Group theory

Peter Rowlett

Recall that we defined $(G, \circ)$ as a set G and a binary operation on that set, $\circ$. Remember that the definition of a binary operation tells us that $(G, \circ)$ has **closure**.

- **Closure**: $\forall a, b \in G \ (a \circ b \in G)$.

1 Group

If $(G, \circ)$ satisfies the following, then it is called a *group*.

- **Associativity**: $\forall a, b, c \in G \ (a \circ (b \circ c) = (a \circ b) \circ c)$.
- **Identity**: $\exists e \in G \ \forall a \in G \ (e \circ a = a \circ e = a)$.
- **Inverse**: $\forall a \in G \ \exists b \in G \ (a \circ b = b \circ a = e)$

2 Abelian group

If $(G, \circ)$ is a group that also satisfies commutativity, it is called an *Abelian group*.

- **Commutativity**: $\forall a, b \in G \ (a \circ b = b \circ a)$.

3 Algebraic structures that are not groups

- If $(G, \circ)$ satisfies closure but none of the other properties, it is called a *magma*, or sometimes a *groupoid*.
- If $(G, \circ)$ satisfies closure and associativity only, it is a *semi-group*.
- If $(G, \circ)$ satisfies closure, associativity, and has an identity, it is a *monoid*.

4 Order of an element and generators

For a group with identity e, the *order* of an element a is the smallest positive integer power of a such that $a^n = e$.

A *generator* is an element g such that every element of the group can be expressed as some power of g.

A group with a generator is called a *cyclic group*.

If g is a generator for a group G, and $g^n = e$, then we write $G = \langle g \rangle$ and can draw a general group table.

	e	g	g^2	g^3	$\dots$	g^{n-2}	g^{n-1}
e	e	g	g^2	g^3	$\dots$	g^{n-2}	g^{n-1}
g	g	g^2	g^3	g^4	$\dots$	g^{n-1}	e
g^2	g^2	g^3	g^4	g^5	$\dots$	e	g
g^3	g^3	g^4	g^5	g^6	$\dots$	g	g^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
g^{n-2}	g^{n-2}	g^{n-1}	e	g	$\dots$	g^{n-4}	g^{n-3}
g^{n-1}	g^{n-1}	e	g	g^2	$\dots$	g^{n-3}	g^{n-2}

5 Subgroups

A *subgroup* of a group $(G, \circ)$ is some $H \subseteq G$ such that $(H, \circ)$ forms a group in its own right.