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MATH 237-54
HW #9

21) $y'' + y = u(t-3) \quad y(0) = 0, \; y'(0) = 1$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u(t-3)\}$

$s^2 Y(s) - sy(0) - y'(0) + Y(s) = e^{-3s}/s$

$s^2 Y(s) - s(0) - 1 + Y(s) = e^{-3s}/s$

$$Y(s) = \frac{e^{-3s}}{s(s^2+1)} + \frac{1}{s^2+1}$$

$$Y(s) = \frac{e^{-3s}}{s} - \frac{se^{-3s}}{s^2+1} + \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{se^{-3s}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \boxed{u(t-3) - \cos(t-3)u(t-3) + \sin(t)}$$

13) $w'' + w = \delta(t-\pi) \quad w(0) = 0, \; w'(0) = 0$

$\mathcal{L}\{w''\} + \mathcal{L}\{w\} = \mathcal{L}\{\delta(t-\pi)\}$

$s^2 W(s) - sw(0) - w'(0) + W(s) = e^{-\pi s}$

$$W(s) = \frac{e^{-\pi s}}{1} \cdot \frac{1}{s^2+1}$$

$\mathcal{L}^{-1}\{W(s)\} = f(t-\pi)u(t-\pi), \quad f(t-\pi) = \sin(t-\pi) \Rightarrow \int_0^t \sin(t-\pi)\,dt = -\sin(t)$

$\Rightarrow \boxed{w(t) = -\sin(t)u(t-\pi)}$

15) $y'' + 2y' - 3y = \delta(t-1) - \delta(t-2) \qquad y(0) = 2, \; y'(0) = -2$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\} - \mathcal{L}\{\delta(t-2)\}$

$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) - 3(Y(s)) = e^{-s} - e^{-2s}$

$s^2Y(s) - 2s + 2 + 2sY(s) - 4 - 3Y(s) = e^{-s} - e^{-2s}$

$(s^2 + 2s - 3)Y(s) = e^{-s} - e^{-2s} + 2s + 2$

$$Y(s) = \frac{e^{-s} - e^{-2s} + 2s + 2}{(s+3)(s-1)} \Rightarrow \underbrace{\frac{e^{-s}}{(s+3)(s-1)}}_{(1)} - \underbrace{\frac{e^{-2s}}{(s+3)(s-1)}}_{(2)} + \underbrace{\frac{2s+2}{(s+3)(s-1)}}_{(3)}$$

$F(s) = \frac{1}{(s+3)(s-1)} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}$

$(1): \mathcal{L}^{-1}\{(1)\} = \mathcal{L}^{-1}\{e^{-s}F(s)\} = f(t-1)u(t-1) = \left(-\frac{1}{4}e^{(-3t+3)} + \frac{1}{4}e^{(t-1)}\right)u(t-1)$

$(2): \mathcal{L}^{-1}\{(2)\} = \mathcal{L}^{-1}\{e^{-2s}F(s)\} = f(t-2)u(t-2) = \left(-\frac{1}{4}e^{(-3t+6)} + \frac{1}{4}e^{(t-2)}\right)u(t-2)$

$(3): \mathcal{L}^{-1}\{(3)\} = \mathcal{L}^{-1}\{2s+2\,F(s)\} = e^{-3t} + e^{t}$

$$\boxed{y(t) = \left[\left(-\frac{1}{4}e^{(-3t+3)} + \frac{1}{4}e^{(t-1)}\right)u(t-1)\right] - \left[\left(-\frac{1}{4}e^{(-3t+6)} + \frac{1}{4}e^{(t-2)}\right)u(t-2)\right] + \left[e^{-3t} + e^{t}\right]}$$

6) $\dfrac{1}{(s+1)(s+2)}$ let $F(s) = 1/(s+1)$

let $G(s) = 1/(s+2)$

$\mathcal{L}^{-1}\{F(s)\} = e^{-t} = f(t)$

$\mathcal{L}^{-1}\{G(s)\} = e^{-2t} = g(t)$

$$f(t) * g(t) = \int_0^t f(t-v)\,g(v)\,dv \Rightarrow \int_0^t e^{-t+v} e^{-2v}\,dv$$

$$\Rightarrow \int_0^t e^{-t} e^{v} e^{-2v}\,dv \Rightarrow e^{-t}\int_0^t e^{-v}\,dv \Rightarrow e^{-t}\left(-e^{-v}\right)\Big|_0^t = \boxed{e^{-t}\left[-e^{-t} - 1\right]}$$

7) $\dfrac{14}{(s+2)(s-5)}$ let $F(s) = 1/(s+2)$

let $G(s) = 1/(s-5)$

$\mathcal{L}^{-1}\{F(s)\} = e^{-2t} = f(t)$

$\mathcal{L}^{-1}\{G(s)\} = e^{5t} = g(t)$

$$\Rightarrow 14\int_0^t f(t-v)\,g(v)\,dv \Rightarrow 14\int_0^t e^{-2(t-v)} e^{5v}\,dv \Rightarrow 14\int_0^t e^{-2t} e^{2v} e^{5v}\,dv$$

$$\Rightarrow 14e^{-2t}\int_0^t e^{7v}\,dv \Rightarrow 14e^{-2t}\left(1/7\, e^{7v}\right)\Big|_0^t \Rightarrow \boxed{14e^{-2t}\left(1/7\, e^{7t} - 1/7\right)}$$

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37) (1): $x' + y = 0 \qquad x(0) = 0$

(2): $x + y' = 1 - u(t-2) \qquad y(0) = 0$

$\mathcal{L}^{-1}\{(1)\} = sX(s) - \cancel{x(0)} + Y(s) = 0 \Rightarrow Y(s) = -s\,X(s)$

$\mathcal{L}^{-1}\{(2)\} = X(s) + sY(s) = 1/s - e^{2s}/s$

$\Rightarrow \cancel{X}(s) + s(-s\,X(s)) = 1/s - e^{2s}/s$

$\cancel{X}(s) - s^2 X(s) = 1/s - e^{2s}/s$

$(1-s^2)X(s) = \dfrac{1-e^{2s}}{s}$

$\boxed{X(s) = \dfrac{1-e^{2s}}{s(1-s^2)}} \qquad Y(s) = -s(X(s)) \Rightarrow \boxed{Y(s) = \dfrac{e^{2s}-1}{(1-s^2)}}$

$\mathcal{L}^{-1}\{X(s)\} = \dfrac{1-e^{2s}}{s} - \dfrac{1-e^{-2s}}{2(s+1)} - \dfrac{1-e^{2s}}{2(s-1)}$

$\Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1-e^{-2s}}{s}\right\} - \mathcal{L}^{-1}\left\{\dfrac{1-e^{-2s}}{2(s+1)}\right\} - \mathcal{L}^{-1}\left\{\dfrac{1-e^{-2s}}{2(s-1)}\right\}$

$\boxed{x(t) = [1 - u(t-2)] - [1/2\,e^{-t} - 1/2\,e^{(-t-2)}u(t-2)] - [1/2\,e^{t} - 1/2\,e^{t-2}u(t-2)]}$

$\mathcal{L}^{-1}\{Y(s)\} = \dfrac{e^{-2s}-1}{2(s+1)} - \dfrac{e^{-2s}-1}{2(s-1)}$

$\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}-1}{2(s+1)}\right\} - \mathcal{L}^{-1}\left\{\dfrac{e^{-2s}-1}{2(s-1)}\right\}$

$\boxed{y(t) = [-1/2\,e^{-t} + 1/2\,e^{(-t+2)}u(t-2)] - [-1/2\,e^{t} + 1/2\,e^{t-2}u(t-2)]}$