ECEN5322: Search Engines and Analysis of High Dimensional Datasets 09/27/2015

Lab Report 2

The Group Structure of S^{n-1}

Author: Prasanth Prahladan(100817764)

1. Prove that O(n) is a group when equipped with matrix multiplication.
Let $U_1, U_2 \in O(n)$. Therefore, by definition we have $U_i^T * U_i = U_i * U_i^T = I_n$. Define $P = U_1 * U_2$ and $Q = U_2 * U_1$.

$$P^T * P = (U_1 * U_2)^T * (U_1 * U_2) = (U_2^T * U_1^T) * (U_1 * U_2) = U_2^T * (U_1^T * (U_1) * U_2 = I_n$$

Similarly, we can prove $P * P^T = I_n$ and $Q * Q^T = Q^T * Q = I_n$. Therefore, $P, Q \in O(n)$. Hence, O(n) is a group when equipped with the matrix multiplication operation.

---

2. Prove that $U \in O(n) \implies det(U) = \pm 1$.
Since, $U_i^T * U_i = U_i * U_i^T = I_n$ and $det(MN) = det(M)det(N)$ we have

$$det(U)^2 = det(I_n) = 1 \implies det(U) = \pm 1.$$

---

3. Prove that SO(n) is a subgroup of O(n).
Let $U_1, U_2 \in SO(n)$. Therefore, $det(U_i) = 1$ and $U_i^T * U_i = U_i * U_i^T = I_n$.Define $P = U_1 * U_2$ and $Q = U_2 * U_1$.

$$det(P) = det(Q) = det(U_1) * det(U_2) = 1$$
$$P^T * P = P * P^T = I_n = Q^T * Q = Q * Q^T$$

Therefore, $P, Q \in SO(n)$. Further, we know that $U_i^{-1} \in SO(n)$ such that $U_i^{-1} * U_i = I_n \in SO(n)$. Therefore, $\tilde{Q} = U_1^{-1} * U_2^{-1} \in SO(n)$ and $\tilde{P} = U_2^{-1} * U_1^{-1} \in SO(n)$.

$$P * \tilde{P} = (U_1 * U_2) * (U_2^{-1} * U_1^{-1}) = U_1 * (I_n) * U_1^{-1} = I_n$$
$$Q * \tilde{Q} = I_n$$

which proves that, $\tilde{P} = P^{-1}$ and $\tilde{Q} = Q^{-1}$. Thus, SO(n) is a subgroup of O(n).

---

4. Consider the subset $\mathcal{G}$ of SO(n) defined by

$$\mathcal{G} = U \in SO(n), Ue_1 = e_1$$

where e_1 is the first element of the canonical basis in $\mathcal{R}^n$. Prove that $\mathcal{G}$ is a subgroup of SO(n).
Let $G_1, G_2 \in \mathcal{G} \subset SO(n)$. Therefore, $G_i e_1 = e_1$. Define $P = G_1 * G_2 \in SO(n)$.

$$Pe_1 = (G_1 * G_2)e_1 = G_1(e_1) = e_1$$

Therefore, $P \in \mathcal{G}$. Further, we observe that

$$e_1 = G_i e_1$$
$$G_i^{-1} e_1 = G_i^{-1} * G_i e_1 = I_n e_1 = e_1$$

Therefore, $G_i^{-1} \in \mathcal{G}$ and

$$P^{-1}e_1 = (G_1 * G_2)^{-1}e_1 = G_2^{-1} * G_1^{-1}e_1 = G_2^{-1}e_1 = e_1$$

This implies that $P \in \mathcal{G}$ and hence, $\mathcal{G}$ is a Subgroup of SO(n).

---

5. Prove that every element $U \in \mathcal{G}$ can be written as below with $W \in SO(n-1)$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} \tag{1}$$

Let $\{u_1, u_2, u_3, \cdots u_n\}$ be the n-column vectors defining $U = [u_1 u_2 \cdots u_n] \in \mathcal{G}$. We have, $Ue_1 = e_1 \implies u_1 = e_1$. Since, rank($U$) $= n$, and $\{u_i\}$ are all orthogonal vectors, we know that $\{u_2, \cdots u_n\}$ form a set of (n-1) independent vectors. Its possible to determine a rotation($K \in SO(n)$) of these (n-1) vectors such that their first-component is 0. Therefore, $K[u_i] = [0 v_i]'$ where $\{v_i\}$ form a set of (n-1) linearly independent orthogonal (n-1) dimensional vectors.

$$U = \begin{bmatrix} 1 & \bar{x}' \\ \bar{0} & M \end{bmatrix} \xrightarrow{K \in SO(n)} \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W \end{bmatrix} = V \in \mathcal{G}$$

where $\bar{x} \in \mathcal{R}^{n-1}$. Further, note that

$$\begin{aligned} det(V) &= det(KU) = det(K) * det(U) = 1 \\ &= 1.det(W) \\ V^T * V &= \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W^T W \end{bmatrix} = I_n \end{aligned}$$

Therefore, $W \in (n-1) \times (n-1)$ real matrix with $det(W) = 1$ and $W^T * W = W * W^T = I_{n-1}$. Hence, $W \in SO(n-1)$.

---

6. Prove that $\Psi : \mathcal{G} \to SO(n-1)$ such that $\Psi(U) = W$ in (??) is an 'isomorphism' between $\mathcal{G}$ and SO(n-1).

Consider elements $U_1, U_2 \in \mathcal{G}$ and $\Psi(U_i) = W_i$ where $W_i \in SO(n-1)$, $U_i \in \mathcal{G}$ and

$$U_i = \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W_i \end{bmatrix}. \tag{2}$$

. We can make the following observations

$$\begin{aligned} U_1 * U_2 &= \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W_1 \end{bmatrix} * \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W_1 * W_2 \end{bmatrix} \\ \Psi(U_1 * U_2) &= W_1 * W_2 = \Psi(U_1) * \Psi(U_2) \end{aligned}$$

1. *Surjective:* $\forall W \in SO(n-1)$ we can construct U such that

$$U = \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W \end{bmatrix}. \tag{3}$$

   Note that, $det(U) = 1.det(W) = 1$, $U^T U = U U^T = I_n$ and $Ue_1 = e_1$. Therefore, $U \in \mathcal{G}$ and $\Psi(\cdot)$ is a surjective function.

2. *Injective:* Let $W_1 = W_2$ where $W_i = \Psi(U_i)$, $W_i \in SO(n-1)$ and $U_i \in \mathcal{G}$. Therefore,

$$\begin{gathered} \Psi(W_1) = \Psi(W_2) \\ \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W_1 \end{bmatrix} = \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W_2 \end{bmatrix} \\ U_1 = U_2 \\ \Psi(U_1) = \Psi(U_2) \implies U_1 = U_2 \end{gathered}$$

Hence, $\Psi(\cdot)$ is an injective function.

From the discussion above, we prove that $\Psi(\cdot)$ is a bijective and an Isomorphism between $\mathcal{G}$ and SO(n-1).

---

7. Prove that $\forall x \in S^{n-1} \exists U \in SO(n)$ such that $Ue_1 = x$. Is U unique?
Let $\{u_1, u_2, \cdots u_n\}$ be the columns of the matrix $U \in SO(n)$. If $u_1 = x$, then we obtain $Ue_1 = x$. Therefore, we also require that the n-1 column vectors $\{u_2, \cdots u_n\}$ be orthogonal to x.

$$U = \begin{bmatrix} x & u_2 & \cdots & u_n \end{bmatrix} \in SO(n)$$

Note, that any rotation of the matrices $u_i \neq x$ can be used to construct U. Hence, $U \in SO(n)$ is not unique.

---

8. Prove that if $U, V \in SO(n)$ such that $Ue_1 = Ve_1 = x$, then there exits $W \in SO(n-1)$ such that

$$U = V * \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W \end{bmatrix}$$

Let $G \in \mathcal{G} \subset SO(n)$. From above, we know that $\exists W \in SO(n-1)$

$$\begin{aligned} Ge_1 &= e_1 \\ G &= \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W \end{bmatrix} \end{aligned}$$

We make the following observations

$$\begin{aligned} x &= Ve_1 = V * (Ge_1) \\ &= Ue_1 \\ U &= V * G = V * \begin{bmatrix} 1 & \bar{0}' \\ \bar{0} & W \end{bmatrix} \end{aligned}$$

---

9. Prove that if U_1 and U_2 are two coset representatives for the same coset then $U_1\mathcal{G} = U_2\mathcal{G}$.
$U\mathcal{G}$ is the coset defined by $U \in SO(n)$ and contains $V = UG, \forall G \in \mathcal{G}$. The set of all cosets $\{U\mathcal{G}\} = SO(n)/\mathcal{G}$, is called the 'quotient-group'.

Let $U\mathcal{G}$ be the coset under consideration, with coset-representations $V_1, V_2 \in U\mathcal{G}$. Therefore, by definition we have for $G_1, G_2 \in \mathcal{G}$

$$U = V_1G_1 = V_2G_2$$

Further,

$$\begin{aligned} V_1\mathcal{G} &= \{X|V_1 = XG, \forall X \in SO(n), \forall G \in \mathcal{G} \text{ and } V_1 \in SO(n)\} \\ V_2\mathcal{G} &= \{Y|V_2 = YG, \forall Y \in SO(n), \forall G \in \mathcal{G} \text{ and } V_2 \in SO(n)\} \end{aligned}$$

We note the following

$$\begin{aligned} V_1\mathcal{G} &= \{X|V_1 = XG, \forall X \in SO(n)\} \\ &= \{X|V_1G_1 = XGG_1, \forall X \in SO(n)\} \\ &= \{X|V_2G_2 = XGG_1, \forall X \in SO(n)\} \\ &= \{X|V_2 = XGG_1G_2^{-1}, \forall X \in SO(n)\} \\ &= \{X|V_2 = X\tilde{G}, \forall X \in SO(n)\} \\ &= V_2\mathcal{G} \end{aligned}$$

---

10. Define the map $\Phi : SO(n)/\mathcal{G} \to S^{n-1}$ such that $U \mapsto \Phi(U) = Ue_1$. Prove that $\Phi(\cdot)$ represents the action of SO(n) i.e. $\forall \Omega \in SO(n), \forall U \in SO(n)/\mathcal{G}$

$$\Phi(\Omega U) = \Omega\Phi(U)$$

Let $x \in S^{n-1}$. We know $\exists U \in SO(n)/\mathcal{G}$ such that $Ue_1 = x$. By definition, $\Phi(U) = Ue_1 = x$

$$\Phi(\Omega U) = (\Omega U)e_1 = \Omega(Ue_1) = \Omega\Phi(U)$$

Hence, $\Phi(\cdot)$ represents the action of SO(n).

---

11. Prove that the map $\Phi(\cdot)$ is bijective and continuous.
To prove that $\Phi(\cdot)$ is bijective, we proceed as follows

1. *Injective* Let $x_1 = x_2 = \tilde{x}$, where $\Phi(U_i) = U_ie_1 = x_i$. Therefore, we have

$$\begin{aligned} x_1 &= x_2 \\ \Phi(U_1) &= \Phi(U_2) \\ U_1e_1 &= U_2e_1 \\ &= U_2Ge_1 \end{aligned}$$

   where $G \in \mathcal{G}$. Therefore, $U_1, U_2 \in U_1\mathcal{G}$ i.e they belong to the same equivalence class. Therefore,there exits a unique $\tilde{U} \in SO(n)/\mathcal{G}$ such that $U_1 = U_2 = \tilde{U}$. Hence, $\Phi(\cdot)$ is injective.

2. *Surjective* For any $x \in S^{n-1}$, there exists non-unique $U \in SO(n)$ such that $Ue_1 = x$. By using the equivalence rule, $U = VG, G \in \mathcal{G}$, there exists a unique $U \in SO(n)/\mathcal{G}\forall x$ such that $Ue_1 = x$. Therefore, $\Phi(\cdot)$ is a surjective function.

From the above discussion, we learn that $\Phi(\cdot)$ is a bijective function.

To prove that $\Phi(\cdot)$ is continuous, we use the notion that images of a converging sequence of points in the domain, form a converging sequence in the range of the map i.e.

$$\lim_{n\to\infty} \Omega_n \to \Omega \implies \lim_{n\to\infty} \Phi(\Omega_n) \text{ exists.}$$

We define convergence in the domain, using the Frobenius norm i.e.

$$\lim_{n\to\infty} \Omega_n \to \Omega \implies \lim_{n\to\infty} ||\Omega_n - \Omega||_F \to 0$$

We observe that

$$\begin{aligned}\lim_{n\to\infty} ||\Phi(\Omega_{n+1}) - \Phi(\Omega_n)|| &= \lim_{n\to\infty} ||\Omega_{n+1}e_1 - \Omega_n e_1|| \\ &\leq \lim_{n\to\infty} ||\Omega_{n+1} - \Omega_n||||e_1|| \to 0\end{aligned}$$

Therefore, $\lim_{n\to\infty} \Phi(\Omega_n)$ exists! Hence, $\Phi(\cdot)$ is a continuous mapping.

Conclusion:

$$\begin{aligned}\mathcal{G} &\xrightarrow{isomorphism} SO(n-1) \\ SO(n)/\mathcal{G} &\leftrightarrow SO(n)/SO(n-1) \\ S^{n-1} &\xrightarrow{homeomorphism} SO(n)/\mathcal{G} \\ S^{n-1} &\leftrightarrow SO(n)/SO(n-1)\end{aligned}$$

We thus conclude that the Sphere S^{n-1} can be identified with the Quotient Group, $SO(n)/SO(n-1)$. And thus properties of Group Action, Group Symmetries can be used to understand it. Further, the topological Group structure of the Sphere, permits us to define a Haar Measure on it, which shall then be used to measure the set of points under study.

Definition 1 (Group). *A group is non-empty set G with an operation * and satisfying the following properties*

1. *for any $f, g \in G$, $h * g \in G$*
2. *for any $f, g, h \in G$, $f * (g * h) = (f * g) * h$*
3. *$\exists e \in G$ such that for any $g \in G$, $g * e = e * g = g$*
4. *for any $g \in G$, $\exists g^{-1} \in G$, such that $g * g^{-1} = g^{-1} * g = e$*

Definition 2 (Subgroup). *A subset H of G is a subgroup if*

1. *for any $g, h \in H$, $h * g \in H$*
2. *for any $g \in H$, $\exists g^{-1} \in H$, such that $g * g^{-1} = g^{-1} * g = e$*

Definition 3 (Orthogonal Group, O(n)). *The orthogonal group $O(n)$ is the subset of $n \times n$ real matrices U such that $U^T * U = U * U^T = I_n$, where I_n is the $n \times n$ Identity Matrix.*

Definition 4 (Special Orthogonal Group, SO(n)). *SO(n) consists of the elements of O(n) with determinant 1 i.e. the rotations.*

Definition 5 (Homomorphism). *A function $\phi : (G, *) \rightarrow (F, \Delta)$ is called a homomorphism if ϕ commutes with Group Operations*

$$\forall G_1, G_2 \in G, \phi(G_1 * G_2) = \phi(G_1)\Delta\phi(G_2)$$

Definition 6 (Isomorphism). *Let $f : (G, *) \rightarrow (F, \Delta)$ be a homomorphism. f is an ismomorphism if f is bijective i.e f^{-1} exits and f^{-1} is also a homomorphism.*

Definition 7 (Equivalence Class). *Two matrices U and V are equivalent under a relation if $Ue_1 = Ve_1 = x$, or equivalently there exists $Q \in \mathcal{G}$ such that $U = VQ$. Let $U \in SO(n)$. The Left-Coset is defined as $U\mathcal{G} = \{V | V \in SO(n-1), V = UQ, Q \in \mathcal{G}\}$ where V is called 'coset representative'.*

Definition 8 (Quotient Group). *The set of left-cosets is called Quotient Group, denoted by $SO(n)/\mathcal{G}$.*