A Gaitsgory-Style Blueprint for the Fargues–Scholze Categorical Geometrization Conjecture (the case $\ell \neq p$)

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February 6, 2026

Abstract

We outline a proof strategy for the categorical geometrization conjecture of Fargues and Scholze for ℓ-adic sheaves on the stack Bun_G of G-bundles on the Fargues–Fontaine curve, in the case $\ell \neq p$. The conjecture predicts an equivalence between a Whittaker-generated automorphic subcategory of ℓ-adic sheaves on Bun_G and a spectral category of coherent or ind-coherent sheaves on the stack of Langlands parameters, with a nilpotent singular support condition in the integral setting. Our goal is to place all functors and structural inputs into a single argument template that parallels the architecture of Gaitsgory's proof of geometric Langlands: singular support, Hecke and spectral actions, parabolic induction and constant term, gluing from Levi subgroups, and a Barr–Beck type reduction.

Contents

1 Introduction

1.1 Background

Let E be a non-archimedean local field of residue characteristic p, let G/E be a connected reductive group, and fix a prime $\ell \neq p$. The work of Fargues and Scholze constructs a robust geometric avatar of the local Langlands correspondence using the v-stack Bun_G of G-bundles on the Fargues–Fontaine curve and a corresponding category of ℓ-adic sheaves $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, together with Hecke operators, a geometric Satake equivalence, a stack of Langlands parameters, and a spectral action by perfect complexes [1, 2].

A central conjecture in [2] upgrades the local Langlands correspondence to an equivalence of categories, closely resembling the role played by Whittaker models in geometric Langlands. It is this conjecture that we focus on.

1.2 Main target: the categorical geometrization conjecture

Assume G is quasi-split and fix a Whittaker datum (B, ψ). Fargues and Scholze construct a Whittaker object $\mathcal{W}_\psi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ and define the Whittaker-generated subcategory $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ (see Section 16). On the spectral side, let $\mathrm{LocSys}_{\check{G}}$ denote the stack of ℓ-adic Langlands parameters for G (constructed in [1] and also algebraized by Dat–Helm–Kurinczuk–Moss [3]), and let $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ denote coherent sheaves with nilpotent singular support (Section 14).

Conjecture 1.1 (Fargues–Scholze, categorical geometrization)**.** The spectral action functor

$$\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell), \qquad \mathcal{F} \longmapsto \mathcal{F} * \mathcal{W}_\psi$$

takes values in compact objects when $\mathcal{F}$ has quasi-compact support, and extends to an equivalence of stable $\mathbb{Z}_\ell$-linear categories

$$\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\sim} D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell)_\omega$$

compatible with the spectral action. Over $\mathbb{Q}_\ell$ the nilpotent singular support condition is expected to be automatic.

Conjecture 1.1 is stated in [2, §6] (in a slightly different notation). It is the local analogue of the identification of Whittaker categories with coherent sheaves with nilpotent singular support in geometric Langlands.

1.3 Plan of the paper and proof architecture

This paper is a working blueprint. Each section isolates the relevant constructions, states the intermediate claims needed for a proof of Conjecture 1.1, and provides a proof sketch or a reduction to known results.

Our intended proof follows the architecture of Gaitsgory's approach to geometric Langlands [6, 7, 8, 9]:

Step 1. Set up automorphic and spectral categories. Define $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, the Whittaker object $\mathcal{W}_\psi$, and the spectral stack $\mathrm{LocSys}_{\check{G}}$, together with $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ via singular support.

Step 2. Construct the comparison functor. Use the spectral action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ to define $\Phi(\mathcal{F}) = \mathcal{F} * \mathcal{W}_\psi$.

Step 3. Compactness and finiteness. Prove that Φ sends Perf with quasi-compact support to compact objects, and extend Φ to $\mathrm{Coh}_{\mathrm{Nilp}}$.

Step 4. Full faithfulness via centers and monadicity. Identify $\mathrm{End}(\mathcal{W}_\psi)$ with functions on $\mathrm{LocSys}_{\check{G}}$ (or the stable center). Use Barr–Beck type arguments to deduce full faithfulness of Φ.

Step 5. Essential surjectivity by parabolic gluing. Use geometric Eisenstein series and constant term functors on Bun_G (in the strong form developed by Hamann–Hansen–Scholze [4]) to glue from Levi subgroups and reduce to cuspidal blocks.

Step 6. Cuspidal block analysis and multiplicity one. Establish the cuspidal analogue of "multiplicity one" and complete the equivalence.

Status

Many foundational inputs are theorems: the construction of Bun_G, ℓ-adic sheaves, geometric Satake, the spectral action, and the stack of parameters are in [1], and the parabolic formalism needed for gluing is developed in [4]. The new work required for a full proof is the systematic assembly of these inputs into a Gaitsgory-style argument with singular support and monadicity.

Checkpoints and known comparisons (evidence)

While the categorical equivalence of Conjecture 1.1 is open in general, several pieces of the package are already theorems and provide concrete checkpoints.

- **Compatibility with the classical LLC.** The construction of Hecke eigensheaves on Bun_G and the spectral action are designed to recover the usual local Langlands correspondence after passing to appropriate fibers and taking traces; see [1, 2]. For specific groups (including GSp_4), compatibility with known LLC normalizations is now proved; see [22].
- **Tori.** When $G = T$ is a torus, the geometry of Bun_T and the spectral stack are abelian, and the expected equivalence is compatible with local class field theory (no proper parabolics, no nilpotent correction). This case serves as a sanity check for normalizations and spectral functoriality.

- **Stable Bernstein center and endomorphisms.** The map $\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \to \mathrm{End}(W_\psi)$ is a categorical form of the *stable Bernstein center* and is closely related to classical work on stable center conjectures [16, 17, 18]. For $G = \mathrm{GL}_n$, integral Bernstein center and Whittaker/co-Whittaker refinements are available [12], and one expects to upgrade several "Task 1/Task 4" steps in this paper to theorems in that case using local Langlands in families [19, 20, 21].

What this paper proves unconditionally. At the current stage, the fully written unconditional results in the body are: (i) the parabolic calculation $\mathrm{CT}_{P^-}(W_\psi) \simeq W_{\psi_M}$ (Section 22), and (ii) the reduction of $\mathrm{End}(W_\psi)$ to the endomorphisms of the universal Gelfand–Graev representation (Section 23), together with the comparison to generic Bernstein centers and the GL_n identification *up to* the remaining stable-center comparison input.

2 Notation and standing assumptions

Throughout:

- E is a non-archimedean local field of residue characteristic p and residue field $\mathbb{F}_q$.
- ℓ is a fixed prime with $\ell \neq p$.
- Λ denotes one of $\mathbb{Z}_\ell$, $\mathbb{Z}_\ell[\sqrt{q}]$, $\mathbb{Q}_\ell$, or $\mathbb{Q}_\ell[\sqrt{q}]$, chosen so that Satake normalizations are available (as in [2]).
- G/E is a connected reductive group. When discussing Whittaker objects, we assume G is quasi-split and fix a Whittaker datum (B, ψ).
- $\check{G}$ denotes the Langlands dual group over Λ, and ${}^L G$ denotes the L-group.

3 The stack Bun_G and its stratification

3.1 Definition of Bun_G

Let $X_{S,E}$ denote the relative Fargues–Fontaine curve over a perfectoid base S. Following [1], the v-stack

$$\mathrm{Bun}_G := \mathrm{Bun}_G(X_{-/E})$$

classifies G-bundles on $X_{S,E}$ in families.

3.2 Harder–Narasimhan strata and local groups

The topological space $|\mathrm{Bun}_G|$ is stratified by the Kottwitz set $B(G)$. For $b \in B(G)$, the stratum Bun_G^b is controlled by a locally profinite group $J_b(E)$.

Proposition 3.1 (Strata and automorphism groups)**.** *For each $b \in B(G)$ there is a natural morphism $\mathrm{Bun}_G^b \to B\underline{J_b(E)}$ which is expected to be an equivalence on semistable points, and for basic b one has*

$$\mathrm{Bun}_G^b \simeq [*/J_b(E)]$$

in the sense of Artin v-stacks.

Sketch of proof. This is part of the geometric structure theory of Bun_G developed in [1, §III] and summarized in [2, §2]. The key input is the classification of G-bundles on the Fargues–Fontaine curve by $B(G)$ and the description of automorphism groups in terms of J_b.

3.3 Gluing along strata

The geometry of the inclusions $\mathrm{Bun}_G^{\leq b} \hookrightarrow \mathrm{Bun}_G$ and of the local charts $M_b \to \mathrm{Bun}_G$ constructed in [1] provides a basis for recollement arguments. For torsion coefficients, a detailed gluing formalism along Harder–Narasimhan strata is developed by Miles [5].

4 Automorphic categories: ℓ-adic sheaves on Bun_G

4.1 The category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$

Following [1], one defines the derived category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ of lisse Λ-sheaves on Bun_G (and variants with constructibility or torsion coefficients). This category admits:

- pullbacks and pushforwards along morphisms of v-stacks in the range needed for Hecke correspondences,
- Verdier duality, and
- compact objects corresponding to "constructible with quasi-compact support" subcategories.

Axiom 4.1 (Compact generation package)**.** The full subcategory of objects supported on a quasi-compact open substack is compactly generated, and the six operation formalism preserves compactness under the finiteness hypotheses proved in [1] and [4].

Comment. In the paper itself, Axiom 4.1 will be replaced by precise statements, extracted from [1, §IV–V] and [4].

5 The spectral stack of parameters and nilpotent singular support

5.1 Algebraic models for the stack of parameters

Fargues and Scholze construct a stack $\mathrm{LocSys}_{\check{G}}$ of Langlands parameters and establish its basic functoriality [1, 2]. An algebraic model for the moduli of Langlands parameters is constructed by Dat–Helm–Kurinczuk–Moss [3], producing a stack locally of finite type over $\mathbb{Z}[1/p]$.

Definition 5.1 (Spectral parameter stack)**.** Fix Λ as in Section 2. We write $\mathrm{LocSys}_{\check{G}}$ for the base change of the DHKM parameter stack to $\mathrm{Spec}(\Lambda)$, equipped with its derived enhancement.

5.2 Derived geometry and singular support

To formulate nilpotent singular support, one needs $\mathrm{LocSys}_{\check{G}}$ to be quasi-smooth in the derived sense (a local complete intersection). The DHKM result that the parameter stack is a reduced local complete intersection provides the expected input for this.

Axiom 5.2 (Quasi-smoothness)**.** The derived enhancement of $\mathrm{LocSys}_{\check{G}}$ is quasi-smooth, so that singular support for ind-coherent sheaves is defined in the sense of Arinkin–Gaitsgory [6].

5.3 Nilpotent cone and the category $\mathrm{Coh}_{\mathrm{Nilp}}$

Let $\mathcal{N}_{\check{G}} \subset \mathrm{Lie}(\check{G})^*$ denote the nilpotent cone, and consider the corresponding conical subset of the singular support space of $\mathrm{LocSys}_{\check{G}}$.

Definition 5.3 (Nilpotent support category)**.** Let $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ denote the full subcategory of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ consisting of objects with quasi-compact support and singular support contained in the nilpotent cone.

Remark 5.4. In the rational coefficient case $\Lambda = \mathbb{Q}_\ell$ (or $\mathbb{Q}_\ell[\sqrt{q}]$), [2] explains that the nilpotent support condition is expected to be automatic. In the integral case, it is a genuine restriction and is the local analogue of the nilpotent singular support condition in geometric Langlands [6].

6 Hecke correspondences, geometric Satake, and the spectral action

6.1 Hecke stacks and Hecke operators

The Hecke correspondence on Bun_G is defined using modifications of G-bundles at the untilt point. This yields for each $V \in \mathrm{Rep}(\check{G})$ a Hecke functor

$$T_V : \; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \;\longrightarrow\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

Sketch of construction. This is [1, §I.6]. The Hecke stack is built from the B_{dR}-affine Grassmannian and the Beauville–Laszlo uniformization, and the Hecke functors are defined by pullback, tensor, and pushforward along this correspondence.

6.2 Geometric Satake

Fargues and Scholze prove a geometric Satake equivalence over the Fargues–Fontaine curve, identifying the spherical Hecke category with $\mathrm{Rep}(\check{G})$ (with coefficients in Λ), compatibly with convolution.

Theorem 6.1 (Geometric Satake over the Fargues–Fontaine curve)**.** There is an equivalence of monoidal categories between the spherical Hecke category on the B_{dR}-affine Grassmannian and $\mathrm{Rep}(\check{G})$.

Sketch of proof. This is established in [1, §I.6]. The argument parallels geometric Satake for algebraic curves, with the untilt point playing the role of the point of modification.

6.3 The spectral action

The geometric Satake equivalence and the parameter stack $\mathrm{LocSys}_{\check{G}}$ together yield a canonical monoidal action

$$\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \;\curvearrowright\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

called the spectral action in [1, §I.10].

Proposition 6.2 (Compatibility of spectral and Hecke actions)**.** *For $V \in \mathrm{Rep}(\check{G}) \subset \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$, the induced endofunctor of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ coincides with the Hecke operator T_V.*

Sketch of proof. This is a formal consequence of geometric Satake and the definition of the spectral action [1, §I.10].

7 Whittaker sheaves and the Whittaker-generated subcategory

7.1 The Whittaker object

Assume G is quasi-split and fix a Whittaker datum (B, ψ) with unipotent radical U. Fargues and Scholze define a Whittaker object $\mathcal{W}_\psi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ as a geometric incarnation of compact induction from $U(E)$ with character ψ [2].

Definition 7.1 (Whittaker category)**.** Let $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ denote the smallest stable full subcategory of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ that contains $\mathcal{W}_\psi$, is closed under colimits, and is stable under the spectral action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$.

7.2 The comparison functor

Define a functor

$$\Phi : \ \mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \ \longrightarrow \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega, \qquad \mathcal{F} \ \longmapsto \ \mathcal{F} * \mathcal{W}_\psi.$$

Conjecture 1.1 asserts that Φ extends to an equivalence from $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$.

8 Parabolic functors and gluing from Levi subgroups

8.1 Geometric Eisenstein series and constant term

Let $P \subset G$ be a parabolic subgroup with Levi quotient M. One has maps of v-stacks

$$\mathrm{Bun}_M \xleftarrow{q} \mathrm{Bun}_P \xrightarrow{p} \mathrm{Bun}_G.$$

Hamann–Hansen–Scholze construct geometric Eisenstein and constant term functors [4]

$$\mathrm{Eis}_P := p_* q^! : \ D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda), \qquad \mathrm{CT}_P := q_* p^! : \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda),$$

prove finiteness theorems for these functors, and establish a geometric form of Bernstein's second adjointness.

Theorem 8.1 (Finiteness and adjointness for parabolic functors)**.** Geometric Eisenstein and constant term functors satisfy finiteness properties analogous to those of parabolic induction and Jacquet modules, and enjoy adjunction and second adjointness statements that are sufficiently strong to support gluing arguments.

Sketch of proof. This is the main content of [4]. The proof uses the geometry of moduli of parabolic bundles, continuity of gluing functors between Harder–Narasimhan strata, and a detailed analysis of the relevant correspondences.

8.2 Cuspidal and Eisenstein decompositions

A crucial structural output of [4] is a decomposition of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ into cuspidal and Eisenstein parts, analogous to the classical Bernstein decomposition in representation theory.

Proposition 8.2 (Cuspidal subcategory)**.** *Define $D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda)$ as the intersection of kernels of all constant term functors for proper parabolics. Then $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is generated under colimits by $D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda)$ and the essential images of Eisenstein series functors from Levi subgroups.*

Sketch of proof. This is proved in [4] as an application of their finiteness results and adjointness.

9 Proof strategy for the categorical geometrization conjecture

This section is the core blueprint: it explains how Conjecture 1.1 should follow from a sequence of intermediate steps that parallel Gaitsgory's proof strategy.

9.1 Compactness and extension from Perf to $\mathrm{Coh}_{\mathrm{Nilp}}$

Proposition 9.1 (Compactness of Whittaker translates)**.** *Let $\mathcal{F} \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have quasi-compact support. Then $\mathcal{F} * \mathcal{W}_\psi$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$.*

Sketch of proof. The compactness statement is the first non-formal obstacle. A proposed approach is:

(a) Use the compatibility of the spectral action with Hecke operators (Proposition 6.2) to reduce compactness to finiteness properties of Hecke correspondences.

(b) Use stratifications of Bun_G and the continuity and finiteness results of [4] to show that Hecke operators preserve compactness on objects supported on a bounded range of Harder–Narasimhan strata.

(c) Use that $\mathcal{F}$ has quasi-compact support on $\mathrm{LocSys}_{\check{G}}$ to obtain uniform bounds on the complexity of the corresponding Hecke operators.

Once this is established, one can extend Φ from Perf to $\mathrm{Coh}_{\mathrm{Nilp}}$ by compact generation and devissage.

9.2 Full faithfulness via endomorphisms and monadicity

Proposition 9.2 (Endomorphisms of the Whittaker generator)**.** *There is a canonical identification*

$$\mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(\mathcal{W}_\psi) \;\cong\; \Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}),$$

compatible with the action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$.

Sketch of proof. Fargues and Scholze construct a map from the spectral Bernstein center to the Bernstein center using the spectral action [1, 2]. The endomorphisms of $\mathcal{W}_\psi$ should recover the same center map. The identification is expected to follow by:

(a) relating $\mathrm{End}(\mathcal{W}_\psi)$ to the center of the compactly generated subcategory $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$,

(b) identifying the latter with functions on $\mathrm{LocSys}_{\check{G}}$ by the spectral action, and

(c) using algebraicity properties of $\mathrm{LocSys}_{\check{G}}$ from [3].

Proposition 9.3 (Barr–Beck reduction)**.** *Assume Propositions 9.1 and 9.2. Then the functor* $\Phi : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ *is fully faithful if the right adjoint of* Φ *is conservative.*

Sketch of proof. Once Φ preserves compact objects and admits a continuous right adjoint, one can apply a Barr–Beck type theorem to compare $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ with modules for the monad $\Phi^R \circ \Phi$. The identification in Proposition 9.2 is expected to identify this monad with the tautological monad on the spectral side, giving full faithfulness.

9.3 Essential surjectivity by parabolic gluing

Proposition 9.4 (Generation by the Whittaker object)**.** *The category* $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ *is generated under colimits by the essential image of* Φ*.*

Sketch of proof. The key point is that $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is by definition the smallest subcategory containing $\mathcal{W}_\psi$ and stable under the spectral action. Thus essential surjectivity reduces to showing that the spectral action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ on $\mathcal{W}_\psi$ is already large enough to generate the whole Whittaker subcategory. Parabolic gluing should be used to reduce this to:

(a) the analogous statement for Levi subgroups (induction on semisimple rank),

(b) the cuspidal case, where Whittaker objects are expected to detect blocks,

(c) compatibility of Φ with constant term and Eisenstein functors (Section 17).

9.4 Cuspidal blocks and multiplicity one

A final step in a Gaitsgory-style proof is a cuspidal uniqueness statement, serving as a local analogue of multiplicity one in geometric Langlands.

Conjecture 9.5 (Cuspidal multiplicity one for the Whittaker category)**.** For a cuspidal Langlands parameter ϕ, the fiber of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Q}_\ell)_\omega$ over ϕ is equivalent to $\mathrm{Perf}(\mathrm{Rep}_{\mathbb{Q}_\ell}(S_\phi))$, and the generic representation in the corresponding packet is unique with respect to the Whittaker datum.

Comment. This is stated in [2, §5] as a "toy model" that the full categorical conjecture refines. In a proof of Conjecture 1.1, Conjecture 9.5 is the natural place where one expects to use the geometry of local shtukas and the known construction of Hecke eigensheaves [1].

10 Consequences and further directions

Assuming Conjecture 1.1, one obtains:

- identification of stable centers and the spectral Bernstein center map [2],
- functoriality for L-morphisms realized by geometric kernels on $\mathrm{Bun}_H \times \mathrm{Bun}_G$ [2],
- structural decompositions of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ compatible with parabolic induction [4],
- a conceptual categorical formulation of local Langlands parameters in families, compatible with the moduli-theoretic construction of Langlands parameters [3].

11 Checklist of inputs and where they enter

This appendix will be expanded to a detailed dependency chart. At the current stage, the intended inputs are:

- [1]: definition of Bun_G, ℓ-adic sheaves, geometric Satake, spectral action, Whittaker object, and the parameter stack.
- [2]: formulation of the categorical conjecture and its consequences.
- [3]: algebraicity and local complete intersection properties of the parameter stack.
- [4]: finiteness and adjointness for Eisenstein series and constant term functors.
- [6]: singular support and the definition of nilpotent support categories.
- [7, 8, 9]: the proof architecture and monadicity patterns to be adapted.
- [5]: detailed gluing along Harder–Narasimhan strata for prime-to-p torsion coefficients.

12 The stack Bun_G and its stratification

This section recalls the geometric object on the automorphic side: the v-stack Bun_G of G-bundles on the Fargues–Fontaine curve, together with its Harder–Narasimhan stratification by the Kottwitz set $B(G)$. We will use three kinds of input from this geometry throughout the paper:

(i) the classification of geometric points of Bun_G by $B(G)$ and the description of strata in terms of groups $J_b(E)$;

(ii) boundedness statements ensuring that natural unions of strata are quasi-compact (needed for compactness and finiteness arguments);

(iii) recollement and gluing along Harder–Narasimhan strata (needed for inductive and parabolic gluing arguments).

All of these statements are proved or explained in [1, 2]; we include them here in a form convenient for later use.

12.1 The relative Fargues–Fontaine curve and G-bundles

Let E be a non-archimedean local field of residue characteristic p. For a perfectoid space S of characteristic p, there is a relative Fargues–Fontaine curve $X_{S,E}$ (and a relative adic space or diamond version), functorial in S. We will not reproduce the construction, but we emphasize two properties:

- for a geometric point $S = \mathrm{Spa}(C, C^+)$ with C an algebraically closed perfectoid field of characteristic p, the category of vector bundles on $X_{C,E}$ (and more generally G-bundles for reductive G) is *discrete up to isomorphism*, in the sense that it is classified by isocrystal-type invariants;
- the construction is functorial enough to define moduli of bundles in families over arbitrary bases S, producing a v-stack.

Definition 12.1 (The v-stack Bun_G)**.** Let G/E be a connected reductive group. The v-stack Bun_G is the functor on perfectoid spaces S (in characteristic p) given by the groupoid of G-bundles on $X_{S,E}$:

$$\mathrm{Bun}_G(S) \; := \; \{G\text{-bundles on } X_{S,E}\}.$$

Remark 12.2. All geometric constructions in [1] are formulated at the level of diamonds and v-stacks. In particular, Bun_G is an Artin v-stack locally of diamond type, and it admits a well-behaved theory of ℓ-adic lisse sheaves for $\ell \neq p$.

12.2 The Kottwitz set $B(G)$ and its structure

Fix an algebraic closure $\overline{E}$ of E and let $\breve{E}$ be the completion of the maximal unramified extension of E inside $\overline{E}$. Let σ denote the Frobenius automorphism of $\breve{E}/E$. Recall that $B(G)$ is the set of σ-conjugacy classes in $G(\breve{E})$:

$$B(G) \; := \; G(\breve{E})/\!\sim, \qquad b \sim b' \iff \exists g \in G(\breve{E}) \text{ with } b' = g^{-1} b \sigma(g).$$

The set $B(G)$ carries two fundamental invariants:

- the *Kottwitz invariant* $\kappa_G : B(G) \to \pi_1(G)_\Gamma$ (where $\Gamma = \mathrm{Gal}(\overline{E}/E)$);
- the *Newton point* $\nu : B(G) \to (X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q})^+$ for a choice of maximal torus T, landing in dominant rational cocharacters.

These maps are compatible with the partial order on $B(G)$ defined by ν (and equality of κ_G). We recall the terminology:

Definition 12.3 (Basic elements)**.** An element $b \in B(G)$ is *basic* if its Newton point is central (equivalently, if the associated G-bundle is semistable). We write $B(G)_{\text{basic}} \subset B(G)$ for the subset of basic elements.

Remark 12.4 (Boundedness and finiteness)*.* A recurring principle is that imposing bounds on ν and fixing κ_G cuts $B(G)$ down to a finite set. We will use this to reduce statements on Bun_G to statements on finite unions of strata.

12.3 Harder–Narasimhan stratification of Bun_G

Fargues and Scholze define a Harder–Narasimhan map from the topological space underlying Bun_G to $B(G)$, which on geometric points recovers the isomorphism class of the associated G-bundle [1].

Definition 12.5 (Harder–Narasimhan strata)**.** For $b \in B(G)$, let $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ denote the locally closed sub-v-stack consisting of those G-bundles whose Harder–Narasimhan invariant equals b. We write $\mathrm{Bun}_G^{\leq b}$ for the open sub-v-stack consisting of bundles whose Harder–Narasimhan invariant is $\leq b$ in the partial order on $B(G)$.

The key structural input is that each stratum is a classifying stack, controlled by a locally profinite group.

Definition 12.6 (The group J_b)**.** Fix $b \in G(\breve{E})$ representing a class in $B(G)$. Define an affine group scheme J_b over E by the functor

$$J_b(R) \; := \; \{g \in G(R \otimes_E \breve{E}) \mid g^{-1} b \sigma(g) = b\},$$

for E-algebras R. Its group of E-points $J_b(E)$ is a locally profinite group.

Proposition 12.7 (Description of strata as classifying stacks)**.** *For each $b \in B(G)$ there is a canonical equivalence of v-stacks*

$$\mathrm{Bun}_G^b \;\simeq\; B\underline{J_b(E)},$$

where $B\underline{J_b(E)}$ denotes the classifying v-stack of the constant sheaf of groups $J_b(E)$.

Sketch of proof. Choose a representative $b \in G(\breve{E})$. Fargues associates to b a G-bundle $\mathcal{E}_b$ on the Fargues–Fontaine curve, and the classification theorem asserts that every geometric G-bundle is isomorphic to a unique $\mathcal{E}_b$. The automorphism group of $\mathcal{E}_b$ is identified with $J_b(E)$. Twisting $\mathcal{E}_b$ by a $J_b(E)$-torsor produces a map $B\underline{J_b(E)} \to \mathrm{Bun}_G$, and one checks that its essential image is exactly the stratum Bun_G^b and that it induces an equivalence onto that stratum. All details are in [1, §III] and [2, §2]. □

Remark 12.8 (Semistable locus)*.* The semistable locus of Bun_G is the union of the basic strata:

$$\mathrm{Bun}_G^{\mathrm{ss}} \;:=\; \coprod_{b \in B(G)_{\mathrm{basic}}} \mathrm{Bun}_G^b.$$

In particular, if b is basic then J_b is an inner form of G and $J_b(E)$ is the group that governs the corresponding inner form block of representation theory.

12.4 Bounded open substacks and quasi-compactness

Most geometric arguments on Bun_G proceed by restricting to a quasi-compact open substack on which the relevant correspondences are proper or of finite cohomological dimension. The Harder–Narasimhan stratification provides a supply of such opens.

Definition 12.9 (Bounded opens)**.** Fix a dominant rational cocharacter μ (or, more generally, a finite subset of $B(G)$ stable under the partial order). Let $\mathrm{Bun}_G^{\le\mu} \subset \mathrm{Bun}_G$ denote the open sub-v-stack consisting of bundles whose Newton point is $\le \mu$ and whose Kottwitz invariant lies in the corresponding finite set. Equivalently, $\mathrm{Bun}_G^{\le\mu}$ is a finite union of strata Bun_G^b.

Proposition 12.10 (Quasi-compactness of bounded opens)**.** *The open sub-v-stack $\mathrm{Bun}_G^{\le\mu}$ is quasi-compact, and it is a finite disjoint union of classifying stacks $B\underline{J_b(E)}$.*

Sketch of proof. By the finiteness principle for $B(G)$ under bounded Newton point and fixed Kottwitz invariant, only finitely many b occur in $\mathrm{Bun}_G^{\le\mu}$. Each stratum $\mathrm{Bun}_G^b \simeq B\underline{J_b(E)}$ is quasi-compact (as a classifying v-stack of a locally profinite group), hence their finite union is quasi-compact. See [2, §2]. □

12.5 Recollement along Harder–Narasimhan strata

The stratification by $B(G)$ induces recollement patterns for categories of ℓ-adic sheaves. This is the geometric mechanism behind "gluing from strata" arguments on the automorphic side.

Proposition 12.11 (Recollement for prime-to-p torsion coefficients)**.** *Let Λ be a torsion ring of characteristic $\ell \neq p$. Let $j : \mathrm{Bun}_G^{\le\mu} \hookrightarrow \mathrm{Bun}_G$ be a bounded open immersion and let $i : \mathrm{Bun}_G \setminus \mathrm{Bun}_G^{\le\mu} \hookrightarrow \mathrm{Bun}_G$ be its complementary closed immersion. Then the derived category of lisse Λ-sheaves on Bun_G admits a recollement diagram with respect to (i, j). Moreover, on bounded opens $\mathrm{Bun}_G^{\le\mu}$ this recollement can be described by gluing sheaves along the Harder–Narasimhan strata.*

Sketch of proof. The gluing formalism along Harder–Narasimhan strata is developed in detail by Miles for prime-to-p torsion coefficients [5], building on the general sheaf theory on v-stacks in [1]. $\square$

Remark 12.12. For $\Lambda = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, one expects the same recollement statements after a suitable finiteness and completeness discussion. In the blueprint approach of this paper, we will systematically reduce compactness questions to bounded opens where the torsion theory and finiteness statements are available.

12.6 Strata as representation theory

The description $\mathrm{Bun}_G^b \simeq B\underline{J_b(E)}$ has an immediate representation-theoretic consequence: lisse ℓ-adic sheaves on a classifying stack are the same as smooth representations of the corresponding locally profinite group.

Proposition 12.13 (Sheaves on strata and smooth representations)**.** *Let H be a locally profinite group and let Λ be a ring in which p is invertible. Then the derived category of lisse Λ-sheaves on $B\underline{H}$ is equivalent to the derived category of smooth Λ-representations of H. In particular, for each $b \in B(G)$ one has a canonical equivalence*

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \;\simeq\; D\big(\mathrm{Rep}_\Lambda^\infty(J_b(E))\big).$$

Sketch of proof. A lisse sheaf on $B\underline{H}$ is a locally constant sheaf on the classifying topos, which amounts to a continuous action of H on a discrete Λ-module; this is equivalent to a smooth representation. The derived equivalence is obtained by passing to complexes. This is the dictionary used throughout [1]. $\square$

Remark 12.14 (Why Bun_G is still interesting)*.* Even though Bun_G is "discrete up to automorphisms" (a disjoint union of classifying stacks), the Hecke correspondences between different strata, and the functoriality of sheaves with respect to these correspondences, encode deep representation theory. All categorical phenomena in the conjecture are driven by these correspondences, not by moduli within a fixed stratum.

13 Automorphic categories: ℓ-adic sheaves on Bun_G

In this section we fix the automorphic category on the Fargues–Fontaine side: the derived category of lisse ℓ-adic sheaves on the v-stack Bun_G. We record the formal properties that will be used in later sections:

(i) a precise definition of $D_{\mathrm{lis}}(X, \Lambda)$ for Artin v-stacks X (in particular $X = \mathrm{Bun}_G$),

(ii) the functoriality needed for correspondences (pullback, pushforward, exceptional functors),

(iii) compact generation and compact objects with bounded support,

(iv) restriction to Harder–Narasimhan strata and the resulting representation-theoretic dictionary.

All of these structures are developed in [1] and used systematically in [2, 4]. We present them here in a way that is adapted to the Gaitsgory-style argument in Section 18.

13.1 Sheaves on diamonds and ℓ-adic coefficients

Let X be a locally spatial diamond. For a coefficient ring Λ with ℓ invertible on X (for us $\ell \neq p$), one has the derived ∞-category $D(X, \Lambda)$ of sheaves of Λ-modules on the pro-étale site of X (equivalently, on the v-site; see [1, §I.2–I.3]).

Definition 13.1 (Lisse sheaves on a diamond)**.** A sheaf $\mathcal{F}$ of Λ-modules on X is *lisse* if it is locally constant on the pro-étale site and its stalks are finite projective Λ-modules (for torsion Λ, "finite projective" means finite of constant rank). Write $D_{\mathrm{lis}}(X, \Lambda) \subset D(X, \Lambda)$ for the full subcategory consisting of complexes whose cohomology sheaves are lisse.

Remark 13.2 (Torsion, integral, and rational coefficients)*.* When $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, the category $D_{\mathrm{lis}}(X, \Lambda)$ is defined directly. For $\Lambda = \mathbb{Z}_\ell$ we adopt the usual derived ℓ-adic formalism:

$$D_{\mathrm{lis}}(X, \mathbb{Z}_\ell) \ := \ \varprojlim_n D_{\mathrm{lis}}(X, \mathbb{Z}/\ell^n\mathbb{Z}),$$

interpreted as an ℓ-adically complete derived category (as in [1, §I.2]). For $\Lambda = \mathbb{Q}_\ell$ one may define $D_{\mathrm{lis}}(X, \mathbb{Q}_\ell) := D_{\mathrm{lis}}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. All later constructions are compatible with these changes of coefficients.

13.2 Six functors for maps of diamonds

Let $f : X \to Y$ be a morphism of locally spatial diamonds. The sheaf theory provides:

- pullback $f^* : D(Y, \Lambda) \to D(X, \Lambda)$ and pushforward $f_* : D(X, \Lambda) \to D(Y, \Lambda)$;
- if f is compactifiable (for example partially proper in the sense of [1, §I.1]), an exceptional pushforward $f_! : D(X, \Lambda) \to D(Y, \Lambda)$;
- if f is representable in locally spatial diamonds and locally of finite cohomological dimension, an exceptional pullback $f^! : D(Y, \Lambda) \to D(X, \Lambda)$;
- tensor products, internal Hom, and Verdier duality on suitable subcategories.

We will only use these functors in the range where they are constructed and shown to satisfy the usual formal properties (base change, projection formula, adjunctions), as developed in [1, §I.2–I.5] and refined for parabolic correspondences in [4].

Proposition 13.3 (Functoriality needed for correspondences)**.** *Let $f : X \to Y$ be a morphism of locally spatial diamonds.*

(a) The functors f^* and f_* preserve D_{lis}.

(b) If f is compactifiable, then $f_!$ is defined on $D_{\mathrm{lis}}(X, \Lambda)$ and preserves D_{lis}.

(c) If f is representable in locally spatial diamonds and cohomologically smooth of relative dimension d, then $f^!$ is defined and satisfies

$$f^!(\mathcal{G}) \simeq f^*(\mathcal{G})(d)[2d]$$

on $D_{\mathrm{lis}}(Y, \Lambda)$ (with the usual Tate twist and shift convention).

Sketch of proof. These statements are standard in the ℓ-adic sheaf theory on diamonds for $\ell \neq p$ and are proved in the setup of [1]. Cohomological smoothness is the diamond analogue of smoothness that guarantees the existence and explicit form of $f^!$; it is discussed systematically in [1, §I.2]. □

13.3 From diamonds to Artin v-stacks

The stack Bun_G is an Artin v-stack locally of diamond type. One defines sheaf categories on such stacks by descent along a smooth atlas.

Definition 13.4 (Lisse sheaves on an Artin v-stack)**.** Let X be an Artin v-stack locally of diamond type. Choose a smooth surjective atlas $p : U \to X$ by a locally spatial diamond U. Let $U_\bullet$ denote the associated Cech nerve, a simplicial diamond. Define

$$D_{\mathrm{lis}}(X, \Lambda) \ := \ \varprojlim_{\Delta} D_{\mathrm{lis}}(U_\bullet, \Lambda),$$

the limit taken in stable presentable ∞-categories. This category is independent of the choice of atlas and has the expected functoriality with respect to morphisms of Artin v-stacks (see [1, §I.3–I.4]).

Remark 13.5. All categories in this paper are stable presentable ∞-categories. When we write an "equivalence of categories", we always mean an equivalence in this sense.

13.4 The automorphic category on Bun_G

Applying Definition 13.4 to $X = \mathrm{Bun}_G$ yields the automorphic category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. We will frequently work with objects supported on bounded opens in the sense of Definition 12.9.

Definition 13.6 (Subcategories with bounded support)**.** Let $j : \mathrm{Bun}_G^{\leq\mu} \hookrightarrow \mathrm{Bun}_G$ be a bounded open immersion. Define $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq\mu}$ to be the full subcategory of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ consisting of objects whose restriction to $\mathrm{Bun}_G \setminus \mathrm{Bun}_G^{\leq\mu}$ vanishes. Equivalently,

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq\mu} \ \simeq \ D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq\mu}, \Lambda)$$

via the restriction functor j^* with inverse given by extension by zero $j_!$.

Proposition 13.7 (Exhaustion by bounded opens)**.** *The v-stack* Bun_G *is the increasing union of bounded opens* $\mathrm{Bun}_G^{\leq\mu}$*, and the category* $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ *is the filtered colimit of the subcategories with bounded support:*

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \ \simeq \ \varinjlim_{\mu} D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq\mu}.$$

Sketch of proof. The union statement is a reformulation of the Harder–Narasimhan stratification: every point lies in some finite union of strata, hence in some bounded open. The colimit statement follows because lisse sheaves satisfy v-descent and extension by zero along open immersions is compatible with filtered unions of opens; see [1, §I.4]. □

13.5 Compact objects and compact generation

To run Barr–Beck type arguments later, we need a manageable supply of compact objects. For $\ell \neq p$ and for bounded opens, compact objects behave as expected.

Definition 13.8 (Compact objects with bounded support)**.** Let μ be a bound. An object $\mathcal{F} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq\mu}$ is called *compact* if $\mathrm{Hom}(\mathcal{F}, -)$ commutes with filtered colimits in the subcategory $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq\mu}$. Write $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{c}}_{\leq\mu}$ for the full subcategory of compact objects.

Proposition 13.9 (Compact generation on bounded opens)**.** *For each bound μ, the category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq\mu}$ is compactly generated. Moreover, for torsion coefficients $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, the compact objects are precisely the objects whose cohomology sheaves are lisse and constructible and whose support is contained in a finite union of Harder–Narasimhan strata inside $\mathrm{Bun}_G^{\leq\mu}$.*

Sketch of proof. For torsion coefficients, compact generation for lisse sheaves on quasi-compact Artin v-stacks follows from the general formalism in [1, §I.4–I.5]. One reduces to the case of a finite union of strata, where each stratum is a classifying stack of a locally profinite group (Proposition 12.7) and compactness can be checked on the corresponding representation category. The constructibility characterization is the diamond analogue of the usual fact that, for a quasi-compact and quasi-separated space, compact objects in the derived category of sheaves are the constructible ones. □

Remark 13.10 (Integral and rational coefficients)*.* For $\Lambda = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, the correct notion of compactness involves derived ℓ-adic completeness and boundedness conditions. In practice, we will reduce compactness statements to the torsion level and then pass to limits; this is the strategy used in [4].

13.6 Restriction to strata and the representation-theoretic dictionary

Let $i_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ be the locally closed immersion of a Harder–Narasimhan stratum. By Proposition 12.7 we may identify Bun_G^b with the classifying v-stack $B\underline{J_b(E)}$.

Proposition 13.11 (Restriction to strata)**.** *For each $b \in B(G)$, restriction along i_b defines functors*

$$i_b^*,\ i_b^! :\ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \ \longrightarrow\ D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda),$$

and there is a canonical identification

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda)\ \simeq\ D\big(\mathrm{Rep}_\Lambda^\infty(J_b(E))\big),$$

where $\mathrm{Rep}_\Lambda^\infty(J_b(E))$ denotes the abelian category of smooth Λ-representations of $J_b(E)$.

Sketch of proof. The existence of i_b^* and $i_b^!$ follows from the six functor formalism on Artin v-stacks (Definition 13.4 and Proposition 13.3). The identification with smooth representations is Proposition 12.13. □

Remark 13.12 (Why global geometry matters)*.* If Bun_G were merely a disjoint union of its strata with no additional structure, then $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ would be a product of representation categories of the groups $J_b(E)$. The geometric content of the Fargues–Scholze program lies in the *correspondences* between strata (Hecke operators, Eisenstein series, constant term), which couple these representation-theoretic fibers. This coupling is what makes a geometric Langlands-type spectral description possible.

13.7 Recollement and gluing along Harder–Narasimhan strata

For later inductive arguments we will need to glue objects from their restrictions to bounded unions of strata.

Proposition 13.13 (Gluing on bounded opens for torsion coefficients)**.** *Assume Λ is torsion of characteristic $\ell \neq p$. Let $\mathrm{Bun}_G^{\leq\mu}$ be a bounded open. Then objects of $D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq\mu}, \Lambda)$ can be reconstructed from their restrictions to the finitely many strata contained in $\mathrm{Bun}_G^{\leq\mu}$, together with the natural gluing data along the closure relations. In particular, there is a recollement formalism for the filtration by closed unions of strata.*

Sketch of proof. This is proved in detail by Miles [5], building on the stratified geometry of Bun_G and the torsion ℓ-adic formalism for v-stacks. The main point is that the closure relations among strata inside a bounded open are finite and admit cohomological control that allows gluing in the derived category. □

Remark 13.14. Later sections will use Proposition 13.13 to reduce compactness and generation statements to calculations on individual strata, where representation theory can be applied.

14 The spectral stack of parameters and nilpotent singular support

This section fixes the spectral geometric object that will control the automorphic category via the spectral action: the stack $\mathrm{LocSys}_{\check{G}}$ of ℓ-adic Langlands parameters, together with the spectral category $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ of coherent sheaves with nilpotent singular support.

There are two complementary viewpoints on $\mathrm{LocSys}_{\check{G}}$:

(i) the v-stack of parameters constructed by Fargues and Scholze in the course of defining the spectral action [1, 2];

(ii) an algebraic model for the moduli of parameters constructed by Dat, Helm, Kurinczuk, and Moss [3].

For the blueprint of this paper, it is convenient to work with the algebraic model of [3] (because it supports the use of derived singular support), while keeping the link to the spectral action of [1] in mind.

14.1 Weil–Deligne parameters and Langlands parameters

Let W_E denote the Weil group of E, and let $I_E \subset W_E$ be the inertia subgroup. Write ${}^LG$ for the L-group of G, defined over Λ:

$$ {}^LG \;=\; \check{G} \rtimes W_E. $$

We recall one standard model for Langlands parameters at $\ell \neq p$.

Definition 14.1 (Weil–Deligne parameter)**.** A *Weil–Deligne parameter* (with coefficients in a Λ-algebra A) is a pair (r, N) where

- $r : W_E \to {}^LG(A)$ is a continuous homomorphism such that the composite $W_E \xrightarrow{r} {}^LG(A) \to W_E$ is the identity and $r(I_E)$ has finite image;
- $N \in \mathrm{Lie}(\check{G}) \otimes_\Lambda A$ is nilpotent and satisfies the usual equivariance relation $r(w)Nr(w)^{-1} = |w|\, N$ for $w \in W_E$.

Two parameters are identified if they are conjugate by $\check{G}(A)$.

Remark 14.2. There are equivalent formulations (for example, via homomorphisms $W_E \times \mathrm{SL}_2 \to {}^LG$). For the purposes of this paper, the particular model is not essential, provided one has an algebraic moduli problem with good finiteness properties.

14.2 The Dat–Helm–Kurinczuk–Moss moduli stack

The foundational input on the spectral side is that Weil–Deligne parameters form an algebraic stack with good geometric properties.

Theorem 14.3 (Dat–Helm–Kurinczuk–Moss)**.** There exists an algebraic stack $\mathcal{X}_{\check{G}}$ locally of finite type over $\mathrm{Spec}(\mathbb{Z}[1/p])$ whose A-points classify $\check{G}$-valued Langlands parameters (equivalently, Weil–Deligne parameters) with coefficients in A. Moreover, $\mathcal{X}_{\check{G}}$ has strong finiteness properties (in particular, it is a reduced local complete intersection stack in the sense of [3]).

Proof sketch. This is the main result of [3]. The construction is by a careful algebraization of the representation-theoretic moduli problem, together with a deformation-theoretic analysis that produces the local complete intersection structure. □

Definition 14.4 (Spectral parameter stack)**.** Fix Λ as in Section 2. We define

$$\mathrm{LocSys}_{\check{G}} \;:=\; \mathcal{X}_{\check{G}} \times_{\mathrm{Spec}(\mathbb{Z}[1/p])} \mathrm{Spec}(\Lambda).$$

We will freely use the same notation for a chosen derived enhancement of this stack (Definition 14.6 below).

Remark 14.5 (Comparison with Fargues–Scholze)*.* Fargues and Scholze construct a stack of ℓ-adic parameters and show that $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ acts on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ [1]. In this paper we view Definition 14.4 as an algebraic incarnation of the same spectral stack, suitable for the use of singular support. Making the comparison between these constructions precise is an important foundational point, but it is not needed for the formal argument template presented here.

14.3 Derived enhancement and deformation theory

Singular support for ind-coherent sheaves requires a quasi-smooth derived stack. We therefore fix a derived enhancement of $\mathrm{LocSys}_{\check{G}}$ compatible with the local complete intersection structure.

Definition 14.6 (Derived enhancement)**.** Let $\mathrm{LocSys}^{\mathrm{cl}}_{\check{G}}$ denote the classical stack of Definition 14.4. A *derived enhancement* of $\mathrm{LocSys}^{\mathrm{cl}}_{\check{G}}$ is a derived stack $\mathrm{LocSys}_{\check{G}}$ whose truncation is $\mathrm{LocSys}^{\mathrm{cl}}_{\check{G}}$ and whose cotangent complex is perfect of Tor-amplitude in $[-1, 0]$ (equivalently, $\mathrm{LocSys}_{\check{G}}$ is quasi-smooth).

Axiom 14.7 (Quasi-smoothness)**.** We fix a quasi-smooth derived enhancement of $\mathrm{LocSys}^{\mathrm{cl}}_{\check{G}}$.

Remark 14.8 (Why this is reasonable)*.* Theorem 14.3 asserts that $\mathrm{LocSys}^{\mathrm{cl}}_{\check{G}}$ is a local complete intersection stack. In derived algebraic geometry, local complete intersection classical stacks admit natural quasi-smooth derived enhancements, and the deformation theory at a point is governed by a two-term complex. In later versions of this paper, Axiom 14.7 will be replaced by a precise reference to the derived enhancement implicit in the local complete intersection statement of [3].

At a point $\phi \in \mathrm{LocSys}_{\check{G}}$ represented by a parameter, the deformation theory is controlled by group cohomology with coefficients in the adjoint local system $\mathrm{ad}(\phi)$.

Proposition 14.9 (Tangent complex at a parameter)**.** *Let $\phi \in \mathrm{LocSys}_{\check{G}}$ be a geometric point corresponding to a Langlands parameter. Then the shifted tangent complex of $\mathrm{LocSys}_{\check{G}}$ at ϕ is canonically identified with a complex computing continuous group cohomology of W_E with coefficients in $\mathrm{ad}(\phi)$:*

$$\mathbb{T}_{\mathrm{LocSys}_{\check{G}},\phi} \;\simeq\; R\Gamma\big(W_E, \mathrm{ad}(\phi)\big)[1],$$

and similarly the cotangent complex is given (up to the same shift) by $R\Gamma(W_E, \mathrm{ad}(\phi)^\vee)$. In particular:

- $H^{-1}(\mathbb{T}_{\mathrm{LocSys}_{\check{G}},\phi})$ *is the Lie algebra of the infinitesimal automorphism group, that is, the Lie algebra of the centralizer of* ϕ*;*
- $H^0(\mathbb{T}_{\mathrm{LocSys}_{\check{G}},\phi})$ *controls first-order deformations of* ϕ*;*
- $H^1(\mathbb{T}_{\mathrm{LocSys}_{\check{G}},\phi})$ *controls obstruction classes.*

Proof sketch. This is the standard deformation theory of representation stacks: the deformation complex is the continuous cochain complex of W_E with coefficients in the adjoint representation associated to ϕ, shifted by one. In the algebraic setting of [3], this deformation theory is built into the construction of the moduli stack and its local complete intersection structure. □

14.4 Ind-coherent sheaves and singular support

Let X be a quasi-smooth derived stack locally of finite type over $\mathrm{Spec}(\Lambda)$. Arinkin and Gaitsgory attach to X a classical stack $\mathrm{Sing}(X)$, called the *singularity stack* (or the stack of singularities), and define for each $\mathcal{F} \in \mathrm{IndCoh}(X)$ its *singular support*

$$\mathrm{SS}(\mathcal{F}) \ \subset\ \mathrm{Sing}(X),$$

a conical Zariski-closed subset [6].

Definition 14.10 (Singularity stack)**.** Let X be quasi-smooth. Write X^{cl} for its classical truncation. The singularity stack $\mathrm{Sing}(X)$ is the relative spectrum

$$\mathrm{Sing}(X) \ :=\ \underline{\mathrm{Spec}}_{X^{\mathrm{cl}}}\Big(\mathrm{Sym}_{\mathcal{O}_{X^{\mathrm{cl}}}}\big(H^1(\mathbb{T}_X)\big)\Big),$$

which is a classical conical stack over X^{cl}.

Remark 14.11. For $X = \mathrm{LocSys}_{\check{G}}$, Proposition 14.9 identifies the fibers of $H^1(\mathbb{T}_X)$ with obstruction spaces for deformations of Langlands parameters. Singular support measures in which obstruction directions an ind-coherent sheaf is allowed to have singularities.

14.5 The nilpotent cone and nilpotent singular support

Let $\mathcal{N} \subset \mathrm{Lie}(\check{G})^*$ denote the nilpotent cone, viewed as a conical $\check{G}$-stable closed subset. Following [6] and the discussion in [2, §6], the nilpotent singular support condition for $\mathrm{LocSys}_{\check{G}}$ is defined by pulling back $\mathcal{N}$ along the natural map from the singularity stack to the adjoint quotient.

Axiom 14.12 (Nilpotent cone map)**.** There exists a canonical morphism of conical stacks

$$\chi:\ \mathrm{Sing}(\mathrm{LocSys}_{\check{G}}) \ \longrightarrow\ [\mathrm{Lie}(\check{G})^*/\check{G}],$$

functorial in G, such that the nilpotent cone condition is defined by the inverse image

$$\mathrm{Nilp}(\mathrm{LocSys}_{\check{G}}) \ :=\ \chi^{-1}\big([\mathcal{N}/\check{G}]\big) \ \subset\ \mathrm{Sing}(\mathrm{LocSys}_{\check{G}}).$$

Remark 14.13. For global geometric Langlands on a curve, the morphism χ is the (global) Hitchin map on the singularity stack, and nilpotent singular support is the condition that the corresponding Higgs field is nilpotent. In the present local setting, Axiom 14.12 should be viewed as the direct analogue: it is the mechanism that cuts the spectral category down to the "Arthur-nilpotent" range required by Whittaker and Eisenstein compatibility [6, 2].

14.6 The category $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$

We now define the spectral category that is predicted to match the Whittaker-generated automorphic category.

Definition 14.14 (Ind-coherent sheaves with nilpotent singular support)**.** Let X be quasi-smooth, and let $\mathcal{Y} \subset \mathrm{Sing}(X)$ be a closed conical substack. Define $\mathrm{IndCoh}_{\mathcal{Y}}(X) \subset \mathrm{IndCoh}(X)$ to be the full subcategory of objects whose singular support is contained in $\mathcal{Y}$:

$$\mathrm{IndCoh}_{\mathcal{Y}}(X) \ := \ \{\mathcal{F} \in \mathrm{IndCoh}(X) \mid \mathrm{SS}(\mathcal{F}) \subset \mathcal{Y}\}.$$

For $X = \mathrm{LocSys}_{\check{G}}$ and $\mathcal{Y} = \mathrm{Nilp}(\mathrm{LocSys}_{\check{G}})$ we write $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$.

Definition 14.15 (Coherent objects with nilpotent singular support)**.** Let $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ be the full subcategory of compact objects in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ (equivalently, the coherent objects of ind-coherent sheaves with nilpotent singular support).

Proposition 14.16 (Basic properties)**.** *Assume Axioms 14.7 and 14.12.*

(a) The subcategory $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ *is stable under colimits and under the action of* $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ *by tensor product.*

(b) Every perfect complex on $\mathrm{LocSys}_{\check{G}}$ *has singular support contained in the zero section, hence belongs to* $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$*.*

(c) If $\Lambda = \mathbb{Q}_\ell$ *(or* $\mathbb{Q}_\ell[\sqrt{q}]$*), then the nilpotent singular support condition is expected to be automatic, in the sense that* $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) = \mathrm{Coh}(\mathrm{LocSys}_{\check{G}})$*.*

Proof sketch. Parts (a) and (b) are general properties of singular support for ind-coherent sheaves on quasi-smooth stacks [6]. The expectation in (c) is explained in [2, §6]: over $\mathbb{Q}_\ell$ the integral correction imposed by nilpotent singular support should disappear. □

Remark 14.17 (Why the integral nilpotent condition matters). Even if (c) holds, the integral category $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ is the natural object for categorical local Langlands: it refines the spectral action integrally and keeps track of congruences and integral structures that are invisible over $\mathbb{Q}_\ell$. This parallels the role of nilpotent singular support in integral formulations of geometric Langlands.

15 Hecke correspondences, geometric Satake, and the spectral action

In this section we recall the geometric Satake formalism over the Fargues–Fontaine curve and the resulting Hecke and spectral actions on the automorphic category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. This is the principal mechanism that couples the different Harder–Narasimhan strata of Bun_G and makes the category sensitive to Langlands parameters.

There are three layers of structure:

(i) the *Hecke correspondence* on Bun_G defined using modifications at the untilt point;

(ii) the *spherical Hecke category* (the Satake category) living on the B^+_{dR}-affine Grassmannian, together with the geometric Satake equivalence;

(iii) the *spectral action* of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, which extends the Hecke action and will be used to define the comparison functor in Section 16.

15.1 The B_{dR}^{+}-affine Grassmannian

Let C be an algebraically closed complete extension of $\breve{E}$. Write $B_{\mathrm{dR}}^{+}(C)$ and $B_{\mathrm{dR}}(C)$ for Fontaine's de Rham period rings, and let $B_{\mathrm{dR}}^{+}(C) \subset B_{\mathrm{dR}}(C)$ denote the canonical inclusion. The B_{dR}^{+}-affine Grassmannian of G is the v-sheaf (in fact a diamond) whose C-points are the double coset space

$$\mathrm{Gr}_G(C) \ := \ G\big(B_{\mathrm{dR}}(C)\big)/G\big(B_{\mathrm{dR}}^{+}(C)\big),$$

and which is functorial in C. Equivalently, Gr_G classifies modifications of the trivial G-bundle on the punctured untilt disc with a trivialization on the formal disc.

Definition 15.1 (B_{dR}^{+}-affine Grassmannian)**.** Let Gr_G denote the B_{dR}^{+}-affine Grassmannian diamond of G over $\mathrm{Spa}(\breve{E})$ defined in [1, §I.3–I.6].

Remark 15.2*.* The diamond Gr_G admits a stratification by Schubert cells indexed by dominant cocharacters. Many geometric features formally parallel those of the usual affine Grassmannian over a curve, but the ambient geometry is mixed characteristic and the stratification is described in the language of diamonds.

15.2 The Hecke correspondence on Bun_G

Fix a untilt point on the relative Fargues–Fontaine curve (equivalently, work with the global version of the Hecke correspondence defined in [1, §I.6]). Informally, the Hecke correspondence parameterizes pairs of G-bundles together with an isomorphism away from the untilt point. Concretely:

Definition 15.3 (Hecke stack)**.** Let Hecke_G be the v-stack classifying triples $(\mathcal{E}_1, \mathcal{E}_2, \beta)$ where $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}_G$ and β is an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ over the complement of the untilt point (equivalently, over the punctured formal disc). There are natural morphisms

$$p_1, p_2 : \mathrm{Hecke}_G \to \mathrm{Bun}_G, \qquad p_i(\mathcal{E}_1, \mathcal{E}_2, \beta) = \mathcal{E}_i.$$

The morphism $(p_1, p_2) : \mathrm{Hecke}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ is representable in diamonds and is locally modeled on the correspondence induced by Gr_G.

Remark 15.4 (Bounded Hecke correspondences)*.* For a dominant cocharacter μ, one defines the closed sub-v-stack $\mathrm{Hecke}_G^{\leq \mu} \subset \mathrm{Hecke}_G$ cut out by the condition that the relative position of the modification lies in the Schubert closure of μ. These bounded Hecke correspondences are the geometric inputs used to define Hecke operators attached to representations of $\check{G}$.

15.3 The spherical Hecke category and convolution

Let Λ be as in Section 2, with $\ell \neq p$. The geometric Satake formalism uses an ℓ-adic sheaf category on Gr_G that is equivariant under the loop group L^+G (defined using B_{dR}^{+}). Fargues and Scholze define a stable monoidal category $\mathrm{Sat}_{G,\Lambda}$ (the spherical Hecke category) whose monoidal structure is given by convolution.

Definition 15.5 (Spherical Hecke category)**.** Let $\mathrm{Sat}_{G,\Lambda}$ denote the full subcategory of $D_{\mathrm{lis}}(\mathrm{Gr}_G, \Lambda)$ consisting of L^+G-equivariant objects satisfying the finiteness conditions that make convolution well-defined. The convolution product $\star$ is defined using the usual correspondence

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{\mathrm{pr}} \widetilde{\mathrm{Gr}}_G \xrightarrow{m} \mathrm{Gr}_G$$

and is associative up to canonical equivalence.

Remark 15.6. For the blueprint of this paper it is not important whether one formulates $\mathrm{Sat}_{G,\Lambda}$ using perverse sheaves, constructible complexes, or the larger lisse derived category with suitable compactness constraints. What matters is that:

- convolution is defined on a monoidal subcategory,
- the unit is the delta sheaf at the base point,
- the geometric Satake equivalence identifies this monoidal category with $\mathrm{Rep}(\check{G})$.

All of this is established in [1, §I.6].

15.4 Geometric Satake over the Fargues–Fontaine curve

Theorem 15.7 (Fargues–Scholze geometric Satake)**.** There is a canonical equivalence of symmetric monoidal categories

$$\mathrm{Sat}_{G,\Lambda} \xrightarrow{\sim} \mathrm{Rep}_{\Lambda}(\check{G}),$$

compatible with standard normalizations (Tate twists and cohomological shifts) as in [1, §I.6] and [2, §4].

Proof sketch. This is proved in [1, §I.6]. The argument follows the geometric Satake strategy for a curve, replacing the usual affine Grassmannian by the B_{dR}^{+}-affine Grassmannian and using the untilt point to define the local geometry of modifications. The key inputs are:

- the stratification of Gr_G by Schubert cells and the geometry of their closures,
- semismallness and purity properties needed for the Tannakian reconstruction,
- compatibility with convolution and duality.

□

15.5 Hecke operators on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$

The Hecke stack Hecke_G produces an action of the spherical Hecke category on Bun_G by integral transforms.

Definition 15.8 (Hecke action of the Satake category)**.** Let $\mathcal{S} \in \mathrm{Sat}_{G,\Lambda}$ and let $\mathcal{F} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. Define an object $T_{\mathcal{S}}(\mathcal{F}) \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ by the correspondence

$$T_{\mathcal{S}}(\mathcal{F}) \;:=\; (p_2)_!\Big(p_1^*(\mathcal{F}) \otimes \mathcal{S}_{\mathrm{Hecke}}\Big),$$

where $\mathcal{S}_{\mathrm{Hecke}}$ is the pullback of $\mathcal{S}$ to Hecke_G along the local model map $\mathrm{Hecke}_G \to \mathrm{Gr}_G$ (as in [1, §I.6]).

Proposition 15.9 (Convolution compatibility)**.** *The assignment $\mathcal{S} \mapsto T_{\mathcal{S}}$ defines a monoidal functor*

$$\mathrm{Sat}_{G,\Lambda} \longrightarrow \mathrm{End}\big(D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)\big),$$

where the target is the monoidal category of colimit-preserving endofunctors under composition. Under the geometric Satake equivalence (Theorem 15.7), this recovers Hecke operators indexed by representations of $\check{G}$.

Proof sketch. This is the standard convolution-of-kernels formalism, once one knows the relevant correspondences are compactifiable and satisfy base change and projection formula statements. The necessary finiteness for these correspondences is established in [1, §I.6]. $\square$

Definition 15.10 (Hecke operators indexed by $\mathrm{Rep}(\check{G})$)**.** For $V \in \mathrm{Rep}_\Lambda(\check{G})$, let $\mathcal{S}_V \in \mathrm{Sat}_{G,\Lambda}$ be the corresponding Satake object. Define the Hecke operator

$$T_V \ := \ T_{\mathcal{S}_V} : \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

15.6 The universal parameter and perfect complexes on $\mathrm{LocSys}_{\check{G}}$

On the spectral stack $\mathrm{LocSys}_{\check{G}}$ there is a tautological "universal Langlands parameter" in the sense that the moduli interpretation produces a universal W_E-representation with values in ${}^L G$. Applying a representation $V \in \mathrm{Rep}(\check{G})$ produces a vector bundle (perfect complex) on $\mathrm{LocSys}_{\check{G}}$.

Definition 15.11 (The Satake-to-spectral functor)**.** Let $\mathcal{V}$ denote the vector bundle on $\mathrm{LocSys}_{\check{G}}$ associated to $V \in \mathrm{Rep}_\Lambda(\check{G})$ by applying V to the universal parameter. This gives a symmetric monoidal functor

$$\mathrm{Rep}_\Lambda(\check{G}) \ \longrightarrow \ \mathrm{Perf}(\mathrm{LocSys}_{\check{G}}), \qquad V \longmapsto \mathcal{V}.$$

Remark 15.12*.* The precise construction depends on which model of $\mathrm{LocSys}_{\check{G}}$ one uses (the v-stack model in [1] or the algebraic model in [3]). In either case, the point is that $\mathrm{LocSys}_{\check{G}}$ is a moduli stack of parameters, hence carries the tautological parameter.

15.7 The spectral action

Fargues and Scholze construct a canonical monoidal action

$$\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \ \curvearrowright \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda),$$

called the *spectral action*. It is uniquely characterized (up to contractible choices) by the requirement that it extends the Hecke action in the sense of Definition 15.11 and Definition 15.10.

Theorem 15.13 (Existence of the spectral action)**.** There exists a canonical symmetric monoidal functor

$$\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \ \longrightarrow \ \mathrm{End}\big(D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)\big)$$

such that, for every $V \in \mathrm{Rep}_\Lambda(\check{G}) \subset \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$, the induced endofunctor is canonically equivalent to the Hecke operator T_V.

Proof sketch. This is constructed in [1, §I.10]. The key input is that Hecke operators satisfy strong compatibilities (factorization and excursion relations) which allow one to extend the assignment $V \mapsto T_V$ from representations to perfect complexes on the parameter stack. One may view this as a categorified form of the fact that excursion operators generate the spectral Bernstein center. $\square$

Proposition 15.14 (Compatibility with Hecke operators)**.** *Let $\mathcal{V} \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ be the vector bundle attached to $V \in \mathrm{Rep}_\Lambda(\check{G})$. Then the action of $\mathcal{V}$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is canonically equivalent to T_V.*

Proof sketch. This is part of the construction of the spectral action and is stated explicitly in [1, §I.10]. $\square$

15.8 Restriction to strata and recovery of the usual spherical Hecke action

Let $b \in B(G)$ and identify $\mathrm{Bun}_G^b \simeq B\underline{J_b(E)}$ (Proposition 12.7). Restriction to the stratum identifies $D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda)$ with smooth representations of $J_b(E)$ (Proposition 13.11). Under this identification, bounded Hecke correspondences recover the usual spherical Hecke operators acting on representation categories.

Proposition 15.15 (Hecke operators on basic strata)**.** *Assume b is basic. Under the equivalence $D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \simeq D(\mathrm{Rep}_\Lambda^\infty(J_b(E)))$, the restriction of T_V to Bun_G^b coincides with the usual (categorified) Hecke operator associated to V acting on smooth representations of the inner form $J_b(E)$.*

Proof sketch. For basic b, the stratum is a classifying stack and the Hecke correspondence restricts to the usual Hecke correspondence for the locally profinite group $J_b(E)$. This is explained in [1, §I.7–I.8], where the restriction of the geometrization functor to basic strata is identified with compact induction and the usual convolution action. □

15.9 A first finiteness principle for later use

Later sections will require that Hecke operators preserve compactness on bounded opens, and that they interact well with the Harder–Narasimhan stratification.

Proposition 15.16 (Boundedness of Hecke operators on bounded opens)**.** *Fix a bound μ on Bun_G and fix $V \in \mathrm{Rep}_\Lambda(\check{G})$. Then there exists a bound μ', depending only on μ and V, such that*

$$T_V\left(D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq \mu}\right) \ \subset \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq \mu'}.$$

In particular, Hecke operators preserve the filtered colimit presentation of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ by bounded support subcategories (Proposition 13.7).

Proof sketch. The modification type corresponding to V is bounded by a dominant cocharacter $\nu(V)$. Applying a bounded modification to a bundle with Harder–Narasimhan polygon bounded by μ yields a bundle with Harder–Narasimhan polygon bounded by a computable function of μ and $\nu(V)$. This is the geometric analogue of the fact that spherical Hecke operators move one between finitely many Bernstein components. The required boundedness statements are discussed in [1, §III] and used in [4]. □

Remark 15.17. Proposition 15.16 is one of the places where the discreteness of $B(G)$ and the finiteness of bounded subsets is essential: it ensures that Hecke operators cannot "run off to infinity" in Bun_G when restricted to objects with bounded support.

16 The Whittaker sheaf and the Whittaker-generated subcategory

The role of this section is to isolate the *Whittaker generator* on the automorphic side and to define the Whittaker-generated subcategory

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega \ \subset \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

that appears in the categorical geometrization conjecture of Fargues and Scholze. This is the precise analogue of the use of the Whittaker category in geometric Langlands, and it is the entry point for the Gaitsgory-style strategy: one uses a Whittaker object to define a comparison functor from the spectral category and then proves that it is fully faithful and essentially surjective by combining Hecke theory, constant term and Eisenstein functors, and gluing.

Throughout this section we assume that G is *quasi-split* over E.

16.1 Whittaker data

Fix a Borel subgroup $B \subset G$ defined over E and let $U \subset B$ be its unipotent radical.

Definition 16.1 (Whittaker datum)**.** A *Whittaker datum* for G (over E) is a pair (B, ψ) where $B \subset G$ is a Borel subgroup and

$$\psi : U(E) \longrightarrow \Lambda^\times$$

is a *generic* smooth character of the locally profinite group $U(E)$.

Definition 16.2 (Generic character)**.** Let Δ be the set of simple roots determined by B (and a maximal torus $T \subset B$). A smooth character $\psi : U(E) \to \Lambda^\times$ is called *generic* if, for every simple root $\alpha \in \Delta$, the restriction of ψ to the corresponding root subgroup $U_\alpha(E) \subset U(E)$ is nontrivial.

Remark 16.3 (Coefficient ring for ψ)*.* In practice, to arrange the existence of a generic character with values in $\Lambda^\times$ one often enlarges Λ (for example, by passing to the ring of integers of a sufficiently large algebraic extension of $\mathbb{Q}_\ell$, as in [1, Conjecture I.10.2]). For the purposes of this blueprint, we fix such a coefficient ring once and for all and suppress it from the notation.

16.2 The basic stratum and the Whittaker representation

Let $1 \in B(G)$ denote the neutral σ-conjugacy class; it corresponds to the trivial isocrystal and, under Fargues' classification, to the *trivial* G-bundle on the Fargues–Fontaine curve. Let $\mathrm{Bun}_G^1 \subset \mathrm{Bun}_G$ be the corresponding Harder–Narasimhan stratum and let

$$i_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$$

denote the locally closed immersion.

Proposition 16.4 (The neutral stratum is a classifying stack)**.** *There is a canonical equivalence of v-stacks*

$$\mathrm{Bun}_G^1 \;\simeq\; B\underline{G(E)}.$$

Consequently, there is a canonical identification

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^1, \Lambda) \;\simeq\; D\big(\mathrm{Rep}_\Lambda^\infty(G(E))\big),$$

the derived category of smooth Λ-representations of $G(E)$.

Sketch of proof. The equivalence $\mathrm{Bun}_G^1 \simeq B\underline{G(E)}$ is proved in [1, §III.4]. The identification of lisse sheaves on a classifying v-stack with smooth representations is part of the general formalism of [1, §V.1] and was recalled in Proposition 12.13. □

Now fix Whittaker data (B, ψ) as above. Let $\text{c-Ind}_{U(E)}^{G(E)}\psi$ denote the usual compact induction of smooth representations.

Definition 16.5 (The Whittaker representation)**.** Let $\mathcal{W}_\psi^{\mathrm{rep}}$ be the smooth Λ-representation of $G(E)$ given by

$$\mathcal{W}_\psi^{\mathrm{rep}} \;:=\; \text{c-Ind}_{U(E)}^{G(E)}\psi.$$

Via Proposition 16.4, we view $\mathcal{W}_\psi^{\mathrm{rep}}$ also as an object

$$\mathcal{W}_\psi^{\mathrm{str}} \;\in\; D_{\mathrm{lis}}(\mathrm{Bun}_G^1, \Lambda).$$

16.3 Definition of the Whittaker sheaf on Bun_G

The Whittaker sheaf is defined by extension by zero from the neutral stratum.

Definition 16.6 (Whittaker sheaf)**.** The *Whittaker sheaf* is

$$W_\psi \;:=\; (i_1)_!\big(\mathcal{W}_\psi^{\mathrm{str}}\big) \;\in\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda).$$

Equivalently, W_ψ is the object of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ supported on Bun_G^1 whose restriction to $\mathrm{Bun}_G^1 \simeq B\underline{G(E)}$ corresponds to the Whittaker representation $\text{c-Ind}_{U(E)}^{G(E)}\psi$.

Remark 16.7 (Relation to the literature)*.* This is the definition used by Fargues and Scholze in both the survey [2, §6.2] and the main preprint [1, §I.10].

16.4 Non-compactness and the meaning of $F * W_\psi$

The object W_ψ is typically *not* compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. Nevertheless, the spectral action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is colimit-preserving, and one can define $F * W_\psi$ for perfect complexes F with quasi-compact support by writing W_ψ as a filtered colimit of finite-type objects.

Lemma 16.8 (Filtered colimit presentation of the Whittaker sheaf)**.** *There exists a filtered system* $\{W_{\psi,\alpha}\}_\alpha$ *in* $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ *such that:*

(a) each $W_{\psi,\alpha}$ *is supported on* Bun_G^1 *and corresponds to a smooth representation of* $G(E)$ *of finite type;*

(b) there is a canonical identification

$$W_\psi \;\simeq\; \varinjlim_\alpha W_{\psi,\alpha}$$

in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$*.*

Sketch of proof. As a smooth $G(E)$-representation, $\text{c-Ind}_{U(E)}^{G(E)}\psi$ is the filtered union of its $G(E)$-subrepresentations of finite type (for example, the subrepresentations generated by vectors fixed by a chosen compact open subgroup). Transporting this filtration through the equivalence $D_{\mathrm{lis}}(\mathrm{Bun}_G^1, \Lambda) \simeq D(\mathrm{Rep}_\Lambda^\infty(G(E)))$ and applying $(i_1)_!$ gives the required presentation. This is exactly the device used in [2, §6.2]. □

Definition 16.9 (Acting on W_ψ)**.** Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ be a perfect complex with quasi-compact support. Define

$$F * W_\psi \;:=\; \varinjlim_\alpha \big(F * W_{\psi,\alpha}\big),$$

where $F * (-)$ denotes the spectral action of F on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ (Theorem 15.13). This is well-defined because the spectral action preserves filtered colimits.

Remark 16.10. One should think of $F * W_\psi$ as a *non-abelian Fourier transform* of F into the automorphic category, with kernel W_ψ. This is exactly the form of the functor that appears in [1, Conjecture I.10.2] and [2, Conjecture 6.2].

16.5 The Whittaker-generated subcategory

We now define the automorphic subcategory that is expected to match coherent sheaves on the spectral stack.

Definition 16.11 (The Whittaker-generated subcategory)**.** Let $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ be the smallest full stable subcategory of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ satisfying:

(a) it is closed under all (small) colimits;

(b) it contains the Whittaker sheaf W_ψ;

(c) it is stable under the spectral action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ (equivalently, if $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ and $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$, then $F * A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$).

Proposition 16.12 (Generation by spectral translates)**.** *The subcategory $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is generated under colimits by the objects $F * W_\psi$, where F ranges over perfect complexes on $\mathrm{LocSys}_{\check{G}}$ with quasi-compact support.*

Sketch of proof. By definition, $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is the closure of W_ψ under colimits and under the spectral action. Any object obtained from W_ψ by iterating colimits and the action is a colimit of objects of the form $F * W_\psi$ for some F; conversely, all such objects lie in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$. □

Remark 16.13 (Dependence on the Whittaker datum). If (B, ψ) and (B', ψ') are two Whittaker data, they are conjugate under $G(E)$ when G is quasi-split. Conjugation induces an autoequivalence of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ that identifies the corresponding subcategories $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$. For the rest of the paper we fix one Whittaker datum once and for all.

16.6 The categorical geometrization conjecture in Whittaker form

The point of introducing W_ψ and $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is that they allow one to formulate the expected equivalence with the spectral category.

Conjecture 16.14 (Categorical geometrization conjecture, Whittaker form)**.** Assume G is quasi-split and fix Whittaker data (B, ψ).

(a) For every perfect complex $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ with quasi-compact support, the object $F * W_\psi$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$.

(b) The functor

$$\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda), \qquad F \longmapsto F * W_\psi$$

extends (uniquely) to an equivalence of small stable ∞-categories

$$\mathrm{Coh}_{\mathrm{Nilp}}\big(\mathrm{LocSys}_{\check{G}}\big) \xrightarrow{\sim} D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega,$$

compatible with the spectral action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$.

Remark 16.15. Conjecture 16.14 is a slightly repackaged form of [1, Conjecture I.10.2] and [2, Conjecture 6.2] (with our notation $\mathrm{LocSys}_{\check{G}}$ for the stack of parameters). In characteristic zero coefficients (for example over $\mathbb{Q}_\ell$ at banal primes), the nilpotent singular support condition is expected to be automatic, and the conjecture should simplify by replacing $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ with $\mathrm{Coh}(\mathrm{LocSys}_{\check{G}})$.

Remark 16.16 (A first consequence: the stable Bernstein center). If Conjecture 16.14 holds, then applying it to the unit object $\mathcal{O}_{\mathrm{LocSys}_{\check{G}}} \in \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ gives

$$\mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(W_\psi) \;\simeq\; \mathrm{End}_{\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})}(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}) \;\simeq\; \Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}),$$

which is the spectral (stable) Bernstein center acting on Whittaker models. This is the categorical form of the identification of centers explained in [2, §6.3.1].

17 Parabolic functors and gluing from Levi subgroups

This section records the parabolic functoriality on Bun_G that will be used to implement the "gluing from Levi subgroups" step in the proof strategy of Section 18. Concretely, for each parabolic subgroup $P \subset G$ with Levi quotient M, Hamann–Hansen–Scholze construct geometric Eisenstein series and constant term functors

$$\mathrm{Eis}_P : \; D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda), \qquad \mathrm{CT}_P : \; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda),$$

prove strong finiteness properties, and establish a geometric analogue of Bernstein's second adjointness [4]. These functors are the local geometric counterparts of parabolic induction and Jacquet modules.

We divide the discussion into five parts:

(i) the geometry of Bun_P and the basic correspondence $\mathrm{Bun}_M \leftarrow \mathrm{Bun}_P \to \mathrm{Bun}_G$;

(ii) definition of Eis_P and CT_P (including normalization conventions);

(iii) finiteness and boundedness properties needed for compactness arguments;

(iv) adjunction and second adjointness;

(v) the resulting "Eisenstein plus cuspidal" generation statements.

17.1 Parabolic substacks and the correspondence $\mathrm{Bun}_M \leftarrow \mathrm{Bun}_P \to \mathrm{Bun}_G$

Fix a parabolic subgroup $P \subset G$ defined over E, and let M be a Levi quotient of P. Let $N \subset P$ denote the unipotent radical, so that $P = MN$.

Definition 17.1 (Stacks of P- and M-bundles)**.** Let Bun_P (respectively Bun_M) denote the v-stack of P-bundles (respectively M-bundles) on the Fargues–Fontaine curve. There are natural morphisms of v-stacks

$$\mathrm{Bun}_M \xleftarrow{q} \mathrm{Bun}_P \xrightarrow{p} \mathrm{Bun}_G,$$

where:

- p is extension of structure group along $P \hookrightarrow G$ (forgetting the reduction to P);
- q is extension of structure group along $P \twoheadrightarrow M$ (quotienting by the unipotent radical).

Remark 17.2. In analogy with the classical case, the stack Bun_P may be viewed as parameterizing pairs $(\mathcal{E}, \mathcal{E}_P)$ consisting of a G-bundle $\mathcal{E}$ together with a reduction $\mathcal{E}_P$ to P; then $p(\mathcal{E}, \mathcal{E}_P) = \mathcal{E}$ and $q(\mathcal{E}, \mathcal{E}_P)$ is the induced M-bundle $\mathcal{E}_P/N$.

17.2 Definition of geometric Eisenstein series and constant term

We next recall the definitions of the parabolic functors on derived categories of lisse ℓ-adic sheaves. The precise definition requires choosing which of the four operations is appropriate in the correspondence

$$\mathrm{Bun}_M \xleftarrow{q} \mathrm{Bun}_P \xrightarrow{p} \mathrm{Bun}_G.$$

For $\ell \neq p$, the relevant functors exist in the range needed here and satisfy the usual base change and projection formula statements.

Definition 17.3 (Unnormalized parabolic functors)**.** Define the *unnormalized* Eisenstein and constant term functors by

$$\mathrm{Eis}_P^{\mathrm{un}} \;:=\; p_*\, q^! :\; D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda),$$

$$\mathrm{CT}_P^{\mathrm{un}} \;:=\; q_*\, p^! :\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda).$$

Remark 17.4 (Normalization conventions)*.* In representation theory, one often normalizes parabolic induction and Jacquet functors by a power of the modulus character $\delta_P^{1/2}$ so that adjunctions take a symmetric form and duality behaves well. In the geometric setting, the modulus character is replaced by a combination of Tate twists and cohomological shifts depending on relative dimensions of stacks and on the half-sum of roots of N. Hamann–Hansen–Scholze implement this by defining normalized variants Eis_P and CT_P that differ from the unnormalized ones by explicit twists and shifts and that satisfy the cleanest adjunction statements [4, §1.1 and §6]. In what follows, we work with their normalized functors and suppress the explicit normalizing twists.

Definition 17.5 (Normalized parabolic functors)**.** Let Eis_P and CT_P denote the normalized geometric Eisenstein and constant term functors constructed in [4]. They are obtained from $\mathrm{Eis}_P^{\mathrm{un}}$ and $\mathrm{CT}_P^{\mathrm{un}}$ by tensoring with a fixed Tate twist and shifting by a fixed cohomological degree that depends only on P.

17.3 Finiteness and boundedness properties

The key technical input for later sections is that Eis_P and CT_P behave like parabolic induction and Jacquet modules: they preserve boundedness with respect to Harder–Narasimhan strata and satisfy finiteness properties strong enough to control compactness.

Theorem 17.6 (Finiteness theorems for parabolic functors)**.** Let Λ be a torsion ring of characteristic $\ell \neq p$, or let $\Lambda = \mathbb{Z}_\ell$. Then the normalized parabolic functors of Definition 17.5 satisfy:

(a) **Boundedness on bounded opens:** for every bound μ on Bun_M there exists a bound μ' on Bun_G such that

$$\mathrm{Eis}_P\big(D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_{\leq \mu}\big) \;\subset\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq \mu'}.$$

Similarly, for every bound ν on Bun_G there exists a bound ν' on Bun_M such that

$$\mathrm{CT}_P\big(D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq \nu}\big) \;\subset\; D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_{\leq \nu'}.$$

(b) **Cohomological amplitude:** on each bounded open, Eis_P and CT_P have finite cohomological amplitude.

(c) **Compactness preservation:** on bounded opens, Eis_P and CT_P take compact objects to compact objects.

Proof sketch. These statements are the core finiteness results of [4]. The boundedness assertions ultimately come from:

- the Harder–Narasimhan classification of bundles by $B(G)$ and the finiteness of bounded subsets;
- a geometric analysis of extensions of an M-bundle by N-torsors, which controls how instability can grow under extension of structure group $P \hookrightarrow G$ or under taking Levi quotients $P \twoheadrightarrow M$.

Finite cohomological amplitude and compactness preservation are proved by reducing to bounded opens (finite unions of strata), and then proving that the relevant correspondences are "cohomologically proper" in a sense appropriate for lisse ℓ-adic sheaves on v-stacks. □

Remark 17.7 (Why these finiteness statements matter later)*.* The compactness statement in Conjecture 16.14(a) will be reduced to compactness calculations after applying constant term functors. Theorem 17.6 is exactly what makes such a reduction possible.

17.4 Adjunction and Bernstein's second adjointness

In representation theory, parabolic induction and Jacquet modules satisfy:

- Frobenius adjunction: parabolic induction is left adjoint to Jacquet;
- Bernstein's second adjointness: parabolic induction is also right adjoint to Jacquet for the opposite parabolic (after normalization).

Hamann–Hansen–Scholze prove geometric analogues of these statements.

Let $P^- \subset G$ denote a parabolic subgroup opposite to P, with the same Levi quotient M.

Theorem 17.8 (Adjunction and second adjointness)**.** *With normalized conventions, the parabolic functors satisfy:*

(a) **First adjointness:** Eis_P *is left adjoint to* CT_P*.*

(b) **Second adjointness:** Eis_P *is also right adjoint to* CT_{P^-}*.*

Both adjunctions hold on the bounded-support subcategories, and hence extend to the filtered colimit $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$*.*

Proof sketch. This is proved in [4]. The first adjunction is a formal consequence of the correspondence definition of Eis_P and CT_P together with Verdier duality. The second adjunction is substantially deeper: it relies on a geometric analysis of the correspondence $\mathrm{Bun}_P \times_{\mathrm{Bun}_G} \mathrm{Bun}_{P^-}$ and a vanishing statement that is the geometric analogue of the classical second adjointness theorem. □

Remark 17.9*.* Theorem 17.8 is the local geometric input that plays the role of "ambidexterity" in the global proof of geometric Langlands. It is one of the main reasons the $\ell \neq p$ case on Bun_G is a promising environment for a Gaitsgory-style proof.

17.5 Compatibility with Hecke operators and the spectral action

The parabolic functors are expected to be compatible with Hecke operators and, more strongly, to be linear with respect to the spectral action. This is the categorical form of the statement that local Langlands parameters are functorial for Levi inclusions.

Let ${}^L M \hookrightarrow {}^L G$ denote the natural morphism of L-groups induced by $M \hookrightarrow G$. It induces a morphism of parameter stacks

$$\mathrm{res}^G_M : \mathrm{LocSys}_{\check{G}} \longrightarrow \mathrm{LocSys}_{\check{M}}.$$

Proposition 17.10 (Hecke compatibility). *For $V \in \mathrm{Rep}(\check{G})$ and its restriction $V|_{\check{M}} \in \mathrm{Rep}(\check{M})$, there are canonical equivalences of functors:*

$$\mathrm{CT}_P \circ T_V \;\simeq\; T_{V|_{\check{M}}} \circ \mathrm{CT}_P, \qquad T_V \circ \mathrm{Eis}_P \;\simeq\; \mathrm{Eis}_P \circ T_{V|_{\check{M}}}.$$

Proof sketch. This is a geometric form of the compatibility of parabolic induction and Jacquet modules with spherical Hecke operators. It is proved by comparing the corresponding correspondences on Bun_G and Bun_M and using base change. In the v-stack setting, the needed finiteness and base change statements are part of the main results of [4]. □

Proposition 17.11 (Spectral linearity). *The functors Eis_P and CT_P are linear with respect to the spectral action in the sense that there are canonical equivalences*

$$\mathrm{CT}_P\big(F * \mathcal{A}\big) \;\simeq\; \big(\mathrm{res}^G_M F\big) * \mathrm{CT}_P(\mathcal{A}),$$

$$\mathrm{Eis}_P\big(F' * \mathcal{B}\big) \;\simeq\; \big((\mathrm{res}^G_M)^* F'\big) * \mathrm{Eis}_P(\mathcal{B}),$$

for $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$, $F' \in \mathrm{Perf}(\mathrm{LocSys}_{\check{M}})$ and objects $\mathcal{A} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$, $\mathcal{B} \in D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)$.

Proof sketch. By Theorem 15.13, the spectral action is generated (as a monoidal action) by the Hecke operators attached to representations of $\check{G}$. Thus Proposition 17.11 is a formal consequence of Proposition 17.10 and monoidality of the spectral action. □

17.6 Cuspidal objects and Eisenstein generation

We now formulate the "gluing from Levi subgroups" principle on the automorphic side.

Definition 17.12 (Cuspidal subcategory). Define the cuspidal subcategory $D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda) \subset D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ by

$$D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda) \;:=\; \bigcap_{P \subsetneq G} \ker(\mathrm{CT}_P),$$

where P runs over proper parabolic subgroups of G (up to conjugacy).

Definition 17.13 (Eisenstein-generated subcategory). Let $D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda) \subset D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ be the smallest full stable subcategory closed under colimits that contains the essential images of Eis_P for all proper parabolics $P \subsetneq G$ (with varying Levi quotients).

Theorem 17.14 (Eisenstein plus cuspidal generation). The category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is generated under colimits by $D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda)$ and $D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda)$. Equivalently, the smallest colimit-closed stable subcategory containing the cuspidal objects and all Eisenstein series objects is the whole category.

Proof sketch. This is proved in [4]. The strategy is an induction on semisimple rank that uses:

- the finiteness properties of Theorem 17.6 to ensure that constant term functors detect noncuspidal objects without losing control of cohomological degrees;
- adjunction and second adjointness (Theorem 17.8) to construct, from an object with nonzero constant term, a map from an Eisenstein series object detecting it;
- gluing along Harder–Narasimhan strata to reduce global statements on Bun_G to finite unions of strata.

□

Remark 17.15 (Analogy with Bernstein decomposition). In the representation theory of p-adic groups, one decomposes the category of smooth representations into blocks generated from supercuspidal data by parabolic induction. Theorem 17.14 is the geometric local analogue: it gives a structural decomposition of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ into a cuspidal part and a part generated from Levi subgroups by Eisenstein series.

17.7 Restriction to strata and comparison with parabolic induction

Finally, we record the basic compatibility with the representation-theoretic meaning of strata. Fix $b \in B(G)$ and identify $\mathrm{Bun}_G^b \simeq B\underline{J_b(E)}$. Let $b_M \in B(M)$ be the corresponding element induced by the Levi quotient map (this is the element controlling the induced M-bundle on the stratum of Bun_M). Then one expects $J_{b_M}(E)$ to be a Levi subgroup of $J_b(E)$ (in a suitable inner form sense), and the restriction of Eis_P and CT_P to strata recovers parabolic induction and Jacquet functors for the groups $J_b(E)$.

Proposition 17.16 (Parabolic functors on basic strata)**.** *Assume b is basic. Under the identifications*

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \simeq D\big(\mathrm{Rep}_\Lambda^\infty(J_b(E))\big), \qquad D_{\mathrm{lis}}(\mathrm{Bun}_M^{b_M}, \Lambda) \simeq D\big(\mathrm{Rep}_\Lambda^\infty(J_{b_M}(E))\big),$$

the restrictions of Eis_P *and* CT_P *to the basic strata coincide with the derived normalized parabolic induction and normalized Jacquet functors between smooth representation categories of $J_{b_M}(E)$ and $J_b(E)$.*

Proof sketch. This is the local meaning of the geometric definitions: on a basic stratum the moduli problem reduces to a classifying stack, and the correspondence $\mathrm{Bun}_M \leftarrow \mathrm{Bun}_P \rightarrow \mathrm{Bun}_G$ restricts to the usual correspondence defining parabolic induction at the level of groupoids. The identification is discussed in [1, §III and §V] and is used implicitly in the representation-theoretic interpretations of [4]. □

Remark 17.17 (How this will be used later). Proposition 17.16 allows one to reduce certain statements about parabolic functors on Bun_G to statements about classical parabolic induction and Jacquet modules on smooth representations. In particular, it is the basic input for cuspidal block arguments in Section 18.

18 Proof strategy for the categorical geometrization conjecture

In this section we give a detailed blueprint for a proof of Conjecture 16.14. The argument is designed to parallel the structure of the proof of the geometric Langlands conjecture in the work of Arinkin–Gaitsgory and Gaitsgory–Raskin: singular support on the spectral side, a Whittaker generator on the automorphic side, monadicity and endomorphism calculations for full faithfulness, and parabolic gluing (with second adjointness) for essential surjectivity [6, 7, 8, 9].

Throughout we work in the case $\ell \neq p$ and keep the coefficient ring Λ from Section 2. We assume G is quasi-split and Whittaker data (B, ψ) are fixed, so that $W_\psi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ and $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ are defined as in Section 16.

18.1 Two equivalent formulations: compact objects and ind-completions

A persistent technical point is that the spectral category in the conjecture is a *small* category $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$, whereas the Whittaker-generated automorphic category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is defined as a *colimit-closed* subcategory and is therefore presentable. To keep the comparison honest, we separate the "compact" and the "presentable" formulations.

Definition 18.1 (Compact objects in the Whittaker-generated category)**.** Let

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{c}}_\omega \;:=\; \Big(D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega\Big)^{\mathrm{c}}$$

denote the full subcategory of compact objects in the presentable category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$.

Definition 18.2 (Ind-completion on the spectral side)**.** Let $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ be as in Definition 18.2, and note that

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \;\simeq\; \mathrm{Ind}\big(\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})\big)$$

because $\mathrm{Coh}_{\mathrm{Nilp}}$ is, by definition, the full subcategory of compact objects.

Remark 18.3. In later stages of the project it will be useful to prove the equivalence at the presentable level

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \;\simeq\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega,$$

and then recover the equivalence of compact objects by taking compacts. Conversely, it is often easier to prove the equivalence first on compact objects and then ind-complete. We will allow ourselves to move freely between these two formulations.

18.2 The comparison functor and the first main theorem

We define the comparison functor on perfect complexes by the spectral action on the Whittaker sheaf.

Definition 18.4 (The basic comparison functor on perfect complexes)**.** Let Φ_{Perf} be the functor

$$\Phi_{\mathrm{Perf}} :\; \mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \;\longrightarrow\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega, \qquad F \;\longmapsto\; F * W_\psi,$$

where the action on W_ψ is defined as in Definition 16.9 when F has quasi-compact support.

The first substantive input needed for the conjecture is the compactness of these translates.

Theorem 18.5 (Compactness of Whittaker translates)**.** Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have quasi-compact support. Then $\Phi_{\mathrm{Perf}}(F) = F * W_\psi$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$. Equivalently,

$$\Phi_{\mathrm{Perf}} : \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})_{\mathrm{q.c.}} \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega^{\mathrm{c}}$$

lands in the compact objects.

Remark 18.6. Theorem 18.5 is exactly part (a) of Conjecture 16.14. Once it is established, the equivalence of categories becomes a statement about compactly generated module categories and can be attacked by monadicity and gluing.

18.3 Compactness: reduction to torsion coefficients and bounded opens

We record a structured reduction of Theorem 18.5 to finiteness results already available in the literature (notably [4] and [5]).

Lemma 18.7 (Reduction to torsion coefficients)**.** *Assume $\Lambda = \mathbb{Z}_\ell$. To prove Theorem 18.5 for $\mathbb{Z}_\ell$-coefficients, it suffices to prove the torsion analogue: for each $n \geq 1$, and each $F_n \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}} \times_{\mathrm{Spec}(\mathbb{Z}_\ell)} \mathrm{Spec}(\mathbb{Z}/\ell^n\mathbb{Z}))$ of quasi-compact support, the object $F_n * W_{\psi,n}$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n\mathbb{Z})_\omega$, where $W_{\psi,n}$ is the mod ℓ^n reduction of W_ψ.*

Proof sketch. The category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ is an ℓ-adic limit of the torsion categories, and compactness in the ℓ-adic completion can be checked after reduction modulo ℓ^n provided one keeps track of derived ℓ-adic completeness. This is the standard passage from torsion to $\mathbb{Z}_\ell$ in the formalism of [1, §I.2] and is used systematically in [4]. ☐

Lemma 18.8 (Reduction to bounded opens)**.** *Assume Λ is torsion of characteristic $\ell \neq p$. To prove compactness of $F * W_\psi$, it suffices to show:*

(a) there exists a bound μ such that $F * W_\psi$ is supported on $\mathrm{Bun}_G^{\leq\mu}$, and

(b) viewed as an object of $D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq\mu}, \Lambda)$, it is compact.

Proof sketch. This is a formal consequence of the exhaustion $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) = \varinjlim_\mu D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\leq\mu}$ (Proposition 13.7) and the fact that compactness is local on a filtered union of open substacks. The main point is that if an object is supported on a quasi-compact open, compactness can be checked in the quasi-compact subcategory. ☐

Lemma 18.9 (Boundedness of Whittaker translates)**.** *Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have quasi-compact support. Then $F * W_\psi$ is supported on a bounded open $\mathrm{Bun}_G^{\leq\mu}$ for some μ depending on F.*

Proof sketch. By definition of the spectral action, the action of F is built from finitely many Hecke operators T_V with $V \in \mathrm{Rep}(\check{G})$, together with finite colimits and shifts (because F is perfect and has quasi-compact support). By Proposition 15.16, each T_V sends bounded-support objects to bounded-support objects. Since W_ψ is supported on the neutral stratum Bun_G^1, repeated application of finitely many Hecke operators yields bounded support. ☐

Remark 18.10 (Where Miles and Hamann–Hansen–Scholze enter). After Lemmas 18.7–18.9, the remaining content of Theorem 18.5 is: compactness of certain explicit objects in $D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq\mu}, \Lambda)$ for a bounded open $\mathrm{Bun}_G^{\leq\mu}$. On bounded opens, one can glue along Harder–Narasimhan strata [5], and one has strong finiteness properties for Hecke and parabolic correspondences [4]. These are the finiteness inputs that the proof template will use repeatedly.

18.4 Extension from perfect complexes to coherent sheaves

Assuming Theorem 18.5, we explain how to extend Φ_{Perf} to a functor on coherent objects with nilpotent singular support.

Proposition 18.11 (Spectral devissage on $\mathrm{Coh}_{\mathrm{Nilp}}$)**.** *The category* $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ *is the idempotent-complete stable subcategory generated by perfect complexes with quasi-compact support under finite colimits and retracts.*

Remark 18.12. Proposition 18.11 is a purely spectral statement and should follow from the general formalism of ind-coherent sheaves with singular support on quasi-smooth stacks [6]. In later versions of the paper we will extract an explicit reference or include a proof. For the blueprint, we treat it as a standard devissage input.

Proposition 18.13 (Extension of Φ_{Perf} to $\mathrm{Coh}_{\mathrm{Nilp}}$)**.** *Assume Theorem 18.5 and Proposition 18.11. Then there exists a unique exact functor*

$$\Phi^{\mathrm{c}} : \ \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \ \longrightarrow \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{c}}_{\omega}$$

whose restriction to $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})_{\mathrm{q.c.}}$ *is* $F \mapsto F * W_{\psi}$*. Moreover,* Φ^{c} *admits an ind-extension*

$$\widetilde{\Phi} : \ \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \ \longrightarrow \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\omega}$$

which is colimit-preserving and whose restriction to compact objects is Φ^{c}*.*

Proof sketch. By Theorem 18.5, the functor Φ_{Perf} lands in compact objects, hence extends uniquely to the idempotent-complete thick subcategory generated by $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})_{\mathrm{q.c.}}$. By Proposition 18.11 this thick subcategory is $\mathrm{Coh}_{\mathrm{Nilp}}$. Ind-completing yields a colimit-preserving functor on $\mathrm{IndCoh}_{\mathrm{Nilp}}$. □

18.5 Compatibility with parabolic functors: the gluing interface

To follow the geometric Langlands proof architecture, one needs compatibility of the comparison functor with Eisenstein series and constant term functors on both sides.

Automorphic parabolic functors. On the automorphic side, the functors Eis_P and CT_P are constructed in Section 17 and satisfy finiteness and adjunction properties by Theorems 17.6 and 17.8.

Spectral parabolic functors. On the spectral side, the inclusion ${}^L M \hookrightarrow {}^L G$ suggests that parabolic induction should be realized by the correspondence of parameter stacks associated with the dual parabolic $\check{P} \subset \check{G}$:

$$\mathrm{LocSys}_{\check{M}} \xleftarrow{q_{\mathrm{spec}}} \mathrm{LocSys}_{\check{P}} \xrightarrow{p_{\mathrm{spec}}} \mathrm{LocSys}_{\check{G}}.$$

We therefore define:

Definition 18.14 (Spectral Eisenstein and constant term functors)**.** Assume the derived stacks $\mathrm{LocSys}_{\check{G}}$, $\mathrm{LocSys}_{\check{M}}$, and $\mathrm{LocSys}_{\check{P}}$ are quasi-smooth, and the correspondence maps are such that the relevant ind-coherent functors are defined. Define

$$\mathrm{Eis}_P^{\mathrm{spec}} \ := \ (p_{\mathrm{spec}})_* \, (q_{\mathrm{spec}})^! : \ \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}),$$

$$\mathrm{CT}_P^{\mathrm{spec}} \;:=\; (q_{\mathrm{spec}})_*\,(p_{\mathrm{spec}})^! :\; \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}),$$

with the same normalizing twists and shifts as on the automorphic side (suppressed from the notation).

Remark 18.15. Definition 18.14 is the direct local analogue of the spectral Eisenstein and constant term functors in geometric Langlands. The nilpotent singular support condition is expected to be exactly what ensures these functors preserve the nilpotent subcategories and satisfy adjunction properties parallel to the automorphic ones [6, 7].

Conjecture 18.16 (Parabolic compatibility of $\widetilde{\Phi}$)**.** Assume the spectral parabolic functors of Definition 18.14 exist. Then there are canonical equivalences of functors

$$\widetilde{\Phi}_G \circ \mathrm{Eis}_P^{\mathrm{spec}} \;\simeq\; \mathrm{Eis}_P \circ \widetilde{\Phi}_M, \qquad \widetilde{\Phi}_M \circ \mathrm{CT}_P^{\mathrm{spec}} \;\simeq\; \mathrm{CT}_P \circ \widetilde{\Phi}_G.$$

Remark 18.17 (Role in the proof)*.* Conjecture 18.16 is the mechanism by which one reduces the equivalence for G to the equivalence for Levi subgroups. It is the local avatar of the compatibility between geometric Langlands functors and Eisenstein series. In this blueprint, we treat it as a required input to be proved by analyzing the universal Hecke eigensheaf construction of the spectral action and the geometric definitions of Eisenstein series on Bun_G.

18.6 Full faithfulness via monadicity and endomorphisms

Assume we have constructed Φ^c and $\widetilde{\Phi}$ as in Proposition 18.13. To prove that Φ^c is fully faithful we aim to apply a Barr–Beck type theorem. For this, we need:

(i) existence and good behavior of the right adjoint of $\widetilde{\Phi}$;

(ii) conservativity of that right adjoint (on the Whittaker subcategory);

(iii) identification of the induced monad with the tautological monad on the spectral side.

Proposition 18.18 (Existence of the right adjoint)**.** *The colimit-preserving functor* $\widetilde{\Phi} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ *admits a continuous right adjoint* $\widetilde{\Phi}^R$*.*

Proof sketch. Both categories are compactly generated presentable stable categories, and $\widetilde{\Phi}$ preserves colimits. Under standard set-theoretic finiteness hypotheses (which hold in the settings considered by Fargues–Scholze), the adjoint functor theorem produces a right adjoint. In the later write-up, we will isolate a precise compact generation statement and cite the general existence of adjoints in presentable stable ∞-categories. □

Proposition 18.19 (Endomorphisms of the Whittaker generator)**.** *There is a canonical algebra map*

$$\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \;\longrightarrow\; \mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(W_\psi)$$

induced by the spectral action, and this map is expected to be an isomorphism.

Remark 18.20. The map in Proposition 18.19 is the categorical incarnation of the stable Bernstein center: Fargues and Scholze construct a map from functions on the parameter stack to endomorphisms of the identity functor on the relevant automorphic category [2, §6.3]. The Whittaker object is the distinguished test object where this map should be detected sharply, by analogy with classical multiplicity one for Whittaker models.

Proposition 18.21 (Monadicity criterion for full faithfulness)**.** *Assume:*

(a) $\widetilde{\Phi}$ *admits a conservative right adjoint* $\widetilde{\Phi}^R$*;*

(b) the endomorphism map of Proposition 18.19 is an isomorphism; and

(c) $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ *is generated under colimits by the unit object* $\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}$ *under the action of* $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$*.*

Then $\widetilde{\Phi}$ *is fully faithful, hence* Φ^{c} *is fully faithful on compact objects.*

Proof sketch. The functor $\widetilde{\Phi}$ is a morphism of module categories for the monoidal category $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$, sending the unit object $\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}$ to W_ψ. Under (c), the monad $\widetilde{\Phi}^R\widetilde{\Phi}$ is determined by its effect on the unit object. By (b), this monad agrees with the tautological monad on $\mathrm{IndCoh}_{\mathrm{Nilp}}$ (tensoring by $\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O})$), and by (a) Barr–Beck gives full faithfulness. This is the local analogue of the monadic arguments used in [8]. □

18.7 Essential surjectivity: parabolic gluing and induction on semisimple rank

We now explain how, once full faithfulness is established, one should prove essential surjectivity of Φ^{c} by an inductive argument on semisimple rank using parabolic functors.

Proposition 18.22 (Automorphic generation)**.** *The category* $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ *is generated under colimits by the cuspidal objects in* $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ *together with the essential images of* Eis_P *from Levi subgroups, where* P *ranges over proper parabolic subgroups.*

Proof sketch. This is a direct refinement of Theorem 17.14 to the Whittaker-generated subcategory, using the fact that $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is stable under parabolic functors (by spectral linearity and Definition 16.11). The needed stability under Eis_P and CT_P is proved in [4]. □

Proposition 18.23 (Spectral generation)**.** *The category* $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ *is generated under colimits by its cuspidal subcategory together with the essential images of the spectral Eisenstein functors* $\mathrm{Eis}_P^{\mathrm{spec}}$ *from Levi subgroups.*

Remark 18.24. Proposition 18.23 is the spectral counterpart of Theorem 17.14. Its proof should use nilpotent singular support exactly as in global geometric Langlands: compatibility of singular support with parabolic correspondences and a gluing argument along Levi strata [6, 7]. We include it here as a required spectral input for the inductive strategy.

Theorem 18.25 (Inductive essential surjectivity template)**.** Assume:

(a) $\widetilde{\Phi}$ is fully faithful (for G and for all proper Levi subgroups of G);

(b) parabolic compatibility holds (Conjecture 18.16);

(c) automorphic and spectral generation statements (Propositions 18.22 and 18.23);

(d) $\widetilde{\Phi}$ induces an equivalence on cuspidal subcategories.

Then $\widetilde{\Phi}$ is essentially surjective, hence an equivalence of presentable categories, and therefore Φ^{c} is an equivalence on compact objects.

Proof sketch. Using (b), the essential images of Eisenstein series from Levi subgroups match under $\widetilde{\Phi}$, and by induction on semisimple rank these Levi images are already in the essential image. By (c), it remains to treat cuspidal objects; by (d) these are also in the essential image. Thus $\widetilde{\Phi}$ is essentially surjective. □

18.8 Cuspidal blocks and local multiplicity one

The remaining input in Theorem 18.25 is the cuspidal equivalence. Here one expects to use a local analogue of the multiplicity one theorem in the proof of geometric Langlands [9], combined with the "toy model" description of cuspidal blocks described by Fargues and Scholze [2, §5].

Conjecture 18.26 (Cuspidal block description and Whittaker multiplicity one)**.** Let ϕ be a cuspidal Langlands parameter for G and let S_ϕ be the component group of the centralizer of ϕ in $\check{G}$. Then the fiber of the Whittaker-generated category over ϕ is equivalent to $\mathrm{Perf}(\mathrm{Rep}_\Lambda(S_\phi))$, and the Whittaker datum singles out a unique generic object in the packet.

Remark 18.27. Conjecture 18.26 is the local categorical analogue of the description of cuspidal Hecke eigensheaves and the uniqueness of Whittaker models. It is the step at which one expects to use deeper geometry of local shtukas and the construction of Hecke eigensheaves in [1], together with the explicit structure of the parameter stack in [3].

18.9 Summary of the dependency chain

Collecting the discussion, a proof of Conjecture 16.14 can be organized as follows:

Task 1. Prove compactness of Whittaker translates (Theorem 18.5) using boundedness of Hecke correspondences (Proposition 15.16), gluing along Harder–Narasimhan strata [5], and finiteness of parabolic functors [4].

Task 2. Establish spectral devissage (Proposition 18.11) and extend the functor to $\mathrm{Coh}_{\mathrm{Nilp}}$ (Proposition 18.13).

Task 3. Construct spectral parabolic functors and prove parabolic compatibility (Conjecture 18.16), using singular support as in [6].

Task 4. Prove full faithfulness by monadicity: show conservativity of the right adjoint and compute $\mathrm{End}(W_\psi)$ (Proposition 18.19), using the stable Bernstein center formalism [2] and the monadic patterns of [8].

Task 5. Prove essential surjectivity by gluing from Levi subgroups (Theorem 18.25), using second adjointness on the automorphic side (Theorem 17.8) and the spectral generation statement (Proposition 18.23).

Task 6. Analyze the cuspidal case via Conjecture 18.26, the local analogue of multiplicity one [9].

This completes the proof template that will guide the remainder of the project.

19 Compactness of Whittaker translates

In this section we focus on **Task 1** from Section 18.9 and give a proof strategy that reduces compactness of Whittaker translates to a precise representation-theoretic finite generation statement. We then verify the required representation-theoretic input for $G = \mathrm{GL}_n$ using the integral theory of co-Whittaker modules due to Helm.

19.1 The compactness problem

Recall that G/E is quasi-split, $\ell \neq p$, and we fix Whittaker data (B, ψ). Let $W_\psi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ be the Whittaker generator (Definition 16.6), and let $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ be the Whittaker-generated subcategory (Definition 16.11).

Theorem 19.1 (Compactness of Whittaker translates)**.** Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ be a perfect complex whose (cohomological) support is quasi-compact. Then the Whittaker translate

$$F * W_\psi \;\in\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$$

is compact.

The remainder of the section gives a reduction of Theorem 19.1 to a representation-theoretic input on the neutral stratum, and proves that input for $G = \mathrm{GL}_n$.

19.2 Perfect actions preserve compact objects

The spectral action gives, for each $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$, an exact colimit-preserving endofunctor

$$T_F \;:\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda), \qquad \mathcal{A} \longmapsto F * \mathcal{A}.$$

A key formal point is that T_F preserves compact objects whenever F is perfect. This uses only rigidity of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$.

Lemma 19.2 (Rigidity implies preservation of compact objects)**.** *Let $\mathcal{C}$ be a compactly generated presentable stable ∞-category. Let $\mathcal{M}$ be a rigid symmetric monoidal stable ∞-category acting on $\mathcal{C}$ by colimit-preserving exact endofunctors. Then for every $m \in \mathcal{M}$, the endofunctor*

$$T_m : \mathcal{C} \to \mathcal{C}, \qquad c \mapsto m * c$$

preserves compact objects.

Proof. Because $\mathcal{M}$ is rigid, m admits a dual $m^\vee$. Rigidity of the action implies that $T_{m^\vee}$ is both left and right adjoint to T_m. Since the action is by colimit-preserving functors, the right adjoint $T_{m^\vee}$ preserves filtered colimits. By the standard adjointness criterion for compactness (a left adjoint preserves compact objects if and only if its right adjoint preserves filtered colimits), it follows that T_m preserves compact objects. □

Remark 19.3. Lemma 19.2 reduces the compactness problem in Theorem 19.1 to constructing *one* compact object in the Whittaker category on which $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ acts, and then observing that all perfect translates remain compact.

19.3 Quasi-compact support and reduction to finitely many connected components

Write

$$\mathrm{LocSys}_{\check{G}} \;=\; \coprod_{\alpha \in \pi_0(\mathrm{LocSys}_{\check{G}})} \mathrm{LocSys}_{\check{G},\alpha}$$

for the decomposition into connected components (equivalently, open and closed substacks). By Theorem 14.3 these components are algebraic stacks locally of finite type; in particular each $\mathrm{LocSys}_{\check{G},\alpha}$ is quasi-compact.

Lemma 19.4 (Quasi-compact support meets finitely many components). *Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have quasi-compact support. Then there exists a finite subset $I \subset \pi_0(\mathrm{LocSys}_{\check{G}})$ such that*

$$\mathrm{Supp}(F) \;\subset\; \coprod_{\alpha\in I} \mathrm{LocSys}_{\check{G},\alpha}.$$

Equivalently, F is the extension by zero of its restriction to the quasi-compact open and closed substack

$$\mathrm{LocSys}_{\check{G},I} \;:=\; \coprod_{\alpha\in I} \mathrm{LocSys}_{\check{G},\alpha}.$$

Proof. A quasi-compact subset of a disjoint union of open and closed quasi-compact components meets only finitely many components. □

Let $\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ denote the structure sheaf of the open and closed substack $\mathrm{LocSys}_{\check{G},I}$, viewed as a direct summand of the unit object $\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}$. Since $\mathrm{LocSys}_{\check{G},I}$ is open and closed, $\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}}$ is an idempotent algebra object.

Lemma 19.5 (Projector reduction). *Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have support contained in $\mathrm{LocSys}_{\check{G},I}$. Then there is a canonical identification*

$$F * W_\psi \;\simeq\; F * \big(\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} * W_\psi\big).$$

Proof. Since F is supported on $\mathrm{LocSys}_{\check{G},I}$, one has a canonical isomorphism $F \simeq \mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} \otimes F$ in $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$. Using monoidality of the action, this gives

$$F * W_\psi \;\simeq\; (\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} \otimes F) * W_\psi \;\simeq\; F * (\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} * W_\psi).$$

□

Lemma 19.5 shows that, to prove compactness of $F * W_\psi$, it suffices to prove compactness of the *localized Whittaker generator*

$$W_{\psi,I} \;:=\; \mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} * W_\psi$$

for finite sets of components I.

19.4 A representation-theoretic finiteness hypothesis

The localized object $W_{\psi,I}$ is still a priori defined inside $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$. To obtain a usable criterion for compactness, we impose a hypothesis that identifies $W_{\psi,I}$ with an object supported on the neutral stratum and controlled by the usual Bernstein theory of smooth representations.

Let $i_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ be the neutral stratum, so $\mathrm{Bun}_G^1 \simeq B\underline{G(E)}$. Write $\mathrm{Rep}_\Lambda^\infty(G(E))$ for the abelian category of smooth Λ-representations. Recall that $W_\psi = i_{1,!}(\text{c-ind}_{U(E)}^{G(E)}\psi)$.

Axiom 19.6 (Blockwise Whittaker finiteness)**.** For every finite union of connected components $\mathrm{LocSys}_{\check{G},I}$, the object $W_{\psi,I} = \mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} * W_\psi$ satisfies:

(a) (*Neutral support*) $W_{\psi,I}$ is supported on the neutral stratum Bun_G^1, so $W_{\psi,I} \simeq i_{1,!}(V_I)$ for a smooth representation V_I of $G(E)$.

(b) (*Finite generation*) The representation V_I belongs to the thick subcategory generated by compact inductions from compact open subgroups (equivalently, V_I is a compact object of $D(\mathrm{Rep}^\infty_\Lambda(G(E)))$).

Remark 19.7. Axiom 19.6(a) is a compatibility statement between the spectral action and restriction to Harder–Narasimhan strata. It is automatic if the action of the idempotent $\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}}$ is induced by the stable Bernstein center on the neutral stratum. Axiom 19.6(b) is a representation-theoretic finiteness statement: it asks for the blockwise Whittaker object selected by $\mathrm{LocSys}_{\check{G},I}$ to be of finite type in the sense relevant for compact generation of smooth representation categories.

Assuming Axiom 19.6, Task 1 becomes formal.

Proposition 19.8 (Compactness from blockwise finiteness). *Assume Axiom 19.6. Then Theorem 19.1 holds.*

Proof. Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have quasi-compact support. Choose a finite set of components I as in Lemma 19.4. By Lemma 19.5 we have

$$F * W_\psi \;\simeq\; F * W_{\psi,I}.$$

By Axiom 19.6(b), $W_{\psi,I}$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$. By Lemma 19.2, the endofunctor T_F preserves compact objects. Therefore $F * W_{\psi,I}$ is compact, hence so is $F * W_\psi$. □

19.5 Block projectors and the GL_n case

We now specialize to $G = \mathrm{GL}_n$ and explain how the "blockwise Whittaker finiteness" input can be replaced by a concrete theorem using the integral Bernstein center and local Langlands in families.

Definition 19.9 (Bernstein block idempotents). Let $\mathfrak{Z}(G(E))$ be the Bernstein center of $\mathrm{Rep}^\infty_\Lambda(G(E))$ (with Λ of characteristic zero), and write

$$\mathrm{Rep}^\infty_\Lambda(G(E)) \;\simeq\; \prod_{\mathfrak{s}} \mathrm{Rep}^\infty_\Lambda(G(E))_{\mathfrak{s}}$$

for the Bernstein decomposition. Let $e_{\mathfrak{s}} \in \mathfrak{Z}(G(E))$ be the central idempotent cutting out the block $\mathrm{Rep}^\infty_\Lambda(G(E))_{\mathfrak{s}}$.

Lemma 19.10 (Components on the parameter stack and Bernstein blocks for GL_n)**.** *Let $G = \mathrm{GL}_n$. There is a natural finite-to-one assignment from connected components of the parameter stack $\pi_0(\mathrm{LocSys}_{\check{G}})$ to Bernstein blocks $\mathfrak{s}$ such that:*

(a) the induced map on global functions factors the usual Bernstein center action through $\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O})$, and

(b) for any finite union of components $I \subset \pi_0(\mathrm{LocSys}_{\check{G}})$, the idempotent $\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}}$ acts on the neutral stratum as the sum of the corresponding block idempotents $\sum_{\mathfrak{s}\in\mathfrak{S}(I)} e_{\mathfrak{s}}$.

Proof sketch. This is a reformulation of "local Langlands in families" for GL_n: one attaches to each Bernstein block a moduli space/stack of Weil–Deligne parameters (or its ℓ-adic avatar) carrying the universal family, and the Bernstein center identifies with functions on the corresponding parameter space. See [19, 20, 21] for the construction and compatibility with Bernstein blocks; see also [3] for a global algebraic model of the parameter stack. □

Theorem 19.11 (Blockwise Whittaker finiteness for GL_n)**.** Let $G = \mathrm{GL}_n$ and let Λ be $\mathbb{Z}_\ell$ or a finite extension of $\mathbb{Z}_\ell$ (with $\ell \neq p$). For every finite union of connected components $\mathrm{LocSys}_{\check{G},I}$, the localized Whittaker object

$$W_{\psi,I} \;:=\; \mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} * W_\psi$$

is supported on the neutral stratum and corresponds there to a finite direct sum of Helm's universal co-Whittaker modules in the associated generic Bernstein blocks. In particular $W_{\psi,I}$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$.

Proof sketch. By Lemma 19.10, the projector $\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}}$ acts on the neutral stratum as a finite sum of Bernstein block idempotents. The Whittaker generator restricts to the universal Gelfand–Graev representation, and in each generic block Helm constructs a universal co-Whittaker module which is a finitely generated projective generator and whose endomorphism algebra is the integral block center [12]. Thus the projected object is a finite sum of compact generators on the neutral stratum, hence compact after extension by zero. □

Corollary 19.12 (Compactness of Whittaker translates for GL_n)**.** *For $G = \mathrm{GL}_n$, Theorem 19.1 holds (for $\Lambda = \mathbb{Z}_\ell$ and hence after inverting ℓ).*

Proof. Combine Theorem 19.11 with Proposition 19.8 (and the rigidity argument Lemma 19.2). □

19.6 Verification for $G = \mathrm{GL}_n$

We now verify Axiom 19.6 for $G = \mathrm{GL}_n$ (and hence prove Task 1 in that case) using the integral theory of co-Whittaker modules and the integral Bernstein center developed by Helm.

Fix $G = \mathrm{GL}_n$ over E and assume $\Lambda = \mathbb{Z}_\ell$ (or a finite extension of $\mathbb{Z}_\ell$). Let $\mathrm{Rep}^\infty_\Lambda(G(E))_\mathfrak{s}$ be a Bernstein block. Helm constructs in each block a *universal co-Whittaker module* $\mathcal{W}_\mathfrak{s}$ which is a finitely generated projective generator of the generic subcategory of the block and whose endomorphism ring is the integral Bernstein center of the block [12].

Theorem 19.13 (Compactness for GL_n)**.** Let $G = \mathrm{GL}_n$ and let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have quasi-compact support. Then $F * W_\psi$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbb{Z}_\ell)_\omega$.

Proof sketch. By Lemma 19.4, the support of F is contained in a finite union of connected components $\mathrm{LocSys}_{\check{G},I}$. The action of $\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}}$ on W_ψ produces an object $W_{\psi,I}$. For GL_n, the comparison between the connected components of the parameter stack and Bernstein blocks identifies $W_{\psi,I}$ (on the neutral stratum) with the direct sum of the universal co-Whittaker modules $\mathcal{W}_\mathfrak{s}$ for the finitely many generic blocks corresponding to I. By [12], each $\mathcal{W}_\mathfrak{s}$ is finitely generated projective in its block, hence defines a compact object of the derived category of smooth representations, and therefore $i_{1,!}(\mathcal{W}_\mathfrak{s})$ is compact on Bun_G. It follows that $W_{\psi,I}$ is compact.

Finally, by Lemma 19.2, the endofunctor T_F preserves compact objects. Thus $F * W_\psi \simeq F * W_{\psi,I}$ is compact. □

Remark 19.14. The only step that is not yet written in complete detail in this section is the explicit comparison between $\mathcal{O}_{\mathrm{LocSys}_{\check{G},I}} * W_\psi$ and the direct sum of Helm's universal co-Whittaker modules on the neutral stratum. This comparison is precisely the expected compatibility between:

- the spectral action of Fargues and Scholze on the neutral stratum, and
- the integral Bernstein center description of the generic representation theory of GL_n.

Making this comparison completely formal will be one of the main "local calculations" to include in a full write-up.

19.7 Compatibility with constant term and reduction to Levi subgroups

We end by recording that the compactness statement is compatible with parabolic restriction, in a form that uses the constant term computation of Section 22.

Let $P \subset G$ be a parabolic with Levi quotient M, and let P^- be the opposite parabolic. Let $f_M^G : \mathrm{LocSys}_{\check{G}} \to \mathrm{LocSys}_{\check{M}}$ be the induced morphism of parameter stacks.

Proposition 19.15 (Constant term preserves compactness for Whittaker translates)**.** *Assume Theorem 19.1 holds for the Levi M. Then for every $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ of quasi-compact support, the object $\mathrm{CT}_{P^-}(F * W_\psi)$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_\omega$.*

Proof. By Corollary 22.6 one has

$$\mathrm{CT}_{P^-}(F * W_\psi) \;\simeq\; (f_M^G)^*(F) * W_{\psi_M}.$$

The pullback $(f_M^G)^*(F)$ is again perfect with quasi-compact support. By the compactness theorem for M, the right-hand side is compact. □

Remark 19.16. Proposition 19.15 is the first indication that compactness of Whittaker translates should be accessible by induction on semisimple rank, once the representation-theoretic finiteness package is established uniformly for Levi subgroups.

20 Spectral dévissage and extension of the Whittaker functor

In this section we address **Task 2** from Section 18.9:

- establish a workable dévissage principle on the spectral side (a description of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ and its compact objects in terms of explicit generators), and
- explain how, once Task 1 is known, the Whittaker comparison functor extends from perfect complexes to a functor on coherent objects with nilpotent singular support.

The main conceptual point is that the nilpotent singular support condition is formulated inside the category $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$, whereas the spectral action constructed by Fargues and Scholze is initially an action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ on the automorphic category. Thus, to extend the Whittaker functor beyond perfect complexes, one needs a spectral dévissage that produces $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ from explicit compact generators on which we can define images geometrically.

20.1 Ind-coherent sheaves, singular support, and the nilpotent subcategory

We retain the notation and assumptions of Section 14. In particular, we fix a quasi-smooth derived enhancement of the parameter stack $\mathrm{LocSys}_{\check{G}}$ (Axiom 14.7), so that singular support is defined in the sense of Arinkin and Gaitsgory [6].

Definition 20.1 (The nilpotent ind-coherent category)**.** Let $\mathrm{Nilp}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$ be the nilpotent conical substack defined in Axiom 14.12. Define

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \;:=\; \mathrm{IndCoh}_{\mathrm{Nilp}(\mathrm{LocSys}_{\check{G}})}(\mathrm{LocSys}_{\check{G}})$$

to be the full subcategory of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ consisting of objects whose singular support is contained in $\mathrm{Nilp}(\mathrm{LocSys}_{\check{G}})$.

Define

$$\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \;:=\; \Big(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})\Big)^{\mathrm{c}}$$

to be the full subcategory of compact objects (equivalently, coherent objects with nilpotent singular support).

Proposition 20.2 (Compact generation and compacts)**.** *The presentable category* $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ *is compactly generated, and its compact objects are precisely* $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$*.*

Proof sketch. For a quasi-smooth derived stack X locally of finite type, $\mathrm{IndCoh}(X)$ is compactly generated and its compact objects identify with the bounded coherent complexes. The singular support formalism of [6] produces, for each conical closed substack $\mathcal{Y} \subset \mathrm{Sing}(X)$, a reflective (colimit-closed) subcategory $\mathrm{IndCoh}_{\mathcal{Y}}(X)$, and one proves that it is again compactly generated with compact objects those coherent complexes whose singular support is contained in $\mathcal{Y}$. Applying this with $X = \mathrm{LocSys}_{\check{G}}$ and $\mathcal{Y} = \mathrm{Nilp}(\mathrm{LocSys}_{\check{G}})$ gives the claim. □

20.2 Perfect complexes lie in the nilpotent subcategory

The category $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ is the input for the spectral action. We record that perfect complexes have *zero* singular support, hence are automatically nilpotent.

Lemma 20.3 (Perfect implies nilpotent singular support)**.** *The natural functor* $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ *factors through* $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$*. Equivalently, every perfect complex has singular support contained in the zero section, hence in* $\mathrm{Nilp}(\mathrm{LocSys}_{\check{G}})$*.*

Proof sketch. This is a general property of singular support in the sense of [6]: objects coming from QCoh (and in particular perfect complexes) have singular support contained in the zero section of $\mathrm{Sing}(X)$. □

Remark 20.4. Lemma 20.3 shows that the Whittaker functor defined on perfect complexes is automatically a functor into the nilpotent ind-coherent category on the spectral side. The difficulty of Task 2 is not to land in the nilpotent subcategory, but to extend beyond perfect complexes.

20.3 Ind-extension from perfect complexes to quasi-coherent sheaves

The action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ on the automorphic category is defined in Theorem 15.13. Since $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ is a small rigid monoidal category and $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is presentable, the action extends formally to an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) = \mathrm{Ind}(\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}))$ by left Kan extension.

Proposition 20.5 (Extension of the spectral action to QCoh)**.** *There is a canonical extension of the spectral action to a colimit-preserving monoidal action*

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \;\curvearrowright\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

whose restriction to $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ *agrees with the original spectral action.*

Proof sketch. The $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$-action is by exact colimit-preserving endofunctors. Since $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ is the ind-completion of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ as a symmetric monoidal category, the universal property of ind-completion gives a unique extension of the action to QCoh. □

Definition 20.6 (The Whittaker functor on quasi-coherent sheaves)**.** Let $\mathcal{W}_\psi$ be the Whittaker generator on Bun_G (Section 16). Define the colimit-preserving functor

$$\Phi_{\mathrm{QCoh}} \;:\; \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega, \qquad \mathcal{F} \longmapsto \mathcal{F} * \mathcal{W}_\psi,$$

using the QCoh-action from Proposition 20.5.

Remark 20.7. The functor Φ_{QCoh} is completely formal once the spectral action is available. What is nontrivial, and was the content of Task 1, is compactness when one restricts from QCoh to perfect complexes of quasi-compact support.

20.4 What Task 2 really requires: generators for $\mathrm{Coh}_{\mathrm{Nilp}}$

Task 2 asks for an extension

$$\Phi^{\mathrm{c}} \;:\; \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{c}}_\omega$$

whose restriction to perfect complexes is $F \mapsto F * \mathcal{W}_\psi$.

At this point it is important to emphasize that, in general, the inclusion $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ is *not* essentially surjective: coherent objects with nilpotent singular support can have nontrivial singularities and need not be perfect. Thus, extending Φ is *not* a tautological thick-closure argument. Instead, one needs a dévissage that presents $\mathrm{Coh}_{\mathrm{Nilp}}$ by explicit compact generators whose geometry can be related to the automorphic side.

The general singular support formalism provides such a dévissage, but we isolate it as an input to be proved (or imported) in our local setting.

Axiom 20.8 (Spectral dévissage by quasi-smooth maps)**.** The category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ is generated under colimits by the essential images of pushforwards

$$f_* : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}),$$

where $f : Y \to \mathrm{LocSys}_{\check{G}}$ ranges over quasi-smooth maps from quasi-compact quasi-smooth derived schemes Y such that the induced map on singularity stacks satisfies

$$\mathrm{Sing}(Y) \;\longrightarrow\; \mathrm{Sing}(\mathrm{LocSys}_{\check{G}}) \quad \text{has image contained in} \quad \mathrm{Nilp}(\mathrm{LocSys}_{\check{G}}).$$

Equivalently, $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ is the smallest idempotent-complete stable subcategory of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ containing all $f_*(\omega_Y)$ for such maps f (where ω_Y denotes the dualizing object on Y).

Remark 20.9. Axiom 20.8 is a local variant of standard generation results in the ind-coherent theory with singular support (compare [6] and the role of "restricted variation" in [13]). In later versions of this paper, we will replace it by an explicit theorem with precise hypotheses on $\mathrm{LocSys}_{\check{G}}$ (for example, quasi-compactness assumptions on the relevant substacks).

20.5 A general extension lemma for compactly generated targets

We now record the purely categorical mechanism that converts Task 1 plus a spectral dévissage into an extension of the Whittaker functor.

Lemma 20.10 (Extension from compact generators)**.** *Let $\mathcal{S}$ be a compactly generated presentable stable ∞-category and let $\mathcal{A}$ be a presentable stable ∞-category. Assume $\mathcal{S}$ is generated under colimits by a set of compact objects $\mathcal{G} \subset \mathcal{S}^{\mathrm{c}}$. Then:*

(a) any assignment $\mathcal{G} \to \mathcal{A}$ extends uniquely (up to contractible choice) to an exact colimit-preserving functor $\mathcal{S} \to \mathcal{A}$ if and only if it is compatible with all finite colimits among objects of $\mathcal{G}$;

(b) the resulting functor carries compact objects of $\mathcal{S}$ to compact objects of $\mathcal{A}$ if and only if it carries every element of $\mathcal{G}$ to a compact object of $\mathcal{A}$.

Proof sketch. This is the standard universal property of compact generation: $\mathcal{S} \simeq \mathrm{Ind}(\mathcal{S}^{\mathrm{c}})$ and colimit-preserving functors out of $\mathcal{S}$ are determined by their values on compact generators. □

20.6 Extension of the Whittaker functor to $\mathrm{Coh}_{\mathrm{Nilp}}$: reduction to images of generators

We can now isolate the exact remaining content of Task 2.

Proposition 20.11 (Task 2 reduction). *Assume:*

(a) Task 1: for every perfect complex F on $\mathrm{LocSys}_{\check{G}}$ with quasi-compact support, $F * \mathcal{W}_\psi$ is compact in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$;

(b) the spectral dévissage axiom (Axiom 20.8); and

(c) for every generator $f_*(\omega_Y)$ appearing in Axiom 20.8, one has a canonically defined compact object

$$\Phi^{\mathrm{c}}_{\mathrm{gen}}\big(f_*(\omega_Y)\big) \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{c}}_\omega$$

compatible with pullback along maps of generators and with the $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$-action.

Then there exists an exact functor

$$\Phi^{\mathrm{c}} : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{c}}_\omega$$

extending the prescription $F \mapsto F * \mathcal{W}_\psi$ on perfect complexes with quasi-compact support. Moreover, Φ^{c} is uniquely determined by its values on the compact generators $f_*(\omega_Y)$.

Proof sketch. By Proposition 20.2 and Axiom 20.8, the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ is compactly generated by the stated compact objects. By Lemma 20.10, any compatible assignment on these generators extends uniquely to an exact colimit-preserving functor

$$\widetilde{\Phi} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$$

carrying generators to compact objects. Restricting $\widetilde{\Phi}$ to compact objects yields the desired functor Φ^{c} on $\mathrm{Coh}_{\mathrm{Nilp}}$. The agreement with $F \mapsto F * \mathcal{W}_\psi$ on perfect complexes follows from (a) and the required compatibility in (c). □

20.7 A concrete proposal for the generators: parabolic (spectral) objects

Proposition 20.11 shifts the problem of Task 2 from "extend from perfect complexes" to "define images of compact generators of $\mathrm{IndCoh}_{\mathrm{Nilp}}$". In geometric Langlands, a natural and representation-theoretically meaningful choice of such generators comes from parabolic induction on the spectral side.

In our local setting, a corresponding supply of quasi-smooth maps $f : Y \to \mathrm{LocSys}_{\check{G}}$ is expected to be given by parameter stacks for dual parabolics: for each parabolic $P \subset G$ with Levi quotient M, let $\check{P} \subset \check{G}$ be the dual parabolic with Levi $\check{M}$, and consider the natural map

$$f_P : \mathrm{LocSys}_{\check{P}} \longrightarrow \mathrm{LocSys}_{\check{G}}.$$

One expects that the objects $f_{P,*}(\omega_{\mathrm{LocSys}_{\check{P}}})$ have nilpotent singular support and that, together with the cuspidal part, they generate $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ under colimits (compare the role of Eisenstein series and gluing in [6]).

We will return to these generators in Task 3 (construction of spectral parabolic functors) and Task 5 (gluing and essential surjectivity). For Task 2, the outcome is:

Remark 20.12 (Conclusion of Task 2 as a reduction). After Task 1 is proved, the remaining work in Task 2 is to pick a concrete generating family $\{f_*(\omega_Y)\}$ for $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ and to define their images in the Whittaker category in a way compatible with the $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$-action. The categorical extension then follows formally from Lemma 20.10.

21 Spectral parabolic functors and parabolic compatibility

In this section we address **Task 3** from Section 18.9: construct spectral parabolic functors and prove parabolic compatibility of the Whittaker functor.

There are three distinct issues:

(i) **Geometry:** define the spectral correspondence $\mathrm{LocSys}_{\check{M}} \xleftarrow{q} \mathrm{LocSys}_{\check{P}} \xrightarrow{p} \mathrm{LocSys}_{\check{G}}$ and verify the finiteness and quasi-smoothness properties needed to define ind-coherent functors.

(ii) **Microlocal control:** show that the functors preserve nilpotent singular support, so that they descend to $\mathrm{IndCoh}_{\mathrm{Nilp}}$.

(iii) **Compatibility:** show that the (conjectural) Whittaker equivalence intertwines automorphic and spectral parabolic functors. In practice, this is established first on perfect (or quasi-coherent) objects, using linearity and the explicit constant term computation of Section 22, and then extended to $\mathrm{IndCoh}_{\mathrm{Nilp}}$ by the dévissage mechanism of Section 20.

We work throughout with $\ell \neq p$ and coefficients Λ of characteristic zero (for simplicity in the discussion of duality and adjunctions on the spectral side).

21.1 The spectral parabolic correspondence

Fix a parabolic subgroup $P \subset G$ and let M be a Levi quotient. Write $\check{P} \subset \check{G}$ and $\check{M} \subset \check{G}$ for the dual parabolic and dual Levi.

Definition 21.1 (Spectral stack with parabolic structure)**.** Let $\mathrm{LocSys}_{\check{G}}$ and $\mathrm{LocSys}_{\check{M}}$ be the parameter stacks of Langlands parameters as in Section 14 (using the Dat–Helm–Kurinczuk–Moss algebraic model). Define $\mathrm{LocSys}_{\check{P}}$ to be the stack classifying *pairs*

$$(\phi, \mathcal{P}),$$

where ϕ is a $\check{G}$-valued Langlands parameter and $\mathcal{P}$ is a reduction of ϕ to $\check{P}$ (up to $\check{P}$-conjugacy). Equivalently, $\mathrm{LocSys}_{\check{P}}$ is the stack of $\check{P}$-valued parameters, modulo $\check{P}$-conjugacy, together with the natural map to $\mathrm{LocSys}_{\check{G}}$ induced by $\check{P} \hookrightarrow \check{G}$.

Remark 21.2 (Two realizations of $\mathrm{LocSys}_{\check{P}}$)*.* There are (at least) two natural realizations of $\mathrm{LocSys}_{\check{P}}$.

(a) **Parameters valued in $\check{P}$:** define $\mathrm{LocSys}_{\check{P}}$ as the moduli of parameters with values in $\check{P}$, and use $\check{P} \hookrightarrow \check{G}$ and $\check{P} \twoheadrightarrow \check{M}$ to get the maps p and q below.

(b) **Reductions of $\check{G}$-parameters:** define $\mathrm{LocSys}_{\check{P}}$ as the moduli of $\check{G}$-parameters equipped with a $\check{P}$-reduction. This has the advantage that the map $p : \mathrm{LocSys}_{\check{P}} \to \mathrm{LocSys}_{\check{G}}$ is represented by a (relative) flag variety and is therefore quasi-compact and often proper-like, which is convenient for defining pushforwards.

In a future version we will choose one construction and verify its properties in the DHKM framework. For the blueprint, it suffices that *some* reasonable $\mathrm{LocSys}_{\check{P}}$ exists with the expected maps.

Definition 21.3 (The parabolic correspondence)**.** Let

$$\mathrm{LocSys}_{\check{M}} \xleftarrow{q} \mathrm{LocSys}_{\check{P}} \xrightarrow{p} \mathrm{LocSys}_{\check{G}}$$

be the correspondence where:

- p forgets the $\check{P}$-reduction (or composes with $\check{P} \hookrightarrow \check{G}$);
- q remembers only the induced $\check{M}$-parameter (or composes with $\check{P} \twoheadrightarrow \check{M}$).

Axiom 21.4 (Geometric hypotheses for spectral parabolic functors)**.** The correspondence of Definition 21.3 admits derived enhancements making all three stacks quasi-smooth, and the morphisms p and q satisfy the finiteness hypotheses required to define the ind-coherent functors p_*, q_*, $p^!$, $q^!$ and to apply base change and projection formulas in the range used below.

Remark 21.5*.* Axiom 21.4 is the local analogue of the algebro-geometric properties of $\mathrm{LocSys}_{\check{P}}$ in global geometric Langlands. In the present local setting, the existence and finiteness properties should ultimately be extracted from the DHKM description of parameter stacks and the properness of the flag variety of parabolics.

21.2 Definition of spectral constant term and Eisenstein functors

Assuming Axiom 21.4, we define spectral parabolic functors by the same formulas as in (derived) geometric Langlands.

Definition 21.6 (Spectral Eisenstein and constant term functors)**.** Define functors

$$\begin{aligned} \mathrm{Eis}_P^{\mathrm{spec}} &:= p_* \circ q^! : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}) \longrightarrow \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}), \\ \mathrm{CT}_P^{\mathrm{spec}} &:= q_* \circ p^! : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) \longrightarrow \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}), \end{aligned}$$

with the usual normalization conventions (suppressed) matching those used on the automorphic side in Section 17.

Remark 21.7 (Adjunction on the spectral side)*.* Under standard finiteness conditions, p_* is right adjoint to $p^!$ and q_* is right adjoint to $q^!$. It follows that $\mathrm{Eis}_P^{\mathrm{spec}}$ is left adjoint to $\mathrm{CT}_P^{\mathrm{spec}}$. A second adjointness statement on the spectral side is expected to hold after replacing P by the opposite parabolic P^-, mirroring Theorem 17.8 on the automorphic side.

21.3 Nilpotent singular support is preserved

The key microlocal input is that $\mathrm{Eis}_P^{\mathrm{spec}}$ and $\mathrm{CT}_P^{\mathrm{spec}}$ preserve the nilpotent singular support condition. This is the local analogue of the corresponding statements in global geometric Langlands [6].

We first recall the general functoriality of singular support for quasi-smooth maps.

Theorem 21.8 (Functoriality of singular support, schematic form)**.** Let $f : X \to Y$ be a quasi-smooth morphism of quasi-smooth derived stacks. Then there are canonical maps of conical stacks

$$\mathrm{Sing}(X) \xleftarrow{f^{\mathrm{Sing}}} \mathrm{Sing}(X \times_Y X) \xrightarrow{g^{\mathrm{Sing}}} \mathrm{Sing}(Y)$$

and one has the following containment statements for singular support:

(a) (*Pullback*) For $\mathcal{F} \in \mathrm{IndCoh}(Y)$,

$$\mathrm{SS}(f^!\mathcal{F}) \ \subset \ (f^{\mathrm{Sing}})^{-1}\big(\mathrm{SS}(\mathcal{F})\big).$$

(b) (*Pushforward*) For $\mathcal{G} \in \mathrm{IndCoh}(X)$ for which f_* is defined,

$$\mathrm{SS}(f_*\mathcal{G}) \ \subset \ \overline{g^{\mathrm{Sing}}(\mathrm{SS}(\mathcal{G}))}.$$

Remark 21.9. Theorem 21.8 summarizes standard properties in the Arinkin–Gaitsgory theory of singular support for ind-coherent sheaves [6]. In a complete write-up, we will replace this schematic statement by a precise citation (or by a tailored lemma) in the form needed here.

We now apply this formalism to the parabolic correspondence.

Definition 21.10 (Nilpotent cones for Levi and parabolic stacks)**.** Let $\mathrm{Nilp}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$ be the nilpotent cone defined in Section 14. Define $\mathrm{Nilp}(\mathrm{LocSys}_{\check{M}})$ and $\mathrm{Nilp}(\mathrm{LocSys}_{\check{P}})$ analogously, using the corresponding nilpotent cones in $\mathrm{Lie}(\check{M})^*$ and $\mathrm{Lie}(\check{P})^*$.

Proposition 21.11 (Parabolic maps respect nilpotent cones)**.** *Under the morphisms induced by $\check{M} \hookrightarrow \check{G}$ and $\check{P} \hookrightarrow \check{G}$, the nilpotent cones map to nilpotent cones. Equivalently, the pullbacks of* $\mathrm{Nilp}(\mathrm{LocSys}_{\check{G}})$ *to* $\mathrm{Sing}(\mathrm{LocSys}_{\check{M}})$ *and* $\mathrm{Sing}(\mathrm{LocSys}_{\check{P}})$ *contain* $\mathrm{Nilp}(\mathrm{LocSys}_{\check{M}})$ *and* $\mathrm{Nilp}(\mathrm{LocSys}_{\check{P}})$*, respectively.*

Proof sketch. At the level of Lie algebras, the inclusion $\mathrm{Lie}(\check{M}) \hookrightarrow \mathrm{Lie}(\check{G})$ sends nilpotent elements to nilpotent elements. The nilpotent singular support condition is defined by pulling back the nilpotent cone along the map $\chi : \mathrm{Sing}(\mathrm{LocSys}_{\check{H}}) \to [\mathrm{Lie}(\check{H})^*/\check{H}]$ (Axiom 14.12). Compatibility of χ with homomorphisms of groups gives the result. ☐

Theorem 21.12 (Spectral parabolic functors preserve nilpotent singular support)**.** Assume Axiom 21.4. Then the functors of Definition 21.6 restrict to functors

$$\mathrm{Eis}_P^{\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}),$$

$$\mathrm{CT}_P^{\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}).$$

Moreover, these restricted functors preserve compact objects, hence induce functors on coherent subcategories:

$$\mathrm{Eis}_P^{\mathrm{spec}} : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}), \qquad \mathrm{CT}_P^{\mathrm{spec}} : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}).$$

Proof sketch. By Theorem 21.8, singular support behaves functorially for $p^!$, $q^!$, p_*, q_*. The nilpotent condition is defined by containment of singular support in a conical closed subset. Proposition 21.11 ensures that the inverse images and direct images of nilpotent cones under the maps induced by p and q remain nilpotent. Therefore the composition $p_*q^!$ and $q_*p^!$ preserve nilpotent support. Compactness preservation follows from compact generation of ind-coherent categories and finiteness hypotheses on p and q (part of Axiom 21.4). □

Remark 21.13. Theorem 21.12 is the precise place where the nilpotent singular support condition is used: without restricting to $\mathrm{IndCoh}_{\mathrm{Nilp}}$, the parabolic functors are not expected to satisfy the required adjunction and gluing properties.

21.4 Parabolic compatibility on the Whittaker generator: constant term

We now turn to compatibility with the automorphic parabolic functors. We begin with constant term, because it is accessible on the automorphic side by the explicit computation of Section 22 and because it is the input from which Eisenstein compatibility can be deduced by adjunction.

Let CT_{P^-} be the normalized automorphic constant term functor (Section 17). Let $\mathrm{CT}^{\mathrm{spec}}_{P^-}$ be the spectral constant term functor of Definition 21.6, restricted to $\mathrm{IndCoh}_{\mathrm{Nilp}}$ by Theorem 21.12.

Proposition 21.14 (Compatibility on perfect objects)**.** *Let $F \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ have quasi-compact support. Then there is a canonical equivalence in $D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)$*

$$\mathrm{CT}_{P^-}\big(F * W_\psi\big) \;\simeq\; \big((f^G_M)^* F\big) * W_{\psi_M},$$

where $f^G_M : \mathrm{LocSys}_{\check{G}} \to \mathrm{LocSys}_{\check{M}}$ is the morphism induced by $\check{M} \hookrightarrow \check{G}$.

Proof. This is exactly Corollary 22.6 proved in Section 22. □

Remark 21.15 (Interpretation)*.* Proposition 21.14 says that, on the objects that generate the Whittaker category on the automorphic side, the effect of constant term is *linear* for the spectral action and is governed on the spectral side by pullback along f^G_M. This is the local analogue of the QCoh-linearity of constant term functors in geometric Langlands.

21.5 A parabolic compatibility conjecture for the full nilpotent category

We now state the desired compatibility in its most useful categorical form.

Conjecture 21.16 (Parabolic compatibility of the Whittaker functor)**.** Let

$$\widetilde{\Phi}_G : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega, \qquad \widetilde{\Phi}_M : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_\omega$$

be the ind-extended Whittaker functors from Section 20. Then there are canonical equivalences of functors

$$\mathrm{CT}_{P^-} \circ \widetilde{\Phi}_G \;\simeq\; \widetilde{\Phi}_M \circ \mathrm{CT}^{\mathrm{spec}}_{P^-},$$

$$\mathrm{Eis}_P \circ \widetilde{\Phi}_M \;\simeq\; \widetilde{\Phi}_G \circ \mathrm{Eis}^{\mathrm{spec}}_P.$$

21.6 Reduction of constant term compatibility to generators

Assume Task 2: the functors $\widetilde{\Phi}_G$ and $\widetilde{\Phi}_M$ exist and are defined on $\mathrm{IndCoh}_{\mathrm{Nilp}}$ by specifying their values on a chosen family of compact generators (Axiom 20.8 and Proposition 20.11).

Proposition 21.17 (Generator reduction for constant term compatibility)**.** *Assume:*

(a) Task 2 and Axiom 20.8 for both $\mathrm{LocSys}_{\check{G}}$ *and* $\mathrm{LocSys}_{\check{M}}$*;*

(b) the functors $\mathrm{CT}^{\mathrm{spec}}_{P^-}$ *and* CT_{P^-} *preserve compact objects (Theorem 21.12 and Theorem 17.6);*

(c) the constant term compatibility holds on the chosen compact generators of $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$*.*

Then constant term compatibility holds on all of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$*:*

$$\mathrm{CT}_{P^-} \circ \widetilde{\Phi}_G \;\simeq\; \widetilde{\Phi}_M \circ \mathrm{CT}^{\mathrm{spec}}_{P^-}.$$

Proof sketch. Both sides are colimit-preserving exact functors between compactly generated categories and preserve compact objects. By compact generation, it suffices to check an equivalence on a set of compact generators. □

21.7 Eisenstein compatibility from constant term compatibility

We now explain the standard Gaitsgory-style maneuver: once constant term compatibility is known, Eisenstein compatibility follows by adjunction, provided we have the right adjoints on both sides.

Proposition 21.18 (Eisenstein compatibility from constant term compatibility)**.** *Assume:*

(a) constant term compatibility holds: $\mathrm{CT}_{P^-} \circ \widetilde{\Phi}_G \simeq \widetilde{\Phi}_M \circ \mathrm{CT}^{\mathrm{spec}}_{P^-}$*;*

(b) Eis_P *is right adjoint to* CT_{P^-} *on the automorphic side (second adjointness, Theorem 17.8);*

(c) $\mathrm{Eis}^{\mathrm{spec}}_P$ *is right adjoint to* $\mathrm{CT}^{\mathrm{spec}}_{P^-}$ *on the spectral side (spectral second adjointness, a consequence of Axiom 21.4 plus duality).*

Then Eisenstein compatibility holds:

$$\mathrm{Eis}_P \circ \widetilde{\Phi}_M \;\simeq\; \widetilde{\Phi}_G \circ \mathrm{Eis}^{\mathrm{spec}}_P.$$

Proof sketch. Let $\mathcal{F} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}})$ and $\mathcal{G} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$. Using the adjunctions in (b) and (c) and constant term compatibility in (a), we obtain canonical equivalences:

$$\mathrm{Hom}\big(\mathrm{Eis}_P(\widetilde{\Phi}_M(\mathcal{F})), \widetilde{\Phi}_G(\mathcal{G})\big) \simeq \mathrm{Hom}\big(\widetilde{\Phi}_M(\mathcal{F}), \mathrm{CT}_{P^-}(\widetilde{\Phi}_G(\mathcal{G}))\big)$$

$$\simeq \mathrm{Hom}\big(\widetilde{\Phi}_M(\mathcal{F}), \widetilde{\Phi}_M(\mathrm{CT}^{\mathrm{spec}}_{P^-}(\mathcal{G}))\big) \simeq \mathrm{Hom}\big(\mathcal{F}, \mathrm{CT}^{\mathrm{spec}}_{P^-}(\mathcal{G})\big) \simeq \mathrm{Hom}\big(\mathrm{Eis}^{\mathrm{spec}}_P(\mathcal{F}), \mathcal{G}\big).$$

By Yoneda, this identifies $\mathrm{Eis}_P \circ \widetilde{\Phi}_M$ with $\widetilde{\Phi}_G \circ \mathrm{Eis}^{\mathrm{spec}}_P$. □

Remark 21.19 (What remains to complete Task 3). The core missing ingredient in Task 3 is to establish spectral second adjointness and, more importantly, to verify constant term compatibility on a set of spectral compact generators. The constant term computation of Section 22 already verifies compatibility on the *perfect* generators coming from $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$. To reach all of $\mathrm{Coh}_{\mathrm{Nilp}}$, one needs the explicit dévissage family of Task 2 and a geometric definition of $\widetilde{\Phi}$ on those generators. Once this is in place, Proposition 21.17 and Proposition 21.18 make the remainder of Task 3 formal.

22 A parabolic calculation for the Whittaker generator

22.1 Setup and statement

Throughout this section we work with coefficients in a finite extension Λ of $\mathbb{Q}_\ell$ for a prime $\ell \neq p$.

Fix a Borel subgroup $B \subset G$ over E, with unipotent radical $U \subset B$, and fix a *non-degenerate* (generic) smooth character

$$\psi\colon U(E) \longrightarrow \Lambda^\times.$$

Write $i_1\colon \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ for the basic (neutral) Harder–Narasimhan stratum, so that $\mathrm{Bun}_G^1 \simeq [*/G(E)]$. Let $\mathcal{D}_{\mathrm{sm}}(G(E), \Lambda)$ denote the (derived) ∞-category of smooth Λ-representations of $G(E)$ (with the usual model in terms of ℓ-adic sheaves on $[*/G(E)]$).

Definition 22.1 (Whittaker generator on Bun_G)**.** Let

$$\mathrm{GG}_{G,\psi} \ := \ \text{c-ind}_{U(E)}^{G(E)}(\psi)$$

be the (smooth) Gelfand–Graev representation. We define the *Whittaker generator* on Bun_G by

$$\mathcal{W}_{G,\psi} \ := \ i_{1,!}\big(\mathrm{GG}_{G,\psi}\big) \ \in \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda),$$

where $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is the derived category of lisse Λ-adic sheaves on Bun_G in the sense of Fargues–Scholze.

Now fix a standard parabolic subgroup $P \subset G$ containing B, with Levi factor M and unipotent radical N. Let $P^- = MN^-$ denote the parabolic opposite to P with respect to M. Set $U_M := U \cap M$ and $\psi_M := \psi|_{U_M(E)}$, which is again non-degenerate.

Let

$$\mathrm{Bun}_M \xleftarrow{q} \mathrm{Bun}_{P^-} \xrightarrow{p} \mathrm{Bun}_G$$

be the natural correspondence. We use the *normalized* constant term functor

$$\mathrm{CT}_{P^-} \ := \ q_* \circ p^! \ : \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda),$$

with the normalizations fixed earlier so that CT_{P^-} is right adjoint to the corresponding normalized Eisenstein series functor (as in the construction of Hamann–Hansen–Scholze).

Theorem 22.2 (Constant term of the Whittaker generator)**.** There is a canonical isomorphism in $D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)$

$$\mathrm{CT}_{P^-}\big(\mathcal{W}_{G,\psi}\big) \ \simeq \ \mathcal{W}_{M,\psi_M}.$$

22.2 Reduction to a representation-theoretic Jacquet module computation

We first record a general fact: on objects supported on the neutral stratum, the geometric constant term reduces to the usual normalized Jacquet module functor on smooth representations.

Proposition 22.3 (Constant term on the neutral stratum)**.** *Let $V \in \mathcal{D}_{\mathrm{sm}}(G(E), \Lambda)$ and view $i_{1,!}V \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. Then $\mathrm{CT}_{P^-}(i_{1,!}V)$ is supported on $\mathrm{Bun}_M^1 \simeq [*/M(E)]$, and there is a canonical identification*

$$\mathrm{CT}_{P^-}(i_{1,!}V) \ \simeq \ i_{1,!}^M\big(J_{P^-}(V)\big),$$

where $i_1^M\colon \mathrm{Bun}_M^1 \hookrightarrow \mathrm{Bun}_M$ is the neutral stratum and J_{P^-} denotes the normalized *Jacquet module functor for the parabolic P^-.*

Proof sketch. Because V is supported on Bun_G^1, the object $p^!(i_{1,!}V)$ is supported on the fiber product

$$\mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^1.$$

On the level of Harder–Narasimhan strata, the preimage of the neutral stratum in Bun_G is exactly the neutral stratum in Bun_{P^-}, hence

$$\mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^1 \;\simeq\; \mathrm{Bun}_{P^-}^1 \;\simeq\; [*/P^-(E)].$$

Under this identification, the map $q\colon \mathrm{Bun}_{P^-}^1 \to \mathrm{Bun}_M^1$ is induced by the quotient homomorphism $P^-(E) \twoheadrightarrow M(E)$, and the functor q_* is the usual derived pushforward along classifying stacks, which identifies with derived (smooth) coinvariants for the subgroup $N^-(E)$. With the chosen normalization of CT_{P^-}, the resulting functor on the level of smooth representations is precisely the normalized Jacquet module J_{P^-}. □

22.3 Jacquet module of the Gelfand–Graev representation

The remaining input is a classical representation-theoretic statement: the Jacquet module of the Gelfand–Graev representation along a parabolic is again a Gelfand–Graev representation for the Levi.

Theorem 22.4 (Bushnell–Henniart; see also Matringe)**.** There is a canonical isomorphism of smooth $M(E)$-representations

$$J_{P^-}\Big(\text{c-ind}_{U(E)}^{G(E)}(\psi)\Big) \;\simeq\; \text{c-ind}_{U_M(E)}^{M(E)}(\psi_M).$$

Reference. This is proved in [10, Theorem 2.2] in the language of compactly supported Whittaker functions; an explicit formulation (including normalizations) is recalled and reproved in [11, Theorem 3.10]. □

22.4 Proof of Theorem 22.2

Proof. Apply Proposition 22.3 to $V = \mathrm{GG}_{G,\psi}$ and use Theorem 22.4. By Definition 22.1, we obtain

$$\mathrm{CT}_{P^-}(\mathcal{W}_{G,\psi}) \;\simeq\; i_{1,!}^M(J_{P^-}(\mathrm{GG}_{G,\psi})) \;\simeq\; i_{1,!}^M(\mathrm{GG}_{M,\psi_M}) \;=\; \mathcal{W}_{M,\psi_M},$$

as claimed. □

22.5 Consequences for the Whittaker subcategory and spectral functoriality

Let $D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)_\omega$ denote the Whittaker-generated subcategory of $D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$, that is, the smallest full stable subcategory closed under colimits and containing $\mathcal{W}_{G,\psi}$ and stable under the spectral action of $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$.

Corollary 22.5 (Constant term preserves the Whittaker-generated subcategory)**.** *The functor CT_{P^-} carries $D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)_\omega$ into $D_{\mathrm{lis}}(\mathrm{Bun}_M,\Lambda)_\omega$ and sends the generator $\mathcal{W}_{G,\psi}$ to the generator $\mathcal{W}_{M,\psi_M}$.*

Proof sketch. Theorem 22.2 gives the statement on generators. Compatibility of constant term functors with Hecke operators (hence with the spectral action) implies stability under the action, and closure under colimits finishes the argument. □

Let $f\colon \mathrm{LocSys}_{\check{M}} \to \mathrm{LocSys}_{\check{G}}$ be the natural map induced by $\check{M} \hookrightarrow \check{G}$. Using the standard description of pullback on quasi-coherent sheaves as relative tensor product,

$$f^*(\mathcal{F}) \;\simeq\; \mathcal{O}_{\mathrm{LocSys}_{\check{M}}} \otimes_{\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}} \mathcal{F},$$

we obtain the following concrete formula on the Whittaker-generated objects.

Corollary 22.6 (Formula on Whittaker translates)**.** *For any $\mathcal{F} \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$, there is a canonical isomorphism*

$$\mathrm{CT}_{P^-}\big(\mathcal{F} \star \mathcal{W}_{G,\psi}\big) \;\simeq\; f^*(\mathcal{F}) \star \mathcal{W}_{M,\psi_M}.$$

Proof sketch. By the definition of the spectral action, the operation $\mathcal{F} \mapsto \mathcal{F} \star (-)$ is $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$-linear. Compatibility of constant term functors with the Hecke category identifies the induced action on the Levi side with pullback along f. It therefore suffices to check the claim for $\mathcal{F} = \mathcal{O}_{\mathrm{LocSys}_{\check{G}}}$, which is exactly Theorem 22.2. ☐

Remark 22.7. Corollary 22.6 is one of the points where the analogy with the characteristic zero geometric Langlands proof strategy becomes visible: on the Whittaker-generated part, parabolic functors are forced by linearity and the identification of the Whittaker generator, and the nontrivial content is reduced to a concrete computation on a single distinguished object.

23 Endomorphisms of the Whittaker generator and the Bernstein center

The goal of this section is to make concrete progress on **Task 4** from Section 18.9: compute $\mathrm{End}(W_\psi)$ and explain how this computation feeds directly into the monadic (Barr–Beck) approach to full faithfulness.

The key point is that W_ψ is supported on the neutral stratum $\mathrm{Bun}_G^1 \simeq B\underline{G(E)}$. Hence its endomorphism ring is the same as the endomorphism ring of the universal Whittaker (Gelfand–Graev) representation of the locally profinite group $G(E)$. This endomorphism ring is classical: it is controlled by the Bernstein center, by work of Bushnell and Henniart, with integral refinements in the case $G = \mathrm{GL}_n$ due to Helm.

23.1 Reduction to the neutral stratum

Recall that $i_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ denotes the locally closed immersion of the neutral Harder–Narasimhan stratum, and that $\mathrm{Bun}_G^1 \simeq B\underline{G(E)}$ (Proposition 16.4).

Lemma 23.1 (Extension by zero is fully faithful on endomorphisms)**.** *Let $\mathcal{F} \in D_{\mathrm{lis}}(\mathrm{Bun}_G^1, \Lambda)$ and set $i_{1,!}\mathcal{F} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. Then the canonical map*

$$\mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G^1,\Lambda)}(\mathcal{F}) \;\longrightarrow\; \mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(i_{1,!}\mathcal{F})$$

is an isomorphism.

Proof sketch. A locally closed immersion factors as an open immersion followed by a closed immersion. For open immersions, extension by zero is fully faithful; for closed immersions, pushforward is fully faithful. Composing yields full faithfulness of $i_{1,!}$ on mapping complexes and hence on endomorphisms. ☐

Proposition 23.2 (Endomorphisms of W_ψ as endomorphisms of the Gelfand–Graev representation)**.** *Let*

$$\mathrm{GG}_{G,\psi} \;:=\; \text{c-ind}_{U(E)}^{G(E)}(\psi)$$

and let $W_\psi = i_{1,!}(\mathrm{GG}_{G,\psi})$ be the Whittaker generator as in Definition 16.6 (equivalently Definition 22.1). Then there is a canonical identification

$$\mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(W_\psi) \;\simeq\; \mathrm{End}_{G(E)}(\mathrm{GG}_{G,\psi}).$$

Proof. Combine Lemma 23.1 with the equivalence $D_{\mathrm{lis}}(\mathrm{Bun}_G^1,\Lambda) \simeq D(\mathrm{Rep}_\Lambda^\infty(G(E)))$ (Proposition 16.4). □

23.2 Bernstein center and the generic Bernstein summand

Let $\mathfrak{Z}(G(E))$ denote the Bernstein center of the category of smooth Λ-representations of $G(E)$ (for Λ a field of characteristic zero, so that the classical Bernstein–Deligne theory applies).

Write

$$\mathrm{Rep}_\Lambda^\infty(G(E)) \;\simeq\; \prod_{\mathfrak{s}} \mathrm{Rep}_\Lambda^\infty(G(E))_{\mathfrak{s}}$$

for the Bernstein decomposition indexed by inertial equivalence classes $\mathfrak{s}$. Let $e_{\mathfrak{s}} \in \mathfrak{Z}(G(E))$ denote the corresponding central idempotent.

Definition 23.3 (Generic Bernstein summand)**.** We say that $\mathfrak{s}$ is *generic* if the block $\mathrm{Rep}_\Lambda^\infty(G(E))_{\mathfrak{s}}$ contains at least one irreducible generic representation, equivalently if

$$e_{\mathfrak{s}} \cdot \mathrm{GG}_{G,\psi} \;\neq\; 0.$$

Let $\mathfrak{Z}_{\mathrm{gen}}(G(E)) \subset \mathfrak{Z}(G(E))$ be the direct product of the centers of the generic blocks:

$$\mathfrak{Z}_{\mathrm{gen}}(G(E)) \;:=\; \prod_{\mathfrak{s}\ \mathrm{generic}} \mathfrak{Z}(G(E))_{\mathfrak{s}} \qquad \text{where} \qquad \mathfrak{Z}(G(E))_{\mathfrak{s}} := e_{\mathfrak{s}}\mathfrak{Z}(G(E)).$$

Remark 23.4. The relevance of $\mathfrak{Z}_{\mathrm{gen}}(G(E))$ is that the universal Whittaker representation $\mathrm{GG}_{G,\psi}$ is supported precisely on the generic blocks. In particular, its endomorphism ring is expected to see exactly $\mathfrak{Z}_{\mathrm{gen}}(G(E))$, not necessarily the full Bernstein center.

23.3 Bushnell–Henniart and Helm: endomorphisms of the universal Whittaker representation

The following theorem packages the representation-theoretic input we will use.

Theorem 23.5 (Bushnell–Henniart; Helm)**.** Assume G is quasi-split over E and Λ is a field of characteristic zero.

(a) For each Bernstein idempotent $e_{\mathfrak{s}}$, the block center $\mathfrak{Z}(G(E))_{\mathfrak{s}}$ acts on $e_{\mathfrak{s}}\mathrm{GG}_{G,\psi}$, and the induced map

$$\mathfrak{Z}(G(E))_{\mathfrak{s}} \;\longrightarrow\; \mathrm{End}_{G(E)}(e_{\mathfrak{s}}\mathrm{GG}_{G,\psi})$$

is injective (faithful action of the block center on Whittaker models).

(b) In many cases (in particular for $G = \mathrm{GL}_n$), the map in (a) is an isomorphism for every generic $\mathfrak{s}$.

(c) For $G = \mathrm{GL}_n$, Helm proves an integral refinement: after fixing integral coefficients (for example $\Lambda = W(k)$ with k algebraically closed of characteristic $\ell \neq p$), the corresponding block center is the full endomorphism ring of the universal co-Whittaker object in that block. In particular, after inverting ℓ, one recovers the isomorphism in (b).

References. Part (a) and the characteristic zero form of (b) are contained in [10] (see the discussion in the abstract and the results described there). For $G = \mathrm{GL}_n$, the integral statement (c) and the explicit identification of the block center with the endomorphism ring are proved in [12, Theorem 5.2]. □

Combining Theorem 23.5 with Proposition 23.2 yields the geometric form:

Corollary 23.6 (Endomorphisms of W_ψ)**.** *Assume Λ is a field of characteristic zero. Then there is a canonical injection of commutative algebras*

$$\mathfrak{Z}_{\mathrm{gen}}(G(E)) \;\hookrightarrow\; \mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(W_\psi),$$

and for $G = \mathrm{GL}_n$ this map is an isomorphism.

Remark 23.7. Corollary 23.6 makes **Task 4** precise: computing $\mathrm{End}(W_\psi)$ is equivalent to understanding how the Bernstein center acts on the universal Whittaker representation, which is classical and explicit in important cases.

23.4 The stable center map on W_ψ

By the spectral action of Theorem 15.13 (Section **??**), the commutative algebra of global functions on the spectral stack acts on every object of the automorphic category. In particular, it acts on W_ψ.

Definition 23.8 (Stable center map on the Whittaker generator)**.** Let

$$\mathfrak{Z}^{\mathrm{st}}_{\check{G}} \;:=\; \Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}_{\mathrm{LocSys}_{\check{G}}}).$$

The spectral action gives a canonical algebra homomorphism

$$\zeta_\psi \colon \mathfrak{Z}^{\mathrm{st}}_{\check{G}} \longrightarrow \mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(W_\psi).$$

Proposition 23.9 (Factorization through the generic Bernstein center)**.** *Assume G is quasi-split and Λ is a field of characteristic zero. Then ζ_ψ factors canonically through the generic Bernstein center:*

$$\mathfrak{Z}^{\mathrm{st}}_{\check{G}} \;\longrightarrow\; \mathfrak{Z}_{\mathrm{gen}}(G(E)) \;\hookrightarrow\; \mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(W_\psi).$$

For $G = \mathrm{GL}_n$ the last map is an isomorphism by Corollary 23.6.

Proof sketch. By Proposition 23.2 the endomorphism ring of W_ψ is the endomorphism ring of the universal Whittaker representation. Any central action on the automorphic category restricts to a central action on $\mathrm{Rep}^\infty_\Lambda(G(E))$ on the neutral stratum, hence factors through the Bernstein center. Since the universal Whittaker representation is supported only on the generic blocks, the action factors through $\mathfrak{Z}_{\mathrm{gen}}(G(E))$. □

23.5 Compatibility with parabolic restriction

We now record a compatibility statement that is already accessible using the constant term computation from Section 22. It will be used later as the "gluing interface" between centers for G and for Levi subgroups.

Let $P \subset G$ be a parabolic containing B, with Levi quotient M, and let P^- be the opposite parabolic. Let

$$f_M^G \colon \mathrm{LocSys}_{\check{G}} \longrightarrow \mathrm{LocSys}_{\check{M}}$$

be the morphism induced by $\check{M} \hookrightarrow \check{G}$.

Proposition 23.10 (Parabolic compatibility on endomorphisms)**.** *The constant term functor* CT_{P^-} *induces an algebra homomorphism*

$$\mathrm{CT}^{\sharp}_{P^-} : \mathrm{End}(W_\psi) \longrightarrow \mathrm{End}(W_{\psi_M})$$

and, under the maps ζ_ψ *and* ζ_{ψ_M} *of Definition 23.8, the diagram*

$$\begin{tikzcd}
\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \arrow[r, "\zeta_\psi"] \arrow[d, "(f_M^G)^*"'] & \mathrm{End}(W_\psi) \arrow[d, "\mathrm{CT}^{\sharp}_{P^-}"] \\
\Gamma(\mathrm{LocSys}_{\check{M}}, \mathcal{O}) \arrow[r, "\zeta_{\psi_M}"] & \mathrm{End}(W_{\psi_M})
\end{tikzcd}$$

commutes.

Proof sketch. Apply CT_{P^-} to an endomorphism of W_ψ and use Theorem 22.2 to identify $\mathrm{CT}_{P^-}(W_\psi) \simeq W_{\psi_M}$. Commutativity of the diagram is a reformulation of Corollary 22.6 in degree zero: constant term is linear with respect to the spectral action, and on the spectral side this linearity is precisely pullback along f_M^G. □

23.6 A clean reduction for Task 4

We end with a formulation that makes the remaining content of **Task 4** entirely explicit.

Conjecture 23.11 (Stable center identification on the Whittaker generator)**.** The map ζ_ψ of Definition 23.8 is an isomorphism:

$$\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \xrightarrow{\ \sim\ } \mathrm{End}(W_\psi).$$

Equivalently, the stable Bernstein center coincides with the endomorphism algebra of the Whittaker generator.

Remark 23.12 (Translation to classical representation theory)*.* By Corollary 23.6, Conjecture 23.11 is equivalent to the statement that the canonical map

$$\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \longrightarrow \mathfrak{Z}_{\mathrm{gen}}(G(E))$$

is an isomorphism. For $G = \mathrm{GL}_n$ this is the expected comparison between the stable center and the Bernstein center in the absence of endoscopy, and it is compatible with Helm's interpretation of integral centers in terms of Galois deformation theory [12].

24 Monadicity and full faithfulness

In this section we address **Task 4** from Section 18.9:

- construct and prove *conservativity* of the right adjoint to the Whittaker functor, and
- reduce *full faithfulness* to an explicit computation of the endomorphism algebra of the Whittaker generator, together with a verification on a set of compact generators on the spectral side.

The point is that the Whittaker generator is expected to play the role of a "vacuum" object in a Gaitsgory-style proof: the entire comparison is forced by how the spectral monoidal category acts on this one object, and full faithfulness becomes a monadic statement.

Throughout, we work with coefficients in a finite extension Λ of $\mathbb{Q}_\ell$ with $\ell \neq p$ and assume G is quasi-split with fixed Whittaker data (B, ψ).

24.1 The Whittaker functor and its right adjoint

Assume Tasks 1–2 have been completed, so that the Whittaker assignment extends to a colimit-preserving functor

$$\widetilde{\Phi}_G \;:\; \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$$

whose restriction to perfect complexes with quasi-compact support agrees with $F \mapsto F * \mathcal{W}_{G,\psi}$ (Section 16 and Section 20).

Proposition 24.1 (Existence of a continuous right adjoint)**.** *The functor $\widetilde{\Phi}_G$ admits a continuous (colimit-preserving on compactly generated subcategories) right adjoint*

$$\widetilde{\Phi}_G^R \;:\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega \longrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}).$$

Proof sketch. Both categories are presentable stable ∞-categories, and $\widetilde{\Phi}_G$ preserves colimits by construction (Task 2). The adjoint functor theorem for presentable categories gives the existence of $\widetilde{\Phi}_G^R$. $\square$

The adjunction supplies, for every $\mathcal{F} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ and $\mathcal{A} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$, canonical equivalences

$$\mathrm{Map}\big(\widetilde{\Phi}_G(\mathcal{F}), \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(\mathcal{F}, \widetilde{\Phi}_G^R(\mathcal{A})\big).$$

In particular, evaluating at $\mathcal{F} = \mathcal{O}_{\mathrm{LocSys}_{\check{G}}}$ gives a basic formula relating the right adjoint to maps out of the Whittaker generator:

$$\mathrm{Map}\big(\mathcal{W}_{G,\psi}, \mathcal{A}\big) \;\simeq\; \mathrm{Map}\big(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}, \widetilde{\Phi}_G^R(\mathcal{A})\big). \tag{1}$$

24.2 Conservativity of the right adjoint

The first part of Task 4 is to show that the right adjoint is conservative. This is a formal consequence of the fact that $\mathcal{W}_{G,\psi}$ generates the Whittaker category by construction.

Lemma 24.2 (Generators detect vanishing)**.** *Let $\mathcal{C}$ be a presentable stable ∞-category and let $G \in \mathcal{C}$. Assume $\mathcal{C}$ is the smallest full stable subcategory closed under colimits that contains G. Then the functor*

$$\mathrm{Map}(G, -) \;:\; \mathcal{C} \longrightarrow \mathrm{Sp}$$

is conservative: if $\mathrm{Map}(G, X) \simeq 0$ then $X \simeq 0$.

Proof. Let $\mathcal{C}_X := \{Y \in \mathcal{C} \mid \mathrm{Map}(Y,X) \simeq 0\}$. Because $\mathcal{C}$ is stable and presentable, $\mathcal{C}_X$ is a full stable subcategory closed under colimits. If $\mathrm{Map}(G,X) \simeq 0$ then $G \in \mathcal{C}_X$, hence $\mathcal{C}_X = \mathcal{C}$ by the generation hypothesis. In particular, $X \in \mathcal{C}_X$, so $\mathrm{Map}(X,X) \simeq 0$, which forces $X \simeq 0$. $\square$

Proposition 24.3 (Conservativity of $\widetilde{\Phi}_G^R$)**.** *The right adjoint $\widetilde{\Phi}_G^R$ is conservative on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$.*

Proof. Suppose $\widetilde{\Phi}_G^R(\mathcal{A}) \simeq 0$. Then the right-hand side of (1) vanishes, hence $\mathrm{Map}(\mathcal{W}_{G,\psi}, \mathcal{A}) \simeq 0$. Since $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is, by definition, generated under colimits by $\mathcal{W}_{G,\psi}$ and stability (Definition 16.11), Lemma 24.2 implies $\mathcal{A} \simeq 0$. $\square$

Remark 24.4. Proposition 24.3 is the clean "right adjoint is conservative" input needed for Barr–Beck type arguments. The proof uses no geometry beyond the definition of the Whittaker subcategory.

24.3 Endomorphisms of the Whittaker generator

We now turn to the second part of Task 4: compute $\mathrm{End}(\mathcal{W}_{G,\psi})$ and relate it to the spectral center. We restate the key outcome from Section 23 in a form tailored for monadicity.

Proposition 24.5 (Endomorphisms reduce to the neutral stratum)**.** *There is a canonical identification*

$$\mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(\mathcal{W}_{G,\psi}) \;\simeq\; \mathrm{End}_{G(E)}\left(\text{c-ind}_{U(E)}^{G(E)}(\psi)\right).$$

Proof. This is Proposition 23.2. $\square$

Let $\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O})$ denote the algebra of global functions on the parameter stack. The spectral action (Theorem 15.13) yields an algebra map

$$\zeta_\psi \;:\; \Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \longrightarrow \mathrm{End}(\mathcal{W}_{G,\psi}).$$

Theorem 24.6 (Generic Bernstein center description)**.** Assume Λ has characteristic zero. There exists a canonical factorization

$$\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \;\longrightarrow\; \mathfrak{Z}_{\mathrm{gen}}(G(E)) \;\longrightarrow\; \mathrm{End}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(\mathcal{W}_{G,\psi}),$$

where $\mathfrak{Z}_{\mathrm{gen}}(G(E))$ denotes the product of the Bernstein centers of the generic Bernstein blocks. For $G = \mathrm{GL}_n$ the second arrow is an isomorphism.

Proof sketch. This is Proposition 23.9 together with Corollary 23.6, which in turn use the representation-theoretic input of Bushnell–Henniart and Helm (Theorem 23.5). $\square$

Conjecture 24.7 (Stable center identification on the Whittaker generator)**.** The map ζ_ψ is an isomorphism:

$$\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O}) \;\xrightarrow{\sim}\; \mathrm{End}(\mathcal{W}_{G,\psi}).$$

Remark 24.8. Conjecture 24.7 is the *precise* endomorphism computation needed for the monadic step below. It is the local categorical incarnation of the identification of the stable Bernstein center with functions on the parameter stack.

24.4 A full faithfulness criterion on perfect complexes

Before stating the monadic argument on the full nilpotent category, it is useful to record a concrete "rank-one" full faithfulness statement that already follows formally from Conjecture 24.7.

Let Φ_{Perf} be the functor on perfect complexes with quasi-compact support

$$\Phi_{\mathrm{Perf}} \ : \ \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})_{\mathrm{q.c.}} \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{\mathrm{c}}_{\omega}, \qquad F \longmapsto F * \mathcal{W}_{G,\psi}.$$

Proposition 24.9 (Full faithfulness on perfect complexes from the center)**.** *Assume Conjecture 24.7. Then Φ_{Perf} is fully faithful.*

Proof. Let $F, G \in \mathrm{Perf}(\mathrm{LocSys}_{\check{G}})_{\mathrm{q.c.}}$. Because $\mathrm{Perf}(\mathrm{LocSys}_{\check{G}})$ is rigid, the dual $F^\vee$ exists and the spectral action satisfies

$$\mathrm{Map}\big(F * \mathcal{W}_{G,\psi},\, G * \mathcal{W}_{G,\psi}\big) \ \simeq \ \mathrm{Map}\big(\mathcal{W}_{G,\psi},\, (F^\vee \otimes G) * \mathcal{W}_{G,\psi}\big).$$

Using the adjunction (1) for $\widetilde{\Phi}_G$ restricted to perfect objects, the right-hand side identifies with

$$\mathrm{Map}\big(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}},\, F^\vee \otimes G\big)$$

provided the action of $\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O})$ on $\mathcal{W}_{G,\psi}$ coincides with the tautological action on the unit, i.e. provided $\mathrm{End}(\mathcal{W}_{G,\psi}) \simeq \Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O})$. This is exactly Conjecture 24.7. Finally, because F is dualizable,

$$\mathrm{Map}\big(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}, F^\vee \otimes G\big) \ \simeq \ \mathrm{Map}(F, G),$$

which proves full faithfulness. □

Remark 24.10. Proposition 24.9 is a genuine "monoidal" step: it shows that, on dualizable (perfect) objects, full faithfulness is controlled by a single endomorphism algebra computation. Extending beyond perfect complexes requires the dévissage and monadic arguments below.

24.5 Barr–Beck reduction for full faithfulness on $\mathrm{IndCoh}_{\mathrm{Nilp}}$

We now formulate the monadic reduction that is the real content of Task 4.

Lemma 24.11 (Full faithfulness and the unit map)**.** *Let $L : \mathcal{C} \to \mathcal{D}$ be a functor with right adjoint R. Then L is fully faithful if and only if the unit $\eta : \mathrm{id}_{\mathcal{C}} \to RL$ is an isomorphism. Moreover, if $\mathcal{C}$ is compactly generated and RL preserves colimits, it suffices to check that η is an isomorphism on a set of compact generators of $\mathcal{C}$.*

Proof. The first statement is standard for adjunctions. For the second, the full subcategory of objects on which η is an isomorphism is stable under colimits and cones; if it contains a set of compact generators, it must be all of $\mathcal{C}$. □

Apply Lemma 24.11 to $L = \widetilde{\Phi}_G$ and $R = \widetilde{\Phi}_G^R$. Set

$$\mathbb{T}_G \ := \ \widetilde{\Phi}_G^R \circ \widetilde{\Phi}_G$$

for the resulting monad on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$.

Proposition 24.12 (Monadic reduction for full faithfulness)**.** *Assume:*

(a) $\widetilde{\Phi}_G$ exists and preserves colimits (Task 2),

(b) $\widetilde{\Phi}_G^R$ *is conservative (Proposition 24.3),*

(c) $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ *is compactly generated by a set of compact objects* $\mathcal{G} \subset \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$*, and*

(d) for every $\mathcal{E} \in \mathcal{G}$*, the unit map* $\mathcal{E} \to \mathbb{T}_G(\mathcal{E})$ *is an isomorphism.*

Then $\widetilde{\Phi}_G$ *is fully faithful. In particular, its restriction*

$$\Phi_G^{\mathrm{c}} : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega^{\mathrm{c}}$$

is fully faithful.

Proof. By Lemma 24.11, full faithfulness is equivalent to $\eta : \mathrm{id} \to \mathbb{T}_G$ being an isomorphism. The hypotheses guarantee that it is an isomorphism on compact generators, hence on all objects. □

Remark 24.13 (How the endomorphism computation enters). Evaluating $\mathbb{T}_G$ on the unit object gives

$$\mathbb{T}_G(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}) \;\simeq\; \widetilde{\Phi}_G^R(\mathcal{W}_{G,\psi}),$$

and the algebra of endomorphisms of $\mathcal{W}_{G,\psi}$ controls this object via (1). In particular, Conjecture 24.7 is the statement that the induced map on π_0-endomorphisms agrees with $\Gamma(\mathrm{LocSys}_{\check{G}}, \mathcal{O})$. To upgrade this to condition (d) in Proposition 24.12, one must check the unit isomorphism on a generating set $\mathcal{G}$. Axiom 20.8 from Task 2 provides a candidate generating set, and Task 3 supplies the parabolic functoriality needed to propagate the unit calculation from the unit object to these generators.

24.6 Parabolic functoriality and propagation of the unit calculation

We record one concrete propagation mechanism already available from Section 22: compatibility with constant term functors.

Let $P \subset G$ be a parabolic with Levi quotient M and opposite parabolic P^-. Assume Task 3, so that spectral parabolic functors exist and constant term compatibility holds:

$$\mathrm{CT}_{P^-} \circ \widetilde{\Phi}_G \;\simeq\; \widetilde{\Phi}_M \circ \mathrm{CT}_{P^-}^{\mathrm{spec}}.$$

Proposition 24.14 (Unit isomorphisms descend under constant term)**.** *Assume:*

(a) $\widetilde{\Phi}_G$ *and* $\widetilde{\Phi}_M$ *exist with right adjoints,*

(b) constant term compatibility holds as above, and

(c) the unit map $\mathcal{E} \to \mathbb{T}_G(\mathcal{E})$ *is an isomorphism for some* $\mathcal{E} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$*.*

Then the unit map

$$\mathrm{CT}_{P^-}^{\mathrm{spec}}(\mathcal{E}) \;\longrightarrow\; \mathbb{T}_M\big(\mathrm{CT}_{P^-}^{\mathrm{spec}}(\mathcal{E})\big)$$

is an isomorphism in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}})$*.*

Proof sketch. The compatibility identifies CT_{P^-} with a functor commuting with the Whittaker functors. Passing to right adjoints yields a compatible identification of the monads $\mathbb{T}_G$ and $\mathbb{T}_M$ after applying $\mathrm{CT}_{P^-}^{\mathrm{spec}}$. Thus the unit isomorphism for $\mathcal{E}$ implies the unit isomorphism after constant term. □

Remark 24.15. Proposition 24.14 is the local analogue of the way parabolic functoriality is used in the proof of geometric Langlands to reduce full faithfulness from G to Levi subgroups. In the present setting it provides an induction mechanism for verifying condition (d) in Proposition 24.12.

24.7 Summary of Task 4

We summarize the concrete output of this section as follows.

- The right adjoint $\widetilde{\Phi}_G^R$ exists and is conservative (Propositions 24.1 and 24.3).
- Full faithfulness of $\widetilde{\Phi}_G$ reduces to a unit-isomorphism statement for the monad $\mathbb{T}_G = \widetilde{\Phi}_G^R \widetilde{\Phi}_G$ on a set of compact generators (Proposition 24.12).
- The base case of this unit calculation is controlled by the endomorphism algebra of the Whittaker generator via the stable center map (Conjecture 24.7).
- Parabolic functoriality (Task 3) provides a mechanism to propagate the unit calculation to further generators, reducing full faithfulness to an inductive verification along Levi subgroups (Proposition 24.14).

This completes the Task 4 blueprint.

25 Essential surjectivity by gluing from Levi subgroups

In this section we address **Task 5** from Section 18.9: prove *essential surjectivity* of the Whittaker functor by gluing from Levi subgroups using parabolic functors and second adjointness. This is the local analogue of the "gluing" step in geometric Langlands: one reconstructs the whole category from Eisenstein series from proper Levi subgroups together with a cuspidal part, and then argues by induction on semisimple rank.

We work at the level of presentable categories; at the end we explain how to pass to compact objects.

25.1 The target statement and inductive framework

Assume Tasks 1–2, so that we have a colimit-preserving Whittaker functor

$$\widetilde{\Phi}_G \;:\; \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega.$$

Assume Task 4, so that $\widetilde{\Phi}_G$ is fully faithful (for example by the monadic criteria of Section 24).

Theorem 25.1 (Essential surjectivity template)**.** Assume:

(a) for every proper Levi subgroup M of G, the corresponding functor $\widetilde{\Phi}_M : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_\omega$ is an equivalence (induction hypothesis on semisimple rank),

(b) the functors $\widetilde{\Phi}_H$ are compatible with parabolic constant term and Eisenstein functors for all parabolics (Task 3), and

(c) $\widetilde{\Phi}_G$ induces an equivalence on the cuspidal subcategories (Task 6).

Then $\widetilde{\Phi}_G$ is essentially surjective and hence an equivalence of presentable categories. Consequently its restriction to compact objects

$$\Phi_G^{\mathrm{c}} : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega^{\mathrm{c}}$$

is an equivalence.

The rest of the section explains the gluing mechanism and proves Theorem 25.1 under the stated assumptions.

25.2 Cuspidal and Eisenstein subcategories on the automorphic side

Fix a proper parabolic subgroup $P \subsetneq G$ with Levi quotient M and opposite parabolic P^-. Let

$$\mathrm{CT}_P : \; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda), \qquad \mathrm{Eis}_P : \; D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

be the normalized constant term and Eisenstein series functors of Hamann–Hansen–Scholze (Section 17), so that Eis_P is right adjoint to CT_{P^-} (second adjointness, Theorem 17.8).

Definition 25.2 (Automorphic cuspidal subcategory inside the Whittaker category)**.** Define the cuspidal subcategory of the Whittaker category by

$$D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda)_\omega \;:=\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega \;\cap\; \bigcap_{P \subsetneq G} \ker(\mathrm{CT}_P),$$

where P runs over all proper parabolic subgroups of G (up to conjugacy).

Definition 25.3 (Automorphic Eisenstein-generated subcategory inside the Whittaker category)**.** Let $D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda)_\omega$ be the smallest full stable subcategory of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ that is closed under colimits and contains, for every proper parabolic $P \subsetneq G$ with Levi quotient M, the essential image of the restricted functor

$$\mathrm{Eis}_P : \; D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_\omega \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega.$$

Proposition 25.4 (Automorphic generation in the Whittaker category)**.** *The category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is generated under colimits by $D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda)_\omega$ and $D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda)_\omega$.*

Proof sketch. By Theorem 17.14, the full category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ is generated under colimits by the cuspidal subcategory and by Eisenstein series objects from proper Levi subgroups. Intersecting with the colimit-closed subcategory $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ gives the statement provided one knows that:

- $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is stable under constant term functors and Eisenstein functors from Levi Whittaker categories, and
- the cuspidal condition is detected by vanishing of all constant terms.

The first point follows from the constant term computation for the Whittaker generator (Theorem 22.2 and Corollary 22.5) together with the fact that the Whittaker category is stable under the spectral action and the parabolic functors are compatible with Hecke and hence with the spectral action (Section 17). ☐

25.3 Cuspidal and Eisenstein subcategories on the spectral side

Assume Task 3, so that spectral parabolic functors exist and preserve nilpotent singular support (Theorem 21.12). For a parabolic $P \subset G$ with Levi M, write

$$\mathrm{CT}_P^{\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}), \qquad \mathrm{Eis}_P^{\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

for the spectral constant term and Eisenstein functors.

Definition 25.5 (Spectral cuspidal subcategory)**.** Define the spectral cuspidal subcategory by

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{cusp}} \;:=\; \bigcap_{P \subsetneq G} \ker\left(\mathrm{CT}_P^{\mathrm{spec}}\right).$$

Definition 25.6 (Spectral Eisenstein-generated subcategory)**.** Let $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{Eis}}$ be the smallest full stable subcategory of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ that is closed under colimits and contains the essential images of $\mathrm{Eis}_P^{\mathrm{spec}}$ for all proper parabolics $P \subsetneq G$.

Axiom 25.7 (Spectral generation by cuspidal and Eisenstein parts)**.** The category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$ is generated under colimits by $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{cusp}}$ and $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{Eis}}$.

Remark 25.8. Axiom 25.7 is the local analogue of the gluing statements in global geometric Langlands: nilpotent singular support is expected to be the exact condition that makes parabolic functors behave well enough for such a generation statement to hold, compare [6] and the use of parabolic gluing in the proof strategy of Gaitsgory–Raskin. In a complete write-up, we expect to deduce Axiom 25.7 from a more structural "gluing" theorem on the spectral side (for example using a stratification by Levi types).

25.4 Essential image and closure properties

Let

$$\mathcal{I}_G \ \subset \ D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$$

denote the essential image of $\widetilde{\Phi}_G$.

Lemma 25.9 (The essential image is stable under colimits)**.** *The essential image $\mathcal{I}_G$ is closed under small colimits in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$.*

Proof. The functor $\widetilde{\Phi}_G$ preserves colimits, hence its essential image is closed under colimits. □

Lemma 25.10 (The essential image contains Eisenstein series from Levi images)**.** *Assume Task 3 (parabolic compatibility). Let $P \subsetneq G$ be a parabolic with Levi quotient M. Then*

$$\mathrm{Eis}_P\big(\mathcal{I}_M\big) \ \subset \ \mathcal{I}_G,$$

where $\mathcal{I}_M$ is the essential image of $\widetilde{\Phi}_M$.

Proof. By Eisenstein compatibility (Conjecture 21.16 or its proven form),

$$\mathrm{Eis}_P \circ \widetilde{\Phi}_M \ \simeq \ \widetilde{\Phi}_G \circ \mathrm{Eis}_P^{\mathrm{spec}}.$$

Thus every object in $\mathrm{Eis}_P(\mathcal{I}_M)$ lies in the essential image of $\widetilde{\Phi}_G$. □

25.5 Induction on semisimple rank

Assume the induction hypothesis: for every proper Levi subgroup M of G, the functor $\widetilde{\Phi}_M$ is an equivalence. Then $\mathcal{I}_M = D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_\omega$.

Proposition 25.11 (The essential image contains the Eisenstein-generated subcategory)**.** *Assume Task 3 and the induction hypothesis for all proper Levi subgroups. Then*

$$D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda)_\omega \ \subset \ \mathcal{I}_G.$$

Proof. By Lemma 25.10 and the induction hypothesis, for each proper parabolic $P \subsetneq G$ with Levi quotient M we have

$$\mathrm{Eis}_P\big(D_{\mathrm{lis}}(\mathrm{Bun}_M, \Lambda)_\omega\big) \ = \ \mathrm{Eis}_P(\mathcal{I}_M) \ \subset \ \mathcal{I}_G.$$

Since $\mathcal{I}_G$ is colimit-closed (Lemma 25.9), it contains the colimit-closure of these images, which is exactly $D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda)_\omega$ by definition. □

25.6 Reduction to the cuspidal case

By Proposition 25.4, the Whittaker category is generated under colimits by its cuspidal subcategory and by the Eisenstein-generated subcategory. The previous proposition handles the Eisenstein-generated part. Thus essential surjectivity reduces to showing that cuspidal objects lie in the essential image.

Proposition 25.12 (Essential surjectivity reduces to cuspidal objects)**.** *Assume:*

(a) Task 3 and the induction hypothesis for all proper Levi subgroups, so that $D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda)_\omega \subset \mathcal{I}_G$ *(Proposition 25.11),*

(b) the essential image $\mathcal{I}_G$ *contains* $D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda)_\omega$*, and*

(c) the essential image is closed under colimits (Lemma 25.9).

Then $\widetilde{\Phi}_G$ *is essentially surjective.*

Proof. By (a), $\mathcal{I}_G$ contains $D_{\mathrm{Eis}}(\mathrm{Bun}_G, \Lambda)_\omega$. By (b), it contains the cuspidal subcategory. By (c), it contains the colimit-closed stable subcategory generated by these two subcategories, which is all of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ by Proposition 25.4. Thus $\mathcal{I}_G = D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$. □

Remark 25.13. Condition (b) is exactly the missing input from Task 6: an identification of the cuspidal subcategories on the spectral and automorphic sides. Once that is established, Task 5 completes essential surjectivity by induction.

25.7 Proof of Theorem 25.1

Proof sketch. Assume (a)–(c) of Theorem 25.1. By Task 4, $\widetilde{\Phi}_G$ is fully faithful. By (a) and Task 3, Proposition 25.11 applies, so the essential image contains the Eisenstein-generated subcategory. By (c), the essential image also contains the cuspidal subcategory. Therefore Proposition 25.12 implies essential surjectivity. Hence $\widetilde{\Phi}_G$ is an equivalence of presentable categories.

Finally, since both sides are compactly generated and $\widetilde{\Phi}_G$ preserves compact objects (Task 1), the induced functor on compact objects

$$\Phi_G^{\mathrm{c}} : \mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \to D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega^{\mathrm{c}}$$

is also an equivalence. □

25.8 What remains beyond Task 5

Task 5 reduces the global essential surjectivity problem to two remaining inputs:

- a spectral generation statement (Axiom 25.7, or a theorem replacing it), and
- the cuspidal identification (Task 6), which is expected to be the local analogue of multiplicity one.

Once these are supplied, the combination of Tasks 1–5 yields the full equivalence predicted by Conjecture 16.14.

26 The cuspidal range and multiplicity one

This section addresses **Task 6** from Section 18.9: identify the *cuspidal* subcategories on the spectral and automorphic sides. This is the remaining input in Theorem 25.1 needed to finish essential surjectivity by parabolic gluing (Task 5).

Conceptually, Task 6 is the local analogue of the "multiplicity one" step in the proof of geometric Langlands [9]: one must show that the cuspidal part is rigid enough that it is controlled by a finite amount of stacky spectral data (component groups of centralizers), and that the chosen Whittaker datum singles out the generic member in each packet.

Throughout we work with coefficients in a finite extension Λ of $\mathbb{Q}_\ell$, where $\ell \neq p$.

26.1 Cuspidal parameters and the cuspidal locus on the spectral stack

Let $\mathrm{LocSys}_{\check{G}}$ denote the stack of Langlands parameters used throughout this paper (for example, the Dat–Helm–Kurinczuk–Moss stack, equipped with a quasi-smooth derived enhancement as in Section 14).

Definition 26.1 (Cuspidal parameter)**.** A (geometric) point $\phi \in \mathrm{LocSys}_{\check{G}}(\overline{\Lambda})$ is called *cuspidal* if it does not factor through any proper Levi subgroup of $\check{G}$ (equivalently, it is not induced from a parameter for any proper Levi).

Remark 26.2. In classical local Langlands terminology, one often uses "discrete" or "elliptic" for parameters not factoring through proper Levi subgroups. We use the adjective "cuspidal" to emphasize its interaction with vanishing of constant terms.

Definition 26.3 (Cuspidal locus)**.** Let $\mathrm{LocSys}_{\check{G}}^{\mathrm{cusp}} \subset \mathrm{LocSys}_{\check{G}}$ be the full substack consisting of cuspidal parameters.

A defining expectation is that cuspidality on the spectral side is detected by vanishing of spectral constant terms.

Conjecture 26.4 (Cuspidality detected by spectral constant term)**.** Let $\mathcal{F} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$. Then $\mathcal{F}$ belongs to the spectral cuspidal subcategory

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{cusp}} \;:=\; \bigcap_{P \subsetneq G} \ker(\mathrm{CT}_P^{\mathrm{spec}})$$

if and only if $\mathcal{F}$ is set-theoretically supported on $\mathrm{LocSys}_{\check{G}}^{\mathrm{cusp}}$.

Remark 26.5. Conjecture 26.4 is the spectral counterpart of the automorphic definition of cuspidal objects by vanishing of constant terms (Definition 25.2). It is one place where the nilpotent singular support condition is expected to matter: it should prevent pathological extensions supported on noncuspidal strata from surviving in the cuspidal intersection of kernels.

26.2 Residual gerbes and component groups

Let $\phi \in \mathrm{LocSys}_{\check{G}}^{\mathrm{cusp}}(\overline{\Lambda})$ be a cuspidal parameter. Write $Z_{\check{G}}(\phi)$ for its centralizer in $\check{G}$ and

$$S_\phi \;:=\; \pi_0\big(Z_{\check{G}}(\phi)\big)$$

for the component group.

Let $\mathcal{G}_\phi \hookrightarrow \mathrm{LocSys}_{\check{G}}$ denote the residual gerbe at ϕ. In good situations (for example on a Deligne–Mumford locus) this residual gerbe is isomorphic to the classifying stack $BZ_{\check{G}}(\phi)$.

A particularly clean situation occurs when $Z_{\check{G}}(\phi)$ is finite, or at least has finite component group and trivial Lie algebra in the relevant directions.

Definition 26.6 (Strongly cuspidal point)**.** A cuspidal point ϕ is called *strongly cuspidal* if the residual gerbe $\mathcal{G}_\phi$ is Deligne–Mumford and has finite automorphism group, so that $\mathcal{G}_\phi \simeq BS_\phi$ for a finite group S_ϕ.

Proposition 26.7 (Nilpotent singular support is automatic on a finite residual gerbe)**.** *Let ϕ be strongly cuspidal, so that $\mathcal{G}_\phi \simeq BS_\phi$ with S_ϕ finite. Then the nilpotent singular support condition on* $\mathrm{IndCoh}(\mathcal{G}_\phi)$ *is automatic:*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathcal{G}_\phi) \;=\; \mathrm{IndCoh}(\mathcal{G}_\phi).$$

Moreover, on compact objects one has a canonical identification

$$\mathrm{Coh}(\mathcal{G}_\phi) \;\simeq\; \mathrm{Rep}_\Lambda(S_\phi)^{\mathrm{fd}},$$

and on ind-completions

$$\mathrm{IndCoh}(\mathcal{G}_\phi) \;\simeq\; D\big(\mathrm{Rep}_\Lambda(S_\phi)\big),$$

the derived category of (possibly infinite-dimensional) Λ-representations of S_ϕ.

Proof sketch. If $\mathcal{G}_\phi \simeq BS_\phi$ with S_ϕ finite, then $\mathcal{G}_\phi$ is smooth of dimension zero, so its singularity stack is the zero section. Hence every ind-coherent sheaf has singular support contained in the zero section, which is nilpotent. The identification $\mathrm{Coh}(BS_\phi) \simeq \mathrm{Rep}_\Lambda(S_\phi)^{\mathrm{fd}}$ is standard. Ind-completing yields the derived representation category. □

Remark 26.8. Proposition 26.7 explains why the cuspidal range is expected to be the easiest spectral region: the microlocal condition becomes trivial, and the entire subtlety is encoded by the *stackiness* (component groups).

26.3 Localizing the automorphic category at a cuspidal parameter

Let $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ be the Whittaker-generated automorphic category (Definition 16.11), equipped with the spectral action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ (Proposition 20.5).

Let ϕ be a strongly cuspidal point and let $i_\phi : \mathcal{G}_\phi \hookrightarrow \mathrm{LocSys}_{\check{G}}$ be its residual gerbe. Let $\mathcal{O}_{\mathcal{G}_\phi} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ denote the extension by zero of the structure sheaf of $\mathcal{G}_\phi$.

Definition 26.9 (Cuspidal block of the Whittaker category)**.** Define the *ϕ-block* of the Whittaker category to be the full subcategory

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\omega,\phi} \;:=\; \mathcal{O}_{\mathcal{G}_\phi} * D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega.$$

Let

$$\mathcal{W}_\phi \;:=\; \mathcal{O}_{\mathcal{G}_\phi} * \mathcal{W}_{G,\psi} \;\in\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\omega,\phi}$$

be the localized Whittaker generator.

Lemma 26.10 (Cuspidal blocks are cuspidal)**.** *Assume Task 3 (parabolic compatibility of constant term with the spectral action). If ϕ is cuspidal, then every object of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\omega,\phi}$ is cuspidal in the sense of Definition 25.2.*

Proof sketch. Let $P \subsetneq G$ be a proper parabolic with Levi quotient M. By constant term compatibility on Whittaker translates (Corollary 22.6) one has, for any $\mathcal{A} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$,

$$\mathrm{CT}_P\big(\mathcal{O}_{\mathcal{G}_\phi} * \mathcal{A}\big) \;\simeq\; \big((f_M^G)^*\mathcal{O}_{\mathcal{G}_\phi}\big) * \mathrm{CT}_P(\mathcal{A}),$$

where $f_M^G : \mathrm{LocSys}_{\check{G}} \to \mathrm{LocSys}_{\check{M}}$ is induced by $\check{M} \hookrightarrow \check{G}$. If ϕ is cuspidal, the pullback $(f_M^G)^*\mathcal{O}_{\mathcal{G}_\phi}$ is zero because ϕ does not factor through $\check{M}$. Therefore CT_P vanishes on $\mathcal{O}_{\mathcal{G}_\phi} * \mathcal{A}$ for every $\mathcal{A}$, as claimed. □

Remark 26.11. Lemma 26.10 is the basic mechanism by which cuspidal spectral support forces cuspidality on the automorphic side. It is the local analogue of the idea that cuspidal Hecke eigensheaves have vanishing constant terms.

26.4 A geometric multiplicity one statement

In the classical representation theory of p-adic groups, a generic irreducible representation admits a Whittaker model and the space of Whittaker functionals is one-dimensional. In our categorical setting, the correct replacement is a statement about maps out of the Whittaker generator.

Definition 26.12 (Whittaker coefficient functor)**.** Define a functor

$$\mathrm{Wh}_\psi \;:\; D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega \longrightarrow D(\Lambda)$$

by

$$\mathrm{Wh}_\psi(\mathcal{A}) \;:=\; \mathrm{RHom}_{D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)}(\mathcal{W}_{G,\psi}, \mathcal{A}).$$

By the adjunction in Task 4 (equation (1)), Wh_ψ factors through the right adjoint $\widetilde{\Phi}_G^R$:

$$\mathrm{Wh}_\psi(\mathcal{A}) \;\simeq\; \mathrm{RHom}_{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})}\big(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}, \widetilde{\Phi}_G^R(\mathcal{A})\big).$$

On the ϕ-block, one expects $\widetilde{\Phi}_G^R$ to land in $\mathrm{IndCoh}(\mathcal{G}_\phi) \simeq D(\mathrm{Rep}_\Lambda(S_\phi))$, so the Whittaker coefficient is expected to compute invariants under S_ϕ.

Conjecture 26.13 (Geometric multiplicity one in the cuspidal range)**.** Let ϕ be strongly cuspidal and let $\mathcal{A} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\omega,\phi}$ be an object corresponding under the expected cuspidal equivalence to an irreducible representation ρ of S_ϕ. Then

$$H^0\big(\mathrm{Wh}_\psi(\mathcal{A})\big) \;\cong\; \begin{cases} \Lambda & \text{if } \rho \text{ is the trivial representation,} \\ 0 & \text{otherwise,} \end{cases}$$

and higher cohomology groups of $\mathrm{Wh}_\psi(\mathcal{A})$ vanish.

Remark 26.14. Conjecture 26.13 is the categorical incarnation of:

- uniqueness of Whittaker models, and
- the Whittaker normalization of local Langlands, which predicts that the generic member of an L-packet corresponds to the trivial representation of the component group.

It is also the precise input needed to identify the *cuspidal block* as the regular module category for $\mathrm{Rep}_\Lambda(S_\phi)$.

26.5 Cuspidal block equivalence and its role in Task 5

We now state the desired cuspidal identification that feeds into Task 5.

Conjecture 26.15 (Cuspidal block equivalence)**.** Let ϕ be strongly cuspidal with residual gerbe $\mathcal{G}_\phi \simeq BS_\phi$. Then the Whittaker functor induces an equivalence of presentable stable ∞-categories

$$\mathrm{IndCoh}(\mathcal{G}_\phi) \xrightarrow{\sim} D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\omega,\phi},$$

compatible with the $\mathrm{QCoh}(\mathcal{G}_\phi) \simeq \mathrm{Rep}_\Lambda(S_\phi)$-module structures. Under this equivalence, the unit object $\mathcal{O}_{\mathcal{G}_\phi}$ corresponds to the localized Whittaker generator $\mathcal{W}_\phi$.

Proposition 26.16 (Task 6 implies the cuspidal input in Task 5)**.** *Assume Conjecture 26.15 holds for all strongly cuspidal points ϕ and that every cuspidal object in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_\omega$ is a colimit of objects lying in cuspidal blocks. Then the induced functor*

$$\widetilde{\Phi}_G \;:\; \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{cusp}} \longrightarrow D_{\mathrm{cusp}}(\mathrm{Bun}_G, \Lambda)_\omega$$

is essentially surjective and fully faithful, hence an equivalence.

Proof sketch. By definition, the spectral cuspidal category is generated under colimits by objects supported on cuspidal residual gerbes (compare Conjecture 26.4). On each gerbe $\mathcal{G}_\phi$, Conjecture 26.15 identifies the image of the restriction of $\widetilde{\Phi}_G$ with the full block $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)_{\omega,\phi}$, which lies in the automorphic cuspidal subcategory by Lemma 26.10. Assuming that cuspidal objects decompose into such blocks, this proves essential surjectivity on the cuspidal subcategory. Full faithfulness follows from Task 4. □

Remark 26.17*.* Proposition 26.16 is the precise bridge between Task 6 and Task 5: once the cuspidal blocks are identified, the Levi-gluing mechanism in Section 25 finishes the full equivalence.

26.6 Sanity checks: tori and general linear groups

Remark 26.18 (Tori)*.* If $G = T$ is a torus, there are no proper parabolic subgroups, so every object is cuspidal. The parameter stack $\mathrm{LocSys}_{\check{T}}$ is (up to mild stackiness coming from automorphisms) an algebraic torus, and the nilpotent singular support condition is the zero-section condition. In this case, Task 6 is essentially the entire conjecture, and the expected equivalence reduces to the description of sheaves on Bun_T in terms of characters, which is compatible with the abelian local Langlands correspondence.

Remark 26.19 ($G = \mathrm{GL}_n$)*.* For $G = \mathrm{GL}_n$, the component groups S_ϕ for cuspidal parameters are expected to be trivial. Thus Conjecture 26.15 predicts that each cuspidal block is equivalent to the derived category of Λ-vector spaces. In other words, the cuspidal block should be generated by a single object (the cuspidal Hecke eigensheaf), and geometric multiplicity one becomes the assertion that its Whittaker coefficient is one-dimensional. This is the categorical avatar of the fact that a discrete series L-packet for GL_n is a singleton.

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