What can go wrong? 2: Full PageRank algorithm

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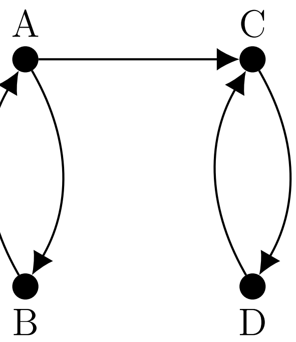
1 Scenario

Here is another voting situation.



Again, we make a table and corresponding matrix.

		From			
		A	B	C	D
To	A	0	1	0	0
	B	$\frac{1}{2}$	0	0	0
	C	$\frac{1}{2}$	0	0	1
	D	0	0	1	0

Now make a matrix using these values as the entries.

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2 What's the problem?

Finding the eigenvector corresponding to $\lambda = 1$, we obtain:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first row: $-w + x = 0$, so $w = x$.
From the second row: $\frac{1}{2}w - x = 0$, so $w = 2x$.
The only way to resolve $w = x$ and $w = 2x$ is with $w = x = 0$.
From the fourth row: $y - z = 0$, so $y = z$.
So let $\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.
This is not ideal:

- we would like to rank A & B;
- person C appears to get no credit from the vote from person A.

3 A solution

In a sense, the votes are pulled across from the left hand side of our graph to the right, never to return. One way to get around this problem is to allow connections between all nodes, though if we allow these to influence the decision too much then they will override the votes and we don't want that.

Say we follow the votes with some probability $1 - \alpha$ and use uniform random allocation with probability α.

Then we can form a new matrix

$$(1-\alpha) \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{4} & 1-\alpha+\frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} \\ \frac{\frac{1-\alpha}{2}+\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} \\ \frac{\frac{1-\alpha}{2}+\alpha}{4} & \frac{\alpha}{4} & \frac{\alpha}{4} & 1-\alpha+\frac{\alpha}{4} \\ \frac{\alpha}{4} & \frac{\alpha}{4} & 1-\alpha+\frac{\alpha}{4} & \frac{\alpha}{4} \end{bmatrix}.$$

Say let $\alpha = \frac{3}{20} = 0.15$, so we mostly follow the votes. Then we have

$$\begin{bmatrix} \frac{3}{80} & \frac{71}{80} & \frac{3}{80} & \frac{3}{80} \\ \frac{37}{80} & \frac{3}{80} & \frac{3}{80} & \frac{3}{80} \\ \frac{37}{80} & \frac{3}{80} & \frac{3}{80} & \frac{71}{80} \\ \frac{3}{80} & \frac{3}{80} & \frac{71}{80} & \frac{3}{80} \end{bmatrix} = \frac{1}{80} \begin{bmatrix} 3 & 71 & 3 & 3 \\ 37 & 3 & 3 & 3 \\ 37 & 3 & 3 & 71 \\ 3 & 3 & 71 & 3 \end{bmatrix}.$$

Notice all columns still sum to 1 and we still have $\lambda = 1$ as an eigenvector.

We find the eigenvector corresponding to $\lambda = 1$ in the usual way. Since this is a 4×4 matrix with no zero entries, this is more laborious than usual and we might make use of software.

Doing so, we obtain an eigenvector

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0.1086 \\ 0.0837 \\ 0.4163 \\ 0.3914 \end{bmatrix}$$

This gives us a ranking C, D, A, B.
The advantage of this is that no person got 0% of the vote and all could be ranked.