sigma_1\sigma_2$; (c) σ_1^{-1}; (d) σ_2^{-1}.

2. For the permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 4 & 1 & 2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 7 & 2 & 1 & 6 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 6 & 1 & 3 & 2 & 4 \end{pmatrix}$$

evaluate (a) σ_2^{-1}; (b) $\sigma_2\sigma_1$; (c) $\sigma_1\sigma_2$; (d) $\sigma_3\sigma_2\sigma_1$; (e) $\sigma_1\sigma_1 = \sigma_1^2$.

3. The symmetries of a square can be looked at as permutations of its vertices. There are 4 vertices so there should be $4! = 24$ symmetries, however there are only 8. Where have the other 16 gone?

4. Recall that the order of an element g in a group is $n \in \mathbb{N}$ such that $g^n = e$. In this way, the order of a permutation is the number of times it takes to get back to the original ordering.

   Determine the order of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 4 & 6 & 2 \end{pmatrix}.$$

5. Break each of the following permutations into disjoint cycles. Calculate the order of each one.

   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$;

   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 5 & 4 & 6 & 8 & 7 \end{pmatrix}$;

   (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 1 & 6 & 11 & 4 & 2 & 7 & 9 & 8 & 5 & 10 \end{pmatrix}$.

6. Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$.

   (a) Show that the order of σ is 4.

   (b) Let $\sigma^2 = \tau$, $\sigma^3 = \rho$, and the identity element be e. Show that $\{e, \sigma, \tau, \rho\}$ forms a group under the composition of permutations, and draw up the group table.

7. (a) How many members does S_3 have? Write them down.

   (b) Show that the subset $\{g \in S_3 \mid g^2 = e\}$ is not a subgroup of S_3.