

# Solving $Ax = 0$ : pivot variables, special solutions

Friday, April 06, 2018 7:45 AM

## Topics

- Computing Null Space  $AX = 0$
- Pivot Variables - free variables
- Special Solutions -  $\text{rref}(A)=R$

## Computing Null Space:

Method to solve a system of linear equations:  $AX=0$

Eg:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

The method of eliminations leaves Null Space unchanged but changed the columns space.

Step 1:  $R_3 = R_3 - R_1 \cdot 3$ ;  $R_2 = R_2 - R_1 \cdot 2$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

The second row pivot element is zero, also row exchange to swap the rows is also futile since  $A(3,2) = 0$  as well. This implies **Col2 is not independent**.

Step 2:  $R_3 = R_3 - R_2 \cdot 2$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The U is in echelon form. i.e., zeros form a stair case form.

NOTE: U has only two pivot elements.

- **RANK of A = Number of pivots (for the eg Rank = 2)**
- The Solution U is the Null space of A

rank of A  
= # of pivots  
= 2

→  $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

2 pivot columns  
free columns

Pivot Columns

Free Columns

Free columns mean,  $X(2,1)$  and  $X(2,4)$  can take any values for solution of A.

For example  $X = [x_1, 1, x_3, 0]^T$

Hence solving the equations by substitution ( $x_2 = 1$  and  $x_4 = 0$ ), we get:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_3 + 4x_4 &= 0 \end{aligned}$$

Solving

$$x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$2x_3 + 4x_4 = 0$$

$$\begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The variables associated with free columns are called free variables ( $x_2$  and  $x_4$ )

NULL Space contains all the combinations of special solution (free variables). One for each free variable.

Free Variables:

For a "m" row and "n" column matrix with rank "r", number of free variables are "n-r"

Similarly, by using another combination of free variables ( $x_2 = 0, x_4 = 1$ ) we get  $X = d[2 \ 0 \ -2 \ 1]^T$

R - Reduced Row Echelon Form (Reduction of U):

In Reduced Row Echelon form, the elements above and below pivots are zero and pivots = 1

In the above example, the 3rd row is zero because,  $R_3$  is a linear combination of  $R_1$  and  $R_2$  and elimination discovered it and eliminated it. Reduced form is in which the upper row is also reduced.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 = R_2/2$

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

In MATLAB  $R = \text{rref}(A)$

Notice  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$  in pivot rows/cols

Inference from R:

- Has an Identity matrix in pivot rows/ columns
- $R_3$  is zero, indicating original row are combinations of other rows.
- Free columns
- Pivot Rows and Columns

Now  $UX$  can be reduced to  $RX$

$$\begin{aligned} UX = 0 \\ x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \\ 2x_3 + 4x_4 = 0 \end{aligned}$$

$$\begin{aligned} RX = 0 \\ x_1 + 2x_2 - 2x_4 = 0 \\ x_3 + 2x_4 = 0 \end{aligned}$$

Here  $UX$   $RX$  and  $AX$  are all equal.  $UX$  can be seen as a combination of Pivot and Free columns

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{matrix} \text{Pivot cols} \\ \text{free cols} \end{matrix}$$

$$x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The free column elements are in the NULL SPACE

General "rref" form:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

r pivot rows

↑ pivot cols      ↑ n-r free cols

For  $RX = 0$ ,

The Nullspace  $N$  matrix, the columns = special solutions.

Also,  $RN = 0$ , where  $N$  is given by

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

Nullspace matrix (columns = special solns)

0

The final solution can be shown as below:

$$RX = 0 \quad \boxed{X_{\text{pivot}} = -FX_{\text{free}}}$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} X_{\text{pivot}} \\ X_{\text{free}} \end{bmatrix} = 0$$