18-Aug-2018

Demorgans law:

Union over $n = \bigcup_n$

1) $\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c$

Suppose $x \in \left(\bigcup_n A_n\right)^c$

$= x \notin \left(\bigcup_n A_n\right)$

$= x \notin A_n \ \forall n$

$= x \in A_n^c \ \forall n$

$= x \in \bigcap_n A_n^c$

|||ly

(2) $\left(\bigcap_n A_n\right)^c = \bigcup_n A_n^c$

Sigma Fields or Sigma Algebra of sets

1) $S \in f$ ← sigma field.

2) If $A \in f$ then $A^c \in f$

3.) If $A_1, A_2 \in f$ then $\bigcup_n A_n \in f$

Consider the experiment, rolling a die

$S = \{1, 2, 3, 4, 5, 6\}$

If $A = \{$No. showing up $>2\}$

$A = \{3, 4, 5, 6\}$

$A^c = \{1, 2\}$

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Consequences of properties 1–(3)

(1) $\phi \in f$ (since $\phi = S^c$)

(2) $A_1, A_2 \ldots \in f$, then $\bigcap_n A_n \in f$

Since $A_1, A_2 \ldots \in f$

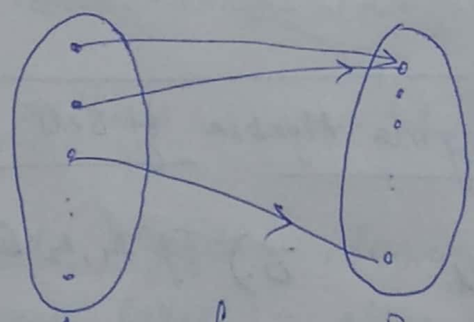
$\Rightarrow A_1^c, A_2^c \ldots \in f$ (II)

$\Rightarrow \bigcup_n A_n^c \in f$ (III)

$\Rightarrow \left(\bigcup_n A_n^c\right)^c \in f$ (II)

$\Rightarrow \bigcap_n A_n \in f$ [By Demorgan's Law]

Function:



A f B

Domain Range / Co-Domain

Probability:

$P : f \longrightarrow [0, 1]$

f – sigma field.

Properties:

P1. $P(S) = 1$

P2. If $A, B \in f$ and $A \cap B = \phi$,

then $P(A \cup B) = P(A) + P(B)$

Corollary:

If $A, B \in f$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Eg:-

Consider a fair die

Event $A : \{x > 2\}$

$P(A) = P(3) + P(4) + P(5) + P(6) = 4\left(\frac{1}{6}\right) = \frac{2}{3}$

$P(B) = P(2) + P(1) = 2\left(\frac{1}{6}\right) = \frac{1}{3}$.

$P(A^c) = 1 - P(A)$ Since $(A \cup A^c = S)$

→ Simplest σ field

$f = \{S, \phi\}$

→ Power Set : (2^S)

$f = \{$All possible subsets of $S\}$

~~If S contains, then number of no of su~~

If 'S' contain n elements, then no of subset $= 2^n$

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Random Variables:

$X : S \longrightarrow \mathbb{R}$ $\{ w \in S \mid X(w) \leq c, \} \in \mathcal{F}$, $c \in \mathbb{R}$

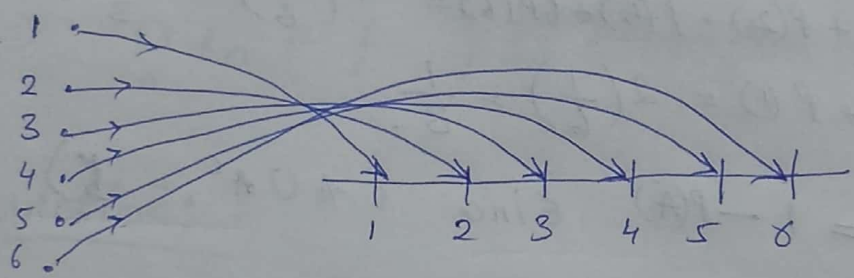
Where 'S' is the sample space i.e., the set that contains all the outcomes of expt

Consider the die rolling experiment:

$X =$ No. the shows up on the face

$S = \{1, 2, 3, 4, 5\}$

$X(1) = 1$; $X(2) = 2$; $X(3) = 3$ . . . . $X(6) = 6$



Eg: $P(X=1) = P(\{w \in S \mid X(w) = 1\})$

$\underline{\underline{w=1}}$ $P(1) = 1/6$.

Conditional Probabilities:

Suppose A & B are two events

$$P(A \mid B) = \begin{cases} \dfrac{P(A \cap B)}{P(B)} & \text{given } P(B) > 0 \\ \text{Undefined.} & \text{otherwise} \end{cases}$$

Consider a die, that is rolled an infinite number of times,

(i) A = A six occurs on every roll

$$P(A) = \lim_{n\to\infty} (1/6)^n = 0 \quad \leftarrow \text{zero prob.}$$

(ii) $P(3 \mid \text{odd outcome}) = \dfrac{P(\{3\} \cap \{1,3,5\})}{P(\{1,3,5\})} = \dfrac{(1/6)}{(1/2)} = \dfrac{P(3)}{3(1/6)} =$

$$= \frac{1/6}{1/2} = \underline{\underline{1/3}}$$

(iii) $P(X \text{ is odd} \mid X \text{ is multiple of } 3)$

$$= \frac{P(A \cap B)}{P(B)} = \frac{P(\{3\} \cap \{3, 6\})}{P(\{3, 6\})} = \frac{(1/6)}{2/6} = \underline{\underline{\frac{1}{2}}}$$

We say that events A & B are independent events if

$$\Rightarrow P(A \mid B) = P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A)\,P(B)}$$

When we have more than 3 events, A, B, & C are independent if;

$$\Rightarrow P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

$$\rightarrow P(A \cap B) = P(A)(B) \; ; \; P(B \cap C) = P(B)\,P(C) \; , \; P(C \cap A) = P(C)\,P(A)$$

$P(A|B) = \frac{P(A\cap B)}{P(B)}$

If A, B & C are 3 events,

$$P(A\cap B | C) = P(A | B\cap C)\cdot P(B|C)$$

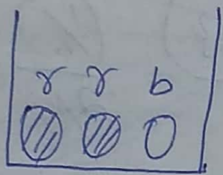
taking LHS

$$\frac{P(A\cap (B\cap C))}{P(C)} = \frac{P(A|B\cap C)\cdot P(B|C)}{P(C)}$$

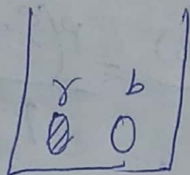
$$= P(A|B\cap C)\cdot P(B|C)$$

Eg:

Urn #1



Urn#2



Expt: Pick one of the urns at random & pick a ball from that urn

$$S = \{1r, 1b, 2r, 2b\}$$

Alternatively, $S = S_1 \times S_2$

where $S_1 = \{1, 2\}$, $S_2 = \{r, b\}$

Let X_1 be out of urn S_1

X_2 be the outcome of ball S_2

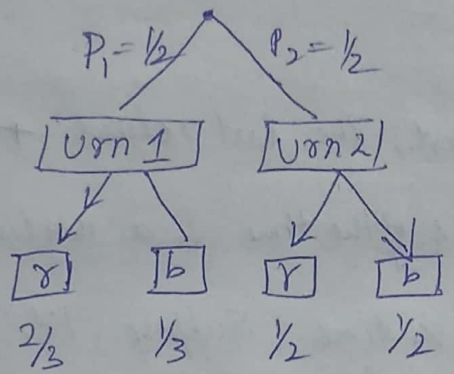
prob of selecting urn

$$P(1r) = P(X_1 = 1, X_2 = r) = P(X_1 = 1)\cdot P(X_2 = r | X_1 = 1)$$

prob of r given Urn 1

$$= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

$$P(2b) = P(X_1 = 2) \cdot P(b \mid X = 2)$$
$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$



Bayes Theorem: Suppose, $A_1, A_2 \ldots$ is a collection of events. such that $A_i \cap A_j = \phi \ \forall i \neq j$

w.k.t $\bigcup_i A_i = S$

Suppose B is another event,

Then
$$P(B) = P(B \cap S)$$
$$= P(B \cap \textstyle\bigcup_i A_i)$$
$$= P(\textstyle\bigcup_i (B \cap A_i)) \quad \text{— } (*)$$

Since A_i are disjoint,
$B \cap A_i$, $i \geq 1$ are also disjoint
$$\textstyle\bigcup_i (B \cap A_i) = B$$

From $(*)$
$$P(B) = \sum_i P(B \cap A_i)$$
$$= \sum P(B \mid A_i) \cdot P(A_i)$$

Consider $P(A_j | B) = \dfrac{P(A_j \cap B)}{P(B)}$

$$= \frac{P(B|A_j) \cdot P(A_j)}{\sum_i \left(P(B|A_i)\, P(A_j)\right)}$$

Example:

A hospital conducts a drug test. The test returns +'ve result for a drug user 99% of the time & a negtive result for a non-user 95% time. Suppose 1% of population uses drug.

Q: What is the prob that an individual is a drug user given that he tests positive.

Soln :-

$$S = \{ \text{User}+, \text{user}-, \text{non-user}+, \text{non user}- \}$$

$P(+|\text{user}) = 0.99$

$P(-|\text{nonuser}) = 0.95$

$$P(u|+) = \frac{P(+|\text{user}) \cdot P(\text{user})}{P(+|\text{user}) \cdot P(\text{user}) + P(+|\text{non-user}) \cdot P(\text{non user})}$$

$$= \frac{0.99 * 0.01}{0.99 * 0.01 + (0.05)(0.99)}$$

$$= \frac{99 \times 1}{99 \times 1 + 99 \times 5} = \frac{1}{6}$$

Expectations:

Suppose X is a random variable,

$E[X]$ of μ_X is called the expected value of mean of X.

For the case of a discrete random value x,

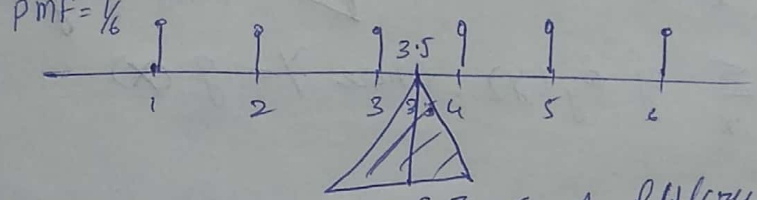
$$E[X] = \sum_{x} x\, P(X=x)$$

<u>Probability mass f^n (PMF)</u> of a random value X

$$PMF_X(x) = P(X=x)$$

1) For a die roll expt, a fair dice,

$$E[X] = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} \;\cdots\; 6\cdot\frac{1}{6} = \frac{21}{6} = 3.5$$



$E[X] \leftarrow$ A fulcrum where it is the center of mass.

2) Given a random variable $X \in \{-2,-1,0,1,2,3,4,5\}$

$\Rightarrow \quad P(X=x) = \frac{1}{8} \quad \forall\, x \in X$

$E[X^2]$ suppose $Y = X^2$

$$E[Y] = \sum_{y} y \cdot P(y)$$

$$P_Y(y) \begin{cases} \frac{1}{8} & y=0 \\ \frac{2}{8} & y=1 \\ \frac{2}{8} & y=4 \\ \frac{1}{8} & y=9 \\ \frac{1}{8} & y=16 \\ \frac{1}{8} & y=25 \end{cases}$$

$$Y = \{0, 1, 9, 16, 25\}$$

$$E[Y] = 0\cdot\frac{1}{8} + 1\cdot\frac{2}{8} + 4\cdot\frac{2}{8} + 9\cdot\frac{1}{8} + 16\cdot\frac{1}{8} + 25\cdot\frac{1}{8}$$

$$= \frac{60}{8}$$

Suppose $Y = g(x)$, $E[Y] = \sum_{y} y \cdot p_y(y)$

Alternatively

$$E[Y] = E[x^2]$$

$$= \sum_{x} x^2 \cdot p_x(x)$$

$$= \frac{1}{8} \sum_{x} \cdot x^2 = \frac{1}{8}(4+1+0+1+4+9+16+25)$$

$$= \cancel{8}\ \frac{60}{8}$$

i.e. $\cancel{g}$ $E[Y] = \sum_{x} g(x) p_x(y)$ ← useful when we know pmf of ~~x~~ y

$$= \sum_{x} y\, p_y(y) \text{ where } Y = g(X)$$

Properties of Expectations:

1. $E[X+C] = E[X] + C$ ($C \in \mathbb{R}$ is some constant)

2. $E[Y+X] = E[X] + E[Y]$

3. $E[aX] = aE[X]$ ($a \in \mathbb{R}$ is some constant)

This shows that E is a linear operator.

Conditional Expectation:

Conditioning is based on an Event. i.e.,

$$E[X \mid Y=y] = \sum_x x \, P(X=x \mid Y=y)$$

Eg 1:
Consider the ~~2~~ two Urn experiment, with additional condition such that,

we get 10 dollars for observing a red ball ~~b~~ $5 for observing a black ball.

$S = \{1r, 1b, 2r, 2b\}$

$$E[gain(X)] = \sum_x gain(x) \cdot p_X(x)$$

$= \cancel{g \cdot 10 * P(X=1r) +}$

$= gain(1r) * P(1r) + gain(1b) * P(1b) +$
$gain(2r) \cdot P(2r) + gain(2b) * p(2b)$

$= 10 * \frac{1}{3} + 5 * \frac{1}{6} + 10 * \frac{1}{4} + 5 * \frac{1}{4} \approx \underline{\underline{7.9167}}$

Eg 2: Suppose we pick ~~urn~~ Urn 1. What is the expected gain.

Soln:

$$E[gain(X) \mid Y=1] = \sum_x gain(X=x) \cdot P(X=x \mid Y=1)$$

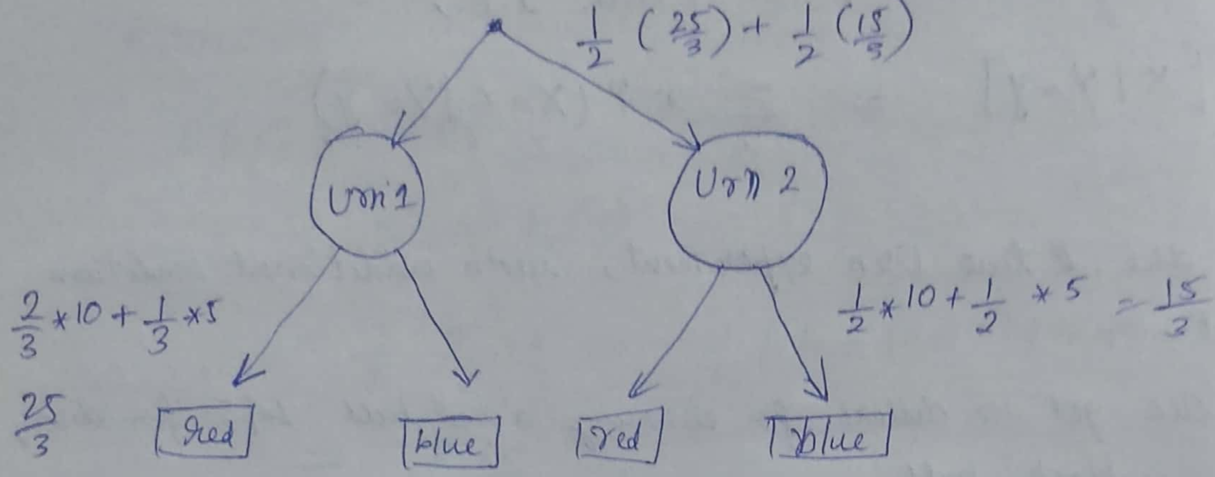
where Y is random variable corresponding to pick up Urn

$= gain(1r) \, P(X=1r \mid Y=1) + gain(1b) \, P(X=1b \mid Y=1) + 0$
$gain(2b) \cancel{P(X=2b \mid Y=1)} + gain(2r) \cdot \cancel{P(X=2b \mid Y=1)}$

$= 10 * \frac{2}{3} + 5 * \frac{1}{3} = \underline{\underline{8.3}}$

$\therefore$ The gain is high if we pick Urn 1 instead of Urn 2

Law of total Expectation:



$$E[X] = \sum_y E[X \mid Y=y]\, P(Y=y) \quad - (2)$$

$$E[X] = \sum_x x\, P(X=x) \quad - (1)$$

W.k.t

$$P(X=x) = P(\{X=x\} \cap S)$$

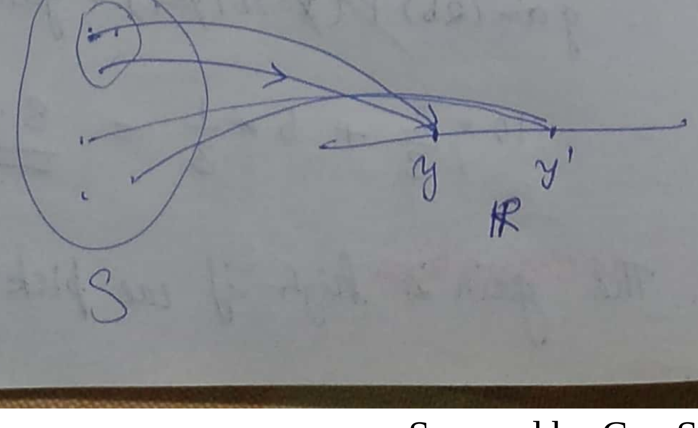
$$= P\left(\{X=x\} \cap \bigcup_y \{Y=y\}\right)$$

$$= P\left(\bigcup_y \{X=x\} \cap \{Y=y\}\right)$$

Events $\{X=x\}$ & $\{Y=y\}$ (are disjoint because Y is a f^n of X).

since

$$\{Y=y\} \Rightarrow \{w \mid Y(w)=y\}$$



continuing ① & ②

$$= \sum_y x \sum_x P(X=x \mid Y=y)\, P(Y=y)$$

$$= \sum_y \left[\sum_x x \; P(X=x \mid Y=y) \right] P(Y=y)$$

$$= \sum_y E[X \mid Y=y] \cdot P(Y=y)$$

Markov Chain:

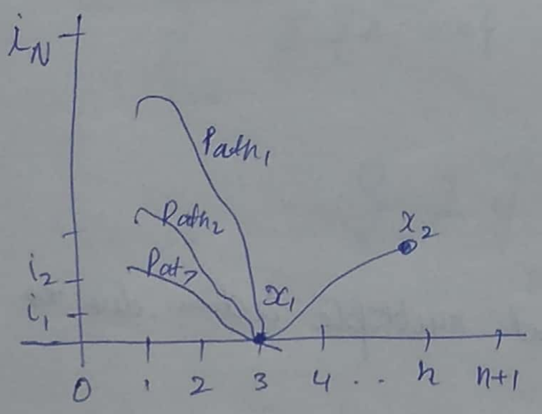
consider a seq $\{X_n\}$ of random variables on a common prob $(S, \mathcal{F}, P)$

$$S = \{ i_1, i_2 \ldots i_N \}$$

$$P(X_{n+1} = j \mid X_n = i_n, X_{n-1} = i_{n-1} \;\ldots\; X_0 = i_0)$$

$= P(X_{n+1} = j \mid X_n = i)$ (next state depends on prev state ~~next sta~~ irrespective of path.)

$\simeq P(i,j)$ transition prob.



x_2 depends on x_1 only.