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MATH-237-SY
HW#7

NB1) $y'' + y = \cos(x) \qquad y(0) = 1, \; y'(0) = -1$

$y_h = C_1\cos(x) + C_2\sin(x)$

$y_p(t) = At\cos(t) + Bt\sin(t)$
$y_p'(t) = (B - At)\sin(t) + (Bt + A)\cos(t)$
$y_p''(t) = (-Bt - 2A)\sin(t) + (2B - At)\cos(t)$

$y_p'' + y_p = \cos(x)$

$(-Bx - 2A)\sin(x) + (2B - Ax)\cos(x) + Ax\cos(x) + Bx\sin(x) = \cos(x)$

$\cancel{-Bx\sin(x)} - 2A\sin(x) + 2B\cos(x) \cancel{- Ax\cos(x)} + \cancel{Ax\cos(x)} + \cancel{Bx\sin(x)} = \cos(x)$

$-2A\sin(x) + 2B\cos(x) = \cos(x)$

$-2A = 0 \quad : \quad 2B = 1$
$A = 0 \qquad\quad B = 1/2$

$y_p = 1/2\, x\sin(x)$

$y = C_1\cos(x) + C_2\sin(x) + 1/2\, x\sin(x)$

$y' = -C_1\sin(x) + C_2\cos(x) + 1/2(\sin(x) + x\cos(x))$

$y(0): \; 1 = C_1\cos(0) + C_2\sin(0) + 1/2\,(0)\sin(0)$
$1 = C_1$

$y'(0): \; -1 = -\sin(0) + C_2\cos(0) + 1/2(\sin(0) + (0)\cos(0))$
$-1 = C_2$

$$\boxed{y = \cos(x) - \sin(x) + 1/2(x\sin(x))}$$

NB2) $y''' - 2y'' + y' = 2 - 24e^{x} + 40e^{5x}$, $y(0) = 1/2$, $y'(0) = 5/2$, $y''(0) = -9/2$

$r^3 - 2r + r = 0$

$r_1 = 0$, $r_2 = 1$, $r_3 = 1$

$y_h = C_1 + C_2 e^{t} + C_3 t e^{t}$

$y_p = At + Bt^2 e^{t} + Ce^{5t}$

$y_p' = A + 2Bte^{t} + Bt^2 e^{t} + 5Ce^{5t}$

$y_p'' = e^{t}(Bt^2 + 4Bt + 2B) + 25Ce^{5t}$

$y_p''' = e^{t}(Bt^2 + 6Bt + 6B) + 125Ce^{5t}$

$y_p''' - 2y_p'' + y_p' = 2 - 24e^{x} + 40e^{5x}$

$e^{x}(Bx^2 + 6Bx + 6B) + 125Ce^{x} - 2(e^{x}(Bx^2 + 4Bx + 2B) + 25Ce^{t}) + (A + 2Bxe^{x} + Bx^2e^{x} + 5Ce^{5x}) = 2 - 24e^{x} + 40e^{5x}$

$A + 2Be^{x} + 80Ce^{5x} = 2 - 24e^{x} + 40e^{5x}$

$A = 2$: $2B = -24$: $80C = 40$

$B = -12$ $C = 1/2$

$y_p(t) = 2t - 12t^2e^{t} + 1/2\, e^{5t}$

$y = C_1 + C_2e^{t} + C_3te^{t} + 2t - 12t^2e^{t} + 1/2\, e^{5t}$

$0 = C_1 + C_2 + 0C_3$

$-2 = 0C_1 + C_2 + C_3$

$7 = 0C_1 + C_2 + 2C_3$

$\boxed{C_1 = 11 \quad C_2 = -11 \quad C_3 = 9}$

$y(0)$: $1/2 = C_1 + C_2 + 1/2 \Rightarrow 0 = C_1 + C_2$

$y' = 1/2(5e^{5t} + (-24t^2 + (2C_3 - 48)t + 2C_2 + 2C_3)e^{t} + 4)$

$y(0)$: $5/2 = 1/2(5 + 2C_2 + 2C_3 + 4) \Rightarrow 5 = 9 + 2C_2 + 2C_3 \Rightarrow -4 = 2C_2 + 2C_3 \Rightarrow -2 = C_2 + C_3$

$y'' = 1/2(25e^{5t} + (-24t^2 + (2C_3 - 96)t + 4C_3 + 2C_2 - 48)e^{t})$

$y(0) = -9/2 = 1/2(25 + 4C_3 + 2C_2 - 48) \Rightarrow -9 = -23 + 4C_3 + 2C_2 \Rightarrow 14 = 4C_3 + 2C_2 = 7 = 2C_3 + C_2$

$$\boxed{y(x) = 11 - 11e^{x} + 9xe^{x} + 2x - 12x^2e^{x} + (1/2)e^{5x}}$$

2) $y'' + y = \sec(t)$

$f(t) = \sec(t)$

$y_h = C_1 \cos(t) + C_2 \sin(t)$

$y_1 = \cos(t)$ $\quad$ $y_2 = \sin(t)$

$y_1' = -\sin(t)$ $\quad$ $y_2' = \cos(t)$

$W = y_1 y_2' - y_1' y_2 \Rightarrow W = (\cos(t) \cdot \cos(t)) - (-\sin(t) \cdot \sin(t))$

$W = \cos^2(t) + \sin^2(t)$

$W = 1$

$v_1 = -\int \frac{f(t)\, y_2}{W}\, dt = -\int \frac{\sec(t)\sin(t)}{1}\, dt = -\int \tan(t)\, dt = \ln|\cos(t)|$

$v_2 = \int \frac{f(t)\, y_1}{W}\, dt = \int \frac{\sec(t)\cos(t)}{1} dt = \int \frac{1}{1} dt = t$

$y_p = y_1 v_1 + y_2 v_2 = \cos(t) \ln|\cos(t)| + \sin(t) \cdot t$

$\boxed{y = C_1 \cos(t) + C_2 \sin(t) + \cos(t) \ln|\cos(t)| + t\sin(t)}$

3) $y'' - 2y' + y = t^{-1}e^{t}$

$r^2 - 2r + 1 = 0$

$r_1 = 1,\ r_2 = 1$

$y_h = C_1 e^{t} + C_2 t e^{t}$

$f(t) = t^{-1}e^{t}$

$y_1 = e^{t}$ $\quad$ $y_2 = te^{t}$

$y_1' = e^{t}$ $\quad$ $y_2' = e^{t} + te^{t}$

$w = y_1 y_2' - y_1' y_2 \Rightarrow w = e^{t}(e^{t} + te^{t}) - e^{t} \cdot te^{t}$

$w = e^{2t}$

$v_1 = -\int \frac{f(t)\, y_2}{w}\, dt \Rightarrow -\int \frac{(t^{-1}e^{t})te^{t}}{e^{2t}}\, dt \Rightarrow -\int \frac{te^{2t}}{te^{2t}}\, dt = -\int dt = -t$

$v_2 = \int \frac{f(t)\, y_1}{w}\, dt \Rightarrow \int \frac{(t^{-1}e^{t})e^{t}}{e^{2t}}\, dt \Rightarrow \int \frac{e^{2t}}{te^{2t}}\, dt \Rightarrow \int 1/t\, dt = \ln|t|$

$y_p = y_1 v_1 + y_2 v_2 \Rightarrow e^{t} \cdot (-t) + te^{t} \cdot \ln|t| \Rightarrow -te^{t} + te^{t}\ln|t| = te^{t}(-1 + \ln|t|)$

$y = C_1 e^{t} + C_2 te^{t} + te^{t}(-1 + \ln|t|)$

7) $y'' + 4y' + 4y = e^{-2t}\ln(t)$

$r^2 + 4r + 4 = 0$

$r_1 = -2 \quad r_2 = -2$

$f(t) = e^{-2t}\ln(t)$

$y_1 = e^{-2t} \qquad y_2 = te^{-2t}$

$y_1' = -2e^{-2t} \qquad y_2' = e^{-2t} + (-2te^{-2t})$

$y_h = C_1 e^{-2t} + C_2 te^{-2t}$

$y_1 v_1' + y_2 v_2' = 0 \Rightarrow e^{-2t} v_1' + te^{-2t} v_2' = 0$

$y_1' v_1' + y_2' v_2' = f(t) \Rightarrow -2e^{-2t} v_1' + (e^{-2t} - 2te^{-2t}) v_2' = e^{-2t}\ln(t)$

$2e^{-2t} v_1' + 2te^{-2t} v_2' = 0$

$-2e^{-2t} v_1' - 2te^{-2t} v_2' + e^{-2t} v_2' = e^{-2t}\ln(t)$

$\Rightarrow e^{-2t} v_2' = e^{-2t}\ln(t) \Rightarrow \int v_2' = \int \ln(t)\,dt \Rightarrow v_2 = t\ln(t) - t$

$\Rightarrow e^{-2t} v_1' + te^{-2t} v_2' = 0 \Rightarrow e^{-2t} v_1' + te^{-2t}\ln(t) = 0 \Rightarrow v_1 = \frac{-t^2(2\ln(t)-1)}{4}$

$\Rightarrow e^{-2t}(v_1' + t\ln(t)) = 0$

$(v_1' + t\ln(t)) = 0$

$v_1' = -t\ln(t)$

$y_p = y_1 v_1 + y_2 v_2 \Rightarrow \left[(e^{-2t})\cdot\left(\frac{-t^2(2\ln(t)-1)}{4}\right)\right] + \left[(te^{-2t})\cdot(t\ln(t) - t)\right]$

$y_p = \frac{-e^{-2t}t^2(2\ln(t)-1)}{4} + \frac{e^{-2t}t^2(4\ln(t)-1)}{4} =$

$$\boxed{y = C_1 e^{-2t} + C_2 te^{-2t} - \frac{e^{-2t}t^2(2\ln(t)-1)}{4} + \frac{e^{-2t}t^2(\ln(t)-1)}{1}}$$

NB3) $y''' + \frac{3}{2x} y'' = x^{-3/2}$

$y_h = C_1 + C_2 x + C_3 \sqrt{x}$

$f(t) = x^{-3/2}$

$y_1 = 1$ $\quad y_2 = x$ $\quad y_3 = \sqrt{x}$

$y_1' = 0$ $\quad y_2' = 1$ $\quad y_3' = \frac{1}{2\sqrt{x}}$

$y_1'' = 0$ $\quad y_2'' = 0$ $\quad y_3'' = -\frac{1}{4x^{3/2}}$

eqn 1) $y_1 v_1' + y_2 v_2' + y_3 v_3' = 0$

eqn 2) $y_1' v_1' + y_2' v_2' + y_3'' v_3' = 0$

eqn 3) $y_1'' v_1' + y_2'' v_2' + y_3'' v_3' = f(t)$

eqn 1:

$(1) v_1' + (x) v_2' + (\sqrt{x}) v_3' = 0$

$v_1' + x v_2' + \sqrt{x}\, v_3' = 0$

$v_1' + (x)(\frac{2}{\sqrt{x}}) + (\sqrt{x})(-4) = 0$

$v_1' + (2\sqrt{x}) - (4\sqrt{x}) = 0$

$\boxed{v_1' = 2\sqrt{x}}$

$\boxed{\frac{4}{3} x^{3/2} = v_1'}$

eqn 2:

$(0) v_1' + (1) v_2' + (\frac{1}{2\sqrt{x}}) v_3' = 0$

$v_2' + (\frac{1}{2\sqrt{x}}) v_3' = 0$

$v_2' + (\frac{1}{2\sqrt{x}})(-4) = 0$

$\frac{2}{\sqrt{x}} = v_2'$

$\boxed{\frac{2}{\sqrt{x}} = v_2'}$

$\boxed{4\sqrt{x} = v_2}$

eqn 3:

$(0) v_1' + (0) v_2' + (\frac{-1}{4x^{3/2}}) v_3' = x$

$(\frac{-1}{4x^{3/2}}) v_3' = x^{-3/2}$

$(-\frac{1}{4}) v_3' = 1$

$\boxed{v_3' = -4}$

$\boxed{v_3 = -4x}$

$y_p = y_1 v_1 + y_2 v_2 + y_3 v_3$

$y_p = (1)(\frac{4}{3} x^{3/2}) + (x)(4\sqrt{x}) + (\sqrt{x})(-4x)$

$y_p = \frac{4}{3} x^{3/2} + 4x\sqrt{x} - 4x\sqrt{x}$

$y_p = \frac{4}{3} x^{3/2}$