Set theory notes

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1 Sets

We call a collection of things a *set*. We think of a set as an object in its own right, and label it with a letter, often a capital letter A, B, C, etc. We define a set by completely describing its members, typically writing these as a comma separated list enclosed in $\{\dots\}$. For example, the following are sets:

1. $A = \{\text{Alice}, \boxdot, \text{yellow}\}$;
2. $B = \{2, 3, 5, 7, 11, \text{pineapple}\}$;
3. $G = \{\text{Noughts and Crosses}, \text{Monopoly}, \text{Poker}, \text{Nim}\}$;

There are no formal requirements for the elements of a set to be connected in any way, though we often consider sets where the objects have some property in common. For example, G is a set of games. The order of elements in a set is irrelevant, so the set $\{\text{Lewis Carroll}, \text{Martin Gardner}\}$ is the same as the set $\{\text{Martin Gardner}, \text{Lewis Carroll}\}$.

Sets may have a finite number of elements (including zero), or an infinite number. If a set A has a finite number of elements, we call this the *cardinality* (or *order*) of A and write this $|A|$. For example, in the list above $|A| = 3$, $|B| = 6$ and $|G| = 4$.

If a is a member of a set A, we indicate this by $a \in A$. This is said "a is in A" or "a is a member of A". Note that we have used a capital 'A' for the set and a lower case 'a' for the element and these are different things.

For example, if $D = \{1, 2, 3, 4, 5, 6\}$ is the set of possible rolls of a standard six-sided die, then 5 is one of the elements of that set. We would write $5 \in D$.

If a is not an element of the set A, we write $a \notin A$ and say "a is not in A" or "a is not a member of A". For example, because you can't roll a seven on a standard die, for the set of standard rolls we could say $7 \notin D$.

We consider an element to be a member of a set only once, so for example the set of letters in the word 'puzzle' is $\{\text{p}, \text{u}, \text{z}, \text{l}, \text{e}\}$.

If the elements of a set follow a pattern, we can indicate this using '$\dots$' like this:

$$\{1, 2, \dots, 99, 100\}.$$

This indicates that we are counting up in ones and this pattern continues up to 100. It is like a child being asked to count to 100 and responding with the rhyme "one, two, miss a few, 99, 100".

Care must be taken to make sure that the '$\dots$' are clear. For example, does the set $\{2, 4, \dots, 64\}$ contain even numbers or powers of 2?

We can also define infinite sets using a '$\dots$' not followed by anything, for example

$$\{2, 4, 6, \dots\}$$

indicates we are starting at 2 and counting through the even numbers forever.

If a set has no elements, we indicate this using $\{\}$ or $\emptyset$. This might seem strange, but the empty set is useful in various ways, as we will see later.

Some frequently-used sets have names, for example

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

is the set of integers. We can indicate the positive integers using $\mathbb{Z}^+$ and the negative integers using $\mathbb{Z}^-$.

There is also a set called the natural numbers written $\mathbb{N}$. Sources differ on whether this is the set $\{0, 1, 2, 3, \dots\}$ or the set $\{1, 2, 3, \dots\}$.

Often, the elements of a set cannot be simply listed or indicated with '…'. In these cases we can use ' | ' which is read "such that" to indicate a condition by which an element is included in the set.

For example, $\{a \in \mathbb{Z} \mid a > 5\}$ is the set of integers greater than 5. It is read "a in $\mathbb{Z}$ such that a is greater than five". We can use the connectives that met in propositional logic, so for example we could define $\{a \in \mathbb{Z} \mid a > 5 \wedge a \text{ is prime}\}$ to indicate the set of integers greater than 5 which are prime.

Here is a list of named sets that might be useful:

Symbol	**Definition**	**Name**
$\mathbb{N}$	$\{1, 2, 3, \dots\}$	Natural numbers
$\mathbb{Z}$	$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$	Integers
$\mathbb{Z}^+$	$\{1, 2, 3, \dots\}$	Positive integers
$\mathbb{Z}^-$	$\{\dots, -3, -2, -1\}$	Negative integers
$\mathbb{Q}$	$\left\{\frac{m}{n} \mid m, n \in \mathbb{Z} \wedge n \neq 0\right\}.$	Rational numbers
$\mathbb{R}$	$\{x \mid x \text{ can be used to mark a position on the number line}\}$	Real numbers
$\mathbb{C}$	$\{a + bi \mid a, b \in \mathbb{R}\}$	Complex numbers

2 Universal set

When dealing with sets, we have a universe of things we are considering, sometimes denoted U. This is the *universal set*, sometimes also called the *universe of discourse*. Often it is clear what the universe is, but sometimes it can help to clarify. For example, we might clarify whether the set $\{1, 4, 7, 21\}$ is drawn from the universal set $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, etc. to explain the wider context.

3 Subsets

We indicate that x is a subset of A using $x \subseteq A$.

If a set A is a subset of B and A is not equal to B, then we say A is a *proper subset* of B and write $A \subset B$.

If $A \subseteq B$ and $B \subseteq A$, then we say A and B are equal. We can say $A \subseteq B \wedge B \subseteq A \iff A = B$. If two sets are not equal, we can say $A \neq B$.

All elements of $B \subseteq A$ are elements of A itself, so we can say $x \subseteq B \implies x \subseteq A$.

A set A is always a subset of itself, so $A \subseteq A$.

Note that the empty set is always a subset of any set, i.e. $\emptyset \subseteq A$. This is because from any set (including the empty set), we can always pick a collection of no elements to form $\emptyset$.

4 Intersection

If A and B are sets, then the *intersection* of A and B, written $A \cap B$, is the elements that are members of both A and B. We can say $x \in A \wedge x \in B \implies x \in A \cap B$.

Note that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. In fact, $A \cap B$ is the largest set which is a subset of both A and B.

If A and B are disjoint (they have no elements in common), then $A \cap B = \emptyset$.

5 Union

If A and B are sets, then the *union* of A and B, written $A \cup B$, is the set containing all elements of A together with all elements of B. We can express this as $x \in A \vee x \in B \implies x \in A \cup B$.

Note that $A \subseteq A \cup B$, $B \subseteq A \cup B$, and $A \cap B \subseteq A \cup B$. In fact, if we simply add the elements of A and the elements of B, we will have added the elements that are in both A and B twice. To find the size of the union, therefore, we use

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

6 Difference and complement

If A and B are sets, then the difference $A - B$ is the set of elements A which are not members of B. We can write this as $x \in A \wedge x \notin B \implies x \in A - B$.

The set $U - A$ is the elements of the universal set that are not in A. We call this the complement of A and write this A'. Note that $A \cup A' = U$ and $A \cap A' = \emptyset$.

7 Product

The *product* of two sets A and B is

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

This way, we can make ordered pairs (x, y) from two sets. The order matters, in the sense that $(x, y) \neq (y, x)$. Two pairs are equal when both components match, that is $(a, b) = (c, d) \iff a = c \wedge b = d$.

If $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, then $|A| = 3$, $|B| = 2$, and $|A \times B| = 2 \times 3 = 6$. The elements of $A \times B$ are

$$(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2).$$

For example, this is often used in coordinate geometry, where the plane is defined by $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$.

The idea can be extended to larger pairings, as $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\}$. The pairing $(x_1, x_2, \ldots, x_n)$ is called an n-tuple.