Gradient Descent: The Foundation of Machine Learning Optimization

From Taylor Series to Modern Deep Learning

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Mathematical Foundations

The Big Picture: Why Optimization Matters

Key Points:

Core ML Problem: Find best parameters $\boldsymbol{\theta}^*$ for our model

Examples everywhere:

- Linear regression: Minimize $(y - \mathbf{X}\boldsymbol{\theta})^2$
- Neural networks: Minimize classification/regression loss
- Logistic regression: Minimize cross-entropy loss

Important: The Challenge

Most ML problems have **no closed-form solution!**

Gradient Intuition: Climbing Mountains

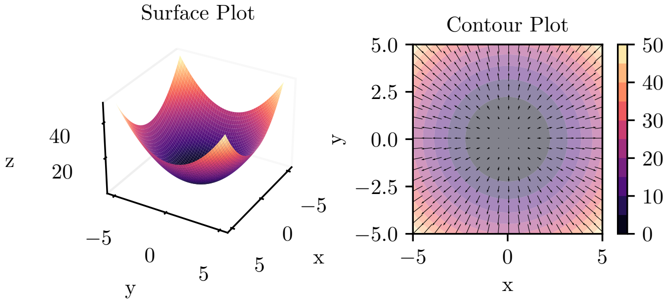
Imagine you're hiking in dense fog and want to reach the valley:

- You can only feel the slope beneath your feet
- **Strategy:** Always step in the steepest downhill direction
- **Gradient** = Direction of steepest uphill (ascent)
- **Negative gradient** = Direction of steepest downhill (descent)

Key Points:

Key insight: Gradient points in direction of steepest ascent
So $-\nabla f$ points in direction of steepest descent!

Geometric Intuition with Level Sets



Mathematical definition: $\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

Taylor Series: The Mathematical Foundation

Why Taylor Series? The Key Insight

Example: The Core Idea

If we can't solve $\min f(\mathbf{x})$ exactly, let's approximate $f(\mathbf{x})$ locally!

Strategy:

- Replace complicated function with simpler approximation
- Optimize the approximation instead
- Move to new point and repeat

Important: Taylor Series Power

Any smooth function can be approximated by polynomials!

Taylor Series: Starting with 1D

Taylor series expansion around point x_0:

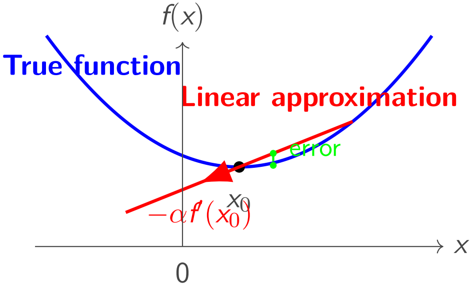
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{6}f'''(x_0)(x - x_0)^3 + \ldots \tag{1}$$

Different orders of approximation:

- **Zero-order:** $f(x) \approx f(x_0)$ (constant)
- **First-order:** $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ (linear)
- **Second-order:** adds $\frac{1}{2}f''(x_0)(x - x_0)^2$ (quadratic)

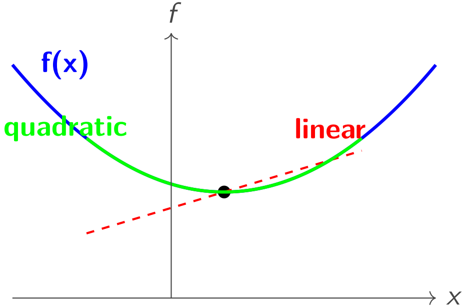
Visual: Tangent Line Approximation

Linear approximation: Use tangent line to approximate function locally



Key insight: Tangent gives best local linear approximation!

Adding Quadratic Term



Key Points:

Higher-order = better approximation, but 1st-order is often sufficient!

Concrete Example: $f(x) = \cos(x)$ at $x_0 = 0$

Let's compute the derivatives:

- $f(0) = \cos(0) = 1$
- $f'(0) = -\sin(0) = 0$
- $f''(0) = -\cos(0) = -1$
- $f'''(0) = \sin(0) = 0$
- $f^{(4)}(0) = \cos(0) = 1$

Taylor approximations:

$$\text{0th order:} \quad f(x) \approx 1 \tag{2}$$

$$\text{2nd order:} \quad f(x) \approx 1 - \frac{x^2}{2} \tag{3}$$

$$\text{4th order:} \quad f(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} \tag{4}$$

Extension to Multiple Variables

For function $f(\mathbf{x})$ around point $\mathbf{x}_0$:

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T\nabla^2 f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \dots \tag{5}$$

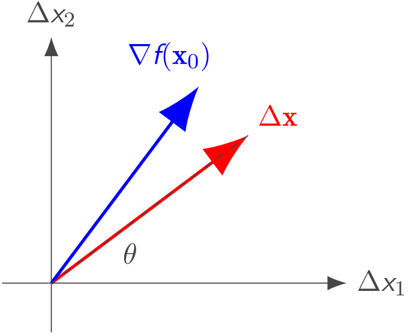
Where:

- $\nabla f(\mathbf{x}_0)$ is the **gradient** (vector of partial derivatives)
- $\nabla^2 f(\mathbf{x}_0)$ is the **Hessian** (matrix of second derivatives)
- $(\mathbf{x} - \mathbf{x}_0) = \Delta\mathbf{x}$ is the step vector

Understanding the Linear Term

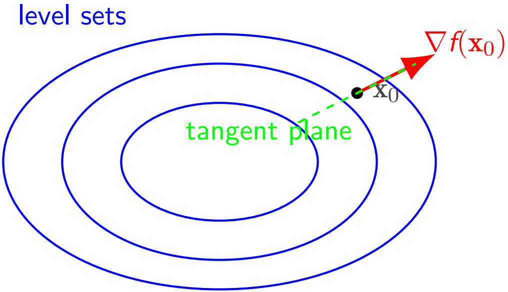
The first-order term: $\nabla f(\mathbf{x}_0)^T \Delta\mathbf{x}$ where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$

For 2D case: $\Delta\mathbf{x} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{0,1} \\ x_2 - x_{0,2} \end{bmatrix}$



Geometric interpretation: $\nabla f(\mathbf{x}_0)^T \Delta\mathbf{x} = |\nabla f||\Delta\mathbf{x}| \cos\theta$

Visual: Multivariate Case with Level Sets



Key Points:

Gradient $\perp$ level sets, tangent plane $\perp$ gradient

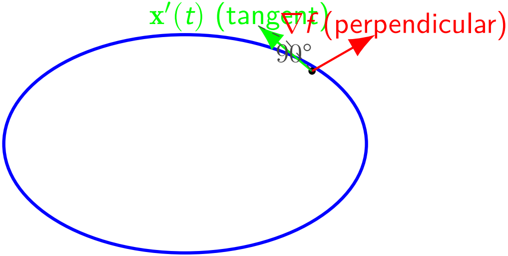
Why is Gradient Perpendicular to Level Sets?

Mathematical insight: Level set $= \{\mathbf{x} : f(\mathbf{x}) = c\}$ for constant c

On level sets: Moving along the level curve keeps $f(\mathbf{x})$ constant

- If $\mathbf{x}(t)$ parameterizes level curve: $f(\mathbf{x}(t)) = c$ (constant)
- Taking derivative: $\frac{d}{dt} f(\mathbf{x}(t)) = \nabla f(\mathbf{x}) \cdot \mathbf{x}'(t) = 0$

Conclusion: $\nabla f(\mathbf{x}) \perp \mathbf{x}'(t)$ for any tangent direction $\mathbf{x}'(t)$



From Taylor Series to Gradient Descent

The Key Question

Goal: Find $\Delta\mathbf{x}$ such that $f(\mathbf{x}_0 + \Delta\mathbf{x}) < f(\mathbf{x}_0)$
Using first-order Taylor approximation:

$$f(\mathbf{x}_0 + \Delta\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^T \Delta\mathbf{x} \tag{6}$$

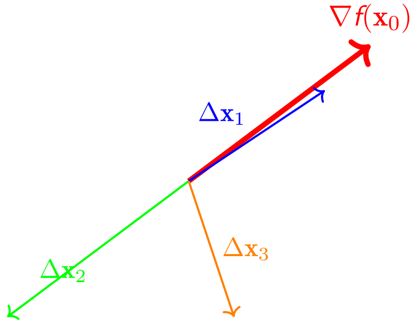
For the function to decrease:

$$\nabla f(\mathbf{x}_0)^T \Delta\mathbf{x} < 0$$

Important: Vector Geometry Reminder

For vectors $\mathbf{a}, \mathbf{b}$: $\mathbf{a}^T\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$
Most negative when: $\cos(\theta) = -1$ (opposite directions!)

Visual Derivation: Finding the Best Direction



Dot products tell us the direction:

- $\nabla f(\mathbf{x}_0)^T \Delta\mathbf{x}_1 > 0$ (increases function)
- $\nabla f(\mathbf{x}_0)^T \Delta\mathbf{x}_2 < 0$ (decreases function - good!)
- $\nabla f(\mathbf{x}_0)^T \Delta\mathbf{x}_3 < 0$ (decreases function)

The Optimal Choice: Direction of Steepest Descent

Definition: Optimal Choice

$$\Delta \mathbf{x} = -\alpha \nabla f(\mathbf{x}_0), \quad \alpha > 0$$

Why this choice?

- $-\nabla f(\mathbf{x}_0)$ points in direction of steepest descent
- $\alpha > 0$ controls the step size
- Guarantees $\nabla f(\mathbf{x}_0)^T \Delta \mathbf{x} < 0$ (function decrease)

Key Points:

This gives us the fundamental gradient descent step!

The Gradient Descent Update Rule

This gives us the gradient descent update:

$$\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}} - \alpha \nabla f(\mathbf{x}_{\text{old}})$$

Definition: Gradient Descent Algorithm

An iterative first-order optimization method for finding local minima

Key properties:

- Uses only first derivatives (gradients)
- Greedy local search
- Guaranteed convergence for convex functions
- Foundation of modern machine learning

Pop Quiz #1: Understanding the Derivation

Answer this!

Consider $f(x) = x^2 + 2$ at point $x_0 = 2$.
Questions:

1. What is $f(x_0)$ and $f'(x_0)$?
2. Write the 1st-order Taylor approximation
3. If we take step $\Delta x = -0.1 \cdot f'(x_0)$, what is our new x?
4. Will the function value decrease?

The Gradient Descent Algorithm

The Complete Algorithm

Algorithm Steps:

1. **Initialize:** Choose starting point $\boldsymbol{\theta}_0$
2. **Repeat until convergence:**
   - Compute gradient: $\mathbf{g}_t = \nabla f(\boldsymbol{\theta}_t)$
   - Update parameters: $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \alpha \mathbf{g}_t$
   - Check stopping criterion

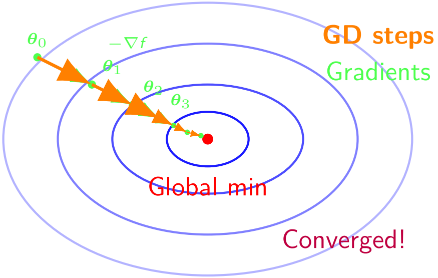
Key hyperparameter: Learning rate α

Key Points:

Learning rate selection is crucial for success!

Animated Gradient Descent in Action

Watch how gradient descent finds the minimum:



Loss surface $f(\boldsymbol{\theta})$

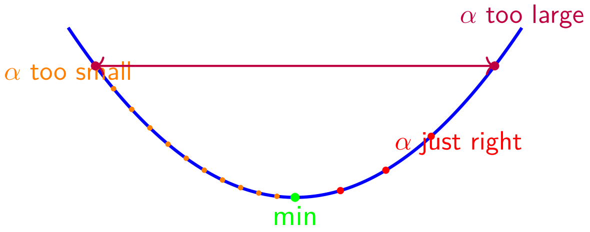
Theorem: Key Insight

Steps get **smaller** as we approach the minimum because $|\nabla f| \to 0$!

Learning Rate: The Step Size

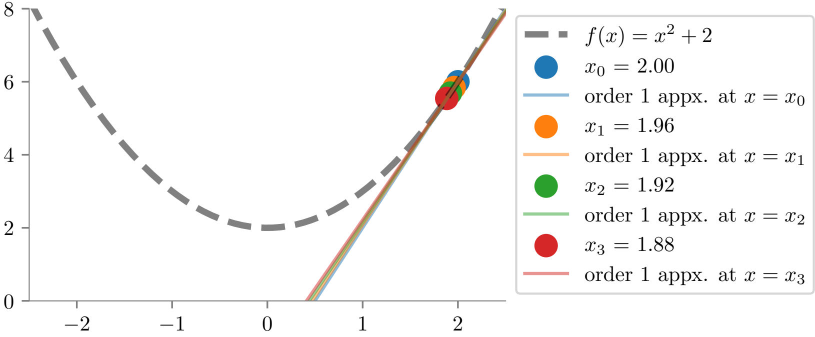
The learning rate α controls how big steps we take:

- **Too small α:** Slow convergence
- **Good α:** Fast, stable convergence
- **Too large α:** Overshooting, instability
- **Way too large α:** Divergence!



Learning Rate Visualization: Too Small

$\alpha = 0.01$: **Convergence is slow but stable**

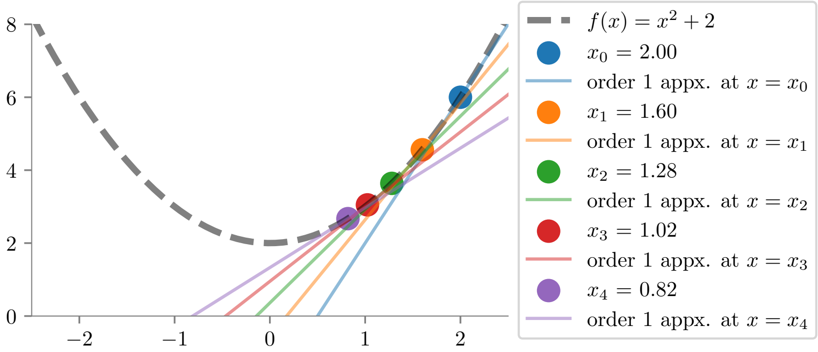


Important: Problem

Takes many iterations to reach the minimum. Computationally expensive!

Learning Rate: Just Right

$\alpha = 0.1$: **Good balance: Fast and stable convergence**

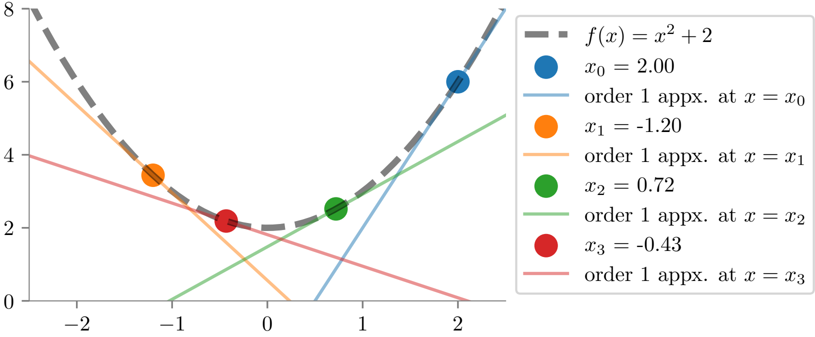


Key Points:

Perfect balance: Fast convergence + Stability

Learning Rate: Too Large

$\alpha = 0.8$: **Fast but may overshoot**

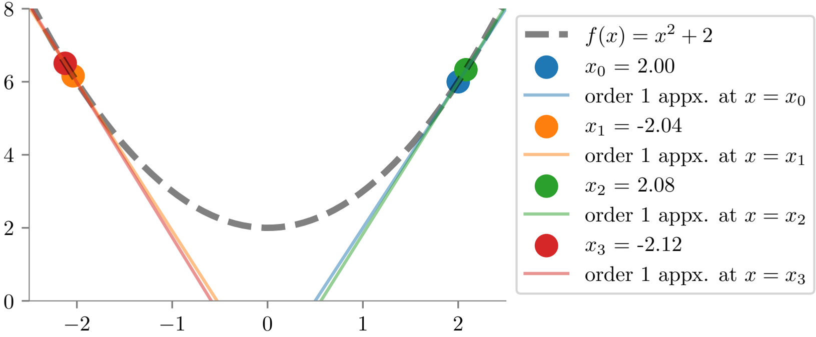


Important: Warning

Quick convergence but risk of instability. Watch out for oscillations!

Learning Rate: Disaster

$\alpha = 1.01$: **Divergence! Function values explode**

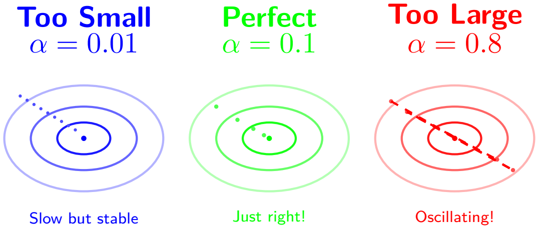


Important: Disaster Zone

The algorithm diverges. Always monitor your loss curves!

Learning Rate Showdown: All Together

Compare different learning rates side by side:



Theorem: Goldilocks Principle

Not too small, not too large - learning rate must be **just right**!

Key Points:

Pro tip: Start with $\alpha \in [0.01, 0.1]$ and adjust based on loss curves

Gradient Descent for Linear Regression

Linear Regression: Our First Application

Problem: Learn $y = \theta_0 + \theta_1 x$ from data

x	**y**
1	1
2	2
3	3

Cost Function (Mean Squared Error):

$$\mathrm{MSE}(\theta_0, \theta_1) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta_0 - \theta_1 x_i)^2$$

Goal: $(\theta_0^*, \theta_1^*) = \arg\min_{\theta_0, \theta_1} \mathrm{MSE}(\theta_0, \theta_1)$

Computing Gradients for Linear Regression

We need: $\nabla \mathrm{MSE} = \begin{bmatrix} \frac{\partial \mathrm{MSE}}{\partial \theta_0} \\ \frac{\partial \mathrm{MSE}}{\partial \theta_1} \end{bmatrix}$

Let's compute each partial derivative:

$$\frac{\partial \mathrm{MSE}}{\partial \theta_0} = \frac{2}{n} \sum_{i=1}^{n} (y_i - \theta_0 - \theta_1 x_i)(-1) \tag{7}$$

$$= -\frac{2}{n} \sum_{i=1}^{n} \epsilon_i \tag{8}$$

$$\frac{\partial \mathrm{MSE}}{\partial \theta_1} = \frac{2}{n} \sum_{i=1}^{n} (y_i - \theta_0 - \theta_1 x_i)(-x_i) \tag{9}$$

$$= -\frac{2}{n} \sum_{i=1}^{n} \epsilon_i x_i \tag{10}$$

where $\epsilon_i = y_i - \hat{y}_i$ is the residual.

Step-by-Step Example: Setup

Initial values: $\theta_0 = 4, \theta_1 = 0$, **Learning rate:** $\alpha = 0.1$
Iteration 1 - Predictions:

- $\hat{y}_1 = \theta_0 + \theta_1 \cdot 1 = 4 + 0 \cdot 1 = 4$
- $\hat{y}_2 = \theta_0 + \theta_1 \cdot 2 = 4 + 0 \cdot 2 = 4$
- $\hat{y}_3 = \theta_0 + \theta_1 \cdot 3 = 4 + 0 \cdot 3 = 4$

Errors (residuals):

- $\epsilon_1 = y_1 - \hat{y}_1 = 1 - 4 = -3$
- $\epsilon_2 = y_2 - \hat{y}_2 = 2 - 4 = -2$
- $\epsilon_3 = y_3 - \hat{y}_3 = 3 - 4 = -1$

Step-by-Step Example: Gradients

Compute gradients:

- $\frac{\partial \, \mathrm{MSE}}{\partial \theta_0} = -\frac{2}{3}(-3 - 2 - 1) = -\frac{2}{3}(-6) = 4$
- $\frac{\partial \, \mathrm{MSE}}{\partial \theta_1} = -\frac{2}{3}(-3 \cdot 1 - 2 \cdot 2 - 1 \cdot 3) = -\frac{2}{3}(-10) = 6.67$

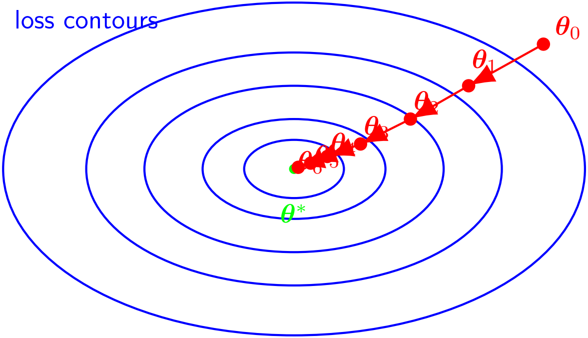
Parameter updates:

- $\theta_0 = 4 - 0.1 \times 4 = 3.6$
- $\theta_1 = 0 - 0.1 \times 6.67 = -0.67$

Key Points:

New parameters: $(\theta_0, \theta_1) = (3.6, -0.67)$
We moved closer to the true solution $(0, 1)$!

Visual Journey: Gradient Descent in Action



Key Points:

Steps get smaller as we approach minimum (gradient magnitude decreases)!

Variants of Gradient Descent

The Gradient Descent Family

Three main variants based on data usage:

Definition: Batch Gradient Descent

Use all training data to compute each gradient

Definition: Stochastic Gradient Descent (SGD)

Use one sample to compute each gradient

Definition: Mini-batch Gradient Descent

Use a small batch of samples to compute each gradient

Comparison: Batch vs SGD vs Mini-batch

Method	Data/update	Updates/epoch	Convergence
Batch GD	n (all)	1	Smooth
SGD	1	n	Noisy
Mini-batch	b	n/b	Balanced

Key Points:

Standard: Mini-batch GD (batches 32-256)

- Balance of stability and efficiency
- Parallel computation on GPUs
- Better estimates than pure SGD

SGD Step-by-Step Example: Setup

Same data, same initial values: $\theta_0 = 4, \theta_1 = 0$, $\alpha = 0.1$

x	**y**
1	1
2	2
3	3

SGD: Use ONE sample per update

- **Iteration 1:** Pick sample $(\mathbf{x}_1, y_1) = (1, 1)$
- $\hat{y}_1 = \theta_0 + \theta_1 \cdot 1 = 4 + 0 \cdot 1 = 4$
- $\epsilon_1 = y_1 - \hat{y}_1 = 1 - 4 = -3$

SGD Step-by-Step Example: Single Sample Gradients

Compute gradients using ONLY sample 1:

- $\frac{\partial \ell_1}{\partial \theta_0} = -2\epsilon_1 = -2(-3) = 6$
- $\frac{\partial \ell_1}{\partial \theta_1} = -2\epsilon_1 x_1 = -2(-3)(1) = 6$

Parameter updates after sample 1:

- $\theta_0 = 4 - 0.1 \times 6 = 3.4$
- $\theta_1 = 0 - 0.1 \times 6 = -0.6$

Key Points:

After sample 1: $(\theta_0, \theta_1) = (3.4, -0.6)$
Compare to batch GD: $(3.6, -0.67)$ - different path!

SGD Step-by-Step Example: Second Sample

Iteration 2: Pick sample $(\mathbf{x}_2, y_2) = (2, 2)$
Using updated parameters: $\theta_0 = 3.4, \theta_1 = -0.6$

- $\hat{y}_2 = 3.4 + (-0.6) \cdot 2 = 3.4 - 1.2 = 2.2$
- $\epsilon_2 = y_2 - \hat{y}_2 = 2 - 2.2 = -0.2$

Gradients for sample 2:

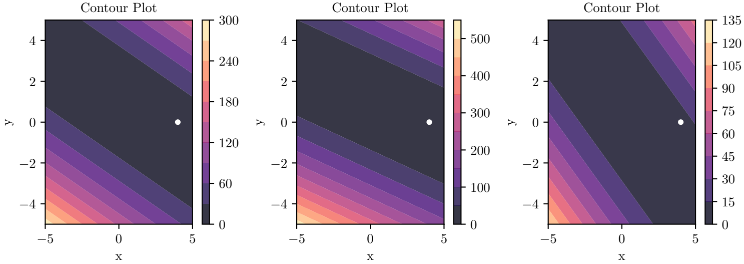
- $\frac{\partial \ell_2}{\partial \theta_0} = -2(-0.2) = 0.4$
- $\frac{\partial \ell_2}{\partial \theta_1} = -2(-0.2)(2) = 0.8$

Parameter updates:

- $\theta_0 = 3.4 - 0.1 \times 0.4 = 3.36$
- $\theta_1 = -0.6 - 0.1 \times 0.8 = -0.68$

SGD: The Noisy Path

SGD uses one sample at a time for updates



Trade-offs:

- **Pro:** Fast updates, can escape local minima
- **Con:** Noisy convergence, may not reach exact minimum
- **Key insight:** Noise can be beneficial for non-convex problems!

Mathematical Properties

Step 1: The Modern ML Computational Challenge

Real-world machine learning problems:

- **Massive datasets:** $n = 1,000,000+$ examples (ImageNet, web data)
- **Large models:** Neural networks with millions of parameters
- **Complex computations:** Each forward pass through model is expensive

The gradient computation bottleneck:

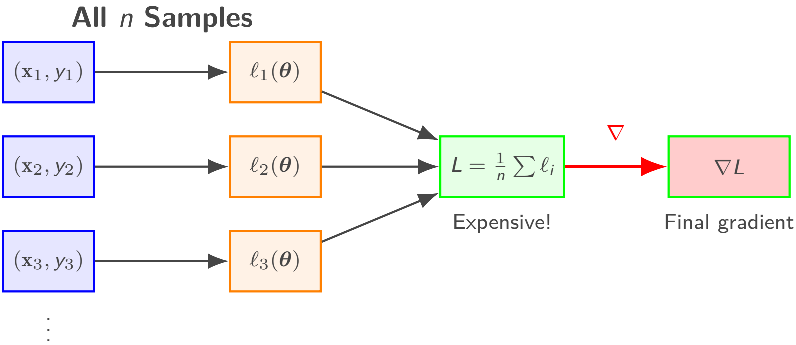
$$\nabla L(\boldsymbol{\theta}) = \nabla \left(\frac{1}{n} \sum_{i=1}^{n} \ell(f(\mathbf{x}_i; \boldsymbol{\theta}), y_i) \right)$$

Important: The Problem

Computing $f(\mathbf{x}_i; \boldsymbol{\theta})$ for ALL n samples is too slow!
Need: Fast approximation that still gives good direction

Step 2: Computational Graph - Can We Break This?

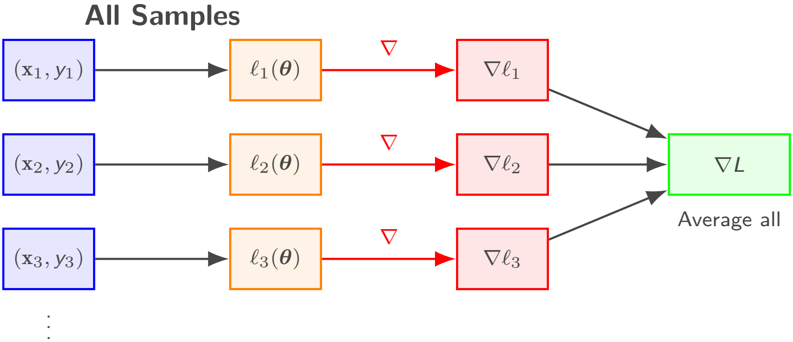
Current approach: Sum first, then take gradient



Key Points:

Problem: Computing losses for all n samples is expensive!

Step 3: The Linearity Insight - What If We Flip the Order?



Theorem: Linearity of Gradient

$\nabla L = \frac{1}{n} \sum_{i=1}^{n} \nabla \ell_i$

Step 4: The Mathematical Equivalence - Linearity of Gradient

Mathematical equivalence:

$$\nabla L(\boldsymbol{\theta}) = \nabla \left(\frac{1}{n} \sum_{i=1}^{n} \ell(f(\mathbf{x}_i; \boldsymbol{\theta}), y_i) \right) \tag{11}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(f(\mathbf{x}_i; \boldsymbol{\theta}), y_i) \tag{12}$$

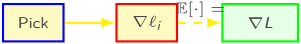
Key Points:

This linearity property is the foundation for all gradient-based optimization!

Step 5: SGD as Unbiased Estimator - The Solution

SGD solution: Sample one gradient instead of all n!

Estimate: $\nabla \tilde{L}(\boldsymbol{\theta}) = \nabla \ell(f(\mathbf{x}_j; \boldsymbol{\theta}), y_j)$ for random j



Important: Unbiased Property

$\mathbb{E}[\nabla \tilde{L}(\boldsymbol{\theta})] = \nabla L(\boldsymbol{\theta})$ - correct direction on average!

The Unbiased Property: Mathematical Proof

Theorem: SGD Unbiased Estimator Property

$$\mathbb{E}[\nabla \tilde{L}(\boldsymbol{\theta})] = \nabla L(\boldsymbol{\theta})$$

$$\begin{aligned}
\mathbb{E}[\nabla \tilde{L}(\boldsymbol{\theta})] &= \mathbb{E}\left[\nabla \ell(f(\mathbf{x}_j; \boldsymbol{\theta}), y_j)\right] && (13)\\
&= \sum_{i=1}^{n} P(\text{sample } i) \cdot \nabla \ell(f(\mathbf{x}_i; \boldsymbol{\theta}), y_i) && (14)\\
&= \sum_{i=1}^{n} \frac{1}{n} \cdot \nabla \ell(f(\mathbf{x}_i; \boldsymbol{\theta}), y_i) && (15)\\
&= \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(f(\mathbf{x}_i; \boldsymbol{\theta}), y_i) \qquad \text{(linearity of expectation)} && (16)\\
&= \nabla L(\boldsymbol{\theta}) \qquad \text{(from previous slide)} && (17)
\end{aligned}$$

Why Unbiasedness Matters

Key Points:

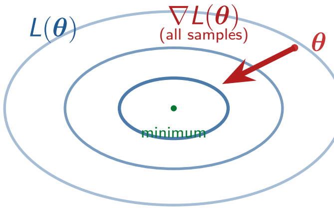
Key insight: On average, SGD points in the correct direction!

Practical implications:

- Individual SGD steps may be “wrong”
- But they average to the correct direction over time
- Theoretical guarantee that justifies SGD’s effectiveness
- The “noise” helps escape local minima in non-convex problems

Visual Intuition 1: Overall Loss Surface

True loss function using all data points:

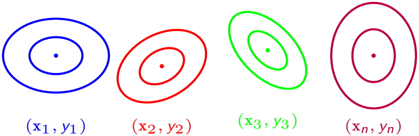


Key Points:

Gradient uses ALL data points for true direction

Visual Intuition 2: Individual Sample Loss Surfaces

Loss for individual data points (different shapes):

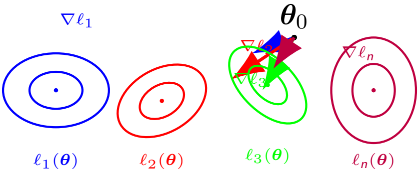


Important: Key Observation

Each individual gradient points in a **different direction** - some variation!

Visual Intuition 3: Gradients from Same Starting Point

What happens when we evaluate gradients from the same point θ_0?

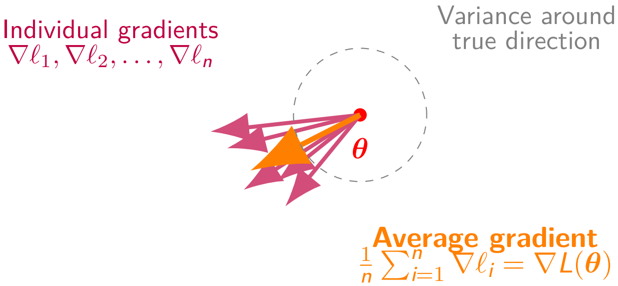


Theorem: Key Insight

From the **same point**, each loss surface gives a **different gradient direction**!

Visual Intuition 4: Averaging Individual Gradients

The magic: Average of individual gradients = True gradient

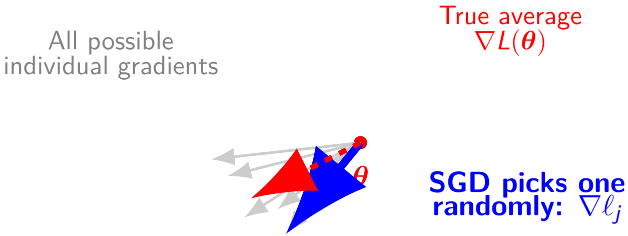


Theorem: Visual Proof of Unbiasedness

Even though individual gradients vary, their average equals the true gradient!

Visual Intuition 4: SGD Sampling Process

SGD randomly picks one gradient at a time:



Key Points:

Key insight: Sometimes SGD goes "wrong" direction, but on average it's correct!

Why Unbiasedness Matters in Practice

Example: Intuitive Analogy

Like asking random people for directions:

- Each person's answer might be slightly off
- But if there's no systematic bias, the average is correct
- SGD does the same with gradient estimates!

Computational Complexity

Normal Equation: The Direct Approach

For linear regression, we can solve directly:

Definition: Normal Equation

$$\hat{\boldsymbol{\theta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

What this means:

- Solves $\nabla_{\boldsymbol{\theta}}||\mathbf{X}\boldsymbol{\theta} - \mathbf{y}||^2 = 0$ directly
- Gives exact solution in one step (no iterations!)
- Requires matrix inversion

Key Points:

One computation gives the optimal $\hat{\boldsymbol{\theta}}$ - no learning rate needed!

Normal Equation: Step-by-Step Complexity

Breaking down $\hat{\theta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$:
Step 1: Compute $\mathbf{X}^T\mathbf{X}$

- $\mathbf{X}$ is $n \times d$, so $\mathbf{X}^T$ is $d \times n$
- $\mathbf{X}^T\mathbf{X}$ is $d \times d$ matrix
- **Cost:** $\mathcal{O}(d^2 n)$ operations

Step 2: Compute $\mathbf{X}^T\mathbf{y}$

- $\mathbf{X}^T$ is $d \times n$, $\mathbf{y}$ is $n \times 1$
- Result is $d \times 1$ vector
- **Cost:** $\mathcal{O}(dn)$ operations

Normal Equation: Matrix Inversion Cost

Step 3: Invert $\mathbf{X}^T\mathbf{X}$
Matrix inversion complexity:

- $\mathbf{X}^T\mathbf{X}$ is $d \times d$ matrix
- Standard algorithms (LU, Cholesky): $\mathcal{O}(d^3)$
- Most expensive step when d is large!

Important: Memory Requirements

Space: Need to store $\mathbf{X}^T\mathbf{X}$ matrix
Size: $d \times d = d^2$ elements $\Rightarrow \mathcal{O}(d^2)$ space

Key Points:

Total time: $\mathcal{O}(d^2 n + d^3)$ dominated by $\mathcal{O}(d^3)$ when d large

Gradient Descent: Iterative Approach

GD update rule for linear regression:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \alpha \nabla L(\boldsymbol{\theta}_t)$$

Where the gradient is:

$$\nabla L(\boldsymbol{\theta}) = \frac{2}{n} \mathbf{X}^T (\mathbf{X}\boldsymbol{\theta} - \mathbf{y})$$

So the update becomes:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \alpha \frac{2}{n} \mathbf{X}^T (\mathbf{X}\boldsymbol{\theta}_t - \mathbf{y})$$

Key Points:

Each iteration requires gradient computation - let's analyze the cost!

Gradient Descent: Per-Iteration Complexity

Breaking down $\nabla L(\boldsymbol{\theta}) = \frac{2}{n}\mathbf{X}^T(\mathbf{X}\boldsymbol{\theta} - \mathbf{y})$:
Step 1: Compute $\mathbf{X}\boldsymbol{\theta}$

- $\mathbf{X}$ is $n \times d$, $\boldsymbol{\theta}$ is $d \times 1$
- Result: $n \times 1$ vector (predictions)
- **Cost:** $\mathcal{O}(nd)$ operations

Step 2: Compute $\mathbf{X}\boldsymbol{\theta} - \mathbf{y}$

- Element-wise subtraction of $n \times 1$ vectors
- **Cost:** $\mathcal{O}(n)$ operations

Gradient Descent: Completing One Iteration

Step 3: Compute $\mathbf{X}^T(\mathbf{X}\boldsymbol{\theta} - \mathbf{y})$

- $\mathbf{X}^T$ is $d \times n$, $(\mathbf{X}\boldsymbol{\theta} - \mathbf{y})$ is $n \times 1$
- Result: $d \times 1$ vector (the gradient!)
- **Cost:** $\mathcal{O}(nd)$ operations

Step 4: Parameter update

- $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \alpha \nabla L(\boldsymbol{\theta}_t)$
- Element-wise operations on $d \times 1$ vector
- **Cost:** $\mathcal{O}(d)$ operations

Key Points:

Per iteration: $\mathcal{O}(nd + n + nd + d) = \mathcal{O}(nd)$

GD vs Normal Equation: Final Complexity Comparison

Important: Normal Equation

$\hat{\boldsymbol{\theta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$
Time: $\mathcal{O}(d^2n + d^3)$
Space: $\mathcal{O}(d^2)$
Iterations: 1 (exact solution)

Key Points: Gradient Descent

$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \alpha\mathbf{X}^T(\mathbf{X}\boldsymbol{\theta}_t - \mathbf{y})$
Time: $\mathcal{O}(T \cdot nd)$ for T iterations
Space: $\mathcal{O}(nd)$
Iterations: T (approximate solution)

GD vs Normal Equation: Final Complexity Comparison

Theorem: Trade-off

Normal equation: Fast but scales poorly with d
Gradient descent: Slower but scales better with d

When to Use Which Method

Key Points:

Modern ML: Gradient descent dominates due to:

- High-dimensional data (d very large)
- Non-linear models (no normal equation exists)
- Large datasets (n very large)

Decision criteria:

- **Few features ($d < 1000$):** Consider normal equation
- **Many features ($d > 10000$):** Gradient descent
- **Non-linear models:** Only gradient descent works
- **Online learning:** Only gradient descent works

Advanced Topics and Extensions

Beyond Basic Gradient Descent

Modern optimizers improve upon vanilla GD:

- **Momentum:** $\mathbf{v}_{t+1} = \beta \mathbf{v}_t + (1 - \beta)\mathbf{g}_t$
- **AdaGrad:** Adaptive per-parameter learning rates
- **Adam:** Combines momentum + adaptive rates
- **RMSprop:** Exponential moving average of squared gradients

Why these improvements?

- Handle different parameter scales automatically
- Accelerate convergence in relevant directions
- Reduce oscillations in narrow valleys

Gradient Descent in Deep Learning

Key Points:

Every deep learning framework uses gradient descent variants!

Key modern extensions:

- **Backpropagation:** Efficient gradient computation
- **Automatic differentiation:** PyTorch/TensorFlow magic
- **GPU acceleration:** Parallel mini-batch processing
- **Mixed precision:** 16-bit + 32-bit arithmetic

Practical Considerations

Learning Rate Selection Strategies

Common approaches:

- **Grid search:** Try $\{0.001, 0.01, 0.1, 1.0\}$
- **Learning rate schedules:** Start high, decay over time
- **Adaptive methods:** Let algorithm adjust automatically
- **Learning rate finder:** Gradually increase and monitor

Warning signs:

- Loss exploding $\rightarrow \alpha$ too high
- Very slow progress $\rightarrow \alpha$ too low
- Oscillating loss $\rightarrow$ Try momentum or smaller α

Convergence Criteria

When to stop training?

- **Gradient magnitude:** $||\nabla f(\boldsymbol{\theta})|| < \epsilon$
- **Function change:** $|f(\boldsymbol{\theta}_{t+1}) - f(\boldsymbol{\theta}_t)| < \epsilon$
- **Parameter change:** $||\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t|| < \epsilon$
- **Maximum iterations:** Always set an upper bound

Key Points:

Best practice: Use multiple criteria + validation performance

Common Pitfalls

Important: Pitfall 1: Poor Initialization

Problem: Bad starting points
Solution: Xavier/He initialization

Important: Pitfall 2: Wrong Learning Rate

Problem: Divergence or slow convergence
Solution: Learning rate schedules, adaptive optimizers

Important: Pitfall 3: Poor Feature Scaling

Problem: Different scales cause poor convergence
Solution: Standardize features: $(x - \mu)/\sigma$

Summary and Key Takeaways

What We've Learned

Key Points:

Gradient descent is the backbone of modern machine learning!

Journey recap:

- **Mathematical foundation:** Taylor series derivation
- **Geometric intuition:** Steepest descent direction
- **Algorithm variants:** Batch, SGD, mini-batch
- **Theoretical properties:** Unbiased estimator guarantees
- **Practical wisdom:** Learning rates, scaling, diagnostics

From Theory to Practice

Next steps for mastery:

- Implement gradient descent from scratch
- Experiment with different learning rates
- Compare batch vs SGD vs mini-batch
- Try advanced optimizers (Adam, momentum)
- Apply to real datasets

Key Points:

Master gradient descent first - it's the foundation for everything else!

Final Pop Quiz #2

Answer this!

True or False?

1. SGD always converges faster than batch GD
2. Learning rates should decrease during training
3. SGD gradient estimates are unbiased
4. Normal equation always beats gradient descent
5. GD guarantees global minimum for any function

Deep Dive: Advanced Theory

For comprehensive mathematical analysis:

Important: Reference Materials

- SGD.pdf: Detailed convergence proofs
- Florian's estimators: https://florian.github.io/estimators/
- Interactive notebooks for hands-on practice

Pop Quiz Solutions

Quiz #1 Solutions:

1. $f(2) = 6$, $f'(2) = 4$
2. $f(x) \approx 6 + 4(x - 2)$
3. New $x = 2 - 0.1 \times 4 = 1.6$
4. Yes, function decreases!

Quiz #2 Solutions:

1. False - SGD faster per epoch, may need more epochs
2. True - schedules often improve convergence
3. True - key theoretical property
4. False - only for linear problems, small d
5. False - only local minima; global for convex only

Thank You!

Questions?

Next: Advanced Optimization Techniques
Practice: Implement GD for your favorite ML model!