331 – Intro to Intelligent Systems
Week 11
Fuzzy Logic

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Bivariate logic

- Bivalent/bivariate (Aristotelian) logic uses two logical values – true and false
- Two major assumptions:
  – For any element and a set, the element is either a member of the set or it is a member of the complement of that set
  – The law of the excluded middle: an element cannot belong to both a set and also its complement
- Fuzzy logic disregards these assumptions

Fuzzy Reasoning

- Multivalent logics use many logical values – often in a range of real numbers from 0 to 1
- It is important to note the difference between multivalent logic and probability
  - P(A) = 0.5 means that A may be true or may be false with a probability of 0.5
  - A logical value of 0.5 means both true and false at the same time (50% true and 50% false)

Why Fuzzy Logic

- Although probability theory is good for measuring randomness of information, it is not good for measuring the *meaning* of information
- Much of the confusion around the use of English (natural) language is related to lack of clarity (vagueness) rather than randomness
- Fuzzy logic then presents a *possibility theory* as a measure of this vagueness as probability theory is measure of randomness

Fuzzy Reasoning

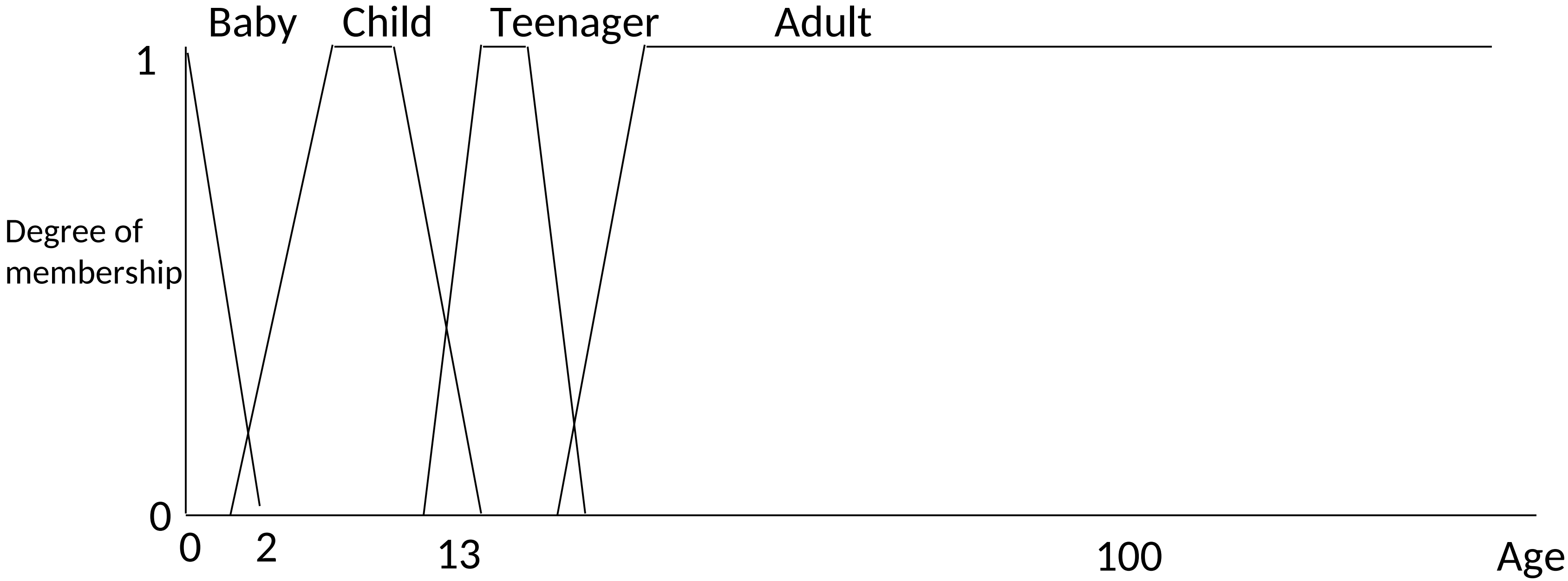
- Variables used in fuzzy systems can express qualities such as height, which can take values such as “tall”, “short” or “very tall”
- These values define subsets of the universe of discourse

Fuzzy Sets

- A *crisp* set is a set for which each value either is or is not contained in the set
- A *fuzzy* set is a set for which each value has a membership value, and so is a member to some extent
- The membership value defines the extent to which a variable is a member of a fuzzy set
- The membership value ranges from 0 (not at all a member of the set) to 1 (absolutely a member of the set)

Fuzzy Set Membership Example

Four sets: Baby, Child, Teenager, and Adult



Note that at age 13 a person might be defined as both a child and a teenager.

Fuzzy Set Membership Functions

We can define a membership function for the fuzzy set Baby (B) and the fuzzy set Child (C) as follows:

$M_B(x) = 1 – (x/2)$ for $x < 2$, 0 for $x >= 2$

$M_C(x) = (x-1)/6$ for $x >= 1$ and $x <= 7$, 1 for $x > 7$ and $x <= 8$,

$(14 – x)/6$ for $x > 8$ and $x <= 14$, 0 for $x > 14$ or $x < 1$

Similarly, we could define membership functions for fuzzy sets Teenager (T) and Adult (A). Note that there is nothing special about these functions – they have been chosen entirely arbitrarily and reflect a subjective view.

Fuzzy Set Membership Functions

To represent a fuzzy set in a computer, we use a list of pairs, where each pair represents a number and the fuzzy membership value for that number:

$$A = \{ (x_1, M_A (x_1)), \ldots, (x_n, M_A (x_n)) \}$$

For example, we could define B, the fuzzy set of Babies as follows:

$$B = \{ (0, 1), (2, 0) \}$$

This can also be thought of as representing the x and y coordinates of two points on the line that represents the set membership function. Similarly, we could define the fuzzy set of Children, C, as follows:

$$C = \{ (1, 0), (7, 1), (8, 1), (14, 0) \}$$

Crisp Set Operators

- Not A – the complement of A, which contains the elements which are not contained in A
- $A \cap B$ – the intersection of A and B, which contains those elements which are contained in both A and B
- $A \cup B$ – the union of A and B which contains all the elements of A and all the elements of B
- Fuzzy sets use the same operators, but the operators have different meanings

Fuzzy Set Operators

- Fuzzy set operators can be defined by their membership functions:

$M_{\neg A}(x) = 1 - M_A(x)$ (complement)

$M_{A \cap B}(x) = \text{MIN}(M_A(x), M_B(x))$ (intersection)

$M_{A \cup B}(x) = \text{MAX}(M_A(x), M_B(x))$ (union)

- We can also define containment (subset operator): $B \subseteq A$ iff $\forall x\, (M_B(x) \leq M_A(x))$

Fuzzy Set Operators

For example, The set of not-Babies, ¬B, is defined as follows:

¬B = { (0, 0), (2, 1) }

Similarly, we can define ¬C = { (1, 1), (7, 0), (8, 0), (14, 1) }.

To determine the intersection of B and C (Babies and Children), we need to have the sets defined over the same values.
Hence, we must *augment* sets B and C:

B = { (0, 1), (1, 0.5), (2, 0), (7, 0), (8, 0), (14, 0) }

C = { (0, 0), (1, 0), (2, 0.166), (7, 1), (8, 1), (14, 0) }

Now find the intersection by using $M_{A \cap B}$ (x) = MIN (M_A (x), M_B (x)):

B ∩ C = { (0, 0), (1, 0), (2, 0), (7, 0), (8, 0), (14, 0) }

Fuzzy Set Operators

But this has not worked! We need to define the set using values that will correctly define the ranges. In other words, define the intersection as follows:

$$B \cap C = \{ (1, 0), (1.75, 0.125), (2, 0) \}$$

where 1.75 was used as the value for x. This was determined by calculating the value of x for which $M_B (x) = M_C (x)$. If a person is in the set $B \cap C$, then she is *both* a Baby *and* a Child.

The union of the fuzzy sets of babies and children is as follows:

$$B \cup C = \{ (0, 1), (1.75, 0.125), (7, 1), (8, 1), (14, 0) \}$$

A person who belongs to the set $B \cup C$ is *either* a Baby *or* a Child.

Fuzzy Set Operators

In traditional set theory, if set A contains set B, then all the elements of set B are also elements of set A. In other words, the union $A \cup B = A$ and the intersection $A \cap B = B$. In this case, B is said to be a subset of A, which is written $B \subset A$.

For example, consider a fuzzy set R which is the fuzzy set of retirees. We will define this set by the following membership function:

$M_R(x) = 0$ for $x <= 55$, $(x - 55)/45$ for $x > 55$ and $x <= 100$, 0 for $x > 100$

Suppose the universe of people range in age from 0 to 100. Then R is a subset of Adult. In fuzzy terms, B is a fuzzy subset of A if B's membership function is always less than (or equal to) the membership function for A, when defined over the same range.

Fuzzy Logic

- Fuzzy logic is a *non-monotonic* logical system that applies to fuzzy variables
  - Non-monotonic: if a new fuzzy fact is added to the database, this fact may contradict conclusions that were already derived
- Each fuzzy variable can take a value from 0 (not at all true) to 1 (entirely true) but can also take on real values in between
  - 0.5 might indicate "as true as it is false"

Fuzzy Logic

- We use logical connectives defined as:

  $A \lor B \equiv MAX(A, B)$

  $A \land B \equiv MIN(A, B)$

  $\neg A \equiv 1 - A$

Note that if $A = 0.5$, then $A = \neg A$ (?)

We cannot write a complete truth table for a fuzzy logical connective because it would have an infinite number of entries. Consider a *trivalent* logical system with three logical values: {0, 0.5, 1}:

A	B	A ∨ B
0	0	0
0	0.5	0.5
0	1	1
0.5	0	0.5
0.5	0.5	0.5
0.5	1	1
1	0	1
1	0.5	1
1	1	1

Fuzzy Logic

- Recall:

  $A \rightarrow B \equiv \neg A \vee B$
- We might define fuzzy implication as:

  $A \rightarrow B \equiv MAX((1 - A), B)$
- This gives the unintuitive truth table shown on the right
- $0.5 \rightarrow 0 = 0.5$, where we would expect $0.5 \rightarrow 0 = 0$

A	B	$A \rightarrow B$
0	0	1
0	0.5	1
0	1	1
0.5	0	0.5
0.5	0.5	0.5
0.5	1	1
1	0	0
1	0.5	0.5
1	1	1

Fuzzy Logic

- An alternative is Gödel implication, which is defined as:

  $A \rightarrow B \equiv (A \leq B) \vee B$

- This gives a more intuitive truth table

A	B	A → B
0	0	1
0	0.5	1
0	1	1
0.5	0	0
0.5	0.5	1
0.5	1	1
1	0	0
1	0.5	0.5
1	1	1

Fuzzy Rules

- A fuzzy rule takes the following form:

  if A op x then B = y

where op is an operator such as >, < or ==

- For example:

  IF temperature > 50 then fan speed = fast

  IF height == tall then trouser length = long

  IF study time == short then grades = poor

Fuzzy Inference

- Inference in fuzzy expert systems uses *Mamdani inference*
- Mamdani inference derives a single crisp value (a recommendation) by applying fuzzy rules to a set of crisp input values (e.g., from a set of sensors, or a human, etc.):

  Step 1: Fuzzify the inputs.

  Step 2: Apply the inputs to the antecedents of the fuzzy rules to obtain a set of fuzzy outputs.

  Step 3: Convert the fuzzy outputs to a single crisp value using de-fuzzification.

Fuzzy Expert Systems

- A fuzzy expert system is built by creating a set of fuzzy rules and applying fuzzy inference
- In many ways this is more appropriate than standard expert systems since expert knowledge is not usually black and white but has elements of gray
- The first stage in building a fuzzy expert system is choosing suitable linguistic variables
- Rules are then generated based on the expert's knowledge using the linguistic variables

Fuzzy Expert Systems

Suppose you are designing an anti-lock braking system for a car, which is designed to cope when the roads are icy and the wheels may lock. The rules for the system might be as follows:

Rule 1: IF pressure on brake is medium
THEN apply the brake

Rule 2: IF pressure on brake is high AND car speed is fast
AND wheel speed is fast THEN apply the brake

Rule 3: IF pressure on brake is high AND car speed is fast
AND wheel speed is slow THEN release the brake

Rule 4: IF pressure on brake is low
THEN release the brake

Fuzzy Expert Systems

The first step is to define the fuzzy sets (the linguistic variables):

- Brake pressure (P)
- Wheel speed (W)
- Car speed (C)

The range of the fuzzy membership values are as follows:

Pressure:

Low (L) = {(0, 1), (50, 0)}

Medium (M) = {(30, 0), (50, 1), (70, 0)}

High (H) = {(50, 0), (100, 1)}

Wheel speed and Car speed:

Slow (S) = {(0, 1), (60, 0)}

Medium (M) = {(20, 0), (50, 1), (80, 0)}

Fast (F) = {(40, 0), (100, 1)}

Fuzzy Expert Systems

In addition, we need the output rules for apply the brake and release the brake:

Brake apply (A) = {(0, 0), (100, 1)}

Brake release (R) = {(0, 1), (100, 0)}

The membership functions for a specific input of {pressure = 60, wheel speed = 55, and car speed = 80} are as follows:

$M_{PL}(60) = 0$	$M_{PM}(60) = 0.5$	$M_{PH}(60) = 0.2$
$M_{WS}(55) = 0.083$	$M_{WM}(55) = 0.833$	$M_{WF}(55) = 0.25$
$M_{CS}(80) = 0$	$M_{CM}(80) = 0$	$M_{CF}(80) = 0.667$

Next, we need to apply these fuzzy values to the antecedent’s of the system’s rules.

Fuzzy Expert Systems

Rule 1: IF pressure on brake is medium
THEN apply the brake

Rule 1, taken on its own, tells us that the degree to which we should apply the brake is the same as the degree to which the pressure on the brake pedal can be described as “medium.”

Since $M_{PM}(60) = 0.5$, Rule 1 gives us a value of 0.5 for the instruction “Apply the brake.”

Fuzzy Expert Systems

Rule 2: IF pressure on brake is high AND car speed is fast AND wheel speed is fast THEN apply the brake

$M_{PH}(60) = 0.2$, $M_{CF}(80) = 0.667$, $M_{WF}(55) = 0.25$

The conjunction of two or more fuzzy variables is the minimum of the membership values, hence the antecedent for Rule 2 has the value of 0.2. Thus Rule 2 gives us a fuzzy value of 0.2 for the instruction “Apply the brake.”

Fuzzy Expert Systems

Rule 3: IF pressure on brake is high AND car speed is fast AND wheel speed is slow THEN release the brake

$M_{PH}(60) = 0.2$, $M_{CF}(80) = 0.667$, $M_{WS}(55) = 0.083$
This gives us a value of 0.083 for "Release the brake."

Fuzzy Expert Systems

<u>Rule 4:</u> IF pressure on brake is low
THEN release the brake

$M_{PL}(60) = 0$. This gives us a value of 0 for "Release the brake."

Now we have four fuzzy values: 0.5 and 0.2 for "Apply the brake" and 0.083 and 0 for "Release the brake."

Fuzzy Expert Systems

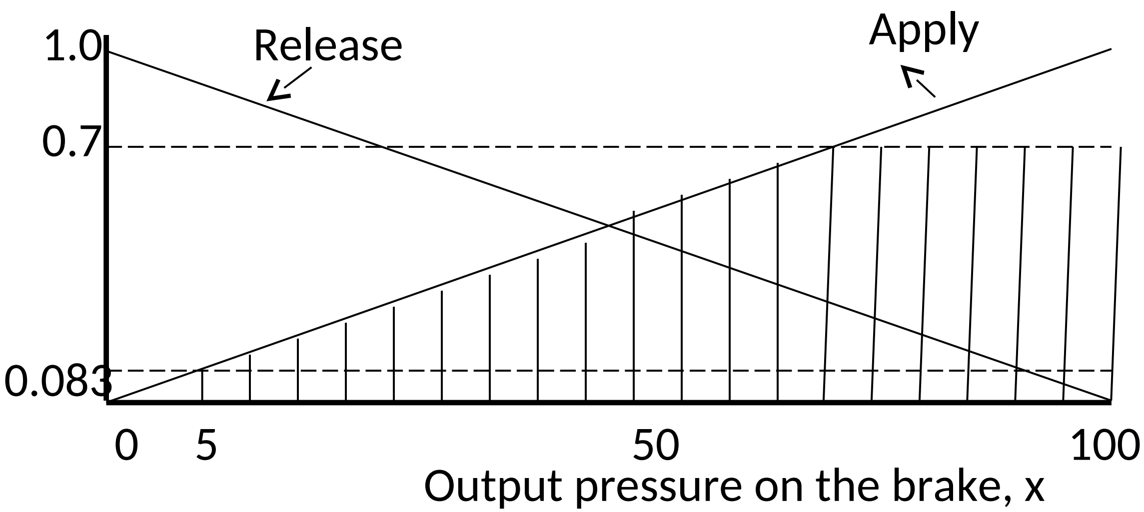
Now we need to combine these values together. We could sum the values, or take the minimum, or take the maximum. The appropriate combination will depend on the nature of the problem being solved. In this case it makes sense to sum the values because the separate rules are giving different reasons for applying or releasing the brakes, and those reasons should combine together cumulatively.

Hence we end up with a value of 0.7 for "Apply the break" and 0.083 for "Release the brake."

Fuzzy Expert Systems

To use this fuzzy output, a crisp output value must now be determined from the fuzzy values. The process of obtaining a crisp value from a set of fuzzy variables is known as *de-fuzzification*. This can be done by obtaining the center of gravity, COG, of the "clipped" membership functions (hashed area in the figure below shows the combined fuzzy output of the four rules).

$$COG = (\sum x\, M(x)) / (\sum M(x))$$



Clip membership in *Apply* to 0.7 and membership in *Release* to 0.083. At each sample point x, find max(M_{Apply}(x), $M_{Release}$(x)) and calculate COG.

Fuzzy Expert Systems

For our example,

$$COG \cong \frac{(5 \times 0.083) + (10 \times 0.1) + (15 \times 0.15) + \ldots + (70 \times 0.7) + \ldots + (100 \times 0.7)}{0.083 + 0.1 + 0.15 + \ldots + 0.7 + \ldots + 0.7}$$

$$\cong \frac{621.415}{9.483}$$

$$\cong 65.53$$

Hence, the crisp output value for this system is 65.53, which can be translated into the value of 65.53 units of pressure applied by the brake to the wheel of the car.