

Tutorial exercise sheet – Determinants and inverses

Peter Rowlett

1. Calculate the determinants of the following matrices.

(a) $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$; (b) $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$; (c) $\mathbf{C} = \begin{bmatrix} 3 & -6 \\ -9 & 12 \end{bmatrix}$;

2. For the matrix $\mathbf{A} = \begin{bmatrix} a & b & c \\ -1 & 1 & 3 \\ 2 & 7 & 9 \end{bmatrix}$, find:

- (a) the minor of a in \mathbf{A} ; (b) the cofactor of a in \mathbf{A} ; (c) the minor of b in \mathbf{A} ;
(d) the cofactor of b in \mathbf{A} ; (e) the minor of c in \mathbf{A} ; (f) the cofactor of c in \mathbf{A} ;
(g) the minor of b in \mathbf{A}^\top ; (h) the cofactor of b in \mathbf{A}^\top ; (i) the cofactor of c in \mathbf{A}^\top .

3. Calculate the determinants of the following matrices.

(a) $\mathbf{A} = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$; (b) $\mathbf{B} = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$; (c) $\mathbf{C} = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 5 & 7 \\ 6 & 3 & 8 \end{bmatrix}$;
(d) $\mathbf{D} = \begin{bmatrix} 2 & 7 & 6 \\ 4 & -1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$; (e) $\mathbf{E} = \begin{bmatrix} 3 & -4 & -3 \\ 2 & 7 & -31 \\ 5 & -9 & 2 \end{bmatrix}$; (f) $\mathbf{F} = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$;
(g) $\mathbf{G} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -3 & 9 \\ 1 & -3 & 9 \end{bmatrix}$; (h) $\mathbf{H} = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$; (j) $\mathbf{J} = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$.

4. Find the values of t for which $\begin{vmatrix} t & 1 & -2 \\ 2 & t+1 & 1 \\ -1 & -4 & t-2 \end{vmatrix} = 17$.

5. In each case, determine whether the statement is true or false, and justify your answer.

- (a) The value of the determinant is independent of which row or column the cofactor expansion takes place along.
(b) The minor M_{ij} of \mathbf{A} is the same as the cofactor A_{ij} if and only if $i + j$ is even.
(c) The determinant of a matrix whose elements are all positive must be positive.
(d) If a matrix has a row or column of zeros, then its determinant is zero.
(e) If each element in a matrix doubles, its determinant also doubles.
(f) For a square matrix \mathbf{A} , $\det(\mathbf{A}) = \det(\mathbf{A}^\top)$.

6. What is the maximum number of zeros a 3×3 matrix can have without having a zero determinant?

7.

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -4 & 7 \\ 3 & -5 \end{bmatrix}.$$

- (a) Compute $3\mathbf{A} - \mathbf{B}$.
- (b) What is \mathbf{A}^2 ?
- (c) Find \mathbf{A}^{-1} and \mathbf{B}^{-1} .
- (d) Show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- (e) The matrix \mathbf{A} satisfies $\mathbf{A}^2 - 5\mathbf{A} + \mathbf{I} = \mathbf{0}$. Use this to find a formula for \mathbf{A}^{-1} in terms of \mathbf{A} , and show that the matrix \mathbf{A}^{-1} obtained equals the matrix \mathbf{A}^{-1} in part (c).

8. (a) Find the matrix \mathbf{X} , given that: $\begin{bmatrix} 1 & 3 \\ 5 & 5 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 13 & 23 \\ 25 & 45 \end{bmatrix}.$

(b) Find the matrix \mathbf{Y} , given that: $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \mathbf{Y} = \begin{bmatrix} 17 & 8 \\ 14 & 6 \end{bmatrix}.$

(c) Find the matrix \mathbf{Z} , given that: $\begin{bmatrix} -4 & 2 \\ 8 & -2 \end{bmatrix} \mathbf{Z} = \begin{bmatrix} 0 & 7 \\ 1 & -6 \end{bmatrix}.$

9. Find the inverse of

$$\mathbf{Y} = \begin{bmatrix} -1 & 0 & 2 \\ -10 & -1 & 0 \\ 3 & 8 & 2 \end{bmatrix}.$$

10.

Consider $\mathbf{K} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 3 & 1 & 2 \end{bmatrix}$. Given that $\mathbf{K}^{-1} = -\frac{1}{a} \begin{bmatrix} b & -3 & -4 \\ -10 & -1 & c \\ d & 5 & -4 \end{bmatrix},$

determine the values of a, b, c and d .

11.

Consider $\mathbf{R} = \begin{bmatrix} 0 & 8 & 6 \\ 1 & 0 & -4 \\ 2 & 3 & 1 \end{bmatrix}$. Given that $\mathbf{R}^{-1} = \frac{1}{a} \begin{bmatrix} b & -5 & 16 \\ \frac{9}{2} & 6 & c \\ d & -8 & 4 \end{bmatrix},$

determine the values of a, b, c and d .