Paths and cycles

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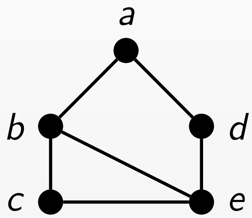
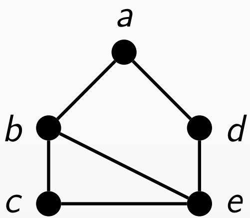
Paths

- Some sources use more specific definitions, but we'll say a *path* is a sequence of edges which join a sequence of vertices.
- A path's *length* is the number of edges in it.
- For example, there is a path of length 3 from a to e in the graph below, it goes $a \to b \to c \to e$.



Cycles

- A cycle is a path that ends at its starting vertex.
- For example, there is a cycle of length 4 from a in the graph below, it goes $a \to b \to e \to d \to a$.



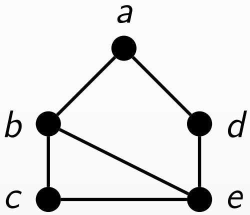
Adjacency matrices

- Diagrams of dots and lines are not the only way to represent graphs. One way that can be useful is to represent the graph as a matrix.
- The *adjacency matrix* for a graph, $G = (V, E)$, with n vertices is an $n \times n$ matrix $\mathbf{M}$ with entries

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \qquad m_{ij} = \begin{cases} 1 \text{ if } (i,j) \in E; \\ 0 \text{ otherwise.} \end{cases}$$

Example

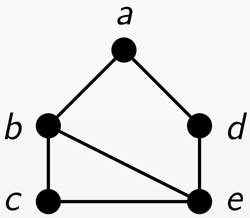
- The following are representations of the same graph.



$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Example

- We can perform matrix operations on $\mathbf{G}$.
- For example, entries of $\mathbf{G}^k$ count the number of paths of length k between vertices.
- i.e. the (i, j)th entry of $\mathbf{G}^k$ gives the number of paths of length k from vertex i to vertex j.



$$\mathbf{G}^2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 & 3 \end{bmatrix}.$$

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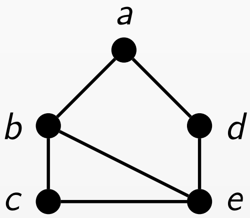

$$\mathbf{G}^3 = \begin{bmatrix} 0 & 5 & 2 & 4 & 1 \\ 5 & 2 & 4 & 1 & 6 \\ 2 & 4 & 2 & 2 & 4 \\ 4 & 1 & 2 & 0 & 5 \\ 1 & 6 & 4 & 5 & 2 \end{bmatrix}.$$

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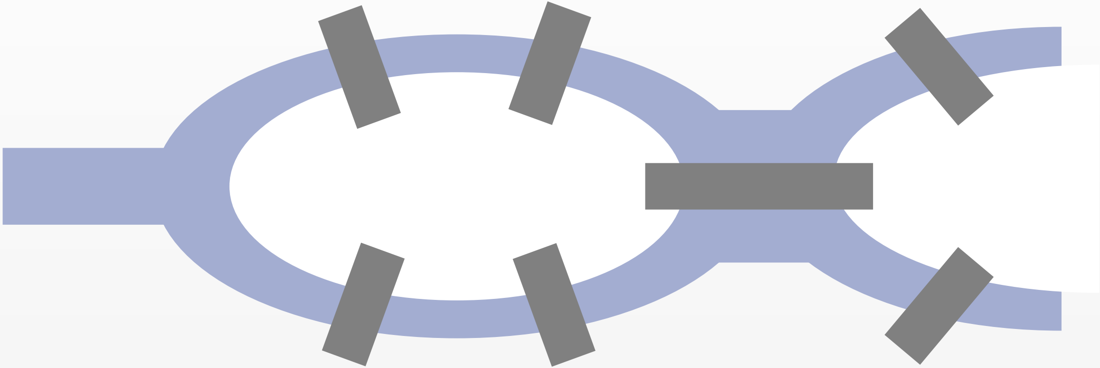
$$\mathbf{G}^{20} = \begin{bmatrix} 10270848 & 14562688 & 11949760 & 9746560 & 15086976 \\ 14562688 & 22220608 & 17699904 & 15086976 & 21696320 \\ 11949760 & 17699904 & 14267264 & 11949760 & 17699904 \\ 9746560 & 15086976 & 11949760 & 10270848 & 14562688 \\ 15086976 & 21696320 & 17699904 & 14562688 & 22220608 \end{bmatrix}.$$

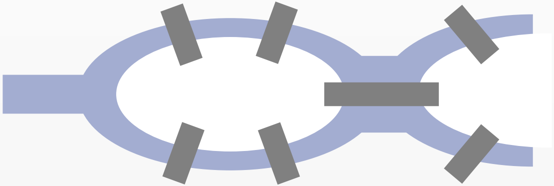
Königsberg Bridges Problem

- In the town of Königsberg there were seven bridges across the river Pregel. Is it possible to go for a walk, crossing each bridge once, but not crossing any bridge twice?



Königsberg Bridges Problem

- This was solved by Euler in 1736 using an essentially topological argument.
- We might redraw the map as a graph.



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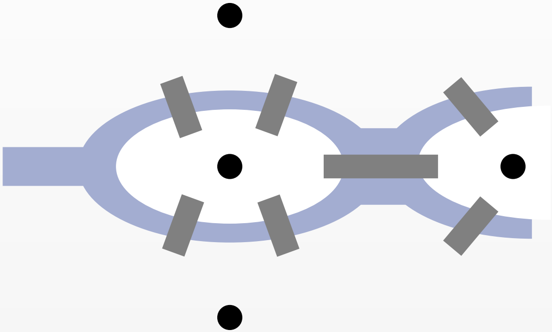

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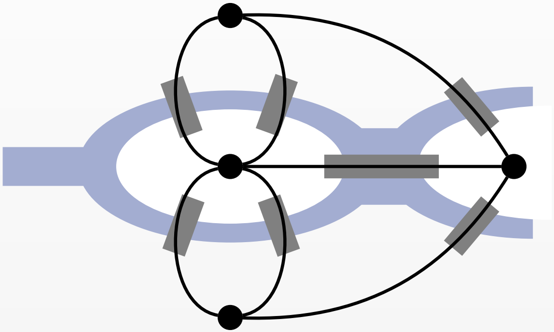

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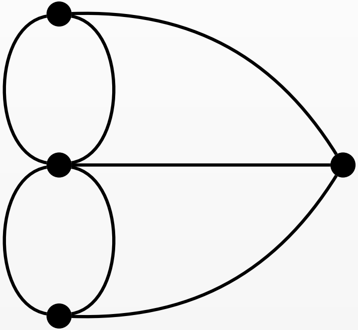
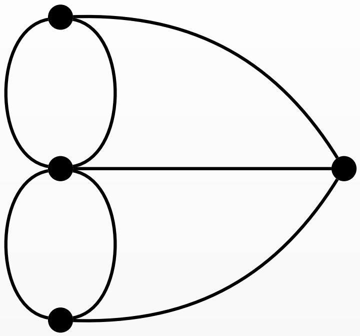


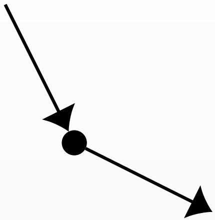
Eulerian cycles

- Now we ask whether there is an *Eulerian cycle* – a cycle that includes every edge of the graph exactly once. (It may visit vertices more than once.)



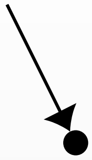
Eulerian cycles

- For a vertex with two edges, we can enter the vertex along one and exit along the other.



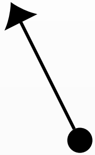
Eulerian cycles

- For a vertex with two edges, we can enter the vertex along one and exit along the other.
- However, for a vertex with one edge, we enter the vertex and cannot proceed.



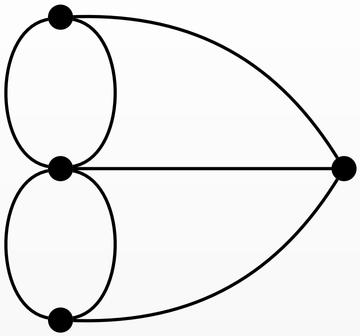
Eulerian cycles

- For a vertex with two edges, we can enter the vertex along one and exit along the other.
- However, for a vertex with one edge, we enter the vertex and cannot proceed.
- Or if we started at that vertex, we leave and cannot return at the end of the cycle.



Eulerian cycles

- So, for a Eulerian cycle, we require that every vertex has even degree.
- A graph which contains an Eulerian cycle is called an *Eulerian graph*.



Eulerian path

- A related concept is an *Eulerian path*.
- This is a path that includes every edge of the graph but does not return to the starting point.
- A graph that contains an Eulerian cycle also contains an Eulerian path, since a cycle is a type of path.
- A graph that contains an Eulerian path but not an Eulerian cycle is called *Semi-Eulerian*.

Useful theorems

Theorem

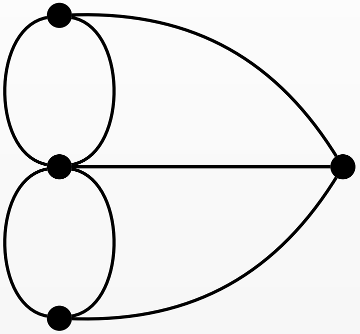
A graph is Eulerian if and only if it is connected and every vertex has even degree.

Theorem

A graph is semi-Eulerian if and only if it is connected and exactly two vertices have odd degree. An Eulerian path must start at one odd vertex and finish at the other.

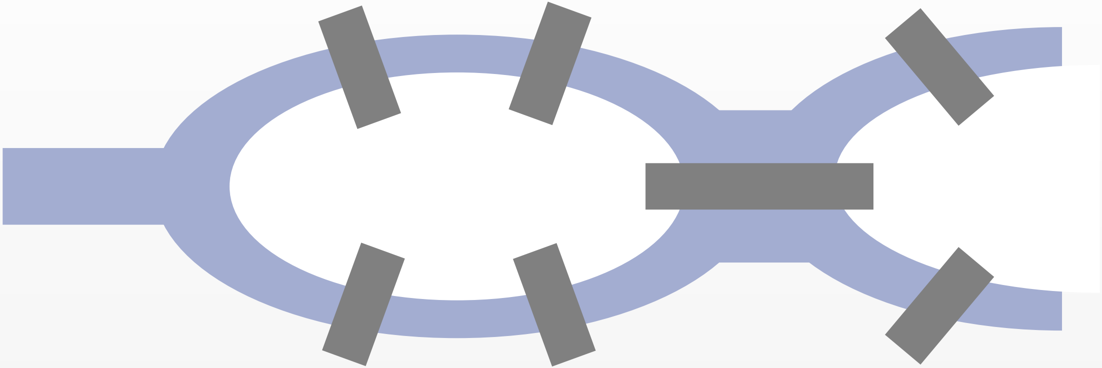
Königsberg Bridges Problem

- In the Königsberg Bridges graph, all four vertices are odd.
- This graph is neither Eulerian nor semi-Eulerian.



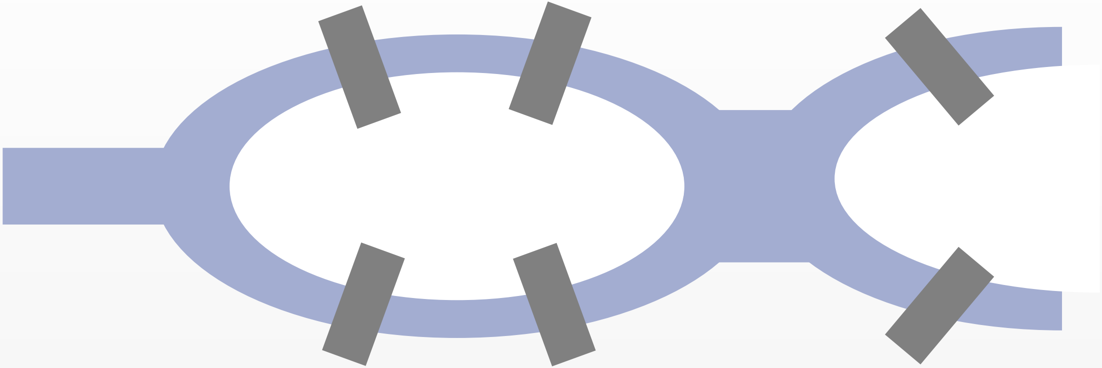
Königsberg Bridges Problem

- This means we cannot cross every bridge exactly once, we must cross at least one bridge more than once.



Different bridges

- If we remove the central bridge, we obtain a different graph.



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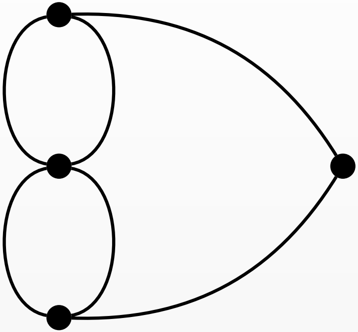

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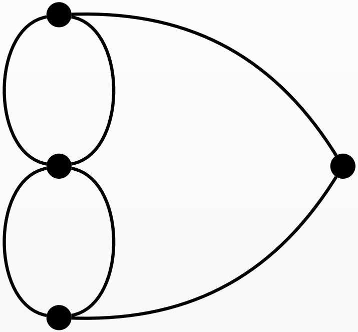


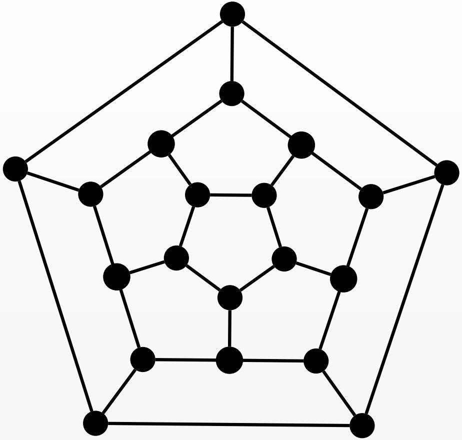
- Since we now have exactly two vertices of odd degree, we can make an Eulerian path on this graph.

Hamiltonian graphs

- A path that visits every vertex exactly once is called a *Hamiltonian path*.
- A cycle that visits every vertex once and then returns to the starting vertex is called a *Hamiltonian cycle*.
- A graph that includes a Hamiltonian cycle is called a *Hamiltonian graph*, and a graph that includes a Hamiltonian path is called a *semi-Hamiltonian graph*.
- A graph that contains an Hamiltonian cycle also contains an Hamiltonian path, since a cycle is a type of path.
- (Note this is about visiting vertices, rather than edges.)

The Icosian Game

- The name comes from this game invented by mathematician William Rowan Hamilton in 1857.
- The game is based on finding cycles on this graph.



Example

- This graph contains a Hamiltonian cycle (e.g. $a \to b \to c \to e \to d \to a$).
- And therefore also a Hamiltonian path (e.g. $a \to b \to c \to e \to d$).