Group theory – exercises

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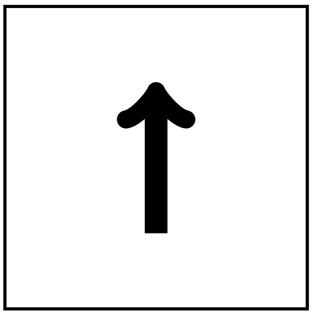
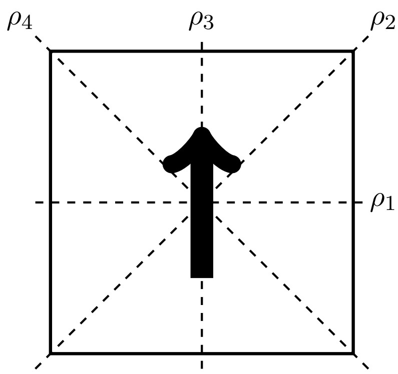
1. Take a square piece of paper or card and draw an arrow on each side, both pointing in the same direction, which we will take as 'up'. It would be best to draw the arrows with some different (e.g. colour) so they are easy to distinguish. Now the card can be rotated, including turning it over, in several different ways.



Let r, r^2 and r^3 represent rotations clockwise through 90°, 180°and 270°respectively, and ρ_1, ρ_2, ρ_3 and ρ_4 represent reflections as denoted on the diagram below.



(a) Form a table of these symmetries, where the operation is simply 'followed by'.

(b) Decide which of the following properties hold: closure, commutativity, identity. You may assume it is associative. Find all the inverses which exist.

(c) Does this form a group?

(d) If so, is it Abelian?

(e) Find the order of each symmetry in this group.

(f) Is the group cyclic? If so give the generators.

(g) Do the following subsets form groups under the same operation?

i. $\{e, r\}$;

ii. $\{e, r^2\}$;

iii. $\{e, r, r^2, r^3\}$;

iv. $\{e, \rho_1\}$.

2. A binary operation $\circ$ is defined on the set $C = \{\text{Green}, \text{Yellow}\}$ by

$$\begin{aligned} \text{Green} \circ \text{Green} &= \text{Green} \\ \text{Green} \circ \text{Yellow} &= \text{Yellow} \\ \text{Yellow} \circ \text{Green} &= \text{Yellow} \\ \text{Yellow} \circ \text{Yellow} &= \text{Green} \end{aligned}$$

3. Define $(\mathbb{R}, \circ)$ using the rule $x \circ y = x + y + x \cdot y$. Prove that $(\mathbb{R}, \circ)$ is a monoid, with identity element 0. Show that $(\mathbb{R} - \{-1\}, \circ)$ is a group.

4. Draw up tables for the following sets with the binary operation given and in each case decide which of the group properties hold. Give the identity if there is one and find the inverses for each element where one exists. Hence decide if it is a group. Also decide if it is commutative.
   (a) $\{0, 1, 2, 3, 4\}$ under addition modulo 5;
   (b) $\{0, 1, 2, 3, 4, 5\}$ under multiplication modulo 6;

5. Show that the set
$$\left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \wedge x^2 + y^2 = 1 \right\}$$
forms an Abelian group under matrix multiplication.

6. Considering $\{1, 2, 3, 4, 5, 6\}$ under multiplication modulo 7.
   (a) Find the order of each element.
   (b) Decide which elements are generators.
   (c) Draw up the table as powers of one of your generators. You might like to reorder the columns and rows to make this as simple as possible.

7. Consider $\{1, 2, 3, \ldots, 10\}$ under multiplication modulo 11.
   - Find one generator for this group.
   - Can you discover a rule to decide which of the powers of your generator will also be generators?

8. Draw up the table for the group generated by $i = \sqrt{-1}$ under multiplication. Find the order of each element in this group. Are any of the other elements generators?