MATH 351–004 – Assignment #6

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Part 2

Problem 1 - Prove that if v is a cut-vertex of a graph G, then v is not a cut-vertex of the complement $\bar{G}$ of G

Suppose there exists two subgraphs $G_1 and G_2$ and a cut-vertex v such that $V(G)\ v = V(G_1) \bigcup V(G_2)$. There exists a uw-path such that $u \in V(G_1)$ and $w \in V(G_2)$, then by Corollary 5.4, v is a cut-vertex of G. Then when we take the complement there will be another vertex y that will be in the uw path. This leads to the uw path in the complement not containing v while containing y. Thus v is not a cut-vertex in $\bar{G}$.

Problem 2 - Prove that a 3-regular graph G has a cut-vertex if and only if G has a bridge

Forwards: A 3-regular graph has a cut-vertex if it has a bridge.
Suppose that G is a 3-regular graph and has cut-vertex u. $G - u$ would create 2 or 3 connected components$(G_1, G_2, (G_3))$ because u has 3 neighbors.
3 Connected components: Each of these components much have exactly 1 neighbor of u.
2 Connected components: One connected component would have exactly 1 neighbor of u and the other have 2 neighbors of u.
Removing the one neighbor u would disconnect the graph. (Theorem 5.4 Corollary 5.4)
Backwards: A 3-regular graph has a bridge, then that graph has a cut-vertex
Suppose that G is a 3-regular graph with vertices u and v. Suppose that uv is a bridge in G. Since uv is a bridge, then either vertex u or v is a cut-vertex. Therefore G has a cut-vertex. (Theorem 5.1)

Problem 3 -
(a) Let G be a nontrivial connected graph. Prove that if v is an end-vertex of a spanning tree of G, then v is not a cut-vertex of G.

(b) Use (a) to give an alternative proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

(c) Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

(d) Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path. [Recall that if a tree contains a vertex of degree exceeding 2, then T has more than two end-vertices.]

a) Since we know that every graph has a spanning tree (Theorem 4.10), an end-vertex of a spanning tree is a leaf node. If we were to remove v from the spanning tree and we observe that $G - v$ is still connected, then v is not a cut-vertex, otherwise $G - v$ would be disconnected.

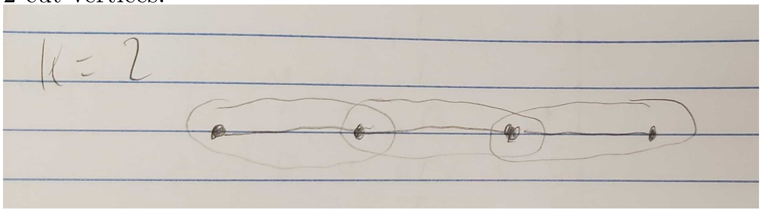
b) Since we know that every tree must have at least two leaf vertices(Theorem 4.3), using part (a) tells us that these leaf nodes are not cut-vertices.

c) Suppose you create a tree via BFS starting at node v. Since BFS visits every node individually without creating a cycle, a spanning tree can be created that contains all of the edges that are incident with v.

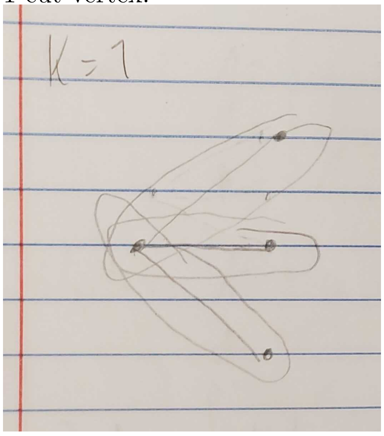
d) Since we have exactly 2 non-cut-vertices on uv path P, these can only be the endpoints of P. Knowing this, adding another vertex to P would lead to one of many cases:
Case 1: Added vertex w has only an edge to a non-cut-vertex.u.
No issue, then w becomes a leaf node and u becomes a cut-vertex.
Case 2: Added vertex w only an edge to a cut-vertex.
This would create a total of 3 non-cut-vertices, which would make G no longer a path.
Case 3: Added vertex w with edges to a non-cut-vertex and a cut-vertex.
This would create a total of 3 non-cut-vertices, which would make G no longer a path.
Case 4: Added vertex w with edges to only non-cut-vertices.
Adding w would create a cycle, where no vertices are cut-vertices.

Problem 4 - If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

The amount of cut vertices we can have is the number of blocks - 1; in this case, we have at most 2 cut-vertices. This is possible because if the amount of blocks equaled to the number of blocks, then the cut vertices of each block would make a cycle, which would not make them cut-vertices. Since we have 3 blocks and a max of 2 cut vertices, we have a total of 3 possible cases:
2 cut-vertices:



1 cut-vertex:



0 cut-vertices:
Since G is connected, and there are 3 blocks, then at least two of the blocks would have a cut-vertex in common, having 0 cut-vertices would make G disconnected.