OMMC 2023 Main Round

OMMC Staff

May 2023

HOW TO SUBMIT

1. During the testing period, go to our official test portal: `https://ommc-test-portal.vercel.app/`. **One person submits per team.** Teams can have 1 to 4 people, inclusive.

2. Register an account using Google or Discord.

3. Fill out team registration information and answers.

4. Submit your answers and all answers and information will be saved to your account. **The submission and your registration information are to be sent in TOGETHER.**

5. You can access this information, and **send in an updated submission if you choose,** if you sign back in to your account later within the testing period. **We will take the last set of answers submitted.**

NO, you do not need to "sign up" beforehand in the conventional sense. Just sign into the test portal **ANYTIME DURING THE TESTING PERIOD and SEND IN A RESPONSE.**

RULES

1. This is a **25-question, untimed examination** that can be worked on by a team of up to 4 people. Only one person from each team has to submit answers. If we receive multiple submissions from a team, the most recent submission will be graded.

2. All answers are positive rational numbers inputted as "a/b" for relatively prime positive integers a, b where $b > 1$, or nonnegative integers. Your score will be the number of correct answers; there is neither partial credit nor a penalty for wrong answers. Top scoring teams will move onto the final round.

3. No aids other than writing utensils, scratch paper, rulers, compasses, erasers, and a **four-function/scientific calculator** are allowed. The use of **graphing calculators, smartphones, smartwatches, and/or outside websites is NOT ALLOWED**. In particular, **Desmos, Geogebra, Wolfram Alpha, and other similar websites are forbidden**. Failure to follow this rule will result in your test score being voided.

4. Discussion of any aspect of the test outside your team is **not allowed** until submissions are closed and discussion is opened. Failure to follow this rule will result in your test score being voided.

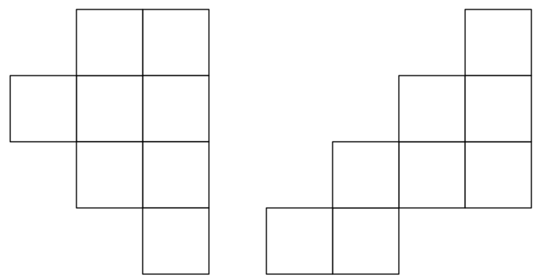
5. Submissions open at **May 12th** and close at **May 24th, 11:59 PM EST, 2023**.

6. Diagrams are not necessarily to scale.

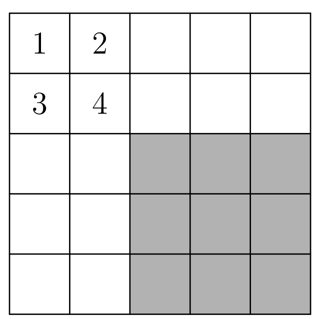
Any questions on the above should be emailed to ommcofficial@gmail.com. With that, good luck! The OMMC team has spent a lot of time on this contest and we hope that you enjoy your experience.

PROBLEMS

1. John has cut out these two polygons made out of unit squares. He joins them to each other to form a larger polygon (but they can't overlap). Find the smallest possible perimeter this larger polygon can have. He can rotate and reflect the cut out polygons.



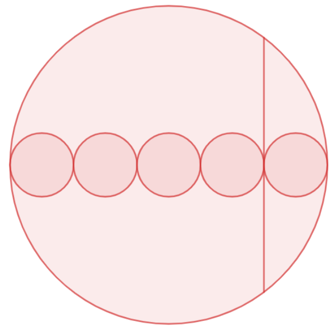
2. Amy places positive integers in each of these cells so each row and column contains each of $1, 2, 3, 4, 5$ exactly once. Find the sum of the numbers in the gray cells.



3. Two cars travel along a circular track n miles long, starting at the same point. One car travels 25 miles along the track in some direction. The other car travels 3 miles along the track in some direction. Then the two cars are once again at the same point along the track. If n is a positive integer, find the sum of all possible values of n.

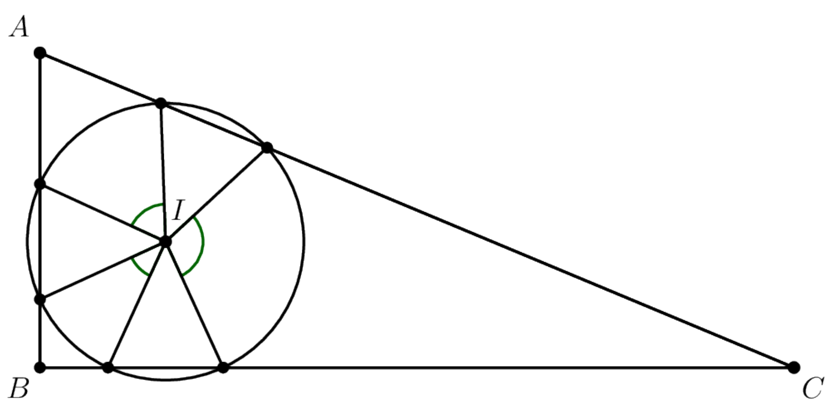
4. Find the number of ways to order the integers $1, 2, 3, 4, 5, 6, 7$ from left to right so that each integer has all its divisors besides itself appearing to the left of it.

5. Five identical circles are placed in a line inside a larger one as shown. If the shown chord has length 16, find the radius of the large circle.



6. Find the unique integer $\overline{CA7DB}$ with nonzero digits so that $\overline{ABCD} \cdot 3 = \overline{CA7DB}$.

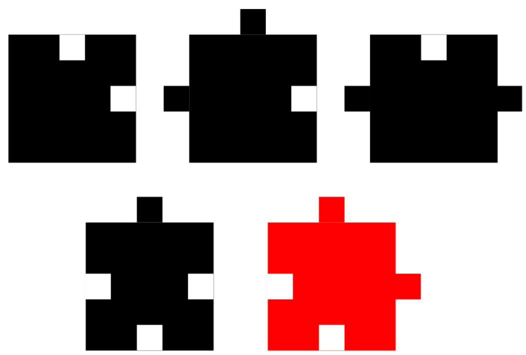
7. Define $\triangle ABC$ with incenter I and $AB = 5$, $BC = 12$, $CA = 13$. A circle ω centered at I intersects ABC at 6 points. The green marked angles sum to $180°$. Find ω's area divided by π.



8. Alice and Bob are each secretly given a real number between 0 and 1 uniformly at random. Alice states, "My number is probably greater than yours." Bob repudiates, saying, "No, my number is probably greater than yours!" Alice concedes, muttering, "Fine, your number is probably greater than mine." If Bob and Alice are perfectly reasonable and logical, what is the probability that Bob's number is actually greater than Alice's?

9. An ant lies on each corner of a 20×23 rectangle. Each second, each ant independently and randomly chooses to move one unit vertically or horizontally away from its corner. After 10 seconds, find the expected area the convex quadrilateral whose vertices are the positions of the ants.

10. Ryan uses 91 puzzle pieces to make a rectangle. Each of them is identical to one of the tiles shown. Given that pieces can be flipped or rotated, find the number of pieces that are red in the puzzle. (He is not allowed to join two “flat sides” together.)



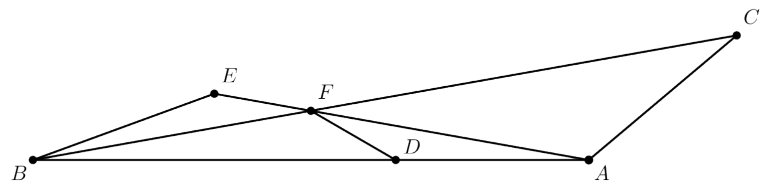
11. Positive real numbers x, y satisfy

$$\lfloor xy \rfloor - \lfloor x \rfloor \lfloor y \rfloor = 8.$$

Find the sum of all possible values of the quantity $\lfloor 2xy \rfloor - \lfloor 2x \rfloor \lfloor y \rfloor$.

12. Initially five variables are defined: $a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0$. On a turn, Evan can choose an integer $2 \leq i \leq 5$. Then, the integer a_{i-1} will be added to a_i. For example, if Evan initially chooses $i = 2$, then now $a_1 = 1, a_2 = 0 + 1 = 1, a_3 = 0, a_4 = 0, a_5 = 0$. Find the minimum number of turns Evan needs to make a_5 exceed $1,000,000$.

13. In triangle ABC, let D lie on AB such that $AD = AC$ and $\angle ADC = 20°$. Let l be a line through B parallel to CD. Let E lie on l with $BE = AD$ so that AE intersects segment BC at F. If $\angle ABC = 10°$, find the degree measure of $\angle FDC$.

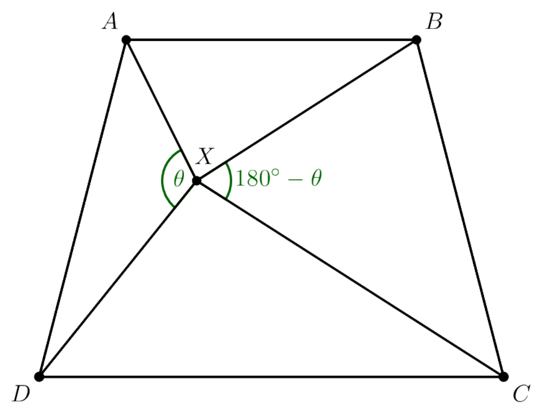


14. Find

$$\left\lfloor \frac{10000}{1} \right\rfloor + \left\lfloor \frac{10000}{2} \right\rfloor + \cdots + \left\lfloor \frac{10000}{100} \right\rfloor - \left\lfloor \frac{10000}{101} \right\rfloor - \cdots - \left\lfloor \frac{10000}{10000} \right\rfloor.$$

15. James the naked mole rat is hopping on the number line. He starts at 0 and jumps exactly 2^n either forward or backward at random at time n seconds, his first jump being at time $n = 0$. What is the expected number of jumps James takes before he is on a number that exceeds 8?

16. Let $ABCD$ be an isosceles trapezoid with $AB = 5$, $CD = 8$, and $BC = DA = 6$. There exists an angle θ such that there is only one point X satisfying $\angle AXD = 180^\circ - \angle BXC = \theta$. Find $\sin(\theta)^2$.



17. Let $a_1, a_2, \ldots$ be a sequence such that $a_1 = a_2 = \frac{1}{5}$, and for $n \geq 3$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{1 + a_{n-1}a_{n-2}}.$$

Find the smallest integer n such that $a_n > 1 - 5^{-2022}$.

18. Kevin writes a nonempty subset of $S = \{1, 2, \dots 41\}$ on a board. Each day, Evan takes the set last written on the board and decreases each integer in it by 1. He calls the result R. If R does not contain 0 he writes R on the board. If R contains 0 he writes the set containing all elements of S not in R. On Evan's nth day, he sees that he has written Kevin's original subset for the 1st time. Find the sum of all possible values of n.

19. Let $\triangle ABC$ be a triangle with $AB = 7$, $AC = 8$, and $BC = 3$. Let P_1 and P_2 be two distinct points on line AC (A, P_1, C, P_2 appear in that order on the line) and Q_1 and Q_2 be two distinct points on line AB (A, Q_1, B, Q_2 appear in that order on the line) such that $BQ_1 = P_1Q_1 = P_1C$ and $BQ_2 = P_2Q_2 = P_2C$. Find the distance between the circumcenters of BP_1P_2 and CQ_1Q_2.

20. Liam writes the number 0 on a board, then performs a series of turns. Each turn, he chooses a nonzero integer so that for every nonzero integer N, he chooses N with $3^{-|N|}$ probability. He adds his chosen integer N to the last number written on the board, yielding a new number. He writes the new number on the board and uses it for the next turn. Liam repeats the process until either 8 or 9 is written on the board, at which point he stops. Given that Liam eventually stopped, find the probability the last number he wrote on the board was 9.

21. Define the minimum real C where for any reals $0 = a_0 < a_1 < \cdots < a_{1000}$ then

$$\min_{0 \le k \le 1000} (a_k^2 + (1000 - k)^2) \le C(a_1 + \cdots + a_{1000})$$

holds. Find $\lfloor 100C \rfloor$.

22. Find the number of ordered pairs of integers (x, y) with $0 \le x, y \le 40$ where

$$\frac{x^2 - xy^2 + 1}{41}$$

is an integer.

23. Define the Fibonacci numbers such that $F_1 = F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k > 2$. For large positive integers n, the expression (containing n nested square roots)

$$\sqrt{2023F_{2^1}^2 + \sqrt{2023F_{2^2}^2 + \sqrt{2023F_{2^3}^2 \cdots + \sqrt{2023F_{2^n}^2}}}}$$

approaches $\frac{a+\sqrt{b}}{c}$ for positive integers a, b, c where $\gcd(a, c) = 1$. Find $a + b + c$.

24. Define acute $\triangle ABC$ with circumcenter O. The circumcircle of $\triangle ABO$ meets segment BC at $D \neq B$, segment AC at $F \neq A$, and the Euler line of $\triangle ABC$ at $P \neq O$. The circumcircle of $\triangle ACO$ meets segment BC at $E \neq C$. Let $\overline{BC}$ and $\overline{FP}$ intersect at X, with C between B and X. If $BD = 13$, $EC = 8$, and $CX = 27$, find DE.

(The Euler line of a triangle passes through its orthocenter, circumcenter, and centroid.)

25. A clock has a second, minute, and hour hand. A fly initially rides on the second hand of the clock starting at noon. Every time the hand the fly is currently riding crosses with another, the fly will then switch to riding the other hand. Once the clock strikes midnight, how many revolutions has the fly taken?

(Observe that no three hands of a clock coincide between noon and midnight.)