

Counting up to symmetry

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1 Symmetries as permutations

Recall that $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ represents a function that maps

$$\sigma(1) = 2$$

$$\sigma(2) = 4$$

$$\sigma(3) = 3$$

$$\sigma(4) = 1.$$

We can write this as a product of disjoint cycles.

In general:

- $(\dots xy \dots)$ means x shifts to y ;
- $(x \dots y)$ means y shifts to x .

For example

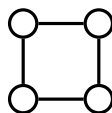
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (124)(3);$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23);$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234);$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4).$$

We can write the symmetries of a square as permutations. Consider the square



First we consider the rotations.

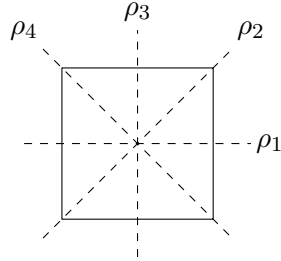
$$e: \begin{array}{cc} \textcircled{1} & \textcircled{2} \\ | & | \\ \textcircled{4} & \textcircled{3} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4).$$

$$r: \begin{array}{cc} \textcircled{4} & \textcircled{1} \\ | & | \\ \textcircled{3} & \textcircled{2} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1234).$$

$$r^2: \begin{array}{cc} \textcircled{3} & \textcircled{4} \\ | & | \\ \textcircled{2} & \textcircled{1} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24).$$

$$r^3: \begin{array}{cc} \textcircled{2} & \textcircled{3} \\ | & | \\ \textcircled{1} & \textcircled{4} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234).$$

Next we consider the reflections, labelled as follows.



$$\rho_1: \begin{array}{cc} \textcircled{4} & \textcircled{3} \\ | & | \\ \textcircled{1} & \textcircled{2} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23).$$

$$\rho_2: \begin{array}{cc} \textcircled{3} & \textcircled{2} \\ | & | \\ \textcircled{4} & \textcircled{1} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (13)(2)(4).$$

$$\rho_3: \begin{array}{cc} \textcircled{3} & \textcircled{2} \\ | & | \\ \textcircled{4} & \textcircled{1} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34).$$

$$\rho_4: \begin{array}{cc} \textcircled{1} & \textcircled{4} \\ | & | \\ \textcircled{2} & \textcircled{3} \end{array} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (1)(24)(3).$$

2 Cycle notation

Let x_i represent a cycle of length i .

Then, for example, we can represent

- (1) as x_1 ;
- (12) as x_2 ;

- (1234) as x_4 ;
- (13)(24) as x_2^2 ;
- (1)(2)(3)(4) as x_1^4 ;
- (1)(2)(34) as $x_1^2 x_2$.

We can represent the symmetries of a square in cycle notation.

$$\begin{aligned} e: & x_1^4 \\ r: & x_4 \\ r^2: & x_2^2 \\ r^3: & x_4 \\ \rho_1: & x_2^2 \\ \rho_2: & x_1^2 x_2 \\ \rho_3: & x_2^2 \\ \rho_4: & x_1^2 x_2 \end{aligned}$$

In a colouring problem, each member of a cycle must be coloured the same way, since the elements are mapped onto each other by the symmetry in question.

For example, (13) means $1 \rightarrow 3$ and $3 \rightarrow 1$, so 1 and 3 must be the same colour.

So if there are m colours, each x_i represents a set of elements that can be coloured m different ways.

3 Burnside's Theorem

Theorem. For G a group of permutations of $\{1, 2, \dots, n\}$, the cycle index of G is

$$P(x_1, x_2, \dots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} (\text{cycle representation of } \pi).$$

Then the number of m -colourings of configurations acted upon by G is $P(m, m, \dots, m)$.

3.1 Example

For colouring the vertices of a square, we have

$$\begin{aligned} P(x_1, x_2, x_3, x_4) &= \frac{1}{8} (x_1^4 + x_4 + x_2^2 + x_4 + x_2^2 + x_1^2 x_2 + x_2^2 + x_1^2 x_2) \\ &= \frac{1}{8} (x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2 x_2). \end{aligned}$$

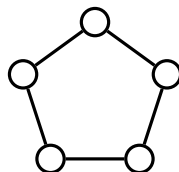
We can 2-colour this in

$$P(2, 2, 2, 2) = \frac{1}{8} (2^4 + 2 \times 2 + 3 \times 2^2 + 2 \times 2^2 \times 2) = 6$$

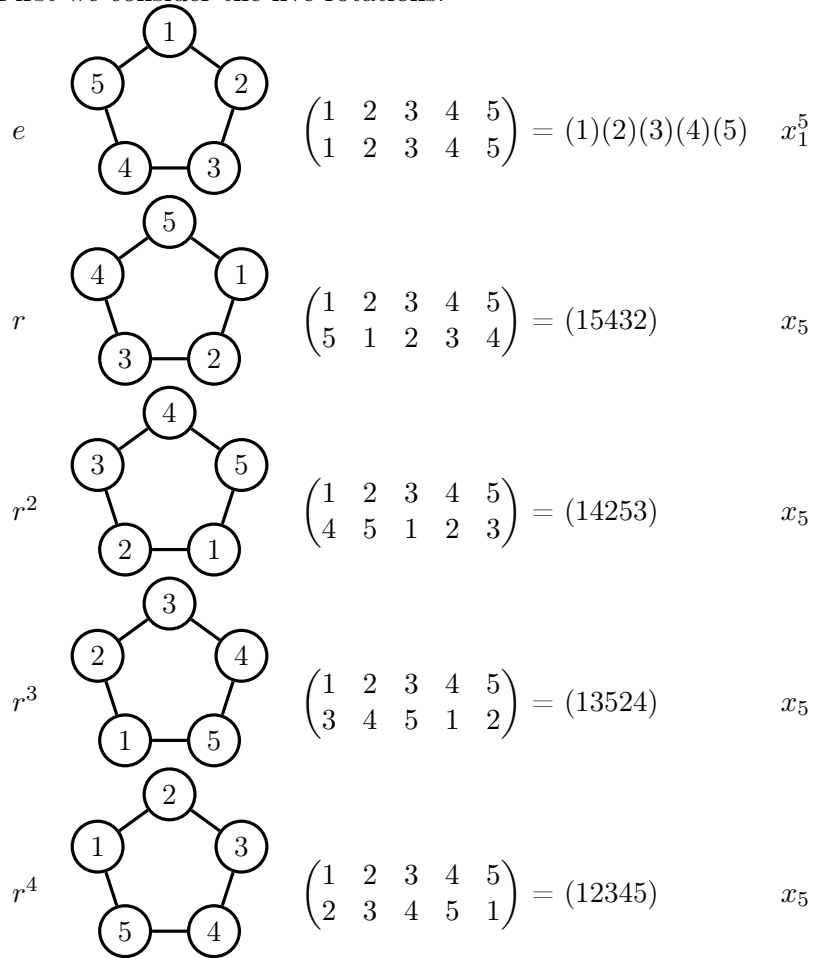
ways.

4 Example

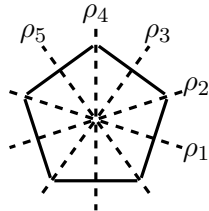
How many ways are there to colour the vertices of a regular pentagon with up to 5 colours?

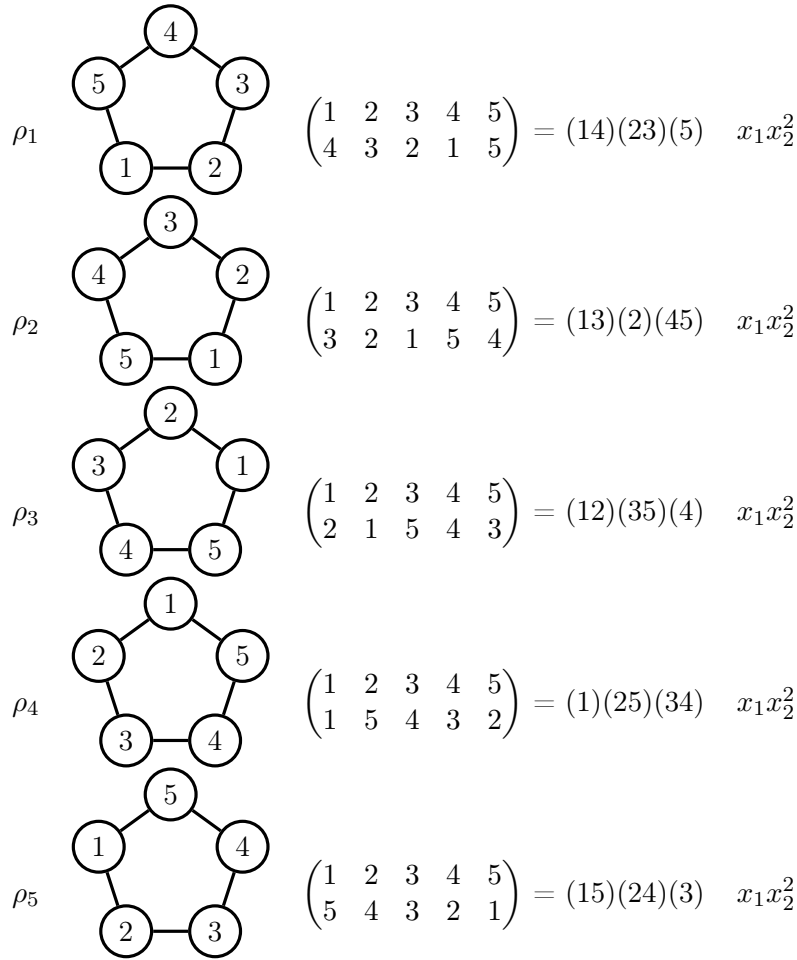


First we consider the five rotations.



Next we consider the following five reflections.





Collecting the cycle representations together, we have

$$P(x_1, x_2, x_3, x_4, x_5) = \frac{1}{10} (x_1^5 + 4x_5 + 5x_1 x_2^2).$$

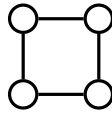
Thus we can 5-colour the shape in

$$P(5, 5, 5, 5, 5) = \frac{1}{10} (5^5 + 4 \times 5 + 5 \times 5 \times 5^2) = 377$$

ways.

5 Pólya enumeration

For colouring the vertices of a square like this



we obtained the function

$$P(x_1, x_2, x_3, x_4) = \frac{1}{8} (x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2 x_2).$$

Elements in x_1 (e.g. (1)) can be coloured either

- blue: $1 + b$;
- green: $1 + g$.

So together these can be coloured $(1 + b)(1 + g) = 1 + b + g + bg$ ways.

There is one element in x_1 and we require that it has a colour, therefore its colours can be expressed as $b + g$.

For x_2 (e.g. (12)), we have

- blue: $1 + b + b^2$;
- green: $1 + g + g^2$.

This gives $(1 + b + b^2)(1 + g + g^2) = 1 + g + g^2 + b + bg + bg^2 + b^2 + b^2g + b^2g^2$.

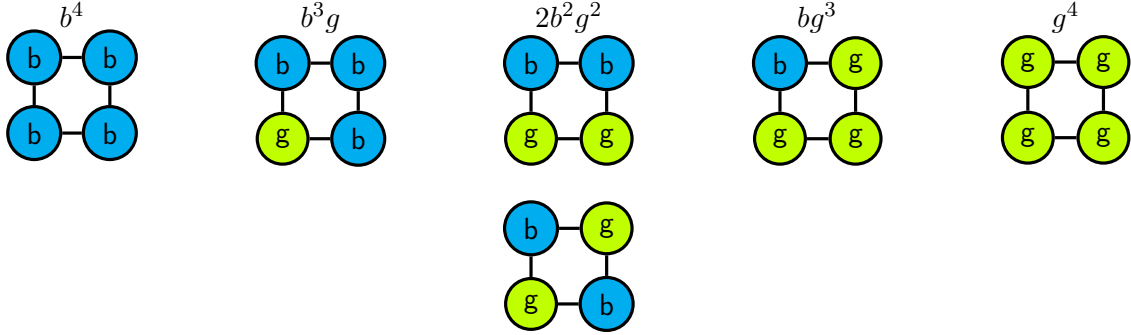
Those terms which represent two elements coloured with the same colour are: $b^2 + g^2$.

In general, x_i can be coloured in $b^i + g^i$ ways.

Now consider

$$\begin{aligned} P(b + g, b^2 + g^2, b^3 + g^3, b^4 + g^4) &= \frac{1}{8} ((b + g)^4 + 2(b^4 + g^4) + 3(b^2 + g^2)^2 + 2(b + g)^2(b^2 + g^2)) \\ &= b^4 + b^3g + 2b^2g^2 + bg^3 + g^4. \end{aligned}$$

These terms generate the 6 ways of colouring the square.



5.1 Example

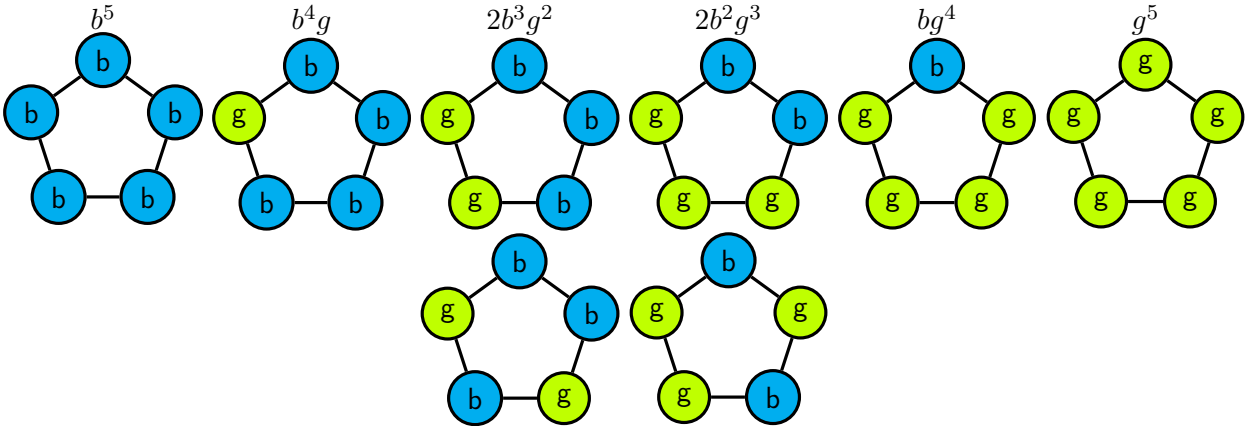
For colouring the vertices of the regular pentagon we found

$$P(x_1, x_2, x_3, x_4, x_5) = \frac{1}{10} (x_1^5 + 4x_5 + 5x_1x_2^2).$$

Then

$$\begin{aligned} P(b + g, \dots, b^5 + g^5) &= \frac{1}{10} ((b + g)^5 + 4(b^5 + g^5) + 5(b + g)(b^2 + g^2)^2) \\ &= b^5 + b^4g + 2b^3g^2 + 2b^2g^3 + bg^4 + g^5. \end{aligned}$$

These terms generate the 8 ways to 2-colour the regular pentagon.



6 Colour Match puzzle

The problem of how many different tiles are used in the Colour Match puzzle can be answered as part of the wider question of four-colouring the vertices of a square.

Here we only have the identity and the rotations, since the tiles cannot be flipped over (the coloured dots don't show through to the back of the tiles). This means

$$P(x_1, x_2, x_3, x_4) = \frac{1}{4} (x_1^4 + 2x_2^2 + x_3^2) .$$

Elements of a cycle x_i of length i can be coloured the same in $b^i + g^i + r^i + y^i$ ways. To generate the ways of colouring, we use

$$\begin{aligned} & P(b + g + r + y, r^2 + b^2 + g^2 + y^2, r^3 + b^3 + g^3 + y^3, r^4 + b^4 + g^4 + y^4) \\ &= \frac{1}{4} ((r + b + g + y)^4 + 2(r^4 + b^4 + g^4 + y^4) + (r^2 + b^2 + g^2 + y^2)^2) \\ &= b^4 + b^3g + b^3r + b^3y + 2b^2g^2 + 3b^2gr + 3b^2gy + 2b^2r^2 + 3b^2ry + 2b^2y^2 + bg^3 + 3bg^2r \\ &\quad + 3bg^2y + 3bgr^2 + 6bgry + 3bgy^2 + br^3 + 3br^2y + 3bry^2 + by^3 + g^4 + g^3r + g^3y + 2g^2r^2 \\ &\quad + 3g^2ry + 2g^2y^2 + gr^3 + 3gr^2y + 3gry^2 + gy^3 + r^4 + r^3y + 2r^2y^2 + ry^3 + y^4. \end{aligned}$$

The relevant term is $6bgry$, the coefficient of which tells us there are six ways of colouring the tiles with each colour used for exactly one vertex.