Proof by induction – exercises

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1. Prove that
$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1).$$

2. Consider the statement $p(n) : 2^n < 2^{n-1}$.
   (a) Show that the inductive step holds, i.e. $p(k-1) \implies p(k)$.
   (b) This means we need a base case to show that the statement is true. Show that we do not have a base case.

3. Prove that $2n \leq 2^n$ for all $n \in \mathbb{N}$.

4. Prove that $3^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

5. Prove that 17 divides $3^{4n} + 4^{3n+2}$ for all $n \in \mathbb{N}$.

6. Prove that $\sin(nx) \leq n\sin(x)$ for all $n \in \mathbb{N}$ and $0 \leq x \leq \frac{\pi}{2}$.

7. Prove that the number of edges in the complete graph K_n is $\frac{n(n-1)}{2}$.

8. Using the fact that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for all $0 \leq r \leq n$, prove the Binomial Theorem, that
$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \quad \forall n \in \mathbb{N}.$$

9. Prove that $(1+x)^n \geq 1 + nx$ for all $n \geq 0$, where $x \in \mathbb{R}^+$.

10. Let $a \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Find $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^2$ and $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^3$. Use your results to guess a formula for $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^n$. Prove by induction that your formula is valid for all $n \geq 1$.