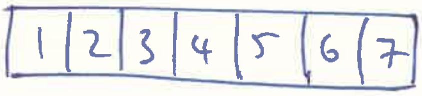
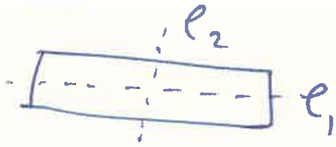
Scarves.





one rotation.

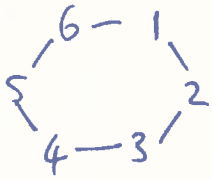
$e: x_1^7.$ $\quad r: \begin{pmatrix} 1&2&3&4&5&6&7 \\ 7&6&5&4&3&2&1 \end{pmatrix} = (17)(26)(35)(4) \quad x_2^3 x_1$

$e_1: \begin{pmatrix} 1&2&3&4&5&6&7 \\ 1&2&3&4&5&6&7 \end{pmatrix} \quad x_1^7 \qquad e_2: \begin{pmatrix} 1&2&3&4&5&6&7 \\ 7&6&5&4&3&2&1 \end{pmatrix} \quad x_2^3 x_1$

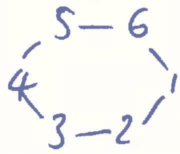
$$P(x_1, x_2) = \tfrac{1}{4}\left(2x_1^7 + 2x_2^3 x_1\right) = \tfrac{1}{2}\left(x_1^7 + x_2^3 x_1\right)$$

$$P(3,3) = \tfrac{1}{2}\left(3^7 + 3^3 \times 3\right) = 1,134$$

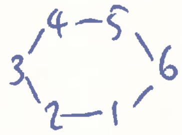
Tantrix



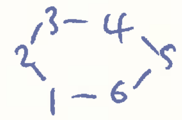
$e: x_1^6.$ $\quad r:$



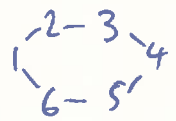
$\begin{pmatrix} 1&2&3&4&5&6 \\ 6&1&2&3&4&5 \end{pmatrix} = (165432) \quad x_6^1$

$r^2:$ 

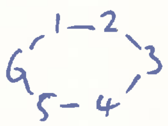
$\begin{pmatrix} 1&2&3&4&5&6 \\ 5&6&1&2&3&4 \end{pmatrix} = (153)(264) \quad x_3^2$

$r^3:$ 

$\begin{pmatrix} 1&2&3&4&5&6 \\ 4&5&6&1&2&3 \end{pmatrix} = (14)(25)(36) \quad x_2^3$

$r^4:$ 

$\begin{pmatrix} 1&2&3&4&5&6 \\ 3&4&5&6&1&2 \end{pmatrix} = (135)(246) \quad x_3^2$

$r^5:$ 

$\begin{pmatrix} 1&2&3&4&5&6 \\ 2&3&4&5&6&1 \end{pmatrix} = (123456) \quad x_6$

$$P(x_1, \dots, x_6) = \tfrac{1}{6}\left(x_1^6 + x_2^3 + 2x_3^2 + 2x_6\right)$$

How many ways to colour x_1^6? $\binom{6}{2}\binom{4}{2} = 90$
(so that we have 3 pairs of vertices each a different colour)
How many ways to colour x_2^3? $\binom{3}{1}\binom{2}{1} = 6.$
How many ways to colour x_3^2 (2 cycles of length 3)?
None that have 3 pairs.
Also 0 for x_6.
So $\frac{1}{6}(90+6+0+0) = 16$. Choosing 3 colours from 4: $\binom{4}{3} \times 16 = 64.$

Combinatorics exercises.

1. There are $\binom{4}{1}$ ways of choosing a king,
$\binom{48}{5}$ ways of choosing 5 non-kings,
out of $\binom{52}{6}$ ways of dealing 6 cards.

$$\frac{\binom{4}{1}\binom{48}{5}}{\binom{52}{6}} \approx 0.336$$

2. $\binom{7}{3}$ best friend choices
$\binom{13}{9}$ for the 9 non-best friends
out of $\binom{20}{12}$ ways of choosing 12 classmates.

$$\frac{\binom{7}{3}\binom{13}{9}}{\binom{20}{12}} \approx 0.20$$

3. (a) 18 types, choosing 3: $18^3 = 5,832$.

(b) Choosing r flavours from 25: $\binom{25}{r}$.
We want $r = 2, 3$ or 4, so

$$\binom{25}{2} + \binom{25}{3} + \binom{25}{4} = 15,250.$$

(c) 25 flavours, choosing $r = 2, 3$ or 4 with repetition:

$$25^2 + 25^3 + 25^4 = 406,875.$$

(d) Choosing r from 25 then rearranging, for $r = 2, 3$ or 4:

$$2!\binom{25}{2} + 3!\binom{25}{3} + 4!\binom{25}{4} = 318,000.$$

4. a) 52 cards in a pack, 4 suits.

Deal 13: $\binom{52}{13}$. All one suit: $\binom{4}{1} / \binom{52}{13}$

Then deal 13 from the remaining 3 suits: $\binom{3}{1} / \binom{39}{13}$.

" " 2 " : $\binom{2}{1} / \binom{26}{13}$

" " 1 " : $\binom{1}{1} / \binom{13}{13}$

All together:

$$\frac{\binom{4}{1}}{\binom{52}{13}} \times \frac{\binom{3}{1}}{\binom{39}{13}} \times \frac{\binom{2}{1}}{\binom{26}{13}} \times 1 = \frac{1}{2235197406895366368301559999}$$

So the number is correct.

(b) For discussion.

5. 4 A's : $1+a+a^2+a^3+a^4$

2 B's : $1+b+b^2$

1 C : $1+c$

Order 3 terms from $(1+a+a^2+a^3+a^4)(1+b+b^2)(1+c)$ are

Term	Ways of rearranging identical terms	
a^3	$3!$	$3!/3! = 1$
a^2b	$2!$	$3!/2! = 3$
a^2c	$2!$	"
ab^2	$2!$	"
b^2c	$2!$	"
abc	-	$3! = 6$

Number of codes: $1+3+3+3+3+6 = 19$.

6. ~~13~~ : 1: $1+o$

2: $1+t$

3: $1+r$

4: $1+f+f^2+f^3$

5: $1+v+v^2+v^3+v^4$

Order 3 terms	Number of rearrangements
f^3	$3!/3! = 1$
$f^2 o$	$3!/2! = 3$
$f^2 r$	3
$f^2 t$	3
$f^2 v$	3
$f o r$	$3! = 6$
$f o t$	6
$f o v$	6
$f r t$	6
$f r v$	6
$f t v$	6
$f v^2$	3
v^3	1
$v^2 t$	3
$v^2 o$	3
$v^2 r$	3
$r t v$	6
$o t v$	6
$o r t$	6
$o r v$	6

$$10 \times 6 + 2 \times 1 + 8 \times 3 = 86$$

7. 2 blue strips: $1+b+b^2$

1 red strip: $1+r$

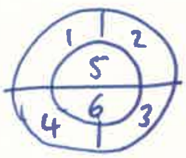
2 yellow strips: $1+y+y^2$

Order 3 terms from

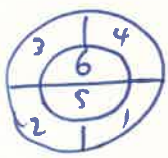
$$(1+b+b^2)(1+r)(1+y+y^2)$$

Term	Ways to rearrange
b^2r	$\frac{3!}{2!}=3$
b^2y	3
by^2	3
ry^2	3
bry	$3!=6$

Total: $6+4\times 3=18$.

8. 

$e: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = (1)(2)(3)(4)(5)(6) \quad x_1^6$

r: 

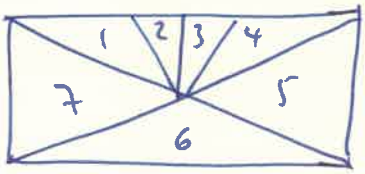
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix} = (13)(24)(56) \quad x_2^3$

$P(x_1, x_2) = \frac{1}{2}(x_1^6 + x_2^3)$.

Then ~~$P(2,2) = \frac{1}{2}(2^6+2^3) = 36$~~

$P(3,3) = \frac{1}{2}(3^6+3^3) = 378$

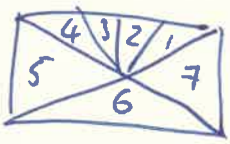
There are 378 ways to 3-colour the tile.

9. 

No rotations possible.
One reflection:

ρ: 

$e: \; x_1^{\,7}$.

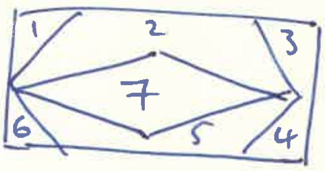
ρ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 1 & 7 & 6 & 5 \end{pmatrix}$$

$$= (14)(23)(57)(6) \qquad x_2^{\,3} x_1$$

$$P(x_1, x_2) = \tfrac{1}{2}\left(x_1^{\,7} + x_2^{\,3} x_1\right)$$

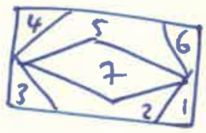
$$P(6,6) = \tfrac{1}{2}\left(6^7 + 6^3 \times 6\right) = 140{,}616.$$

10. 

One rotation, two reflections

ρ_2  $-\rho_1$

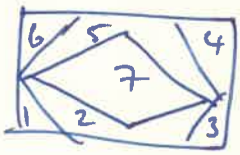
$e: \; x_1^{\,7}$.

r: 

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 1 & 2 & 3 & 7 \end{pmatrix}$$

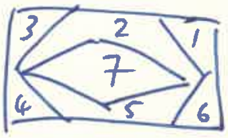
$$= (14)(25)(36)(7)$$

$$x_2^{\,3} x_1$$

ρ_1: 

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 4 & 3 & 2 & 1 & 7 \end{pmatrix} = (16)(25)(34)(7)$$

$$x_2^{\,3} x_1$$

ρ_2: 

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 6 & 5 & 4 & 7 \end{pmatrix} = (13)(2)(46)(5)(7)$$

$$x_2^{\,2} x_1^{\,3}$$

$$P(x_1, x_2) = \tfrac{1}{4}\left(x_1^{\,7} + 2x_2^{\,3} x_1 + x_2^{\,2} x_1^{\,3}\right)$$

$$P(6,6) = \tfrac{1}{4}\left(6^7 + 2 \times 6^3 \times 6 + 6^2 \times 6^3\right) = 72576$$