Nullnorms and Uninorms as Generators of Unions and Intersections of Balanced Fuzzy Sets

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Abstract

In everyday life, we collect large amounts of data that describe different states: positive, neutral or negative. This data can describe redundancies, uncertain information and deficiencies. So, gathering, analysing, and processing data have three-state forms, i.e., true, false, and no information, which requires developing extensions of fuzzy sets that allow their description and processing. One such extension is balanced fuzzy sets. Their development resulted in studying the concept of new fuzzy operators and their various properties. Nevertheless, many operators that have been described for classical fuzzy sets can indeed be extended to the range $[-1, 1]$. To support fuzzy operations' bipolarity, we propose using extended to interval $[-1, 1]$ uninorms as unions and nullnorms as intersections. The work also presents the relationships between these operations and fuzzy balanced norms and conorms.

Keywords: balanced fuzzy sets, balanced fuzzy operators, uninorms, nullnorms.

1. Introduction

The problem of transforming data into numerical or symbolic data is still an important topic. However, many existing mathematical models do not consider negative information or cannot handle missing data. Assuming that the knowledge provided by experts is uncertain and incomplete, it may also convey information that carries a positive message (or describes an excess) as well as a negative message (or describes a deficit). The focus should be on exploring new mathematical concepts. Particular attention should be paid to those models that enable modelling a three-state form of information.

Undoubtedly, the fuzzy sets allow us to better analyse the reality around us [16]. It is a fact that many concepts are conceived in a fuzzy way, and the introduction of the degree of membership facilitates the description, processing and interpretation of many ongoing processes. Several works on fuzzy sets have been written describing their theory and applications. Groups of operators on fuzzy sets, particularly triangular norms, triangular conorms (t-norms, s-norms) uninorms and nullnorms, were characterised and studied [9]. However, the problem of not belonging to a set has also become very significant. Thus, intuitionistic fuzzy sets [10] or bipolar fuzzy sets [17] have been introduced in which the degree of set membership and non-membership is given. However, elements and events in the surrounding reality do not have this duality. In 2006, a new idea was introduced: balanced fuzzy sets (BFS) [7]. In balanced fuzzy sets, three states are assumed: yes, no, and neutral (or unknown).

In current data analysis, in addition to the degree of affiliation and membership of the element, 0 has an important role, as it does not mean that it does not belong to a set. However, it is difficult to determine (due to missing, contradictory, or insufficient data) whether an element belongs to a given set or not. When trying to model the belonging and non-belonging of elements to fuzzy sets using classic triangular operations, problems show that they do not properly reflect the human decision-making process. For example, values such as 0.1 should indicate strong disaffiliation because they are closer to the 0 end than the neutral 0.5. Therefore, the union of $\{a|0.6\}$ and $\{b|0.1\}$ should be 0.1, representing a stronger non-membership than 0.6. Thus, when there are strong positive and weak negative stimuli, or vice versa, these values should have different signs and reflect the strength of the message of a given argument. Thus, it recognises the need to use BFS in various data analyses. Work on the properties of operators in BFS is necessary. In this work, we have proved that uninorms and nullnorms can be used as union and intersection operators.

2. Balanced fuzzy sets

Based on [7] and [13], let us recall the basic concepts of balanced fuzzy sets and their operators. The definition of a balanced fuzzy set A in X has the following form:

$$\eta_A(x) = \begin{cases} \mu_A^1(x), & x \text{ in set } X, \\ 0, & \text{when } x \text{ is in and out the set in the same degree} \\ \mu_A^2(x), & x \text{ not in set } X, \end{cases} \tag{1}$$

where $\mu_A^1(x) \in (0, 1]$ and $\mu_A^2(x) \in [-1, 0)$, where $x \in X$.

2.1. Balanced t-norms and t-conorms

Based on the definition of triangular norms and triangular conorms, the definition of balanced norms and conorms was introduced.

Definition 1 ([7], Definition 5)**.** The balanced t-norm TB and t-conorm SB are mappings P : $[-1, 1]^2 \to [-1, 1]$, where P stands for both the balanced norm and balanced conorm satisfy associativity, commutativity, monotonicity, and boundary conditions (i.e.: $S(0, x) = x$ and $T(1, x) = x$ for $x \in [0, 1]$), symmetry (i.e.: $P(x, y) = N(P(N(x), N(y)))$ for $x \neq -y$). The operation $N(x) = -x$ is a reversal operation.

The obvious example is the equivalent of the classic operations: max and min. Papers [7] and [12] discuss the construction and give the formula of balanced operators, hereafter referred to as MAX and MIN, which are respectively balanced conorm and balanced norm.

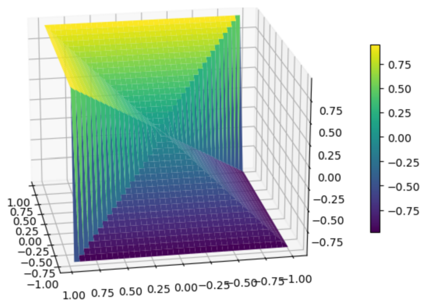
$$MAX(a, b) = \begin{cases} \max(a, b) & a, b \in [0, 1], \\ \min(a, b) & a, b \in [-1, 0), \\ b & |a| < |b|, \ a \cdot b < 0, \\ a & |b| < |a|, \ a \cdot b < 0, \\ a & |b| = |a|, \ a \cdot b < 0. \end{cases} \tag{2}$$

The graph of the function MAX is shown in the Figure 1a and MIN in the Figure 1b.

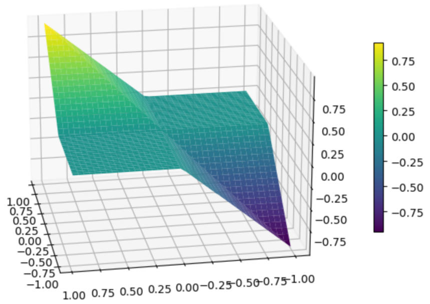
$$MIN(a, b) = \begin{cases} \min(a, b) & a, b \in [0, 1], \\ \max(a, b) & a, b \in [-1, 0], \\ 0 & a \cdot b < 0. \end{cases} \tag{3}$$

2.2. The operation on balanced fuzzy sets

The supplement operator I is defined based on formulas (4) describing fuzzy negation in the interval $(0, 1]$ it determines the complement to the full membership of the set, while for values



(a) Balanced t-conorm MAX.



(b) Balanced t-norm MIN.

Fig. 1. MAX and MIN operations.

$[-1, 0)$, it determines the complement to the full non-membership of the set.

$$I(x) = \begin{cases} n(x) & x > 0, \\ N(n(N(x))) & x < 0, \\ 0 & x = 0. \end{cases} \tag{4}$$

The supplement of the fuzzy balanced set A is the set denoted by A^S and:

$$\eta_A^S(x) = I(\eta_A(x)), \quad \text{for} \ \ x \in X. \tag{5}$$

To introduce the notion of a complement fuzzy balanced set, we first introduce the idea of balanced negation.

Definition 2. A decreasing function $NB : [-1, 1] \to [-1, 1]$ that satisfies the following conditions: $NB(0) = 0$, $x \cdot NB(x) \leq 0$, for all $x \in X$ is called balanced fuzzy negation. NB is strict negation when it is a bijection, and a strong negation when NB is an involution.

The complement of fuzzy balanced set A having the membership functions η_A is the set A^C having the membership function described by the formula:

$$\eta_A^C(x) = NB(\eta_A(x)), \quad \text{for} \ \ x \in X. \tag{6}$$

The union of two fuzzy balanced sets A and B having, respectively, the membership functions η_A and η_B is the set $C = A \cup B$ having the membership function described by the formula:

$$\eta_{A \cup B}(x) = SB(\eta_A(x), \eta_B(x)), \quad \text{for} \ \ x \in X. \tag{7}$$

where SB is balanced conorm.

The intersection of fuzzy balanced sets A and B having, respectively, the membership functions η_A and η_B is the set $C = A \cap B$ having the membership function described by the formula:

$$\eta_{A \cap B}(x) = TB(\eta_A(x), \eta_B(x)), \quad \text{for} \ \ x \in X. \tag{8}$$

where TB is balanced norm.

3. Uninorms

Definition 3 ([15])**.** Operation $U : [0, 1]^2 \to [0, 1]$ is called a uninorm if it is commutative, associative, increasing and has a neutral element $e \in [0, 1]$. A uninorm with neutral element $e = 1$ is called a triangular norm and a uninorm with neutral element $e = 0$ is called a triangular conorm.

The following Theorems can be used to give the general structure of the uninorm (cf. Fig. 2).

Theorem 1 ([5]). *If a uninorm U has a neutral element $e \in (0,1)$, then there exist a triangular norm T and a triangular conorm S such that*

$$U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2. \end{cases}$$

Moreover, $\min \leq U \leq \max$ *in* $A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e)$ *and* $U(0,1), U(1,0) \in \{0,1\}$.

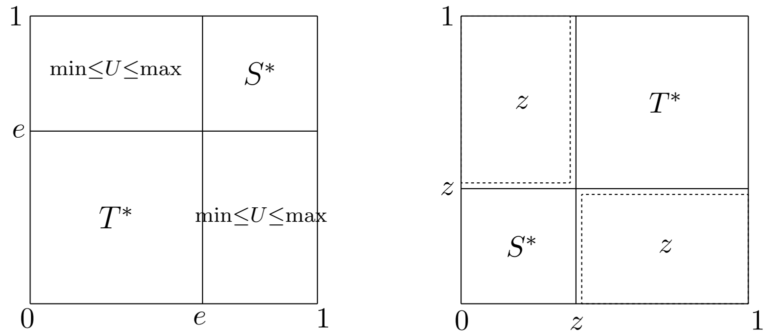


Fig. 2. Diagram of a uninorm (left) and nullnorm (rigth)

The most studied classes of uninorms are:

• Uninorms in $\mathcal{U}_{\min}$ (respectively $\mathcal{U}_{\max}$), those given by minimum (respectively maximum) in $A(e)$, that were characterized in [5].

• Representable uninorms, those that have additive generators.

• Uninorms continuous in the open unit square $(0,1)^2$, that were characterized in [8, 3].

• Idempotent uninorms, those such that $U(x,x) = x$ for all $x \in [0,1]$.

• Locally internal uninorms, those such that $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$.

In what follows, we recall some results about the structure of several classes of uninorms. More uninorm classes and details can be found in [4].

Theorem 2. *([5]) Let* $U : [0,1]^2 \to [0,1]$ *be a uninorm with neutral element* $e \in (0,1)$. *Then, the sections* $x \mapsto U(x,1)$ *and* $x \mapsto U(x,0)$ *are continuous at each point except perhaps at* e *if and only if* U *is given by one of the following formulas:*

(a) If $U(1,0) = 0$, *then*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } x,y \in [0,e] \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } x,y \in [e,1] \\ \min(x,y) & \text{if } x,y \in A(e) \end{cases}$$

(b) If $U(0,1) = 1$, *then the same structure holds, changing minimum by maximum in* $A(e)$.

Example 1. Let $e \in (0,1)$. The following operations are uninorms satisfying the conditions of Theorem 2.

$$U_1(x,y) = \begin{cases} \min(x,y) & \text{if } x,y \leq e, \\ \max(x,y) & otherwise \end{cases} \tag{9}$$

$$U_2(x,y) = \begin{cases} \max(x,y) & \text{if } x,y \geq e, \\ \min(x,y) & otherwise \end{cases} \tag{10}$$

As it turns out, there are no continuous uninorms with the neutral element $e \in (0,1)$. Therefore, there is considered continuity on some subsets.

Theorem 3 ([5]). *Let $U : [0,1]^2 \to [0,1]$ be a binary operation and $e \in (0,1)$. The following statements are equivalent:*

(i) U is a uninorm with neutral element e that is strictly increasing on $]0,1[^2$ and continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$.

(ii) There exists a strictly increasing bijection $u : [0,1] \to [-\infty,+\infty]$ with $u(e) = 0$ such that for all $(x,y) \in [0,1]^2$ it holds that $U(x,y) = u^{-1}(u(x)+u(y))$, where in case of a conjunctive uninorm U, we adopt the convention $(+\infty)+(-\infty) = -\infty$, while in case of a disjunctive uninorm, we adopt the convention $(+\infty)+(-\infty) = +\infty$.

If this representation holds, then the function u is uniquely determined by U up to a positive multiplicative constant, and it is called an additive generator of the uninorm U.

Example 2. Let $e \in (0,1)$. The following operations are uninorms satisfying the conditions of the above Theorem.

$$U_3(x,y) = \begin{cases} 0 & \text{if } x \text{ or } y = 0, \\ \frac{\left(\frac{1}{e}-1\right)^2 xy}{\left(\frac{1}{e}-1\right)^2 xy + \left(\frac{1}{e}-1\right)(1-x)(1-y)} & \text{otherwise,} \end{cases}$$

$$U_4(x,y) = \begin{cases} 1 & \text{if } x \text{ or } y = 1, \\ \frac{\left(\frac{1}{e}-1\right)^2 xy}{\left(\frac{1}{e}-1\right)^2 xy + \left(\frac{1}{e}-1\right)(1-x)(1-y)} & \text{otherwise.} \end{cases}$$

4. Union operations

By analyzing the definition of the union of BFS and uninorms, we present how the latter are used to define the union.

Lemma 1 (see [4]). *Let T be a strict t-norm with additive generator $t : [0,1] \to [0,+\infty]$ and S its dual t-conorm with additive generator $s : [0,1] \to [0,+\infty]$ given by $s(x) = t(1-x)$. Then a function $u : [0,1] \to [-\infty,+\infty]$ given by*

$$u(x) = \begin{cases} -t\,(2x) & if\ x \in [0,0.5] \\ s\,(2x-1) & if\ x \in (0.5,1] \end{cases} = \begin{cases} -t\,(2x) & if\ x \in [0,0.5] \\ t\,(2-2x) & if\ x \in (0.5,1] \end{cases} \tag{11}$$

is an additive generator of representable uninorm with neutral element $e = 0.5$.

Theorem 4. *Let U be a representable uninorm with an additive generator u constructed as in Lemma 1. Let $f : [-1,1] \to [0,1]$ be linear transformation of both intervals given by $f(x) = 0.5x + 0.5$. Then the operations $SB_{-1}, SB_1 : [-1,1]^2 \to [-1,1]$ given by*

$$SB_{-1}(x,y) = \begin{cases} -1 & if\ (x,y) \in \{(-1,1),(1,-1)\} \\ f^{-1} \circ u^{-1}\left(u \circ f(x) + u \circ f(y)\right) & otherwise \end{cases}, \tag{12}$$

$$SB_1(x,y) = \begin{cases} 1 & if\ (x,y) \in \{(-1,1),(1,-1)\} \\ f^{-1} \circ u^{-1}\left(u \circ f(x) + u \circ f(y)\right) & otherwise \end{cases} \tag{13}$$

are balanced t-conorm.

Proof. We only prove that SB_1 is balanced t-conorm. First observe, that $([-1,1], SB_1)$ and $([0,1], U)$ are isomorphic structures. So, SB_1 is associative, commutative and monotonic. Moreover, 0 is its neutral element. To prove condition 5 from Definition 1, due to symmetry we can assume that $y < -x$ and we have, for $x, y < 0$

$N(SB_1(N(x), N(y))) = -SB_1(-x,-y) = -f^{-1}\left(u^{-1}\left(u(f(-x)) + u(f(-y))\right)\right) =$
$-f^{-1}\left(u^{-1}\left(u\left(-\frac{1}{2}x+\frac{1}{2}\right) + u\left(-\frac{1}{2}y+\frac{1}{2}\right)\right)\right) = -f^{-1}\left(u^{-1}\left(t\left(1+x\right)+t\left(1+y\right)\right)\right) =$
$-f^{-1}\left(1-\frac{1}{2}t^{-1}\left(t\left(1+x\right)+t\left(1+y\right)\right)\right) = t^{-1}\left(t\left(1+x\right)+t\left(1+y\right)\right) - 1 =$
$f^{-1}\left(\frac{1}{2}t^{-1}\left(t\left(1+x\right)+t\left(1+y\right)\right)\right) = f^{-1}\left(u^{-1}\left(-t\left(1+x\right)-t\left(1+y\right)\right)\right) =$
$f^{-1}\left(u^{-1}\left(u\left(\frac{1}{2}x+\frac{1}{2}\right)+u\left(\frac{1}{2}y+\frac{1}{2}\right)\right)\right) = f^{-1}\left(u^{-1}\left(u(f(x))+u(f(y))\right)\right) = SB_1(x,y)$

and for $x < 0 < y$

$N(SB_1(N(x), N(y))) = -SB_1(-x,-y) = -f^{-1}\left(u^{-1}\left(u(f(-x)) + u(f(-y))\right)\right) =$
$-f^{-1}\left(u^{-1}\left(u\left(-\frac{1}{2}x+\frac{1}{2}\right) + u\left(-\frac{1}{2}y+\frac{1}{2}\right)\right)\right) = -f^{-1}\left(u^{-1}\left(t\left(1+x\right)-t\left(1-y\right)\right)\right) =$
$-f^{-1}\left(1-\frac{1}{2}t^{-1}\left(t\left(1+x\right)-t\left(1-y\right)\right)\right) = t^{-1}\left(t\left(1+x\right)-t\left(1-y\right)\right) - 1 =$
$f^{-1}\left(\frac{1}{2}t^{-1}\left(t\left(1+x\right)-t\left(1-y\right)\right)\right) = f^{-1}\left(u^{-1}\left(-t\left(1+x\right)+t\left(1-y\right)\right)\right) =$
$f^{-1}\left(u^{-1}\left(u\left(\frac{1}{2}x+\frac{1}{2}\right)+u\left(\frac{1}{2}y+\frac{1}{2}\right)\right)\right) = f^{-1}\left(u^{-1}\left(u(f(x))+u(f(y))\right)\right) = SB_1(x,y)$

the case when $y < 0 < x$ is analogous to the previous one. So, operation SB_1 is balanced t-conorm.

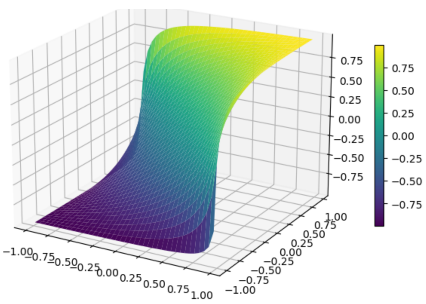
Let us denote the additive generator of SB by $f_s = u \circ f$. □

Example 3. Let $f_s(x) = arc\tanh(x)$, so the corresponding balanced conorms has the following form: $S(x,y) = f_s^{-1}(f_s(x), f_s(y))$, Using that $f_s(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$ and $f_s^{-1}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, where $x \neq -1$ and $x \neq 1$ we obtain: $S(x,y) = f_s^{-1}\left(\frac{1}{2}\ln\frac{1+x}{1-x} + \frac{1}{2}\ln\frac{1+y}{1-y}\right) = f_s^{-1}\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}\right)\right) = \frac{e^{\frac{1}{2}\ln\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}} - e^{-\frac{1}{2}\ln\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}}}{e^{\frac{1}{2}\ln\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}} + e^{-\frac{1}{2}\ln\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}}} = \frac{\sqrt{\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}} - \sqrt{\frac{1-x}{1+x}\cdot\frac{1-y}{1+y}}}{\sqrt{\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}} + \sqrt{\frac{1-x}{1+x}\cdot\frac{1-y}{1+y}}}$.

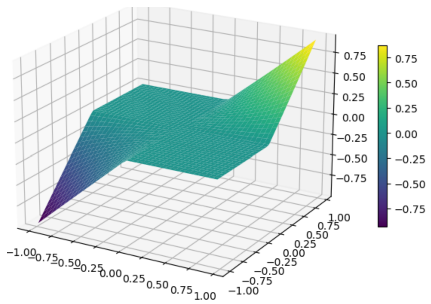
After the reduction we get conorm:

$$SB_{ahtg}(x,y) = \frac{x+y}{1+xy} \tag{14}$$

The balanced conrom from (14) and t-nrom from (18) are illustrated in Figures 3a and 3b.



(a) Balanced t-conorm SB_{ahtg}.



(b) Balanced t-norm TB_{ahtg}.

Fig. 3. Balanced triangular operations obtained from $f_s(x) = arc\tanh(x)$.

Example 4. Let U be a uninorm of the class $\mathcal{U}_{\min}$. Then the transformation f given in Theorem 4 does not lead to a balanced t-conorm. Let us consider the uninorm given by the formula (10). Then $SB(x,y) = f^{-1}(U_2(f(x), f(y)))$ is given by the formula

$$SB(x,y) = \begin{cases} \max(x,y) & \text{if } x, y \geq 0, \\ \min(x,y) & otherwise \end{cases}$$

and $SB\left(-\frac{1}{4}, \frac{1}{2}\right) = -\frac{1}{4} \neq \frac{1}{2} = -SB\left(\frac{1}{4}, -\frac{1}{2}\right)$. So, the symmetry condition from Definition 1 is not fulfilled.

5. Nullnorms

Definition 4 ([11, 1])**.** Increasing binary operation $V : [0,1]^2 \to [0,1]$ is called nullnorm (t-operator) if it is commutative, associative, has a zero element $z \in [0,1]$, and satisfies

$$V(0,x) = x \text{ for } x \le z, \quad V(1,x) = x \text{ for } x \ge z. \tag{15}$$

Triangular norms and conorms are examples of nullnorms with $z \in \{0,1\}$. In general, we have the case of the sum of ordered semigroups with common zero element (cf. Figure 2):

Theorem 5 ([1])**.** *Let $z \in (0,1)$. A binary operation V is a nullnorm with zero element z if and only if there exist triangular norm T and triangular conorm S such that*

$$V(x,y) = \begin{cases} zS\left(\frac{x}{z}, \frac{y}{z}\right) & \text{if } x,y \in [0,z] \\ z + (1-z)T\left(\frac{x-z}{1-z}, \frac{y-z}{1-z}\right) & \text{if } x,y \in [z,1] \\ z & \text{otherwise} \end{cases} \tag{16}$$

Example 5 (cf. [2])**.** Let $z \in [0,1]$. A nullnorm V with zero element z given by

$$V = \begin{cases} \max & \text{in } [0,z]^2 \\ \min & \text{in } [z,1]^2 \\ z & \text{otherwise} \end{cases}$$

is idempotent nullnorm. This is the only idempotent nullnorm with zero element z.

6. Intersection operators

By analyzing the definition of the intersection of BFS and nullnorms, we present how the nullnorms are used to define the intersection. Let V be a nullnorm with zero element $z = 0.5$, underlying t-norm T and its dual t-conorm S given by $S(x,y) = 1 - T(1-x, 1-y)$. Then we have the following relationship between the nullnorm and the balanced norm.

Theorem 6. *Let V be a nullnorm with dual underlying operations T and S. Let $f : [-1,1] \to [0,1]$ be linear transformation of both intervals given by $f(x) = 0.5x + 0.5$. Then the operations $TB : [-1,1]^2 \to [-1,1]$ given by*

$$\begin{aligned} TB(x,y) &= \begin{cases} f^{-1}\left(zS\left(f\left(\frac{x}{z}\right), f\left(\frac{y}{z}\right)\right)\right) & \text{if } x,y \in [-1,0) \\ f^{-1}\left(z + (1-z)T\left(f\left(\frac{x-z}{1-z}\right), f\left(\frac{y-z}{1-z}\right)\right)\right) & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} S(x+1, y+1) - 1 & \text{if } x,y \in [-1,0] \\ T(x,y) & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{17}$$

is balanced norm.

Note that by extending Example 5 to $[-1,1]$ and using $z = 0$ we obtain MIN.

Example 6. Let us determine the balanced norm, for the norm of the Example 3, for $(x,y) \in [0,1]^2$, we get $1 - S(1-x, 1-y) = 1 - \frac{1-x+1-y}{1+(1-x)(1-y)} = \frac{x \cdot y}{2-x-y+x \cdot y}$ and for $(x,y) \in [-1,0]^2$ we obtain $-1 + S(1+x, 1+y) = -1 + \frac{2+x+y}{1+(1+x)(1+y)} = \frac{-x \cdot y}{2+x+y+x \cdot y}$. So, we have

$$TB_{ahtg}(x,y) = \begin{cases} \frac{x \cdot y}{2-x-y+x \cdot y} & \text{in } [0,1]^2 \\ \frac{-x \cdot y}{2+x+y+x \cdot y} & \text{in } [-1,0]^2 \\ 0 & \text{otherwise} \end{cases}. \tag{18}$$

7. Conclusion

BFS is currently the subject of research. Due to the possibility of representativeness of the three-state description of reality, the degree of positivity, neutrality, and negativity which becomes more critical. However, only the necessary mathematical foundations of these sets have been introduced so far. Nevertheless, the first works indicating the potential of BFS have already been published (cf. [14], [6]). Undoubtedly, before building real models, we need to develop a coherent theory that guarantees the correctness and explainability of empirical research. However, research on operators and their properties may take many years (like for fuzzy sets). Thus, showing that we can extend certain known operators to a form suitable as connectives in BFS will speed up work on both the theory of BFS and its applicability. This work, aiming to indicate the validity of using extended uninorms and triangular nullnorms, advances research on the concept of BFS. This work also shows that not all triangular operations can be used as operators on BFS. Further work will be focused on the characterisation of balanced operations.

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