Alex Jacob
Prof. Barlow

MATH-237-S4
Final

1) $\frac{dy}{dt} + ty = t$, $y(0) = 0$ (separation of variables)

$\frac{dy}{dt} = t - ty \Rightarrow \frac{dy}{dt} = t(1-y) \Rightarrow \int \frac{dy}{1-y} = \int t\,dt \Rightarrow -\ln|1-y| = \frac{t^2}{2} + \tilde{c}$

$\Rightarrow \ln|1-y| = -\frac{t^2}{2} - \tilde{c} \Rightarrow e^{\ln|1-y|} = e^{-\frac{t^2}{2} - \tilde{c}} \Rightarrow 1 - y = e^{-t^2/2}\tilde{c}$

$\Rightarrow 1 - e^{-t^2/2}\tilde{c} = y \Rightarrow y = 1 - e^{-t^2/2}\tilde{c}$

$y(0) = 0 \Rightarrow 0 = 1 - e^0\tilde{c} \Rightarrow 0 = 1 - 1\tilde{c} \Rightarrow \tilde{c} = 1$

$\Rightarrow \boxed{y = 1 - e^{-t^2/2}}$

2) $\frac{dy}{dt} + ty = t$, $y(0) = 0$ (variation of parameters)

$\frac{dy_h}{dt_h} + t_h y_h = 0 \Rightarrow \frac{dy_h}{dt_h} = -t_h y_h \Rightarrow \int \frac{dy_h}{y_h} = \int -t_h\,dt_h \Rightarrow \ln|y_h| = -\frac{t^2}{2} + \tilde{c}$

$\Rightarrow e^{\ln|y_h|} = e^{-t^2/2 + \tilde{c}} \Rightarrow \boxed{y_h = e^{-t^2/2}\tilde{c}}$

$y_p = v(x)\,e^{-t^2/2}$ $\qquad y_p' + t y_p = t$

$y_p' = v'(x)\,e^{-t^2/2} + v(x)\left(-t e^{-t^2/2}\right)$

$\Rightarrow \left(v'(x)e^{-t^2/2} + \cancel{v(x)\left(-t e^{-t^2/2}\right)}\right) + \cancel{t\left(v(x)e^{-t^2/2}\right)} = t \Rightarrow v'(x)e^{-t^2/2}$

$v'(x) = t/e^{-t^2/2} \Rightarrow v(x) = e^{t^2/2} \Rightarrow y_p = e^{t^2/2} \cdot e^{-t^2/2} \Rightarrow 1$

$y = y_h + y_p \Rightarrow \tilde{c}e^{-t^2/2} + 1$: $y(0) = 0 \Rightarrow 0 = 1 + 1\tilde{c} \Rightarrow \tilde{c} = -1 \Rightarrow \boxed{y = 1 - e^{-t^2/2}}$

3) $y'' = t\cos(t)$ (method of undetermined coefficients)

$y_h'' = 0$

$r^2 = 0$

$r_1 = 0, r_2 = 0$ (case 2 : $y = C_1 e^{0t} + tC_2 e^{0t} \Rightarrow \boxed{C_1 + tC_2 = y_h}$

$y_p = t\cos(t)$
$y_p' = 1\cos(t) - t\sin(t)$
$y_p'' = -t\cos(t) - 2\sin(t)$
$y_p''' = -3\cos(t) + t\sin(t)$

$y_p = At\cos(t) + Bt\sin(t)$
$y_p' = (Bt + A)\cos(t) + (B - At)\sin(t)$
$y_p'' = (2B - At)\cos(t) + (-Bt - 2A)\sin(t)$

$y_p'' = t\cos(t)$

$\Rightarrow (2B - At)\cos(t) + (-Bt - 2A)\sin(t) = t\cos(t)$

$2B\cos(t) - At\cos(t) - Bt\sin(t) - 2A\sin(t) = t\cos(t)$

$2B - At = t$: $-Bt - 2A = 0$
$2B = t(1 - A)$ $\quad -Bt = 2A$
$t = 2A/-B$

$2B = \frac{2A}{-B}(1 - A)$

$2B = \frac{2A}{-B} - \frac{2A^2}{-B} \Rightarrow 2B = \frac{2A - 2A^2}{-B} \Rightarrow -2B^2 = 2A - 2A^2 \Rightarrow -B^2 = A - A^2$

4) $y'' = t\cos(t)$ (variation of parameters)

$$\boxed{y_h = C_1 + tC_2}$$

$f(t) = t\cos(t)$ $\qquad W = y_1 y_2' - y_1' y_2 \Rightarrow (1\cdot 1) - (0 \cdot t) = 1$

$y_1 = 1 \quad y_2 = t$

$y_1' = 0 \quad y_2' = 1$

$$v_1 = -\int \frac{f(t)\,y_2}{W}\,dt \Rightarrow -\int t\cos(t)\,(t)\,dt = -t^2\sin(t) + 2\sin(t) - 2t\cos(t)$$

$$v_2 = \int \frac{f(t)\,y_1}{W}\,dt \Rightarrow \int \frac{t\cos(t)(1)}{1}\,dt \Rightarrow \int t\cos(t)\,dt = t\sin(t) + \cos(t)$$

$y_p = y_1 v_1 + y_2 v_2$

$y_p = 1\left(-t^2\sin(t) + 2\sin(t) - 2t\cos(t)\right) + t\left(t\sin(t) + \cos(t)\right)$

$y_p = \cancel{-t^2\sin(t)} + 2\sin(t) - 2t\cos(t) + \cancel{t^2\sin(t)} + t\cos(t)$

$y_p = 2\sin(t) - t\cos(t)$

$y = y_h + y_p \Rightarrow \boxed{y = C_1 + tC_2 - t\cos(t) + 2\sin(t)}$

5) $y'' + 4y = \delta(t-2) \qquad y(0)=0 \;,\; y'(0)=0$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\}$

$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = e^{-2s}$

$s^2Y(s) - \cancel{s(0)} - \cancel{0} + 4Y(s) = e^{-2s}$

$(s^2+4)Y(s) = e^{-2s}$

$Y(s) = e^{-2s}/s^2+4 \Rightarrow e^{-2s} \cdot \frac{1}{s^2+4} \qquad (c=2)$

$Y(s) = e^{-2s} \cdot \mathcal{L}\{\tfrac{1}{2}\sin(2t)\}$

$\hookrightarrow f(t) = \tfrac{1}{2}\sin(2t)$

$\tilde{f}(t-c) = \tfrac{1}{2}\sin(2(t-c))$

$= \tfrac{1}{2}\sin(2t-2c) \qquad (c=2)$

$= \tfrac{1}{2}\sin(2t-4)$

$\Rightarrow \tilde{f}(t-c)u(t-2) \Rightarrow \boxed{y = \tfrac{1}{2}\sin(2t-4)u(t-2)}$

6) (1) $x'' + y = 0 \quad x(0) = 0, \; x'(0) = 0$

(2) $y'' + x = 4 \quad y(0) = 0, \; y'(0) = 0$

$\mathcal{L}^{-1}\{(1)\} = s^2 X(s) - \cancel{sx(0)} - \cancel{x'(0)} + Y(s) = 0 \Rightarrow s^2 X(s) = -Y(s) \Rightarrow X(s) = -Y(s)/s^2$

$\mathcal{L}^{-1}\{(2)\} = s^2 Y(s) - \cancel{sy(0)} - \cancel{y'(0)} + X(s) = 4/s \Rightarrow s^2 Y(s) + \frac{-Y(s)}{s^2} = 4/s$

$$\frac{s^4 Y(s)}{s^2} - \frac{Y(s)}{s^2} = \frac{4}{s} \Rightarrow \frac{Y(s)(s^4-1)}{s^2} = \frac{4s}{s^2} \Rightarrow Y(s)(s^4-1) = 4s$$

$$Y(s) = \frac{4s}{(s^4-1)} \Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4s}{(s^2+1)(s+1)(s-1)}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)} + \frac{1}{(s+1)} + \frac{1}{(s-1)}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{-2s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\Rightarrow 2\cos(t) + e^{-t} + e^{t}$$

$$\mathcal{L}^{-1}\{Y(s)\} = 2\cos(t) + e^{-t} + e^{t} \Rightarrow \boxed{y(t) = 2\cos(t) + e^{-t} + e^{t}}$$

7) $\frac{dy}{dt} + ty = t$

a) Classification:

1) indep var: t
2) dep var: y
3) order: first order
4) Ordinary differential equation
5) linear
6) variable coefficient
7) non-homogeneous t

b) $\frac{dy}{dt} + ty^2 = t$

8) 1) True
2) True
3) False
4) True
5) False