

Complete Solution of $AX=b$

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Consider a set of linear equations as below, it can be represented as RHS

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 &= b_3 \end{aligned}$$

unique

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array} \right]$$

Augmented matrix = $[A \ b]$

On elimination of $[A \ b]$,

1. $R_2 = R_2 - 2R_1$;
2. $R_3 = R_3 - 3R_1$;
3. $R_3 = R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right]$$

Pivot columns

We can see that for R_3 ,

$0 = b_3 - b_2 - b_1 \rightarrow b_1 + b_2 = b_3$. This is the **condition for solvability**,

If $b = [1 \ 5 \ 6]$, this b is acceptable.

Condition for Solvability:

- $AX = b$ is solvable, if b is in $C(A)$
- For a combination of rows of A gives a zero row, then the same combination of entities of b must give 0.

To find Complete Solution of $AX = b$

1. $X_{\text{particular}}$: Set free variables to zero and Solve $AX = b$ for Pivot Variables.
For the above equation, x_2 and x_4 are zero, on solving, we get: $b = [2 \ 0 \ 1.5 \ 0]^T$
2. $X_{\text{nullspace}}$: Find the null space

Complete Solution is $X_{\text{particular}} + X_{\text{nullspace}}$

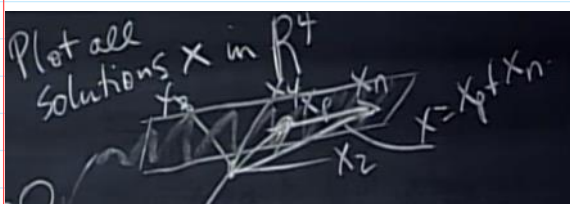
$$\begin{aligned} Ax_p &= b \\ Ax_n &= 0 \\ \hline A(x_p + x_n) &= b \end{aligned}$$

Complete Solution for above example

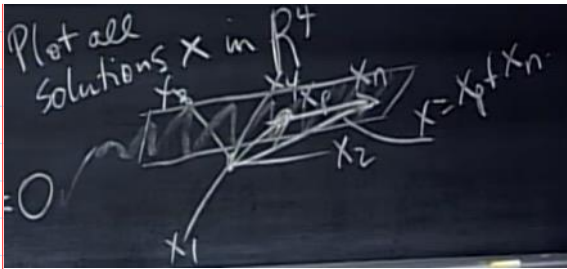
$$X_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Note: X_p is a single solution while X_n is subspace hence c_1 and c_2 .

The solution in R^4 can be shown as follows:



The solution is 2D figure **not a subspace** that passes through X_p .



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Algorithm to find Complete Solution

1. Find the particular solution'
2. Find the null space.

The Bigger Picture:

For any,

m by n matrix A of rank r

(Know $r \leq m, r \leq n$)

Since each column can have one pivot and **rank** cannot be more than number of rows

Maximum Ranks:

- Full column Rank: $r = n$

This implies all the columns have pivot, thus no free variables. Thus Null Space is only zero vector. $N(A) = 0$ (pivots because I and free variables (F) lead to null space)

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \xrightarrow{R = \text{rref}(A)} R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

There are 4 equations but only two in X , hence there is not always a solution except $X = [0 \ 0 \ 0 \ 0]^T$. A particular combination of b would have a solution (a linear combination of A) say $b = [4 \ 3 \ 7 \ 6]^T$ then $X = [1 \ 1 \ 1 \ 1]^T$, that is X_p

Complete Solution:

$X = X_p$ i.e., only unique solution if it exists. (0 or 1 solutions)

- Full row Rank: $r = m$

Can solve $AX = b$ for every b . Solutions exists.

And we have $n-r$ free variables i.e., $n-m$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & -7 & -4 \\ 0 & 1 & -5 & -2 \end{bmatrix}$$

- For $r = m = n$

$$\begin{bmatrix} r = m = n \\ A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \\ R = I \end{bmatrix}$$

Null space is zero vector only.

$AX=b$ is possible for every b , and has unique solution.

Solutions

$$r=m=n$$

$$R=I$$

1 solution to $AX=b$

$$r=n < m$$

$$R=\begin{bmatrix} I \\ 0 \end{bmatrix}$$

((0 or 1 solution))

$$r=m < n$$

$$R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

((1 or ∞ solution))

$$r < m, r < n$$

$$R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

((0 or ∞ solution))

THE RANK represents all the information about the solution