MATH 351–004 – Assignment #4

Alex Iacob
ai9388

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Problem 1 - Prove that every connected graph all of whose vertices have even degrees contains no bridges

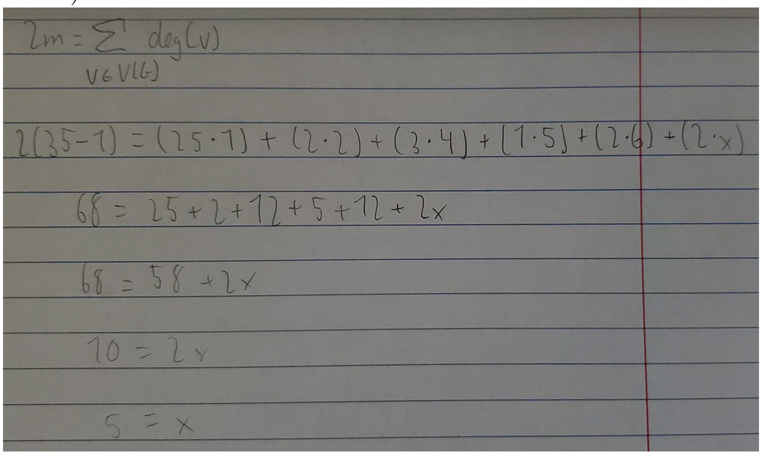
If we were to consider Graph G with all even degree vertices. Somewhere in this graph, there is this bridge that joins vertex x to vertex y. Upon removing this edge, there are two subgraphs of G, G_x and G_y. If we were to consider G_x, then G_x has exactly one vertex with an odd degree and the rest have even degrees. This creates a contradiction of Corollary 2.3, thus every connected graph all of whose vertices have even degrees contains no bridges.

Problem 2 - Let G be a connected graph that e_1 and e_2 be two edges of G. Prove that $G - e_1 - e_2$ has three components if and only if both e_1 and e_2 are bridges in G.

Forward:
By definition, G is connected, therefore it has one connected component.
Since e_1 is a bridge, by definition, $G - e_1$ would contain two connected components.
However, we do not know if e_2 is a bridge in $G - e_1$. Using Theorem 4.1, we know that if e is a bridge in G and $e \in H \subseteq G$, then e is a bridge of H. Therefore we know that e_2 is a bridge in $G - e_1$, which would make the total connected components equal to 3.
Backward:
If $G - e_1 - e_2$ has 3 connected components, then edges $e - 1$ and e_2 are bridges.
Using the formula $\kappa(G) \leq \kappa(G - e) \leq \kappa(G) + 1$. Suppose that e1 is not a bridge, then we know that
$\kappa(G - e_1) = 1$
$1 \leq \kappa((G - e_1) - e_2) \leq 2$
Contradiction Suppose that e2 is not a bridge, then we know that
$\kappa(G - e_1) = 2$
$2 \leq \kappa((G - e_1) - e_2) \leq 3$
$\kappa((G - e_1) - e_2) = 3$

Problem 3 - A certain tree T of order 35 is known to have 25 vertices of degree 1, two vertices of degree 2, three vertices of degree 4, one vertex of degree 5 and two vertices of degree 6. It also contains two vertices of the same (unknown) degree x. What is x?

Via the Handshake Lemma and knowing that size of tree is its order - 1 (and also featuring my artistic talent)



$$2m = \sum_{v \in V(G)} \deg(v)$$

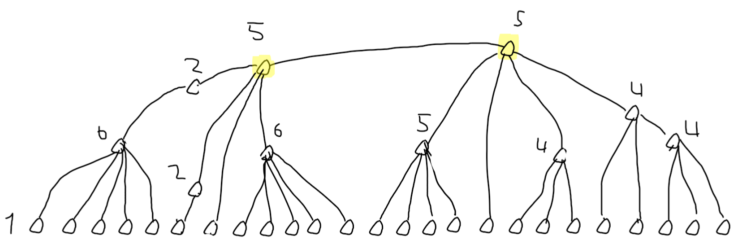
$$2(35-1) = (25 \cdot 1) + (2 \cdot 2) + (3 \cdot 4) + (1 \cdot 5) + (2 \cdot 6) + (2 \cdot x)$$

$$68 = 25 + 2 + 12 + 5 + 12 + 2x$$

$$68 = 58 + 2x$$

$$10 = 2x$$

$$5 = x$$



Problem 4 - A certain tree T of order n contains only vertices of degree 1 and 3. Show that T contains $(n-2)/2$ vertices of degree 3

Let $x =$ the amount of vertices with degree 1.
Let $y =$ the amount of vertices with degree 3.
$x + y = n$
$x + 3y = 2n - 2$