331 – Intro to Intelligent Systems
Week06
Propositional Logic
R&N Chapter 7.1 – 7.5

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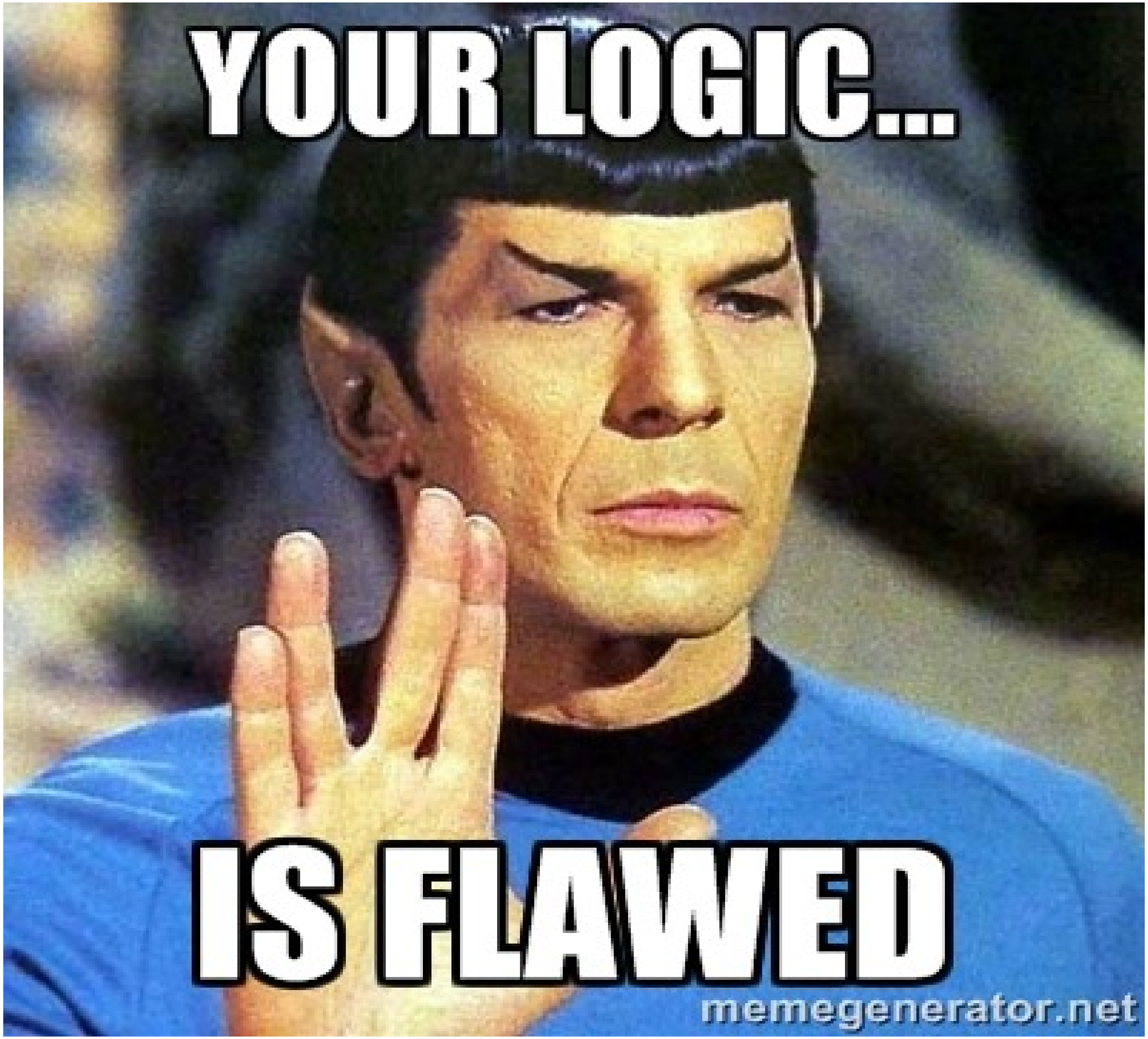
YOUR LOGIC...
IS FLAWED
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What is Logic?

- Reasoning about the validity of arguments.
- An argument is **valid** if its conclusions follow logically from its premises – even if the argument doesn't actually reflect the real world:
  - All lemons are blue
  - Mary is a lemon
  - Therefore, Mary is blue

How is Logic Used in Intelligent Systems?

- Logic is used as a representational method for communicating concepts and theories
- Logic allows us to reason about negatives (“the book is not red”) and disjunctions (“he’s either a soldier or a sailor”)
- Logic is used in systems that attempt to understand and analyze human language

Weaknesses of Logic

- Formal logics are unable to deal with uncertainty
  - Logical statements must be expressed in terms of truth or falsehood, not possibilities
- Formal logics are not well suited to deal with change
- Formal logics are not well suited to deal with events unfolding over time

Logical Operators

And	$\wedge$
Or	$\vee$
Not	$\neg$
Implies	$\rightarrow$ (if… then…)
Iff	$\Leftrightarrow$ (if and only if)

Truth Tables

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

Truth Tables

- Truth table demonstrating the equivalence of $P \rightarrow Q$ and $\neg P \vee Q$:

P	Q	$\neg P$	$P \rightarrow Q$	$\neg P \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Truth Tables

- Truth table demonstrating the non-equivalence of A ∧ (B ∨ C) and (A ∧ B) ∨ C:

A	B	C	A ∧ (B ∨ C)	(A ∧ B) ∨ C
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

English vs. Logic

- Facts and rules need to be translated into logical notation
- For example:
  - It is raining and it is Thursday:
  - R means “It is raining”, T means “it is Thursday”
  - $R \land T$

English vs. Logic

- Sentences in predicate calculus are created using predicates along with logical operators and quantifiers
- For example, the English sentence, “Whenever he eats sandwiches that have pickles in them, he ends up either asleep at his desk or singing loud songs” can be expressed as:

  $s(Y) \wedge e(X, Y) \wedge p(Y) \rightarrow a(X) \vee (s(X, Z) \wedge o(Z))$

  s(Y) refers to the sandwich (Y)

  e(X, Y) means that he (X) eats the sandwich (Y)

  p(Y) means that the sandwich (Y) has pickles in it

  a(X) means that he (X) ends up asleep at his desk

  s(X, Z) means that he (X) sings songs (Z)

  o(Z) means that those songs (Z) are loud

Entailment

- Entailment means that one thing follows from another:
- KB $\models \alpha$

- A knowledge base KB entails sentence α if and only if α is true in all worlds where KB is true
  - KB is a subset of α
  - KB is a stronger assertion than α since it rules out more worlds
    - Entailment is a relationship between sentences (i.e., syntax) that is based on semantics
    - For example, the KB containing “the Giants won” and “the Bills won” entails “the Giants won and the Bills won”

Tautology

- The expression $A \vee \neg A$ is a tautology.
- This means the expression is always true, regardless of the value of A
- A is a tautology is written as A
- A tautology is true under any interpretation
- An expression which is false under any interpretation is contradictory

A	$A \vee \neg A$
false	true
true	true

Properties of Logical Systems

- Completeness: Every tautology is a theorem
- Soundness: Every theorem is valid
- Decidability: An algorithm exists that will determine if a well-formed formula is valid
- Monotonicity: A valid logical proof cannot be made invalid by adding additional premises or assumptions

Logical Equivalence

- Two expressions are equivalent if they always have the same logical value under any interpretation:
  - $A \wedge B \equiv B \wedge A$
- Equivalences can be proven by examining truth tables
- Two sentences are logically equivalent iff they are true in the same models (knowledge base):
  - $\alpha \equiv$ ß iff $\alpha \models \beta$ and $\beta \models \alpha$

Logical Equivalence

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) && \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) && \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) && \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) && \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) && \text{de Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) && \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) && \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) && \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

Propositional Logic

- A proposition is a statement that is either true or false, given some state of the world
- Propositional logic is a logical system that deals with propositions
- Propositional calculus is the language we use to reason about propositional logic
- A legal sentence in propositional logic is called a well-formed formula (wff)

Propositional Logic

The following are wff's:

P, Q, R…

true, false

(A)

$\neg A$

$A \wedge B$

$A \vee B$

$A \rightarrow B$

$A \Leftrightarrow B$

Propositional Logic: Syntax

- Propositional logic is the simplest logic
  - It illustrates basic ideas
- Rules for constructing legal sentences (well-formed formulae) in propositional logic (the proposition symbols S_1 and S_2 are sentences):
  - If S is a sentence, $\neg S$ is also a sentence (negation)
  - If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)
  - If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)
  - If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)
  - If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Propositional Logic: Semantics

A specific model is an assignment of true or false to each proposition symbol.

For example, assume A , B, and C are statements in propositional logic. With these 3 symbols, $2^3 = 8$ possible models can be enumerated automatically

One possible model assigns each statement a specific value:

A = false   B = true   C = false

A simple recursive process can now evaluate a sentence:

$\neg A \wedge (B \vee C) = \textit{true} \wedge (\textit{true} \vee \textit{false}) = \textit{true} \wedge \textit{true} = \textit{true}$

Deduction

- Deduction is the process of deriving a conclusion from a set of assumptions
- If we deduce a conclusion C from a set of assumptions (facts), we write:

  $\{A_1, A_2, \ldots, A_n\} \vdash C$

- To derive a conclusion from a set of assumptions, we apply a set of *inference rules*
- To distinguish an inference rule from a set of assumptions, we often write $A \vdash B$ as $\dfrac{A}{B}$

Inference Rules

- ¬ ¬ Elimination : if we have a sentence that is negated twice, we can conclude the sentence itself, without the negation:

$$\frac{\neg\ \neg\ A}{A}$$

Inference Rules

- And-Introduction (Conjunction): given sentences A and B, we can deduce $A \land B$:

$$\frac{A,\ \ B}{A \land B}$$

- And-Elimination (Simplification): given $A \land B$, we can deduce A and we can deduce B separately:

$$\frac{A \land B}{A} \qquad \frac{A \land B}{B}$$

Inference Rules

- Or-Introduction (Addition): given sentence A, we can deduce the disjunction of A with *any* other sentence:

$$\frac{A}{A \vee B}$$

- Modus Ponens (M.P.): given sentence A and the fact that A implies B, we can derive sentence B:

$$\frac{A \rightarrow B,\ \ A}{B}$$

Inference Rules

- Hypothetical Syllogism (H.S.)

$$\frac{A \to B \wedge B \to C}{A \to C}$$

- Disjunctive Syllogism (D.S.)

$$\frac{A \vee B,\ \neg A}{B}$$

Inference Rules

- $\longrightarrow$ Introduction: if, in carrying out a proof, we start from assumption A and derive a conclusion C, then we can conclude that $A \longrightarrow C$:

$$\frac{A \;.\;.\;.\; C}{A \longrightarrow C}$$

Indirect Proof

- Reductio Ad Absurdum: if we assume A is incorrect (negate A) and this leads to a contradiction, then we can conclude that A is correct (proof by contradiction):

$$\neg A \;\; \ldots \;\; \frac{\perp}{A}$$

$\perp$ is called *falsum*

Careful!

- An invalid argument that looks similar to M.P. is as follows:

$$\frac{A \rightarrow B\ ,\ \ B}{A}$$

- This is known as the "Fallacy of Affirming the Consequent"

Deduction Example 1

- Prove the following: $\{A, \neg A\} \vdash \perp$

First, note that , $\neg A \equiv (A \rightarrow \perp)$ This can be seen by comparing the truth tables for $\neg A$ and for $A \rightarrow \perp$. Hence we can take as our set of assumptions $\{A, A \rightarrow \perp\}$. Thus, our proof using modus ponens is as follows:

$$\frac{A \quad , \quad A \rightarrow \perp}{\perp}$$

Deduction Example 2

- Prove the following: $\{A \wedge B\} \vdash A \vee B$

$$\dfrac{\dfrac{A \wedge B}{A}\quad \text{by And-Elimination}}{A \vee B}\quad \text{by Or-Introduction}$$

Deduction Example 3

- Prove the following:

$$\{\neg A, \neg A \rightarrow B, \neg B\} \vdash (\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$$

$\neg A \quad \neg A \rightarrow B$	assumptions
$B \quad \neg B$	modus ponens
$B \quad B \rightarrow \bot$	rewriting $\neg B$
$\bot$	modus ponens
A	reductio ad absurdum
$\neg B \rightarrow A$	$\rightarrow$ introduction
$(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$	$\rightarrow$ introduction