OMMC 2024 Main Round

OMMC Staff

May 2024

HOW TO SUBMIT

1. During the testing period, go to our official test portal: `https://ommc-test-portal.vercel.app/`. **One person submits per team.** Teams can have 1 to 4 people, inclusive.

2. Register an account using Google or Discord.

3. Fill out team registration information and answers.

4. Submit your answers and all answers and information will be saved to your account. **The submission and your registration information are to be sent in TOGETHER.**

5. You can access this information, and **send in an updated submission if you choose,** if you sign back in to your account later within the testing period. **We will take the last set of answers submitted.**

NO, you do not need to "sign up" beforehand in the conventional sense. Just sign into the test portal **ANYTIME DURING THE TESTING PERIOD and SEND IN A RESPONSE.**

RULES

1. This is a **25-question, untimed examination** that can be worked on by a team of up to 4 people. Only one person from each team has to submit answers. If we receive multiple submissions from a team, the most recent submission will be graded.

2. All answers are positive rational numbers inputted as "a/b" for relatively prime positive integers a, b where $b > 1$, or nonnegative integers. Your score will be the number of correct answers; there is neither partial credit nor a penalty for wrong answers. Top scoring teams will move onto the final round.

3. No aids other than writing utensils, scratch paper, rulers, compasses, erasers, and a **four-function/scientific calculator** are allowed. The use of **graphing calculators, smartphones, smartwatches, and/or outside websites is NOT ALLOWED**. In particular, **Desmos, Geogebra, Wolfram Alpha, and other similar websites are forbidden**. Failure to follow this rule will result in your test score being voided.

4. Discussion of any aspect of the test outside your team is **not allowed** until submissions are closed and discussion is opened. Failure to follow this rule will result in your test score being voided.

5. Submissions open on **May 19th** and close at **May 26th, 11:59 PM EST, 2024**.

6. Diagrams are not necessarily to scale.

Any questions on the above should be emailed to ommcofficial@gmail.com. With that, good luck! The OMMC team has spent a lot of time on this contest and we hope that you enjoy your experience.

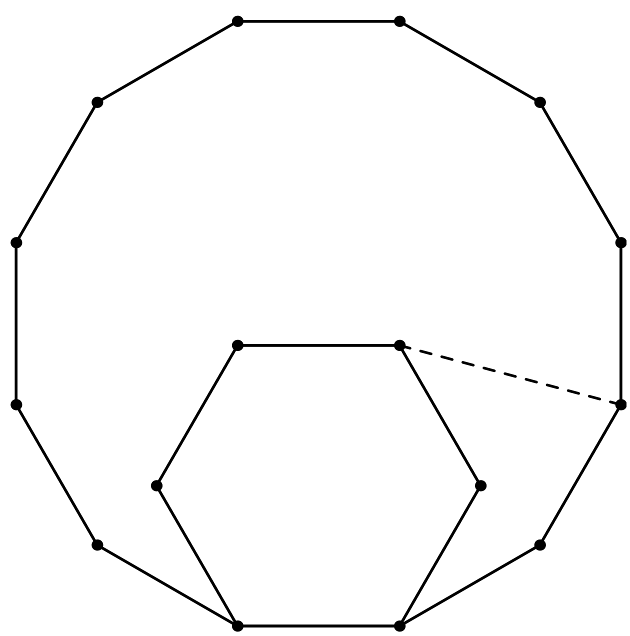
PROBLEMS

1. Lotad writes down five two-digit positive integers, no two of which share a common digit. What is the least possible value of their range?

2. Distinct positive integers a, b have the property that $a! + b!$ ends in three zeroes. Find the smallest possible value of $a + b$.

3. Andy is playing a game in which he has 2024 turns and begins at 0 points. Each turn, he flips a coin. If he gets heads, his score multiplies by -1. Otherwise, his score increases by 1. What is Andy's expected score?

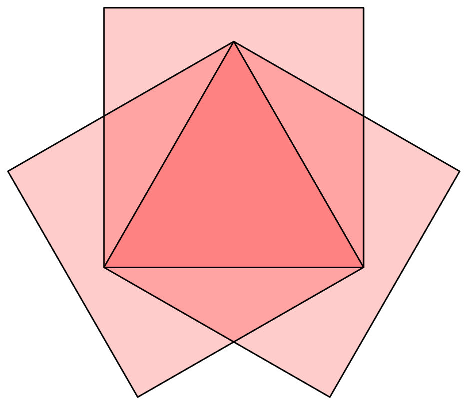
4. A regular hexagon and a regular dodecagon have side length 24 as shown. Find the length of the dotted line squared.



5. Find the number of ways a king and rook can be placed on an 8×8 chess board such that neither piece attacks each other.

   (A rook can attack any square in the same row or column. A king can attack any square directly adjacent or diagonally adjacent to it.)

6. The center triangle has side length 2, and unit squares are constructed using each side of the triangle as the base as shown in the diagram. If the shaded area equals $a - \sqrt{b}$ for positive integers a, b find $a + b$.



7. Find the number of ordered pairs (a, b) of real numbers such that both a and b are roots of the polynomial $x^2 + ax + b = 0$.

8. Suppose quadrilateral $BEAM$ has side lengths $EA = 20$, $BM = 24$, and $BE = AM$. If $\angle B = 50^\circ$ and $\angle M = 40^\circ$, what is the area of $BEAM$?

9. A point randomly selected inside some rectangle is closer to a side than any diagonal with probability $\frac{19}{49}$. If the diagonals have length 6, find the area of the rectangle.

10. The first 2024 positive integers are written in a random order on a circle. Alice starts from 1 and traverses the circle clockwise in the order $1, 2, 3, \ldots, 2024$. What is the expected number of times she passes 1 (not including her start there)?

11. Find the sum of all positive integers $n \leq 100$ such that $n + \lfloor \sqrt{n} \rfloor$ is a perfect square.

12. Let real numbers α, β, and γ satisfy

$$\begin{aligned}\tan\alpha &= 2\cos\beta,\\ \tan\beta &= 3\cos\gamma,\\ \tan\gamma &= 7\cos\alpha.\end{aligned}$$

Find $|\cos 2\alpha|$.

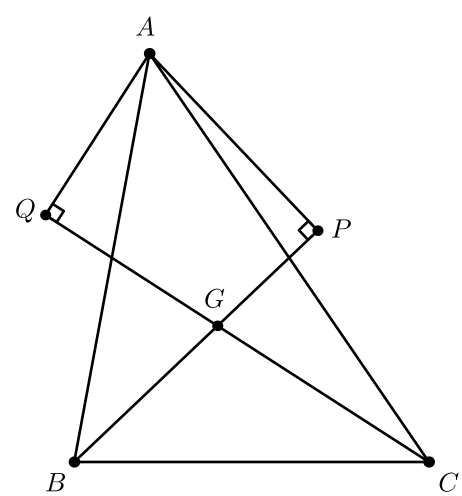
13. Evan constructs a number $r \in [0,1]$ with the following process:

- The 1st digit after the decimal point of r is 9.
- If the n^{th} digit after the decimal point of r has been chosen as k, the $n+1^{\text{th}}$ digit is randomly chosen from the digits between 0 to k inclusive.

Find the expected value of r.

14. Given that a, b, c, and d are positive reals satisfying $ab+bc+cd+da=25$, find the minimum possible value of $108a+27b^2+4c^3+d^4$.

15. In $\triangle ABC$, suppose segments $\overline{BP}$ and $\overline{CQ}$ bisect sides $\overline{AC}$ and $\overline{AB}$, respectively, such that $\angle APB = \angle AQC = 90^\circ$. Let $\overline{BP}$ and $\overline{CQ}$ meet at G.



If $BG = 354$, $CG = 492$, and $QG = 433$, find PG.

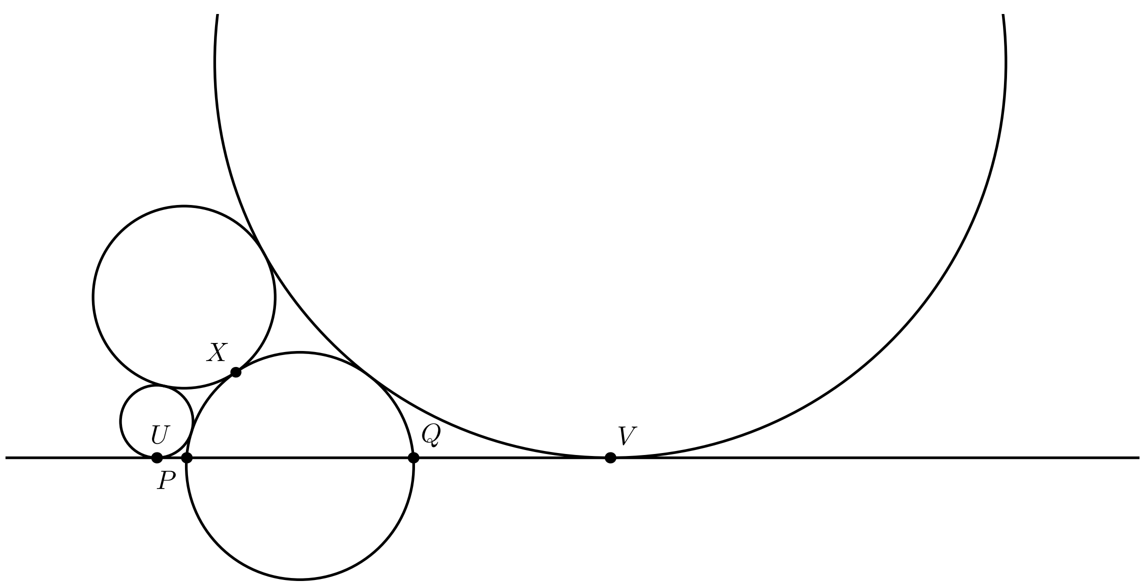
16. A frog is initially situated at the point $(0,0)$. Every day, one of the four points $(1,1),(-1,1),(-1,-1),(1,-1)$ is chosen at random, then the frog's current position is rotated 90 degrees clockwise around the chosen point. The probability that the frog's position after 10 days is on one of the coordinate axes is $\frac{m}{4^{10}}$ for a positive integer m. Find $m \pmod{100}$.

17. Let $p = 2024^2 + 3275^2$ be a prime. Find the smallest positive integer n where $p \mid n^2 + 1$.

18. In an acute triangle ABC, let H denote the orthocenter. Let D be the midpoint of minor arc BC and suppose ray AH meets the circumcircle again at X. Given $AH = 2HX = HD = 2$, compute $(AB + AC)^2$.

19. Let $f(x)$ be the unique polynomial of minimal degree for which $f(k) = 0$ for $1 \le k \le 30$ and $f(k) = k - 30$ for $31 \le k \le 61$. Find $f(62) \pmod{67}$.

20. Define circles $\omega_1, \omega_2, \omega_3, \omega_4$ so ω_i is externally tangent to ω_{i+1} for all $1 \le i \le 4$ ($\omega_5 = \omega_1$). Also, ω_1 and ω_3 are externally tangent at X. A common external tangent to ω_2 and ω_4 at U and V intersects ω_1 at P, Q so U, P, Q, V lie in that order. If $PQ = 9, UV = 18, PX = 4, XQ = 8$, then $UP = a - \sqrt{b}$ for positive integers a, b. Find $a + b$.



21. Find the smallest real number r where for any 99 real numbers $-1 \leq a_1, a_2, \dots a_{99} \leq 1$ that sum to 0, there exists a permutation $b_1, b_2, \dots b_{99}$ of those numbers where for any integers $1 \leq i \leq j \leq 99$ we have $|b_i + \cdots + b_j| \leq r$.

22. Rohan has a set of 2024 identical bags each with 2023 marbles; each marble is either red or white and no two bags contain the same number of marbles of each color. He chooses a random subset S of $\{1, 2, \dots, 2023\}$, and for each $i \in S$, he does the following:

    (a) he chooses a random bag of the 2024 bags he has,

    (b) he draws i marbles at random from the bag, recording the number of red marbles

    (c) and then puts all the marbles back into the bag.

    The probability that the total number of red marbles he drew is even can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $\nu_2(n)$.

23. Find the sum of the two positive integers n where $2023n^2 - n^3 + 1$ is a perfect square.

24. Find the squared volume of tetrahedron $ABCD$ with $AB = 7, AC = 8, AD = 3\sqrt{5}, CD = 2\sqrt{5}, BD = \sqrt{5}, BC = 2\sqrt{6}$.

25. There are 4913 cells in a 17×289 grid, of which k are randomly marked. Alice scans each row left to right, going from top to bottom. Bob scans each column top to bottom, going from left to right. They start at the same time and both check one cell per second. For how many $1 \leq k \leq 4913$ is the chance of Alice finding a marked cell first equal to the chance of Bob finding a marked cell first?