PREVIOUS YEAR EXAMINATION
Question Paper-2017 (Fully Solved)

Name of the Paper : **Numerical Methods**
Name of the Course : **Mathematics; Generic Elective for Honours**
Semester : **IV**

Q.1. (a) Explain Significant digits and Local Truncation Errors with examples. An approximate value of π is given by 3.1428571 and its true value is 3.1415926. Find the absolute and relative errors.

Ans. Significant digits: The Digits 1, 2, 3, ..., 9 that are used to express a number are called Significant digits or Significant figures. '0' is also a significant figure except when it is used to fix the decimal point or fill the places of unknowns or discarded digits. Example: The number 0.00123 has only three significant digits, viz. 1, 2 and 3. The numbers 0.66753, 3.1416 and 3.0687 each contains five significant digits.

Local Truncation Error: Errors which are caused by using approximate formulae in computation or replacing an infinite process by a finite process, are called truncation errors.

Consider on exponential series:

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\infty = X(\text{say})$$

It is replaced by

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!} = X'\ (\text{say})$$

Then the truncation error $= X - X'$.

Given, True value = 3.1415926

Approximate Value = 3.1428571

Error = True value – Approximate value

$$= 3.1415926 - 3.1428571$$

$$= -0.0012645$$

$$E_A\ (\text{Absolute error}) = |\text{Error}| = 0.0012645$$

$$\text{Relative Error } (E_R) = \frac{E_R}{\text{True Value}}$$

$$= \frac{0.0012645}{3.1415926}$$

$$= 0.000402502$$

(b) Explain the Bisection Method for computing the roots of equation $f(x) = 0$. Perform three iterations of the Bisection method in the interval (1, 2) to obtain roots of the equation $f(x) = x^3 - x - 1 = 0$.

Ans. Bisection method is based on the repeated application of the Intermediate value theorem. If a root $f(x) = 0$ lies in the interval $I_0 = (a_0, b_0)$, then we bisect I_0

at the point $m_1 = (a_0 + b_0)/2$. Denote by I_1 the interval (a_0, m_1) if $f(a_0)\,f(m_1) < 0$ or the interval (m_1, b_0) if $f(m_1)\,f(b_0) < 0$.

So, the Interval I_1 also contains the roots. Now bisect the interval I_2 and get a subinterval I_2 at whose end points $f(x)$ takes the values of opposite signs and therefore it contains the root. Continuing in this manner, a sequence of nested sets of sub-intervals $I_0 \supset I_1 \supset I_2$ such that each subinterval contains the root. Repeating the Bisection process q times, we either find the root or find the interval I_q of length $(b_0 - a_0)/2^q$ which contains the root. The midpoint of the last sub-interval in taken as the desired approximation to the root.

Here $f(x) = x^3 - x - 1 = 0$ (Bisection method)

Here, $f(1) < 0$ and $f(2) > 0$

$$\Rightarrow \qquad x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$f(x_0) = \frac{15}{8} > 0$$

Therefore, the root lies between 1 and 1.5 and we obtain $x_1 = 1.25$.

Now, $f(x_1) < 0$, so the root will lies between 1.25 and 1.5.

Therefore,
$$x_2 = \frac{x_0 + x_1}{2} = \frac{1.5 + 1.25}{2} = 1.375.$$

Continuing in this manner we get

$$x_3 = 1.3125,\ x_4 = 1.34375,\ x_5 = 1.328128, \text{ etc.}$$

(c) Define the rate of Convergence. Determine the rate of convergence for the Secant method.

Ans. An Iterative method is said to be of order 'p' or has the rate of convergence 'p' if p is the largest positive real number for which there exists a finite constant $c \neq 0$ such that

$$|\varepsilon_k + 1| \le c\,|\varepsilon_k|^p$$

where $\varepsilon_k = x_k - \xi$ is the error in the k^{th} iterate.

The constant C is called the **asymptotic error constant.**

Rate of convergence of secant method.

We assume that ξ is a simple root of $f(x) = 0$.

Substituting $x_k = x + \varepsilon_k$ in the formula

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{fx_k - fx_{k-1}}\, fx_k,\quad k = 1, 2, \ldots \qquad \ldots(1)$$

Then we obtain,

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k - \varepsilon_{k-1})\, f(\xi + \varepsilon_k)}{f(\xi + \varepsilon_k) - f(\xi + \varepsilon_{k-1})} \qquad \ldots(2)$$

Expanding $f(\xi + \varepsilon_k)$ and $f(\xi + \varepsilon_{k-1})$ in Taylor's series about the point ξ and putting $f(\xi) = 0$, we get,

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k - \varepsilon_{k-1})\left[\varepsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \ldots\right]}{(\varepsilon_k - \varepsilon_{k-1}) f'(\xi) + \frac{1}{2}(\varepsilon_k^2 - \varepsilon_{k-1}^2) f''(\xi) + \ldots}$$

$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\varepsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \ldots\right]\left[1 + \frac{1}{2}(\varepsilon_{k-1} + \varepsilon_k)\frac{f''(\xi)}{f'(\xi)} + \ldots\right]^{-1}$$

or $$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k \varepsilon_{k-1} + o(\varepsilon_k^2 \varepsilon_{k-1} + \varepsilon_k \varepsilon_{k-1}^2)$$

or $$\varepsilon_{k+1} = C\varepsilon_k \varepsilon_{k-1} \qquad \ldots(3)$$

or $$C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$$ and higher power of ε_k are neglected.

Equation (3) is called the error equation. From the definition of the rate of convergence, we seek a relation of the form $\varepsilon_{k+1} = A\,\varepsilon_k^p$ ...(4)

We have to determine A and p.

From (4) we have

$$\varepsilon_k = A\varepsilon_{k-1}^p \text{ or } \varepsilon_{k-1} = \varepsilon^{\frac{-1}{p}} \varepsilon_k^{\frac{1}{p}}$$

Substituting the values of ε_{k+1} and ε_{k-1} in (3) we get

$$\varepsilon_k^p = CA^{-\left(1+\frac{1}{p}\right)} \varepsilon_k^{1+\frac{1}{p}}$$

Comparing the power of ε_k on both sides, we get

$$P = 1 + \frac{1}{P}$$

$$\Rightarrow \qquad P^2 - P + 1 = 0 \Rightarrow P = \frac{1}{2}(1 \pm \sqrt{5})$$

Neglecting the minus sign, we find the rate of convergence for the secant method is $P = 1.618$.

Q.2. (a) Perform four iterations of the Regula-Falsi Method to obtain a root of the equation:

$$f(x) = 3x + \sin x - e^x = 0$$

Ans. Given

$$f(x) = 3x + \sin x - e^x = 0$$

Here $$x = b - \frac{b-a}{y_b - y_a} \cdot y_b$$

or $$x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) f(x_1)$$

When $$x_0 = 0, f(x_0) = -1 < 0$$

When $x_1 = 1, f(x_1) = 0.2991 > 0$

Therefore, $x_2 = 1 - \dfrac{1-0}{0.299152-(-1)}(0.299152)$

$= 0.7697$

Similarly, we can obtain

$x_3 = 0.6928;\ x_4 = 0.6347;\ x_5 = 0.6172$, etc.

(b) Perform three iteration of Newton's method to find the root of the equation $x^4 - x - 10 = 0$ and starting the approximation as 1.5.

Ans. Given $f(x) = y = x^4 - x - 10 = 0$

We know, by Newton's method

$$x_{k+1} = x_k - \frac{y_k}{y_k'}, k = 0, 1, 2, ...$$

Given $x_0 = 1.5, f(x_0) = y_0 = -6.4375$

$$f'(x_0) = 4x^3 - 1$$

$$f'(x_0) = y'_0 = 12.5$$

$$x_1 = x_0 - \frac{y_0}{y_0'} = 1.5 - \frac{(-6.4375)}{12.5}$$

$$x_1 = 2.015$$

$f(x_1) = y_1 = 4.47$ which gives

$$y_1' = 31.725$$

Similarly, $x_2 = x_1 - \dfrac{y_1}{y_1'} = 2.015 - \dfrac{4.47}{31.725}$

$= 1.8741.$

and $x_3 = 1.8558$, etc.

(c) Perform two iterations of Newton's method to solve the non-linear system of equation with initial approximation (1,1):

$$f(x, y) = x^2 + y - 11 = 0 \text{ and}$$

$$g(x, y) = x^2 + y^2 - 7 = 0$$

Ans. Here $fx = 2x, fy = 1, g_x = 1, g_y = 2y$

$$f_x(x_0, y_0)_r + f_y(x_0, y_0)_s = -f(x_0, y_0)$$

$$g_x(x_0, y_0)_r + g_y(x_0, y_0)_s = -g(x_0, y_0)$$

Substituting in the formula we get

$$2r + s = 9$$

$$r + 2s = 5$$

Solving these equations we get

$$r = \frac{13}{3} \text{ and } s = \frac{1}{3}$$

$$\Rightarrow \qquad x_1 = x_0 + s = 1 + \frac{1}{3} - \frac{4}{3} = 1.33$$

$$y_1 = y_0 + r = 1 + \frac{13}{3} = \frac{16}{3} = 5.33$$

Now, $f_x(x_1, y_1)_r + f_y(x_1, y_1)_s = -f(x_1, y_1)$

$$g_x(x_1, y_1)_r + g_y(x_1, y_1)_s = -g(x_1, y_1)$$

$$2\left(\frac{4}{3}\right)r + s = 4.34$$

$$2.66r + s = 4.34 \text{ and}$$

$$r + 10.66s = -22.77$$

$$\Rightarrow \qquad s = -2.373 \text{ and } r = 2.526$$

$$\Rightarrow \qquad x_2 = 1.33 + 2.526 = 3.856$$

$$y_2 = 5.33 - 2.373 = 2.957$$

Q.3. (a) Solve the linear system $AX = b$, using Gaussian elimination with pivoting:

$$A = \begin{bmatrix} 6 & 2 & 2 \\ 6 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

Ans. The augmented matrix is given by $AX = b$

$$\begin{bmatrix} 6 & 2 & 2 \\ 6 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \text{ Perform } R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 6 & 2 & 1 \\ 6 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

Now, $R_2 \to R_2 - 6R_1$

$R_3 \to R_3 - 6R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -10 & 7 \\ 0 & -10 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

Perform $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -10 & 7 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$$

Therefore, we get

$$x + 2y - z = 0$$
$$-10y + 7z = 5 \Rightarrow z = -5$$

Now, by backward substitution

$$-10y + 7(-5) = 5 \Rightarrow y = -4$$
$$x + 2(-4) - (-5) = 0 \Rightarrow x = 3$$
$$\Rightarrow \qquad x = 3;\ y = -4;\ z = -5.$$

(b) Starting with initial vector $(x, y, z) = (0, 0, 0)$ perform three iterations of gauss-seidal method to solve the following system of equations:

$$2x - y = 7;\ -x + 2y - z = 1;\ -y + 2z = 1$$

Ans. Here

$$x^{(k+1)} = \frac{1}{2}[7 + y^{(k)}]$$

$$y^{(k+1)} = \frac{1}{2}[1 + x^{(k)} + z^{(k)}]$$

$$z^{(k+1)} = \frac{1}{2}\left[1 + y^{(k)}\right]$$

When $k = 0$; then $\quad x^{(1)} = \frac{1}{2}[7 + 0] = 3.5$

$$y^{(1)} = \frac{1}{2}[1 + 3.5 + 0] = 2.25$$

$$z^{(1)} = \frac{1}{2}[1 + 2.25] = 1.625$$

When $k = 1$,

$$x^{(2)} = 4.625$$
$$y^{(2)} = 3.625$$
$$z^{(2)} = 2.3125$$

When $k = 2$,

$$z^{(3)} = 5.3125$$
$$y^{(3)} = 4.3125$$
$$z^{(3)} = 2.6563$$

(c) Explain Thomas Algorithm and solve the following Tridiagonal system $AX = b$ using the Thomas Method:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 10 \end{bmatrix}, b = \begin{bmatrix} 10 \\ 17 \\ 22 \end{bmatrix}$$

Ans. Thomas Algorithm:

Step 1: For the first equation form the new elements a_1 and r_1

$$a_1 = \frac{a_1}{d_1}; r_1 = \frac{r_1}{d_1}$$

Step 2: For each of the equations, from $i = 2, ..., n-1$

$$a_i = \frac{a_i}{d_i - b_i a_{i-1}}; r_i = \frac{r_i - b_i r_{i-1}}{d_i - b_i a_{i-1}}$$

Step 3: For the Last equation

$$r_n = \frac{r_n - b_n r_{n-1}}{d_n - b_n a_{n-1}}$$

Step 4: Solve by back – substitution

$x_n = r_n; x_i = r_i - a_i x_{i+1}$ (where $i = n-1, n-2, ..., 2.1$)

The given matrix is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 10 \end{bmatrix}, b = \begin{bmatrix} 10 \\ 17 \\ 22 \end{bmatrix}$$

Now,

$$a = (2 \quad 3 \quad 0)$$
$$r = (10 \quad 17 \quad 22)$$
$$b = (0 \quad 1 \quad 3)$$
$$d = (1 \quad 3 \quad 10)$$

$$a_1 = \frac{a_1}{d_1} = \frac{2}{1} = 2; \; r_1 = \frac{r_1}{d_1} = \frac{10}{1} = 10$$

$$a_2 = \frac{a_2}{d_2 - b_2 a_1} = \frac{3}{3 - 1.2} = 3; \; r_2 = \frac{r_2 - b_2 r_1}{d_2 - b_2 a_1} = \frac{17 - 1.10}{3 - 1.12} = 7$$

$$r_3 = \frac{22 - 3 \times 7}{10 - 3.3} = 1$$

Therefore

$$x_3 = r_3 = 1$$
$$x_2 = r_2 - a_2 x_3 = 4$$
$$x_1 = r_1 - a_1 x_2 = 10 - 2.4 = 2$$

Therefore

$$x_1 = 2, x_2 = 4, x_3 = 1.$$

Q.4. (a) Find the unique polynomial $P(x)$ of degree 2 or less using Lagrange Interpolating Formula for the following data:

$$x = [1, 3, 4]$$
$$F(x) = [1, 27, 64]$$

Also, estimate $P(1.5)$.

Ans. We have $x_0 = 1, x_1 = 3, x_2 = 4$

$f_0 = 1, f_1 = 27, f_2 = 64$

The lagrange's fundamental polynomial are given by $L_0 = ((x - x_1)(x - x_2))/((x_0 - x_1)(x_0 - x_2))$

$$= ((x - 3)(x - 4))/(-2)(-3)$$
$$= 1/6\,(x^2 - 7x + 12)$$
$$L_1(x) = ((x - x_0)(x - x_2))/((x_1 - x_0)(x_1 - x_2))$$
$$= ((x - 1)(x - 4))/2(-1)$$
$$= 1/2(5x - x^2 - 4)$$
$$L_2(x) = ((x - x_0)(x - x_1))/((x_2 - x_0)(x_2 - x_1))$$
$$= ((x - 1)(x - 3))/3.1$$
$$= 1/3(x^2 - 4x + 3)$$

Hence, the lagrange quadratic interpolating polynomial is given by

$$P_2(x) = L_0(x).f_0 + L_1(x).f_1 + L_2(x).f_2$$

$= [1/6(x^2 - 7x + 12)].1 + ([1/2(5x - x^2 - 4)].27) + ([1/3(x^2 - 4x + 3)]).64$

$= 8x^2 - 19x + 12.$

Estimate, $P_2(1.5) = 8(1.5)^2 - 19(1.5) + 12$

$= 1.5$

(b) Prove the following relations:

(i) $\mu = \sqrt{1 + \dfrac{\delta^2}{4}}$

(ii) $\Delta = \dfrac{1}{2}\delta^2 + \delta\sqrt{1 + \dfrac{1}{4}\delta^2}$

Ans.

(i) We have $\delta = E^{1/2} - E^{-1/2}$

$$\delta^2 = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 = E + E^{-1} - 2$$

Also, $$\left[1 + \frac{\delta^2}{4}\right]^{1/2} = \left[1 + \frac{1}{4}(E + E^{-1} - 2)\right]^{1/2}$$

$$= \frac{1}{2}\left\{\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]^2\right\}^{1/2} = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) = \mu$$

(ii) From R.H.S

$$\Delta = \frac{1}{2} \times \delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$$

We have $\delta = E^{1/2} - E^{-1/2}$

$$\delta^2 = E + E^{-1} - 2$$

From above, $\left[1+\frac{\delta^2}{4}\right]^{\frac{1}{2}} = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

$$= \frac{1}{2}\delta^2 + \delta\left[1+\frac{\delta^2}{4}\right]^{\frac{1}{2}}$$

$$= \frac{1}{2}[E + E^{-1} - 2] + \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right]\left[\frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)\right]$$

$$= \frac{1}{2}[E + E^{-1} - 2] + \frac{1}{2}[E - E^{-1}]$$

$$= \frac{1}{2}(E + E^{-1} - 2 + E - E^{-1}) = E - 1 = \Delta$$

Hence Proved.

(c) By use of Richardson extrapolation, find $f'(3)$ using the approximate formula:

$$f'(x_0) = (f(x_0 + h) - f(x_0 - h))/2h$$

with $h = 4$, 2 and 1, from the following values:

x	$f(x)$
–1	1
1	1
2	16
3	81
4	256
5	625
7	2401

Ans. Using the formula $f'(x_0) = (f(x_0 + h) - f(x_0 - h))/2h$ which can be re-written as

$$f'(x_1) = (f(x_2) - f(x_0))/2h$$

When $h = 4$,

$$f(h) = (2401 - 1)/(2 \times 4) = 300$$

When h=2,

$$f(h/2) = (625 - 1)/4 = 156$$

When $h = 1$,

$$f(h/4) = (256 - 16)/2 = 120$$

Also

$$f'(h) = (4f(h/2) - f(h))/3 = 108$$

$$f'(h) = (4f(h/2^2) - f(h/2))/3 = 108$$

$$f^2(h) = (4^2 f'(h/2) - f'(h))/(4^2 - 1) = 108.$$

Extrapolation Table is given by

Order	$O(h^2)$	$O(h^4)$	$O(h^6)$
h	$f(h)$		
		$f'(h)$	
$h/2$	$f(h/2)$		$f''(h)$
		$f'(h/2)$	
$h/2^2$	$f(h/2^2)$		

Using above calculations, extrapolation table can be used to obtain $f'(3)$.

h	$O(h^2)$	$O(h^4)$	$O(h^6)$
4	300		
		108	
2	156		108
		108	
1	120		

Thus $f'(3) = 108$, must be the exact solution as $f'(4) = f'(2) = f''(4)$.

Q.5. (a) The following data represents the function $f(x) = e^x$:

x	$f(x)$
1	2.718
1.5	4.4817
2.0	7.3891
2.5	12.1825

Estimate the value of $f(2.25)$ using the Newton's forward difference Interpolation and then compare with the exact value.

Ans. The forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	2.718			
		1.764		
1.5	4.4817		1.144	
		2.9074		0.742
2.0	7.3891		1.886	
		4.7934		
2.5	12.1825			

Newton divided difference interpolating polynomial is given by

$$P(x) = f(x_0) + \frac{(x-x_0)}{h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f(x_0)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{3!h^3}\Delta^3 f(x_0)$$

Where $h = x_i - x_{i-1} \Rightarrow h = 0.5$

$$P(x) = 2.7183 + \frac{(x-1)}{0.5}(1.7634) + \frac{(x-1)(x-1.5)}{2(0.5)^2}(1.144)$$

$$+ \frac{(x-1)(x-1.5)(x-2)}{6(0.5)^3}(0.742)$$

$$= \frac{1}{0.75}[0.742x^3 - 1.623x^2 + 3.1781x - 0.258375]$$

$$\Rightarrow \quad P(x) = 0.9893x^3 - 2.164x^2 + 4.2374x - 0.3445$$

Estimate $P(2.25)$

$$P(2.25) = 9.50675$$

Given $f(x) = e^x \Rightarrow$ Exact value of $f(2.25) = 9.487795$.

(b) Obtain the piecewise linear interpolating polynomial for the function $f(x)$ defined by the given data:

x	$f(x)$
0	**1**
1	**2**
2	**5**
3	**10**

and interpolate at $x = 0.5$ and 1.5.

Ans. In the interval [0, 1] we have,

In $[x_i, x_{i+1}]$

$$P_{i+1,1}(x) = \frac{[x - x_{i+1}]}{[x_i - x_{i+1}]} f(x_i) + \frac{[x - x_i]}{[x_{i+1} - x_i]} f(x_{i+1})$$

For $i = 0$, $[x_0, x_1] = [0, 1]$

$$P_{1,1}(x) = \frac{[x-1]}{[0-1]}(1) + \frac{[x-0]}{[1-0]}(2)$$

$$= x + 1$$

For $i = 1$, $[x_1, x_2] = [1, 2]$

$$P_{2,1}(x) = \frac{[x-2]}{[1-2]}(2) + \frac{[x-1]}{[2-1]}(5)$$

$$= 3x - 1$$

For $i = 2$, $[x_2, x_3] = [2, 3]$

$$P_{3,1}(x) = \frac{[x-3]}{[2-3]}(5) + \frac{[x-2]}{[3-2]}(10)$$

$$= 5x - 5$$

Hence, the piecewise interpolating polynomials are given as

$$P_1(x) = \begin{cases} x+1, & 0 \le x \le 1 \\ 3x-1, & 1 \le x \le 2 \\ 5x-5, & 2 \le x \le 3 \end{cases}$$

Using, $P_{1,1}(x)$ we obtain $f(0.5) = 0.5+1 = 1.5$ and $P_{2,1}(x)$; $f(1.5) = 3(1.5)-1 = 3.5$.

(c) Find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$ Using

(i) Trapezoidal Rule

(ii) Simpson's Rule.

Ans. (i) Using the Trapezoidal rule, we have

$$\int_a^b f(x)\,dx = \frac{b-a}{2}[f(a)+f(b)]$$

We have, $I \equiv \frac{1}{2}\left(1+\frac{1}{2}\right) = 0.75$

(ii) Using Simpson's Rule , we have

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$I \equiv \frac{1}{6}\left[1+\frac{8}{3}+1/2\right] = \frac{25}{36} = 0.69444$$

Q.6. (a) Apply Euler's modified method to approximate the solution of the initial value problem and calculate $y(1.3)$ by using $h = 0.1$,

$$\frac{dy}{dx}+\frac{y}{x} = \frac{1}{x^2}, \; y(1)=1$$

Ans. Euler's modified method

$$\frac{dy}{dx} = \frac{1}{x^2}-\frac{y}{x} = f(x,y)$$

Here, $h = 0.1$;

$$x_0 = 1;\; x_1 = 1.1;\; x_2 = 1.2.$$

Formula: $\quad y_{i+1}^{(1)} = y_i + h\,f(x_i, y_i)$

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(1)})]$$

For $i = 0$; $\quad y_1^{(1)} = 0.995$

$$y_1 = 0.9605$$

For $i = 1$; $\quad y_2^{(1)} = 0.9856$

$$y_2 = 0.9857$$

For $i = 2$;
$$y_3^{(1)} = 0.971$$
$$y_3 = 0.9716$$

So,
$$y_1 = 0.9605;\ y_2 = 0.9857;\ y_3 = 0.9716.$$

(b) Apply mid-point method (R.K. Second order) to solve the initial value problem:

$$\frac{dy}{dx} = yx^3 - 1.5y$$

from $x = 0$ to 2 where $y(0) = 1$ by using h=1.

Ans. By R.K. 2[nd] order mid-point method

$$\frac{dy}{dx} = yx^3 - 1.5y;\ y(0) = 1;\ h = 1$$

Therefore $x_0 = 0$ and $y_0 = 1$

$\Rightarrow x_1 = 1.$

Formula for **R.K. 2[nd] order mid-point method** is

$$y_{i+1} = y_i + K_2 h$$
$$K_1 = f(x_i, y_i)$$
$$K_2 = f\left(x_i + \frac{h}{2}, y_i + K_1\frac{h}{2}\right)$$

For $i = 0$, we have

$$K_1 = y_0 x_0^3 - 1.5y_0 = -1.5$$
$$K_2 = f\left(x_0 + \frac{1}{2}, y_0 + \frac{k}{2}\right) = f(0.5, 0.25)$$
$$= -0.7875$$
$$y_1 = 1 + (-0.71875).1 = 0.28125$$

For $i = 1$, we have

$$K_1 = y_1 x_1^3 - 1.5y_1$$
$$= 0.140625$$
$$K_2 = f\left(x_1 + \frac{h}{2}, y_1 + k_1/2\right)$$
$$= f(1.5, 0.3515625)$$
$$= 0.659179$$
$$y_2 = y_1 + k_2.1$$
$$= y_1 + k_2$$
$$= 0.28125 + 0.659179 = 0.940429$$

Therefore,
$$y_1 = 0.28125;\ y_2 = 0.940429.$$

(c) Apply finite difference method to solve the given problem:

$$\frac{d^2y}{dx^2} = y + x(x-4),\ 0 \le x \le 4,$$

$$y(0) = 0;\ y(4) = 0 \text{ with } h = 1.$$

Ans. Given $\quad y^{11} = y + x(x-4);\ 0 \le x \le 4.$

We have $\quad y(0) = y(4) = 0$

and $n = 4$;

So $\quad y_0 = y_4 = 0$

Now, here $\quad x_1 = 1;\ x_2 = 2;\ x_3 = 3.$

Therefore, by **Central difference formula**

$$y^{11}(x_i) = \frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} = y_i + x_i(x_i - 4)$$

where $i = 1, 2, 3$

Therefore,
$$y_2 - 2y_1 + 0 = y_1 + 1(1-4)$$
$$y_3 - 2y_2 + y_1 = y_2 + 2(2-4)$$
$$0 - 2y_3 + y_2 = y_3 + 3(3-4)$$

Therefore,
$$-3y_1 + y_2 = -3$$
$$y_1 - 3y_2 + y_3 = -4$$
$$y_2 - 3y_3 = -3$$

Solving the above equations, we get

$$y_1 = \frac{13}{7};\ y_2 = \frac{18}{7};\ y_3 = \frac{13}{7}.$$

PREVIOUS YEAR EXAMINATION
Question Paper-2018 (Fully Solved)

Name of the Paper : **Numerical Methods**
Name of the Course : **Generic Elective Mathematics**
Semester : **IV**

Q.1. (a) Define floating-point representation, Global error and Truncation error with examples.

Ans. Floating point representation is also known as real point representation. It uses the number with fractional parts as operands. Example: If we consider a number 2.366, in this 2 is the fixed point and 0.366 is the Fractional part known as Floating point.

$$F = (\beta, k, m, M) = \text{Floating point system}$$

Here

where β = base, k = no. of digits in basic expansion, m = min. exponent, M = max. exponent.

Global Error of a multi-step method is the difference:

$$E_n(x) = y(x) - y(x_n)$$

where $x = x_n$ is fixed and n is a variable.

Truncation error is the error which is caused by using approx. values on computation of an infinite process.

If we take exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty = X(\text{say})$

And if it is replaced by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = X'$ (say) here we are approximating a mathematical procedure), then the truncation error $= E_T = X - X'$.

(b) Explain the of Newton's method for computing the roots of equation $f(x) = 0$. Perform *three* iterations of the of Newton's method to find the smallest positive roots of the equation $f(x) = x^3 - 5x + 1 = 0$.

Ans. (i) If a function is derivable (i.e. its derivate exists) then real roots of $f(x) = 0$ can be computed by N-R method: (Newton-Raphson method)

Let $f(x) = 0$ which implies that $f'(x)$ exists.

Let x_0 = approx. value of x.

h = correction which is added to x_0 to get exact value

Then $x = x_0 + h$. So if $f(x) = 0$ then $f(x_0 + h) = 0$

Now expand the function by putting Taylor's theorem:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ldots + \frac{h^n}{n!} f^n(x_0) = 0$$

If h is very small, then we can neglect powers of h so $f(x_0) + hf'(x_0) = 0$

$$\Rightarrow \qquad h = \frac{-f(x_0)}{f'(x_0)}$$

Now $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(ii) $f(x) = x^3 - 5x + 1 = 0$ [Given]

$$f'(x) = 3x^2 - 5$$

$$\left.\begin{array}{r} f(0)=1 \\ f(1)=-3 \end{array}\right] \rightarrow \text{Smallest +ve root}$$

$$f(2) = -1$$

So interval is between 0 and 1 for smallest positive root

$$x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -3$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{(-3)}{(-2)}$$

$$x_2 = -\frac{1}{2} \text{ [1st iteration]}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -\frac{1}{2} - \frac{3.375}{-4.25}$$

$$= -0.5 + 0.79$$

$$= 0.29 \text{ [2nd iteration]}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.29 - \frac{(+0.425)}{(+4.75)} = 0.29 - 0.089 = 0.201$$

So smallest +ve root = 0.201

(c) Define rate of convergence. Determine the rate of convergence for the Regula-Falsi method.

Ans. The speed at which a convergent sequence approaches its limit is called the rate of convergence.

$$\lim_{n\to\infty} \frac{|\in_{n+1}|}{|\in_n|^{\alpha}} = k$$

i.e.

Where, $\in_n$ = Error = $q_n - q$ that becomes less as $n \to \infty$.

Generally, $\lim_{n\to\infty} \frac{\|\in_{n+1}\|}{\|\in_n\|^{\alpha}} = k < 1$

Here α determines the order of convergence.

(ii) Rate of Convergence of Falsi Method: Let the sequence $\{q_0, q_1, ...\}$ converges to q

Error term = $\in_n = q_n - q$. The sequence converges iff $|\in_n| \to 0$ as $n \to \infty$.

Now if we consider a function $f(q) = 0$ in the interval $\{q_0, q_1\}$ containing the root, then in false position method or regula falsi method, one of the given two points q_0 or q_1 is always fixed and the other varies. If x_0 is fixed then the function is approximated by the straight line passing through (x_0, f_0) and (x_n, f_n), $n = 1, 2, ...$

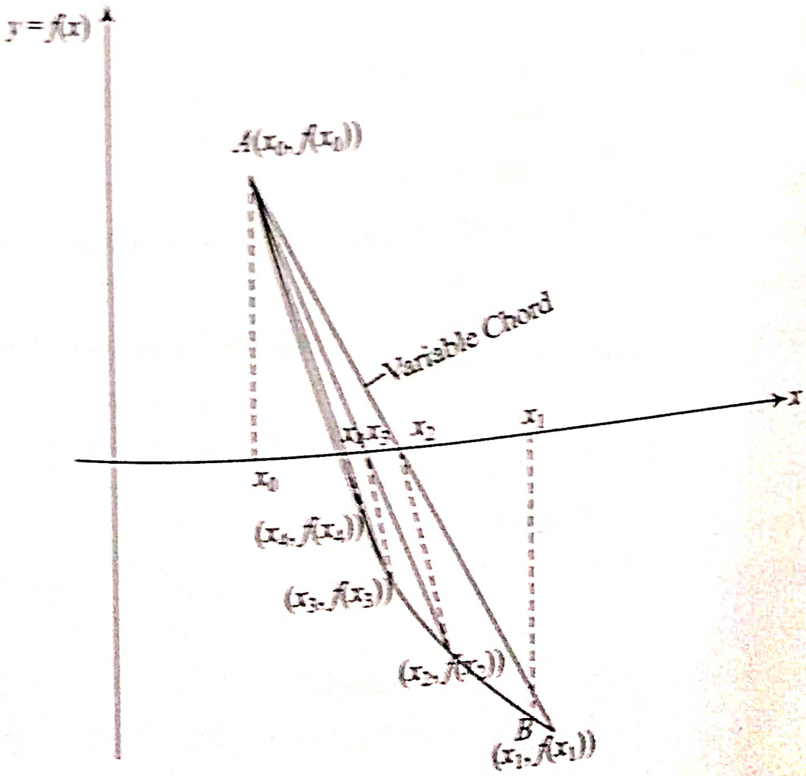
Let λ and α be the positive constants,

$$\lim_{n\to\infty} \frac{\|\in_{n+1}\|}{\|\in_n\|^{\alpha}} = \lambda = \text{Rate of convergence}$$

Now,
$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad ...(1)$$

$$x_n = \frac{x_{n-2}f(x_{n-1}) - x_{n-1} f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})} \quad ...(2)$$

$$\in_n = x_n - x \Rightarrow x_n = \in_n + x$$
$$\in_{n-1} = x_{n-1} - x \Rightarrow x_{n-1} = \in_{n-1} + x$$
$$\in_{n-2} = x_{n-2} - x \Rightarrow x_{n-2} = \in_{n-2} + x$$



Substitute these values x_n, x_{n-1} and x_{n-2} in equation (2) we get,

$$\in_n + x = \frac{(\in_{n-2}+x)f(\in_{n-1}+x)-(\in_{n-1}+x)f(\in_{n-2}+x)}{f(\in_{n-1}+x)-f(\in_{n-2}+x)}$$

Expand it by Taylor's expansion,

$$\in_n = \frac{\in_{n-2}\left[f(x)+\in_{n-1}f'(x)+\frac{(\in_{n-1})^2}{2!}f''(x)+...\right]}{\left[f(x)+\in_{n-1}f'(x)+\frac{(\in_{n-1})^2}{2!}f''(x)+...\right]}$$

$$\frac{-\in_{n-1}\left[f(x)+\in_{n-2}f'(x)+\frac{(\in_{n-2})^2}{2!}f''(x)+....\right]}{-\left[f(x)+\in_{n-2}f'(x)+\frac{(\in_{n-2})^2}{2!}f''(x)+...\right]}$$

$$\in_n = \frac{\in_{n-2}\left[\frac{(\in_{n-1})^2}{2!}f''(\text{x})\right]-\in_{n-1}\left[\frac{(\in_{n-2})^2}{2!}f''(\text{x})\right]}{[\in_{n-1}-\in_{n-2}]f'(\text{x})}$$

$$\in_n = \frac{1}{2}\frac{f''(\text{x})}{f'(\text{x})}\in_{n-1}\in_{n-2}$$

$$\frac{1}{2}\frac{f''(\text{x})}{f'(\text{x})} = C = \text{Asymptotic error constant} < 1$$

$$\Rightarrow \qquad \in_n < \in_{n-1} - \in_{n-2}$$

So it has linear rate of convergence.

Q.2. (a) Perform four iterations of the Bisection method to obtain a root in the interval (0, 1) of the equation:

$$f(x) = \cos x - xe^x = 0.$$

(b) Perform three iterations to find the cube root of 17 by Newton's method with the initial approximation $x_0 = 2$.

(c) Perform two iterations of Newton's method to solve the non-linear system of equations with initial approximation (1, 1):

$$f(x, y) = x^2 + y^2 - 4 = 0 \text{ and}$$
$$g(x, y) = x^2 + y^2 - 16 = 0.$$

Ans. 2. (a) $\qquad f(x) = \cos(x) - xe^x = 0$

$$f(0) = \cos(0) - 0e^0 = 1, f(1) = \cos(1) - 1e^1 = -1.72 \rightarrow \text{one root}$$

Let $a = 0$ and $b = 1$

$$x_1 = 0.5 \text{ [1st iteration]}$$
$$f(0.5) = 0.99 - 0.82 = 0.17$$

$$x_2 = \frac{0.5+1}{2} = 0.755 \text{ [2nd iteration]}$$

Now,

$$f(0.75) = 0.99 - 1.58 = -0.59$$

$$x_3 = \frac{0.5+0.75}{2} = 0.625 \text{ [2nd iteration]}$$

Now,

$$f(0.625) = 0.99 - 1.16 = -0.17$$

So

$$x_1 = \frac{0.5+0.625}{2} = 0.5625 \quad \text{Ans. (upto 4}^{\text{th}}\text{ iteration)}$$

(c) $f(x, y) = x^2 + y^2 - 4 = 0$

$$g(x, y) = x^2 + y^2 - 16 = 0$$

By taking initial approx. as,

$$x_0 = 1, y_0 = 1$$

$$J = \begin{bmatrix} f_x(x_n, y_n) & f_y(x_n, y_n) \\ g_x(x_n, y_n) & g_y(x_n, y_n) \end{bmatrix}$$

$$f_x(x_n, y_n) = 2x_n, f_y(x_n, y_n) = 2y_n, g_x(x_n, y_n) = 2x_n, g_y(x_n, y_n) = 2y_n$$

So,

$$J = \begin{bmatrix} 2x_n & 2y_n \\ 2x_n & 2y_n \end{bmatrix}$$

$$D^n = |Jn|$$
$$= 4x_n y_n - 4x_n y_n$$
$$= 0$$

Q.3. (a) Solve the linear system $Ax = b$ using Gaussian elimination with pivoting:

$$A = \begin{bmatrix} 6 & 2 & 2 \\ 6 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

(b) Start with initial vector $(x, y, z) = (0, 0, 0)$ perform three iterations of Gauss-Seidel method to solve the following of the equations:

$$6x + 15y + 2z = 72,$$
$$x + y + 54z = 110,$$
$$27x + 6y - z = 85.$$

(c) Explain Thomas Algorithm and solve the following Tridiagonal system $Ax = b$ using the Thomas method:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 10 \end{bmatrix} \text{ and } b = \begin{bmatrix} 10 \\ 17 \\ 22 \end{bmatrix}$$

Ans. (a) Here $A = \begin{bmatrix} 6 & 2 & 2 \\ 6 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$

In form of $AX = B$

$$\begin{bmatrix} 6 & 2 & 2 \\ 6 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - \frac{1}{6}R_1$

$$\begin{bmatrix} 6 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 5/3 & -4/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 5/3 & -4/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

Now, $-z = 5$ Therefore, $z = -5$

Now, $(5/3)y - (4/3)z = 0$

$$(5/3)y = (4/3)*(-5)$$

$$y = -4$$

and $3x + y + z = 0$, putting values of z and y in $x = 3$.

Hence, $x = 3, y = -4\ \&\ z = -5$.

(b) Given System of equations:

$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$

Can be written as:

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Taking initial approx.:

$$X_0 = Y_0 = Z_0 = 0$$
$$X_1 = (1/27)\times(85 - 6Y_0 + Z_0) = 3.14$$
$$Y_1 = (1/15)\times(72 - 6X_1 - 2Z_0) = 3.541$$
$$Z_1 = (1/54)\times(110 - X_1 - Y_1) = 1.913$$

Now,

$$X_2 = (1/27)\times(85 - 6\times(3.541) + 1.913) = 2.43$$
$$Y_2 = (1/15)\times(72 - 6X_2 - 2\times(1.913)) = 3.573$$
$$Z_2 = (1/54)\times(110 - 2.43 - 3.573) = 1.926$$

Now,

$$X_3 = (1/27) \times (85 - 6 \times (1.915) + 1.956) = 2.795$$
$$Y_3 = (1/15) \times (72 - 6X_3 - 15Y_2) = 1.767$$
$$Z_3 = (1/54) \times (110 - X_3 - Y_3) = 1.952$$

So,

Finally we have:

$$X = 2.759,\ Y = 1.767, \text{ and } Z = 1.952$$

(c) (i) Algorithm for Gauss Thomas Method:

The system of equations is:

$$d_1 x_1 + a_1 x_2 = r_1$$
$$b_2 x_1 + d_2 x_2 + a_2 x_3 = r_2$$
$$b_n x_{n-1} + d_n x_n = r_n$$

Step 1: For the first equation, form the new elements a_1 and r_1 by new

$$a_1 = (a_1/d_1);\ r_1 = (r_1/d_1)$$

Step 2: For each of the new equations:

From $i = 2, \ldots, n-1$

$$a_i = a_i / (d_i - b_i a_{i-1});\ r_i = (r_i - b_i r_{i-1})/(d_i - b_i a_{i-1})$$

Step 3: For the last equation

$$r_n = (r_n - b_n r_{n-1})/(d_n - b_n a_{n-1})$$

Step 4:

(i) Solve by back substitution:

$$x_n = r_n \qquad (i = n-1, n-2, \ldots, 2, 1)$$
$$x_i = r_i - a_i x_{i+1}$$

Given tridiagonal system of equations:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 10 \end{bmatrix} \text{ and } r = \begin{bmatrix} 10 \\ 17 \\ 22 \end{bmatrix}$$

So,

$$r = [10 \quad 17 \quad 22]$$
$$a = [2 \quad 3 \quad 0] \qquad \text{(above diagonal)}$$
$$b = [0 \quad 1 \quad 3] \qquad \text{(below diagonal)}$$
$$d = [1 \quad 3 \quad 10] \qquad \text{(diagonal)}$$

Step 1:

$$\text{New } a_1 = a_1/d_1 = 2/1 = 2$$
$$\text{New } r_1 = r_1/d_1 = 10/1 = 10$$

Step 2:

$$a_1 = a_2/d_2 - b_2 a_1 = 3/3 - 1 \times 2 = 3$$
$$r_2 = r_2 - b_2 r_1/d_2 - b_2 a_1 = 17 - 1 \times 10/3 - 1 \times 2 = 7/1 = 7$$

Step 3: For the last Equation

$$r_3 = (r_3 - b_3 r_2)/(d_3 - b_3 a_2) = (22 - 3 \times 7)/(10 - 9) = 1$$

$$x_3 = r_3 = 1$$

$$x_2 = r_2 - a_2 x_3 = 7 - 3 = 4$$

$$x_1 = r_1 - a_1 x_2 = 10 - 2 \times 4 = 2$$

So, $\quad x_1 = 2, x_2 = 4 \;\&\; x_3 = 1$

Q.4. (a) Construct the divided difference table for the data:

x	$f(x)$
–1	–2
1	0
4	63
7	342

Hence, find the Newton divided difference interpolating polynomial and an approximation to the value of $f(6)$.

(b) Prove the following relations:

(i) $\Delta = \left(1 + \frac{1}{2}\Delta\right)\sqrt{(1+\Delta)}$

(ii) $\Delta = \left(\frac{1}{1-\nabla}\right) - 1$

(c) For the following data, obtain the backward difference polynomial using Gregory-Newton backward difference interpolation. Also, interpolate at $x = 0.35$.

x	$f(x)$
0.1	1.40
0.2	1.56
0.3	1.76
0.4	2.00
0.5	2.28

Ans. (a)

X	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
–1	–2	$(0-(-2))/(1-(-1)) = 1$	$(21-1)/(4-(-1)) = 4$	
1	0	$(63-0)/(4-1) = 21$		$(12-4)/7-(-1) = 1$
4	63	$(342-63)/(7-4) = 93$	$(93-21)/(7-1) = 12$	
7	342			

Now,

$$x_0 = -1, x_1 = 1, x_2 = 4, x_3 = 7$$

So, $f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)\,(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)$
$(x - x_2)\,\Delta^3 f(x)$

$= -2 + (x+1) + (x+1)(x-1) \times 4 + (x-1)(x+1)(x-4)$

$= -2 + x + 1 + 4(x^2-1) + (x-4)(x^2-1)$

$$f(x) = x^3 - 1$$

So,
$$f(6) = 6^3 - 1$$
$$= 216 - 1$$
$$= 215$$

(b) $\Delta = \left(\frac{1}{1-\nabla}\right) - 1$

So,
$$\Delta = \frac{1-1+\nabla}{1-\nabla}$$
$$= \frac{\nabla}{1-\nabla}$$
$$= \frac{1-E^{-1}}{E^{-1}}$$
$$= \frac{1}{E^{-1}} - 1$$
$$= E - 1$$

Now,
$$\Delta y_x = (E-1)y_x$$
$$\Delta = E - 1$$

So,
$$\Delta = \left(\frac{1}{1-\nabla}\right) - 1$$

(c)

μ	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
–2	0.1	1.40			
–1	0.2	1.56	0.16	0.04	0
0	0.3	1.76	0.20	0.04	
1	0.4	2.00	0.24	0.04	0
2	0.5	2.28	0.28		

Here,
$$x_0 = 0.3, h = 0.1,$$

So
$$\mu = \frac{x-x_0}{h} = \frac{x-0.3}{0.1} = 10x - 3$$

So, $f(\mu) = f(0) + \mu\Delta f(0) + \mu(\mu+1)\frac{\Delta^2 f(0)}{2!} + \mu(\mu+1)(\mu-2)\frac{\Delta^3 f(0)}{3!}$

$= 1.76 + (10x - 3)0.20 + (10x-3)(10x-2)0.04$

$= 1.76 + 2x - 0.60 + (100x^2 - 50x + 6)0.04$

$f(x) = 4x^2 + 1.40$

Now for $x = 0.55$, $u = \dfrac{0.55 - 0.5}{0.1} = \dfrac{0.05}{0.1} = 0.5$

So, $$f(0.55) = 1.76 + (0.5)(0.2) + 0.5(1.5)(0.04)$$
$$= 1.89$$

Q.5. (a) Obtain the piecewise linear interpolating polynomial for the function $f(x)$ defined by the given data:

x	$f(x)$
0	1
1	2
2	5
3	10

and interpolate at $x = 0.5$ and 1.5.

(b) By use of Richardson extrapolation, find $f'(1)$ using the approximate formula:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

with $h = 0.4$, 0.2 and 0.1 from the following values:

x	$f(x)$
0.6	0.707178
0.8	0.859892
0.9	0.925863
1.0	0.984007
1.1	1.033743
1.2	1.074575
1.4	1.127986

(c) Find the approximate value of:

$$I = \int_0^1 \frac{dx}{1+x}$$

(i) Trapezoidal rule

(ii) Simpson's rule.

Ans. (a)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	1		
1	2	3	2	0
2	5	5	2	
3	10			

Here $a = 0$, $h = 1$

So, $$\mu = \frac{x-a}{h} = \frac{x-}{} = x$$

$$\text{Now, } f(\mu) = f(0) + \mu\Delta f(0) + \mu(\mu-1)\frac{\Delta^2 f(0)}{2!} + \mu(\mu-1)(\mu-2)\frac{\Delta^3 f(0)}{3!}$$

$$= 1 + x \times 1 + x \times (x-1)2 + 0$$

$$= x + 2x^2 - 2x + 1$$

$$f(x) = 2x^2 - x + 1$$

Now, if $x = 0.5$ then $\mu = \frac{0.5-0}{1} = 0.5$

$$\text{So, } f(0.5) = f(a) + \mu\Delta f(a) + \mu(\mu-1)\frac{\Delta^2 f(a)}{2!} + \mu(\mu-1)(\mu-2)\frac{\Delta^3 f(a)}{3!}$$

$$f(0.5) = 1 + 0.5 \times 1 + (0.5)(-0.5) \times 2 + 0$$

$$= 1 + 0.5 - 0.5$$

$$= 1.$$

& If $x = 1.5$

$$\mu = \frac{1.5-0}{1} = f(1.5) = f(a) + u\Delta f(a) + u(u-1)\Delta^2 f(a)$$

$$= 1 + 1.5 \times 1 + 1.5\,(0.5) \times 2$$

$$= 1 + 1.5 + 1.5$$

$$f(1.5) = 4.$$

(b) Do Yourself.

(c) (i)

x	0/6	1/6	2/6	3/6	4/6	5/6	6/6
$f(x) = y$	1	0.8567	0.7501	0.6666	0.6002	0.5455	0.5

Here $h = 1/6$.

So, By Trapezoidal Rule:

$$\int_0^1 \frac{1}{1+x}dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6]$$

$$= \frac{1}{12}[1 + 2(0.7501 + 0.8576 + 0.6666 + 0.6002 + 0.5455) + 0.5]$$

$$= \frac{1}{12}[1 + 2(3.42) + 0.5]$$

$$= \frac{1}{12}[8.34]$$

$$= 0.695$$

(ii)

x	0/6	1/6	2/6	3/6	4/6	5/6	6/6
$f(x)=y$	1	0.8567	0.7501	0.6666	0.6002	0.5455	0.5

$$f(x) = \frac{1}{1+x}$$

Here $h = 1/6$.

Now by Simpson's $1/3^{rd}$ rule:

$$\int_0^1 \frac{1}{1+x}dx = \frac{h}{3}[y_0 + 4\{y_1 + y_3 + y_5\} + 2\{y_2 + y_4\} + y_6]$$

$$= \frac{1}{18}[1 + 4(0.8567 + 0.6666 + 0.5455) + 2(0.7501 + 0.6002) + 0.5]$$

$$= \frac{1}{18}[1 + 4(2.0697) + 2(1.3503) + 0.5]$$

$$= \frac{1}{18}[1 + 8.2788 + 2.7006 + 0.5]$$

$$= \frac{1}{18}[12.4794] = 0.6933 \qquad \text{[Ans.]}$$

Q.6. (a) Apply Euler's modified method to approximate the solution of the initial value problem and calculate $y(0.3)$ by using $h = 0.3$:

$$\frac{dy}{dx} = 1 + xy, y(0) = 2.$$

(b) Apply R. K. fourth order to solve the initial value problem and calculate y(0.1) by using h = 0.1

$$\frac{dy}{dx} = x^2 - y, y(0) = 1.$$

(c) Apply finite difference method to solve the given problem:

$$\frac{d^2y}{dx^2} = y + x, y(0) = 2, y(1) = 2.5$$

Ans. (a) Here, $\frac{dy}{dx} = 1 + xy, y(0) = 2$

So, $f(x_0, y_0) = 1 + 0 \times 2; x_0 = 0, y_0 = 2, x_1 = 0.3$

$= 1$

Now, $y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 2 + 0.3 \times 1$

$= 2.3$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 2 + \frac{0.3}{2}[1 + 1 + 0.3 \times 2.3]$$

$$= 2 + \frac{0.3}{2}[2 + 0.69]$$

$$= 2 + 0.40 = 2.4$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 2 + \frac{0.3}{2}[1 + 1 + 0.3 \times 2.4]$$

$$= 2 + \frac{0.3}{2}[2 + 0.72] = 2.408$$

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 2 + \frac{0.3}{2}[1 + 1 + 0.3 \times 2.408]$$

$$= 2 + \frac{0.3}{2}[2.7224]$$

$$= 2 + 0.408 = 2.408$$

So, $y_1^{(3)} = y_1^{(4)} = 2.408$ for $x = 0.3$

(b) Here, $\frac{dy}{dx} = x^2 - y, y(0) = 1$

So, $K_1 = hf(x_0, y_0)$

$$= 0.1 \times (-1) = -0.1$$

Now $K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$

$$= 0.1\ f(0.05, 0.95)$$

$$= 0.1\ [0.0025 - 0.95]$$

$$= 0.1\ [-0.9475] = -0.09475$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\ f(0.05, 0.526)$$

$$= 0.1\ [0.0025 - 0.526] = -0.05235$$

Now, $K_4 = hf[x_0 + h, y_0 + k_3]$

$$= 0.1\ f(0.1, 0.9476)$$

$$= 0.1\ [0.01 - 0.9476]$$

$$= 0.1\ [-.9376] = -0.09376$$

So finally, $$K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1/6(-0.1 + 2(-0.09475 - 0.05235) - 0.09376)$$
$$= -0.01674$$

So, $$y_1 = y(0.1) = y_0 + K = 1 - 0.01674$$
$$= 0.98326$$

(c) Here $\frac{d^2y}{dx^2} = y'' = y + x$

$$y(0) = 2, y(1) = 2.5$$

and $n = 2.5$ with $n = 4$ Sub-intervals

The central difference formula for 2[nd] derivative is:

$$y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

So, this method will find an approx. solution at points $x_1 = 0.25, x_2 = 0.5, x_3 = 0.75$

So, $$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i + x_i$$

So, $$h = 0.25$$

$$y_{i+1} - 2y_i + y_{i-1} = 0.0625[y_i + x_i]$$
$$y_2 - 2y_1 + y_0 = 0.0625[y_1 + x_1]$$
$$y_2 - 2y_1 + y_0 = 0.0625[y_1 + 0.25]$$
$$y_2 - 2y_1 + y_0 = 0.0625y_1 + 0.015625$$
$$y_2 - 2.0625y_1 = 1.984325 \quad ...(1)$$
$$y_3 - 2y_2 + y_1 = 0.0625[y_2 + x_2]$$
$$y_3 - 2y_2 + y_1 = 0.0625y_2 + 0.03125$$
$$y_3 - 2.0625y_2 + y_1 = 0.03125 \quad ...(2)$$
$$y_4 - 2y_3 + y_2 = 0.0625[y_3 + x_3]$$
$$2.5 - 2.0625y_3 + y_2 = 0.046875$$
$$y_2 - 2.0625y_3 = -2.453125 \quad ...(3)$$

So, from Equation (1)

$$y_2 - 2.0625y_1 = 1.984375$$
$$y_1 = \frac{y_2 + 1.984375}{2.0625}$$

From Equation (3)

$$y_2 - 2.0625y_3 = -2.453125$$
$$y_3 = \frac{y_2 + 2.453125}{2.0625}$$

So, putting value of y_1 & y_3 in Equation (2) we get,

$$\frac{y_2 + 1.984325}{2.0625} + \frac{y_2 + 2.453125}{2.0625} - 2.0625y_2 = 0.03125$$

$$\frac{2y_2 + 4.4375 - 4.25y_2}{2.0625} = 0.03125$$

So,
$$-2.25y_2 + 4.4375 = 0.0645$$
$$2.25y_2 = 4.4375 - 0.0645$$
$$y_2 = \frac{4.373}{2.25} = 1.94$$

Now,
$$y_1 = \frac{y_2 + 1.984325}{2.0625} = \frac{3.924375}{2.0625}$$
$$y_1 = 1.902$$
$$y_3 = \frac{y_2 + 2.453125}{2.0625} = 2.13$$

So,
$$y_{(0.25)} = 1.902$$
$$y_{0.5} = 1.94$$
$$y_{0.75} = 2.13$$