Set theory exercises

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1. The game **Addiction** (1978) included thirteen dice with letters on the faces and numbers used in scoring. A player rolls a die and then must play the letter that is showing on top of the die on a 5×5 grid. Once placed, a die cannot be moved. The game proceeds in this way until all thirteen dice are placed on the grid. The player scores points for those dice which form words in a linked intersection formation.

   I rolled eight of the **Addiction** dice and obtained this set of letters: $\{\mathrm{A}, \mathrm{U}, \mathrm{C}, \mathrm{X}, \mathrm{E}, \mathrm{V}, \mathrm{W}, \mathrm{T}\}$. Find from this

   (a) A subset that forms a word representing a four-legged family pet.

   (b) Two subsets, one that spells an underground chamber, and a second that you get when it rains, which have a non-empty intersection.

   (c) Two subsets, one that spells a reduction, and a second that spells a product made by bees, which have an empty intersection.

2. What is the cardinality of the following sets?

   (a) $\{1, 2, 5, 4, 6\}$;
   (b) $\{3, 4, \mathrm{cat}\}$;
   (c) $\{3, \{4, \mathrm{cat}\}\}$;
   (d) $\{\pi, 6, \{\pi, 5, 8, 10\}\}$;
   (e) $\{\pi, 6, \{\pi, 5, 8, 10\}, \{\mathrm{dog}, \mathrm{cat}, \{5\}\}\}$;
   (f) $\emptyset$;
   (g) $\mathbb{N}$;
   (h) $\emptyset$;
   (i) $\{\emptyset\}$;
   (j) $\{\emptyset, \{\emptyset\}\}$.

3. Are the following statements true or false?

   (a) $1 \in \mathbb{Z}$; (b) $0 \in \emptyset$; (c) $\frac{3}{2} \in \mathbb{N}$; (d) $\frac{3}{2} \in \mathbb{R}$; (e) $\pi \notin \mathbb{C}$; (f) $\frac{\pi}{2} \in \mathbb{Q}$.

4. Consider $A = \{2, 3, 7, 15\}$. Give rules that define subsets of three elements which exclude each of the members in turn from A. For example, $B = \{x \mid x > 2\}$ would exclude 2, though you could try to be more creative.

5. Consider $A = \{x \in \mathbb{N} \mid 1 \leq x \leq 20\}$ and $B = \left\{x \mid x, \frac{x}{2} \in \mathbb{N} \wedge 2 \leq x \leq 30\right\}$.

   (a) Write out in words what numbers are in sets A and B.

   (b) Write out full lists of the members of A and B.

   (c) Write out the list of elements in these sets:

      i. $\{x \mid x \in A \wedge x \in B\}$;
      ii. $\{x \in B \mid x \text{ is prime}\}$;
      iii. $\{x \in A \mid \sqrt{x} \in \mathbb{Z}\}$.

6. Write out the list of elements in these sets:

   (a) $\{x \in \mathbb{Z} \mid x^2 \leq 25\}$;
   (b) $\{x \in \mathbb{R} \mid x^2 = 2\}$;
   (c) $\{x \in \mathbb{Z} \mid x^2 = 2\}$;
   (d) $\{x \in \mathbb{R} \mid 6 < x < 3\}$.

7. Write definitions of the form $\{x \in \ldots \mid \ldots\}$ for the following sets.

   (a) $\{1, 4, 9, 16, 25, 36, 49, \ldots\}$;
   (b) $\{1, 2, 4, 8, 16, 32, \ldots\}$;
   (c) $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$.

8. Which, if any, of the following sets are equal to each other?

   (a) $A = \{1, 2, 3\}$;
   (b) $B = \{x \in \mathbb{N} \mid x > 0 \wedge x^2 < 10\}$;
   (c) $C = \{x \in \mathbb{N} \mid n^2 < 1\}$;
   (d) $D = \emptyset$.

9. Which of the following statements are true?

   (a) $\mathbb{Z} \subseteq \mathbb{N}$;
   (b) $\mathbb{N} \subseteq \mathbb{Z}$;
   (c) $\{1, 3, 7\} \subset \mathbb{N}$;
   (d) $\{1, 3, 7\} \subset \{1, 3, 7\}$;
   (e) $\{1, 3, 7\} \subseteq \{1, 3, 7\}$;
   (f) $1 \in \mathbb{Z}$;
   (g) $1 \subseteq \mathbb{Z}$;
   (h) $\{1\} \in \mathbb{Z}$;
   (i) $\{1\} \subseteq \mathbb{Z}$;
   (j) $\emptyset \subseteq \mathbb{Z}$;
   (k) $\{0\} \subseteq \emptyset$;
   (l) $\emptyset \subseteq \{1, 2\}$;
   (m) $\{\emptyset\} \subseteq \emptyset$;
   (n) $\emptyset \subseteq \{\emptyset\}$.

10. Let $X = \{0, 1\}$ and $Y = \{1, 2, 3\}$. What are the elements of $X \times Y$?

11. Let $P = \{x \in \mathbb{R} \mid \sin(x) = 0\}$ and $Q = \{n\pi \mid n \in \mathbb{Z}\}$. What is the relationship between P and Q?

12. Consider the sets $A = \{x \in \mathbb{Z} \mid 2 \leq x\}$ and $B = \{x \in \mathbb{Z} \mid x \leq 5\}$. Show that $A \cap B$ is finite. $A \cup B$ has a special name, what is it?

13. Find $\mathbb{Z} \cap \mathbb{Z}$, $\mathbb{Z} \cap \emptyset$, and $\mathbb{Z} \cap \mathbb{R}$.

14. Considering a proposition p acting within some set D, the *truth set* of p is the set of elements $x \in D$ for which p is true, i.e. $\{x \in D \mid p(x)\}$.

    What is the truth set of the following propositions?

    (a) p: "x is a day you currently have classes".
    (b) q: "x is a logical connective studied in this module".
    (c) r: "x is a Sheffield Hallam lecturer who has taught you".
    (d) s: "x is a real number and $x^2 - 4x + 3 = 0$".