Patterns that break

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- A number's prime factorisation is when we write the number as a product of its prime factors.
- For example,
  - $6 = 2 \times 3$;
  - $18 = 2 \times 3 \times 3$.
- Notice 6 has an even number of factors and 18 has an odd number of factors.
- Let $E(n)$ be the size of the set

$$\{k \in \mathbb{Z}^+ \mid 2 < k \leq n \wedge k \text{ has an even number of factors}\}$$

- Let $O(n)$ be the size of the set

$$\{k \in \mathbb{Z}^+ \mid 2 < k \leq n \wedge k \text{ has an odd number of factors}\}$$

n	No. factors	$E(n)$	$O(n)$
1	0	1	0
2	1	1	1
3	1	1	2
4	2	2	2
5	1	2	3
6	2	3	3
7	1	3	4
8	3	3	5
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- In 1962, a counterexample was found.
- The smallest counterexample is $n = 906,150,257$.

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  - There are no $a, b, c \in \mathbb{Z}^+$ which satisfy

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = 4$$

    until

$$a = 154476802108746166441951315019919837485664325669565431700026634898253202035277999$$
$$b = 36875131794129999827197811565225474825492979968971970996283137471637224634055579$$
$$c = 4373612677928697257861252602371390152816537558161613618621437993378423467772036$$

- What pattern starts $1, 1, 2, 3, 5, 8, 13 \ldots$?