

Partizan games

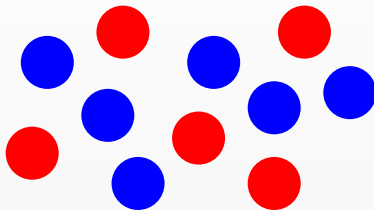
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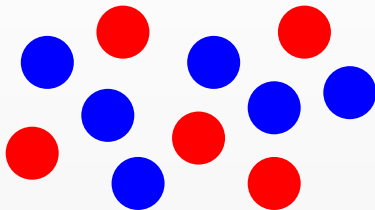
A simple partizan game

- **Pick up your colour:** Playing on a pile of blue and red counters, Left picks up any number of blue counters and Right picks up any number of red counters. The first player who cannot move loses.



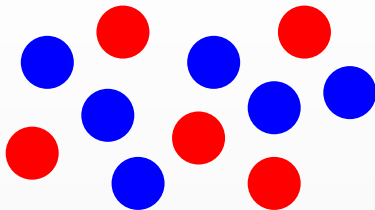
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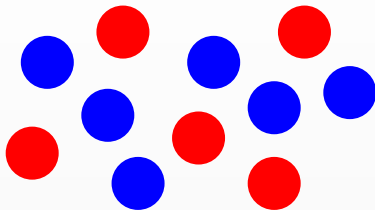
- Note: **L**eft plays **bL**ue and **R**ight plays **R**ed.

A simple partizan game



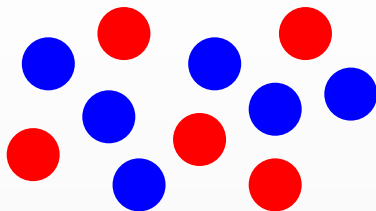
- This is different to the games we have been playing so far because it is a *partizan* game: Left and Right have their own counters which only they can move, and different goals.

A simple partizan game



- It is a simple game because, e.g. in this position it is a win for Left going either first or second because Left has more counters.

Pick up your colour



- ▶ Let's think about the game in terms of the number of counters advantage Left has over Right.
- ▶ So we add $+1$ for each blue counter and -1 for each red counter.
- ▶ The game position shown is

$$6 + (-5) = 1.$$

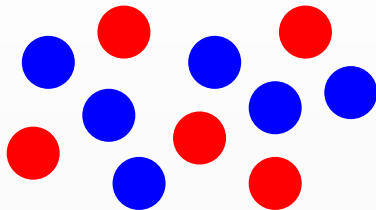
Notation for partizan games

► Let

$$G = \{ \underbrace{a_1, a_2, a_3, \dots}_{L \text{ positions}} \mid \underbrace{b_1, b_2, b_3, \dots}_{R \text{ positions}} \}$$

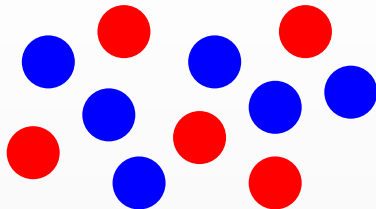
be a game where L represents the positions Left can move to and R represents the positions Right can move to.

Pick up your colour



- ▶ Left must remove between one and six blue counters, so they can move the game to any position in $\{0, -1, -2, -3, -4, -5\}$.
- ▶ Right must remove between one and five red counters, so they can move the game to any position in $\{2, 3, 4, 5, 6\}$.

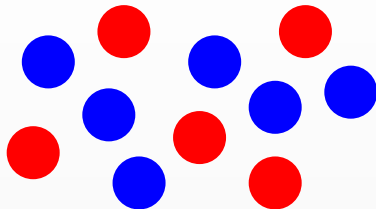
Pick up your colour



- So we can write this game position we called 1 as

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

Pick up your colour



- So we can write this game position we called 1 as

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

- Notice that a move by either player makes their own position worse.

Conway process

- ▶ A strange way of inventing numbers

Conway process

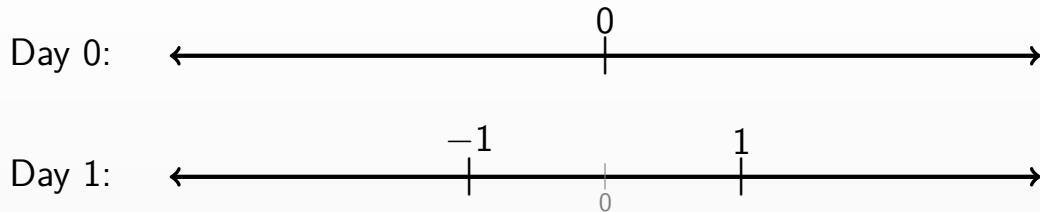
- ▶ A strange way of inventing numbers
- ▶ On day 0 the number 0 is 'invented'.
- ▶ Then on day n there are 2^n new numbers 'invented'.
- ▶ If on a day we have numbers $a_1 < a_2 < \dots < a_k$ then the next day we create:
 - ▶ The largest integer smaller than a_1 ;
 - ▶ The smallest integer larger than a_k ;
 - ▶ for every pair a_i, a_{i+1} with $i \in \{1, \dots, k-1\}$:

$$\frac{a_i + a_{i+1}}{2}.$$

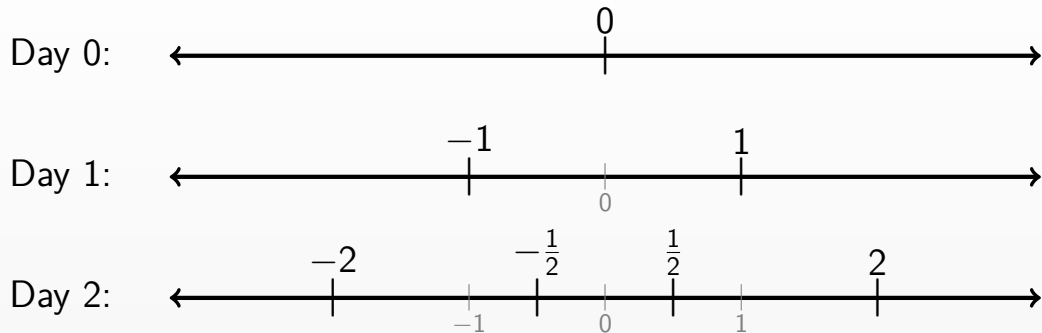
Conway process: early stages



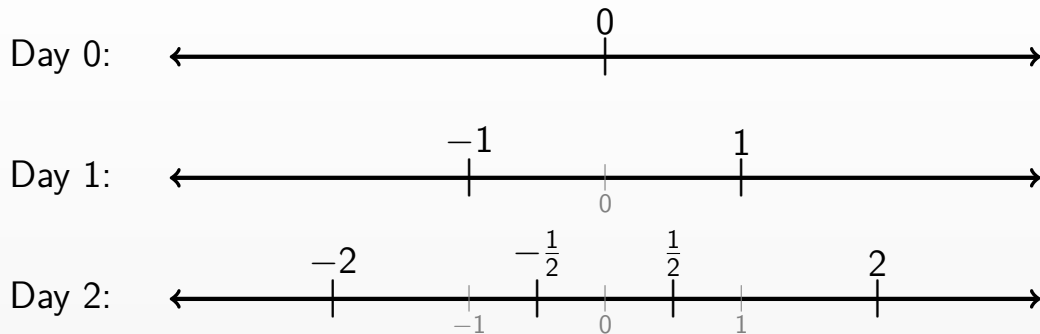
Conway process: early stages



Conway process: early stages

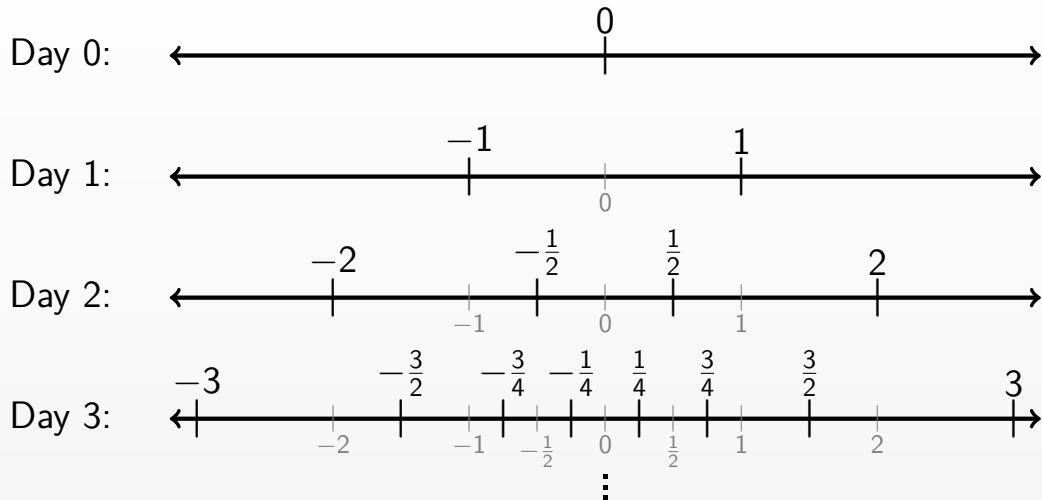


Conway process: early stages



What numbers are invented on day 3?

Conway process: early stages



Consider a game

- ▶ Consider a simple game G in which
 - ▶ Left can change in one move to any of $\alpha > a_1 > a_2 > \cdots > a_n$;
 - ▶ Right can change in one move to any of $\beta < b_1 < b_2 < \cdots < b_n$.
- ▶ We can write this

$$G = \{\alpha, a_1, a_2, \dots, a_n \mid \beta, b_1, b_2, \dots, b_n\}.$$

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- ▶ We can write this

$$G = \{\alpha, a_1, a_2, \dots, a_n \mid \beta, b_1, b_2, \dots, b_n\}.$$

- ▶ Then there are three options:
 1. $\{\alpha, a_1, a_2, \dots, a_n\} = \{\beta, b_1, b_2, \dots, b_n\}$:
 - ▶ Then we don't need to distinguish between Left and Right and can just write $G = \{\alpha, a_1, a_2, \dots, a_n\}$.
 - ▶ Actually, we've seen games like this and labelled them using the $*$ notation.
 - ▶ They are equivalent to Nim heaps.

Consider a game

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 - ▶ Left can change in one move to any of $\alpha > a_1 > a_2 > \cdots > a_n$;
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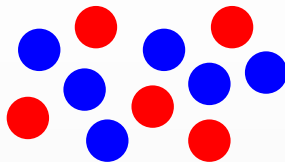
2. $\alpha < \beta$:

- ▶ G is a number;
- ▶ It's the 'simplest' number between α and β ;
- ▶ That is, the first to be 'invented' by the Conway process.
- ▶ We're going to consider games like this today.

Numbers that are games

- ▶ In a lot of partizan games a move by either Left or Right makes their own position worse.
- ▶ So for a game position $\{a \mid b\}$ with $a < b$:
 - ▶ a move by Left to position c will have $c < a$, so we now have position $\{c \mid b\}$ with $c < b$;
 - ▶ a move by Right to position d will have $d > b$, so we now have position $\{a \mid d\}$ with $a < d$.

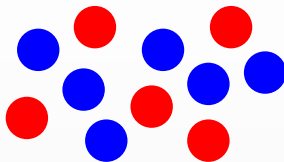
Back to Pick up your colour



► We called this game

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

Back to Pick up your colour

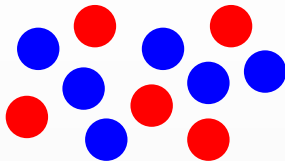


- We called this game

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

- Note that $0 < 2$.

Back to Pick up your colour

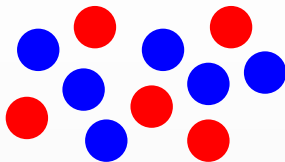


- We called this game

$$\{0, -1, -2, -3, -4, -5 \mid 2, 3, 4, 5, 6\}.$$

- Note that $0 < 2$.
- This game position is the first number between 0 and 2 to be invented by the Conway process.

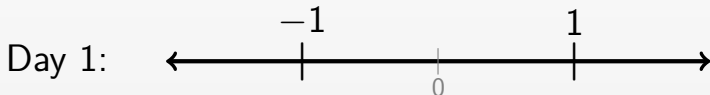
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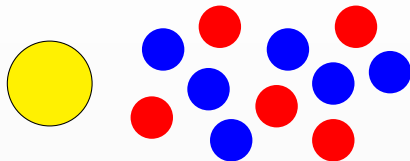
The third option

- ▶ Consider a simple game G in which
 - ▶ Left can change in one move to any of $\alpha > a_1 > a_2 > \dots > a_n$;
 - ▶ Right can change in one move to any of $\beta < b_1 < b_2 < \dots < b_n$.
- ▶ We can write this

$$G = \{\alpha, a_1, a_2, \dots, a_n \mid \beta, b_1, b_2, \dots, b_n\}.$$

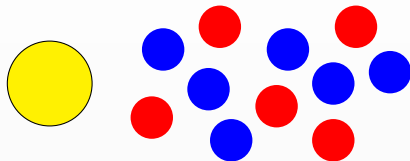
- ▶ Then there are three options:
 3. $\alpha > \beta$: This is called a 'hot game' or a 'switch'.

Pick up your colour with a twist



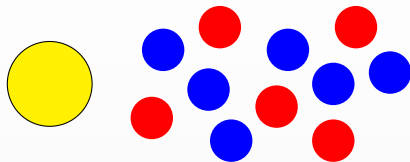
- This is the same game except if you take the yellow token it places five counters of your colour on the board.

Pick up your colour with a twist



- ▶ The options for each player are:
 - ▶ Left can remove between one and six blue counters, moving to one of $\{0, -1, -2, -3, -4, -5\}$, or take the yellow token, adding five blue counters and so moving the game to 6;
 - ▶ Right can remove between one and five red counters, moving to one of $\{2, 3, 4, 5, 6\}$, or take the yellow token, adding five red counters and so moving the game to -4 .

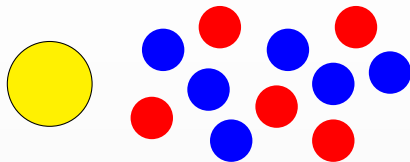
Pick up your colour with a twist



► We can write this as

$$G = \{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

Pick up your colour with a twist

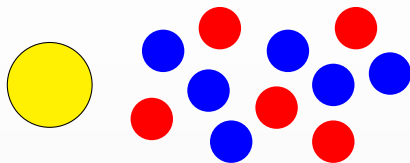


- We can write this as

$$G = \{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

- Note that $6 > -4$.

Pick up your colour with a twist

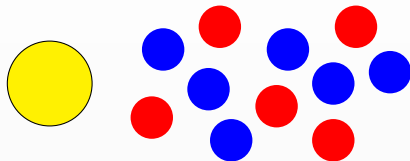


- We can write this as

$$G = \{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

- Note that $6 > -4$.
- This is 'hot': everyone wants to play it as quickly as possible.
- We call other positions 'cold': each player makes their own position worse by playing.

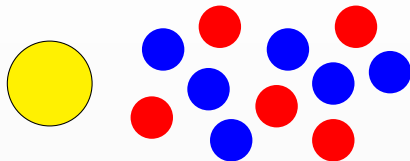
Pick up your colour with a twist



- ▶ Consider the yellow token alone.
- ▶ This is not $*5$ because it does not move the game to the same position regardless of who takes it.
- ▶ It is worth $+5$ to Left and -5 to Right. We can write this

$$\pm 5 = \{5 \mid -5\}$$

Pick up your colour with a twist



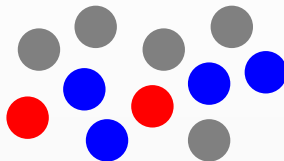
- ▶ Another way of scoring this game, then, is:
- ▶ The blue and red dots were 1.
- ▶ So the blue and red dots with the yellow token are

$$1 \pm 5 = \{6, 0, -1, -2, -3, -4, -5 \mid -4, 2, 3, 4, 5, 6\}.$$

There are more exotic games

- ▶ Same game except in their turn players can:
 - ▶ pick up counters that are their colour or grey; and,
 - ▶ change counters of their opponent's colour to grey.

Pick up or change colour:



There are more exotic games

- ▶ Here, Left can move the game to 0 by removing the counter.
- ▶ Right can move the game to $*1$ by changing the blue counter to grey.

Pick up or change colour:



There are more exotic games

- This game position is therefore

$$G = \{ 0 \mid * 1 \}.$$

Pick up or change colour:



There are more exotic games

- ▶ This game position is therefore

$$G = \{ 0 \mid * 1 \}.$$

- ▶ This is called \uparrow ('up').

Pick up or change colour:



There are more exotic games

- This game position is therefore

$$G = \{ 0 \mid *1 \}.$$

- \uparrow is infinitesimal but positive ($\uparrow > 0$).

Pick up or change colour:



There are more exotic games

- ▶ $\{ *1 \mid 0 \} = \downarrow < 0$ is similarly defined.

Pick up or change colour:



Back to partizan games that are numbers

- In a game

$$G = \{\alpha, a_1, a_2, \dots, a_n \mid \beta, b_1, b_2, \dots, b_n\},$$

say that $\alpha > a_1 > a_2 > a_3 > \dots$ and $\beta < b_1 < b_2 < b_3 < \dots$, i.e. α is the largest option for Left and β is the smallest option for Right.

- Then G is the first number between α and β to be ‘invented’ by the Conway process.

Back to partizan games that are numbers

- ▶ In a game

$$G = \{\alpha, a_1, a_2, \dots, a_n \mid \beta, b_1, b_2, \dots, b_n\},$$

say that $\alpha > a_1 > a_2 > a_3 > \dots$ and $\beta < b_1 < b_2 < b_3 < \dots$, i.e. α is the largest option for Left and β is the smallest option for Right.

- ▶ Then G is the first number between α and β to be ‘invented’ by the Conway process.
- ▶ If there are only options for Left — $\{\alpha, a_1, a_2, a_3, \dots \mid \}$ — the game position is the earliest number greater than α .
- ▶ If there are only options for Right — $\{\mid \beta, b_1, b_2, b_3, \dots\}$ — the game position is the earliest number less than β .

Hackenbush

Hackenbush is played on a graph with blue and red edges. **Left** removes **b**lue edges and **Right** removes **R**ed edges. After a removal, any part of the graph not connected to the ground floats away, out of the game. The last player to make a move wins.

