Boolean algebra

Peter Rowlett

Boolean algebra uses the logic connectives such as $\wedge$, $\vee$ and $\neg$ to describe logical operations involving the value true and false. We represent truth by 1 and falsehood by 0. For example, if p is the proposition "some pigs can fly" we would say $p = 0$.

Boolean algebra follows some basic laws you may be familiar with from other algebras. There are some other laws that are less familiar.

Commutative law	$p \vee q = q \vee p$	$p \wedge q = q \wedge p$
Associative law	$(p \vee q) \vee r = p \vee (q \vee r)$	$(p \wedge q) \wedge r = p \wedge (q \wedge r)$
Identity	$p \vee 0 = p$	$p \wedge 1 = p$
Distributive law	$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
Idempotent law	$p \wedge p = p$	$p \vee p = p$
Negation	$\neg 1 = 0$	$\neg 0 = 1$
Double negation	$\neg\neg p = p$	
Tautology	$p \vee \neg p = 1$	
Contradiction	$p \wedge \neg p = 0$	
Annihilation	$p \wedge 0 = 0$	$p \vee 1 = 1$
Absorption	$p \wedge (p \vee q) = p$	$p \vee (p \wedge q) = p$
De Morgan's Laws	$\neg(p \wedge q) = \neg p \vee \neg q$	$\neg(p \vee q) = \neg p \wedge \neg q$

Example

Find a simpler formula equivalent to $\neg(p \vee \neg q)$.

Solution:

(Recall that $\iff$ is "if and only if" and we are using it here instead of writing "is equivalent to".)

$$\begin{aligned} & \neg(p \vee \neg q) \\ \iff & \neg p \wedge \neg\neg q && \text{(De Morgan's Laws)}, \\ \iff & \neg p \wedge q && \text{(double negation)}. \end{aligned}$$

We can check this by making a truth table.

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p \wedge q$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	0	1	1
0	0	1	1	1	0	0

Since the last two columns are identical, we have confirmed that $\neg(p \vee \neg q) = \neg p \wedge q$.

Typically, we just apply the rules of Boolean algebra instead of using truth tables, though checking your answer via a second method isn't a bad idea.

Example

Find a simpler formula equivalent to $\neg(q \wedge \neg p) \vee p$.

Solution:

$$\begin{aligned} & \neg(q \wedge \neg p) \vee p \\ \iff & (\neg q \vee \neg\neg p) \vee p && \text{(De Morgan's Laws)}, \\ \iff & (\neg q \vee p) \vee p && \text{(double negation)}, \\ \iff & \neg q \vee (p \vee p) && \text{(associative law)}, \\ \iff & \neg q \vee p && \text{(idempotent law)}. \end{aligned}$$

So $\neg(q \wedge \neg p) \vee p = \neg q \vee p$. Again, we could check this via a truth table.