Binary operators – exercises

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1. Complete the following table for the operation $\circ$ acting on the set $\{0, 1, 2, 3, 4\}$ where $\circ$ is defined by the rule $a \circ b = a + b \pmod 5$.

	0	1	2	3	4
0					
1	1				
2					
3		4			
4				3	

   Is the operation $\circ$ closed? Does it obey associativity and commutativity? Are there identity elements and an annihilator?

2. Complete the following table for the operation $\circ$ acting on the set $\{0, 1, 2, 3, 4\}$ where $\circ$ is defined by the rule $a \circ b = a \cdot b \pmod 5$.

	0	1	2	3	4
0					
1	0				
2					
3		3			
4				1	

   Is the operation $\circ$ closed? Does it obey associativity and commutativity? Are there identity elements and an annihilator?

3. Which of the following sets are not closed with respect to the operation given? For those which are not closed, give an example to prove it.

   (a) $\mathbb{Z}, -$

   (b) $\mathbb{Z}^+, \div$

   (c) $\mathbb{Q} - \{0\}, \cdot$

   (d) $\mathbb{Q} - \{0\}, \div$

   (e) $\mathbb{Z} - \{0\}, \div$

   (f) $\mathbb{R}^+, +$

   (g) $\mathbb{R}^+, \div$

   (h) $\mathbb{R}^+, -$

   (i) $S = \{1, 2, 3, 6, 12\}, \cdot$

4. Is the operation $\circ$ closed on $\mathbb{Z}$ when $\circ$ is defined by

   (a) $a \circ b = a^2 b$;

   (b) $a \circ b = a + b + 1$?

5. Let $A = \{2^m \cdot 3^n | m, n \in \mathbb{Z}\}$. Show that $(A, \cdot)$ is closed, where $\cdot$ is ordinary multiplication.

6. Give an example of another binary operation that is closed (and the set on which it is acting).

7. Give an example of another binary operation that is not closed on a set.

8. Decide which of the following operations are commutative on the sets given. For those which are not, give a counter-example.

   (a) $(\mathbb{R}, +)$;
   (b) $\mathbb{Q}, \div$;
   (c) matrix multiplication on 2×2 matrices.

9. Define $(\mathbb{R}, \circ)$ using the rule $x \circ y = x \cdot y + 1$. Show that this is commutative but not associative. Does it have identity and annihilator elements?

10. Is the operation $\circ$ commutative on $\mathbb{Z}$ when $\circ$ is defined by

   (a) $a \circ b = a^2 b$;
   (b) $a \circ b = a + b + 1$.

11. Give an example of another binary operation that is commutative (and the set on which it is acting).

12. Give an example of another binary operation that is not commutative on a set.

13. Decide which of the following operations are associative on the sets given. For those which are not, give a counter-example.

   (a) $(\mathbb{R}, +)$;
   (b) $\mathbb{Q}, \div$;
   (c) matrix multiplication on 2×2 matrices.

14. Is the operation $\circ$ associative on $\mathbb{Z}$ when $\circ$ is defined by

   (a) $a \circ b = a^2 b$;
   (b) $a \circ b = a + b + 1$.

15. Give an example of another binary operation that is associative (and the set on which it is acting).

16. Give an example of another binary operation that is not associative on a set.

17. Decide which of the following have identity elements, and it they do state what the identity is.

   (a) $(\mathbb{R}, +)$;
   (b) $(\mathbb{Q} - \{0\}, \div)$;
   (c) $(\mathbb{Q} - \{0\}, \cdot)$.

18. Under which of the following operations do we have an identity element in $\mathbb{Q}$? For those that have an identity element, find which elements in $\mathbb{Q}$ (if any) have inverses in $\mathbb{Q}$ with respect to $\circ$.

   (a) $x \circ y = x + 2y$;
   (b) $x \circ y = x + y - x \cdot y$;
   (c) $x \circ y = x$.