RICHARDSON EXTRAPOLATION

(1) Order of a Method :-

If a method has an error term that is $O(h^k)$, we say the method is k^{th} order method. For example, if we use a Taylor polynomial to approximate a function at $x = a+h$, we have

$$f(x) = f(a+h)$$
$$= f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(\eta)$$

where $\eta \in [a, a+h)$

then we say that the approximation is $O(h^3)$, or the method is of 3^{rd} order. This is written as $f(h) = O(h^3)$.

Richardson Extrapolation :-

In earlier section, we derived first and second-order finite difference approximation formulas for first derivative by interpolating the data points. Now, we will use different values of h to improve the accuracy of these formulas.

Let us consider the second-order central difference formula for $f'(x_i)$ which gives $O(h^2)$ approximation:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} - \frac{h^2}{6} f^3(\eta) + O(h^4)$$

(this part we have truncated before)

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h} - \frac{h^2}{6} f^3(\eta) + O(h^4) \quad \text{— (1)}$$

where $\eta \in [x_{i-1}, x_{i+1}]$.

Notations:

D = true value of the derivative that is, $f'(x_i)$

$D(h)$ = approximate value of the derivative obtained by using a step size h.

$D(h/2)$ = approximate value of the derivative obtained by using a step size $h/2$.

The equation (1) can be written as

$$D = D(h) - \frac{h^2}{6} f^3(\eta) + O(h^4) \quad \text{— (2)}$$

If we consider the step size $h/2$, instead of h, then from eqn. (1), we get

$$f'(x_i) = \frac{f(x_i + \frac{h}{2}) - f(x_i - \frac{h}{2})}{h} - \frac{h^2}{24} f^{(3)}(\eta) + O(h^4)$$

In other words

$$D = D\left(\frac{h}{2}\right) - \frac{h^2}{24} f^{(3)}(\eta) + O(h^4) \quad \text{--- (3)}$$

Multiplying Equation (3) by 4 (so that the error term cancels), we get

$$4D = 4D\left(\frac{h}{2}\right) - \frac{h^2}{6} f^{(3)}(\eta) + O(h^4) \quad \text{--- (4)}$$

Subtracting Equation (2) from (4), we have

$$3D = 4D\left(\frac{h}{2}\right) - D(h)$$

$$\Rightarrow \boxed{D = \frac{4D\left(\frac{h}{2}\right) - D(h)}{3}}$$

This gives an $O(h^4)$ approximation to $D = f'(x_i)$.

This process of extrapolating from $D(h)$ and $D\left(\frac{h}{2}\right)$ to approximate D with higher order of accuracy is called as Richardson Extrapolation.

Question (4) - Consider the following data

	x_0	x_1	x_2	x_3	x_4
x	1	2	3	4	5
$y = f(x)$	2	4	8	16	32

Estimate $f'(x_2) = f'(3)$ using Richardson Extrapolation with $h = 2$.

Solution:

We will directly calculate $f'(x_2)$ using Richardson Extrapolation. We know

$$D(h) = \frac{f(x_i + h) - f(x_i - h)}{2h}$$

& $$D\left(\frac{h}{2}\right) = \frac{f(x_i + h/2) - f(x_i - h/2)}{h}$$

We have given that $h=2$. This implies

$$D(h) = \frac{f(x_2 + h) - f(x_2 - h)}{2h}$$

$$= \frac{f(3+2) - f(3-2)}{2(2)}$$

$$= \frac{f(5) - f(1)}{4} = \frac{32-2}{4}$$

$$= \frac{30}{4} = \frac{15}{2} = \boxed{7.5}$$

and $$D(h/2) = \frac{f(x_2 + h/2) - f(x_2 - h/2)}{h}$$

$$D(h/2) = \frac{f(3+1) - f(3-1)}{2}$$

$$= \frac{f(4) - f(2)}{2} = \frac{16-4}{2}$$

$$= \frac{12}{2} = \boxed{6}$$

Putting these values in

$$D = \frac{4D(h/2) - D(h)}{3}$$

$$= \frac{4(6) - 7.5}{3}$$

$$= \frac{24 - 7.5}{3} = \frac{16.5}{3}$$

$$= \boxed{5.5}$$

Remark:- from the central difference formula,

we got $\boxed{f'(3) = 6}$

Using Richardson Extrapolation, we get

$\boxed{f'(3) = 5.5}$

Actual function is $f(x) = 2^x$

$\boxed{f'(x) = 2^x \ln(2)}$

$\therefore \quad f'(3) = 2^3 \ln(2) \approx 5.5452$

Clearly, R.E gives more accuracy than C.D.F.

(*) **Higher - Order Accuracy :-**

Rewriting Equation ( ), we have

$$D = D(h) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \cdots \quad \text{(odd terms were cancelled)} \quad -(5)$$

This implies

$$D = D\left(\frac{h}{2}\right) + \frac{a_2 h^2}{4} + \frac{a_4 h^4}{16} + \frac{a_6 h^6}{64} + \cdots \quad -(6)$$

We need to eliminate the h^2 term from eqn. (5) & (6) to get $O(h^4)$ approximations. Hence, multiplying the eqn. (6) by 4 and then subtracting it by eqn. (5), we get

$$3D = 4D\left(\frac{h}{2}\right) - D(h) + O(h^4)$$

This gives

$$D = \frac{4D(h/2) - D(h)}{3} + O(h^4)$$

Therefore, the $O(h^4)$ approximation of D using the step size h and $h/2$ is given by

$$D^1(h) = \frac{4D(h/2) - D(h)}{3}$$

Similarly, the $O(h^4)$ approximation of D using the step size $h/2$ and $h/4$ is given by

$$D^1(h/2) = \frac{4D(h/4) - D(h/2)}{3}$$

and so on.

Now, we will find an $O(h^6)$ approximation of D.

We have already eliminated h^2 term and obtained a new approximated value D^1 of D. Therefore, we have

$$D = D^1(h) + b_4 h^4 + b_6 h^6 + \cdots \quad \text{--- (7)}$$

and

$$D = D^1\left(\frac{h}{2}\right) + \frac{b_4 h^4}{16} + \frac{b_6 h^6}{64} + \cdots \quad \text{--- (8)}$$

Now, we will eliminate h^4 term from equation (7) & (8) to get $O(h^6)$ approximation. So, multiplying the eqn. (8) by 16 and then subtracting it by eqn. (7), we get

$$15 D = 16 D^1(h/2) - D'(h) + O(h^6)$$

$$D = \frac{16 D'(h/2) - D'(h)}{15} + O(h^6)$$

Therefore, the $O(h^6)$ approximation of D using the step size h and $h/2$ is given by:

$$\boxed{D^2(h) = \frac{16 D'(h/2) - D'(h)}{15}}$$

Similarly, the $O(h^6)$ approximation of D using the step size $h/2$ and $h/4$ is given by

$$\boxed{D^2(h/2) = \frac{16\, D'(h/4) - D'(h/2)}{15}}$$

and so on.

In general, the $O(h^{2(m+1)})$ approximation to D with step size h and $h/2$ is given by:

$$\boxed{D^m(h) = \frac{4^m D^{m-1}(h/2) - D^{m-1}(h)}{(4^m - 1)}}$$

where $m = 1, 2, \ldots$ such that $D^0(h/2) = D\left(\frac{h}{2}\right)$ and $D^0(h) = D(h)$.

Similarly, the $O(h^{2(m+1)})$ approximation to D with step size $h/2$ and $h/4$ is given by:

$$\boxed{D^m(h/2) = \frac{4^m D^{m-1}(h/4) - D^{m-1}(h/2)}{(4^m - 1)}}$$

and so on.

The Extrapolation Table is given by

step size	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
h	$D(h)$	$D'(h)$	$D^2(h)$	$D^3(h)$
$h/2$	$D(h/2)$	$D'(h/2)$	$D^2(h/2)$	
$h/4$	$D(h/4)$	$D'(h/4)$		
$h/8$	$D(h/8)$			

Question: The following table of values is given:

x	-1	1	2	3	4	5	7
$f(x)$	1	1	16	81	256	625	2401

Find $f'(3)$ by Richardson Extrapolation with $h=4$, $h=2$ and $h=1$ using the following approximate formula

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h}$$

Solution: Since we are given three values of h, that is, $h, h/2, h/4$, so we will calculate approximation of $O(h^2)$, $O(h^4)$ and $O(h^6)$.

step size	$O(h^2)$	$O(h^4)$	$O(h^6)$
$h = 4$	$D(h)$	$D'(h)$	$D^2(h)$
$h/2 = 2$	$D(h/2)$	$D'(h/2)$	
$h/4 = 1$	$D(h/4)$		

Therefore, we have

$$D(h) = f'(3) = \frac{f(3+h) - f(3-h)}{2h} = \frac{f(7) - f(-1)}{8}$$

$$= \frac{2401 - 1}{8} = \frac{2400}{8} = \boxed{300}$$

$$D(h/2) = f'(3) = \frac{f(3+h) - f(3-h)}{2h}$$

$$= \frac{f(5) - f(1)}{4} = \frac{625 - 1}{4} = \boxed{156}$$

$$D(h/4) = f'(3) = \frac{f(3+h) - f(3-h)}{2h} = \frac{f(4) - f(2)}{2}$$

$$= \frac{256 - 16}{2} = \boxed{120}$$

This gives

$$D'(h) = \frac{4D(h/2) - D(h)}{3} = \frac{4(156) - 300}{3}$$

$$= \frac{324}{3} = \boxed{108}$$

$$D'(h/2) = \frac{4D(h/4) - D(h/2)}{3} = \frac{4(120) - 156}{3}$$

$$= \frac{324}{3} = \boxed{108}$$

and $$D^2(h) = \frac{4^2 D'(h/2) - D'(h)}{(4^2-1)}$$

$$= \frac{16(108) - 108)}{15} = \frac{1620}{15}$$

$$= \boxed{108}$$

Therefore, the table becomes

Step size	$O(h^2)$	$O(h^4)$	$O(h^6)$
4	300		
		108	
2	156		108
		108	
1	120		

Hence, $f'(3) = 108$ with $O(h^6)$ approximation.

Q. 3 Use Richardson Extrapolation, compute $f'(1)$ from the following data:

x	0.6	0.8	0.9	1.0	1.1	1.2	1.4
$f(x)$	0.704178	0.859892	0.925863	0.984007	1.033743	1.074575	1.127986

Apply the approximate formula

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

with $h = 0.4$, $h = 0.2$ & $h = 0.1$.