1 $y = 1 - x + \tan x$

$\frac{dy}{dx} = -1 + \sec^2 x$ **M1** Differentiating **A1** $\frac{dy}{dx}$ ✓

$\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$ **A1** $\frac{d^2y}{dx^2}$ ✓

When $x = \frac{1}{4}\pi$, $y = 2 - \frac{1}{4}\pi$ **B1** ***cao***

$\frac{dy}{dx} = 1$ **B1** ***ft***

$\frac{d^2y}{dx^2} = 4$ **B1** ***ft*** **Dealing with the curve** 6

Let circle have eqn. $(x - a)^2 + (y - b)^2 = r^2$ **M1** At any stage

Then $2(x - a) + 2(y - b)\frac{dy}{dx} = 0$ **M1 A1**

and $2 + 2(y - b)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$ **M1** (Product/Quotient Rule) **A1**

or $\frac{dy}{dx} = \frac{a - x}{y - b} \Rightarrow \frac{d^2y}{dx^2} = \frac{(y-b)(-1) - (a-x)\frac{dy}{dx}}{(y-b)^2}$ **Dealing with the circle** 5

When $x = \frac{1}{4}\pi$, $y = 2 - \frac{1}{4}\pi$, we have **Substitution**

$\left(\frac{1}{4}\pi - a\right)^2 + \left(2 - \frac{1}{4}\pi - b\right)^2 = r^2$ **M1 A1**

$\frac{dy}{dx} = -\frac{(x-a)}{(y-b)} = 1$ **M1**

$\Rightarrow -\frac{1}{4}\pi + a = 2 - \frac{1}{4}\pi - b$ or $a + b = 2$ **A1**

$2 + 2\left(2 - \frac{1}{4}\pi - b\right).4 + 2.(1)^2 = 0$ **M1 A1**

Matching the two up 6

$b = \frac{5}{2} - \frac{1}{4}\pi$ **A1 cao**

$a = \frac{1}{4}\pi - \frac{1}{2}$ **A1 cao**

$r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \Rightarrow r = \frac{1}{\sqrt{2}}$ **A1 cso** **Answers** 3

2 $\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$ **M1** $2x$ & x and sin or $\cos(A + B)$ used

$= (2c^2 - 1)c - 2sc.s$ **M1** Double-angles and $s^2 + c^2 = 1$ used somewhere

$= (2c^2 - 1)c - 2c(1 - c^2)$

$= 4c^3 - 3c$ **A1 (ANSWER GIVEN)**

$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$

$= 2sc.c + (1 - 2s^2)s$

$= 2s(1 - s^2) + s(1 - 2s^2)$

$= 3s - 4s^3$ **A1**

ALT. $\cos 3x + \mathrm{i}\sin 3x = (c + \mathrm{i}\,s)^3$ **M1** *de Moivre* and equating Re. and Im. parts
M1 binomial expansion **A1 A1**

(If 2nd result just quoted, score M0 M0 A0 **A1**) ④

(i) $I(\alpha) = \int_0^\alpha \left(7\sin x - 8\sin^3 x\right)\,\mathrm{d}x$

$\downarrow$ **M1** Use of above result to get rid of s^3

$= \int_0^\alpha \left(\sin x + 2\sin 3x\right)\,\mathrm{d}x$ **A1**

$= \left[-\cos x - \tfrac{2}{3}\cos 3x\right]_0^\alpha$ **A1** ***ft*** for both "$a\cos kx$" terms

$= -\cos\alpha - \tfrac{2}{3}(4\cos^3\alpha - 3\cos\alpha) + 1 + \tfrac{2}{3}$ **M1** Use of $\cos 3x$ to get expression in c

$= -\tfrac{8}{3}c^3 + c + \tfrac{5}{3}$ **A1** legitimately from correct unsimplified form (**ANSWER GIVEN**)

$I(\alpha) = 0$ when $c = 1$ $(\alpha = 0)$ **B1** ⑥

(ii) $J(\alpha) = \left[\tfrac{7}{2}\sin^2 x - \tfrac{8}{4}\sin^4 x\right]_0^\alpha$ **B1** both

$= \tfrac{7}{2}(1 - \cos^2\alpha) - 2(1 - \cos^2\alpha)^2$ **M1** Getting c's only

$= -2c^4 + \tfrac{1}{2}c^2 + \tfrac{3}{2}$ **A1** ✓ MUST be simplified (here or later)

M1 A1 for subst$^{\text{g}}$. $c = -\tfrac{1}{6}$ into both sides: $\tfrac{245}{162}$ (N.B. may be done after following algebra)

M1 Equating two polynomials in c

$I(\alpha) = J(\alpha)$ when $0 = 2c^4 - \tfrac{8}{3}c^3 - \tfrac{1}{2}c^2 + c + \tfrac{1}{6}$ i.e. $0 = 12c^4 - 16c^3 - 3c^2 + 6c + 1$

M1 Full factorisation attempted: $0 = (c - 1)^2(2c + 1)(6c + 1)$ **A1**

$\cos\alpha = -\tfrac{1}{2}$ i.e. $\alpha = \tfrac{2}{3}\pi$ **A1**

$\cos\alpha = -\tfrac{1}{6}$ i.e. $\alpha = \pi - \cos^{-1}(\tfrac{1}{6})$ or $\cos^{-1}(-\tfrac{1}{6})$ and $\alpha = 0$ **A1** both ⑩

N.B. Unfortunately, the $\alpha \in (0, \pi)$ demand disappeared, so please ignore any work towards general solutions.

2 (ii) **Special Scheme for those who use** $\int \sin x \; dx = -\cos x$ rather than Eustace's $\frac{1}{2}\sin^2 x$

$J(\alpha) = \left[-7\cos x - 2\sin^4 x\right]_0^\alpha$ **B0**

$= 7 - 7\cos\alpha - 2(1 - \cos^2\alpha)^2$ **M1** Getting c's only

$= -2c^4 + 4c^2 - 7c + 5$ **A1 ft** MUST be simplified (here or later)

M1 A0 for subst^g. $c = -\frac{1}{6}$ into both sides: $\frac{245}{162} = \frac{4067}{648}$!

M1 Equating two polynomials in c

$I(\alpha) = J(\alpha)$ when $0 = 6c^4 - 8c^3 - 12c^2 + 24c - 10$

M1 Full factorisation attempted

A1 $0 = 2(c-1)^3(3c+5)$

A0 A0 Answers Max ⑥

3 (i) **M1** Subst^{g}. $n = 0, 1, (2), 3$ into given formula

$F_0 = 0 \Rightarrow 0 = a + b$ or $b = -a$ **A1**

$F_1 = 1 \Rightarrow 1 = a(\lambda - \mu)$ **A1**

$[F_2 = 1 \Rightarrow 1 = a(\lambda^2 - \mu^2) \Rightarrow \lambda + \mu = 1]$

$F_3 = 2 \Rightarrow 2 = a(\lambda^3 - \mu^3) = a(\lambda - \mu)(\lambda^2 + \lambda\mu + \mu^2)$ **M1** Difference of 2 cubes

$= 1.(\lambda^2 + \lambda\mu + \mu^2) \Rightarrow \lambda^2 + \lambda\mu + \mu^2 = 2$ **A1** (**ANSWER GIVEN**) ⑤

$(\lambda + \mu)^2 - \lambda\mu = 1 - \lambda\mu \Rightarrow \lambda\mu = -1$

M1 Getting any two suitable eqns.; e.g. any two of $\lambda\mu = -1$, $\lambda - \mu = \dfrac{1}{a}$ and $\lambda + \mu = 1$

M1 Solving simultaneously

A1 for $a = \dfrac{1}{\sqrt{5}}$, $b = -\dfrac{1}{\sqrt{5}}$ **A1** for $\lambda = \dfrac{1}{2}\left(1+\sqrt{5}\right)$, $\mu = \dfrac{1}{2}\left(1-\sqrt{5}\right)$ ④

(ii) **M1** Using the formula $F_n = a\,\lambda^n + b\,\mu^n = \dfrac{1}{2^n\sqrt{5}}\left\{(1+\sqrt{5})^n - (1-\sqrt{5})^n\right\}$ with $n = 6$

M1 Good attempt at a binomial expansion

$$F_6 = \frac{1}{2^6\sqrt{5}}\left\{1 + 6\sqrt{5} + 15.5 + 20.5\sqrt{5} + 15.5^2 + 6.5^2\sqrt{5} + 5^3\right.$$ **A1** $576 + 256\sqrt{5}$

$$\left. -\left(1 - 6\sqrt{5} + 15.5 - 20.5\sqrt{5} + 15.5^2 - 6.5^2\sqrt{5} + 5^3\right)\right\}$$ **M1** Conjugate of previous

$$= \frac{2}{2^6\sqrt{5}}\left(6\sqrt{5} + 100\sqrt{5} + 150\sqrt{5}\right) = \frac{2.2^8\sqrt{5}}{2^6\sqrt{5}} = 8$$ **A1** Legitimately shown ⑤

(iii) $\displaystyle\sum_{n=0}^{\infty}\frac{F_n}{2^{n+1}} = \frac{a}{2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{2}\right)^n - \frac{a}{2}\sum_{n=0}^{\infty}\left(\frac{\mu}{2}\right)^n$ **M1** Use of formula

M1 Split into 2 series (& something useful done with them)

$$= \frac{1}{2\sqrt{5}}\left(\frac{1}{1-\frac{1}{4}\left(1+\sqrt{5}\right)}\right) - \frac{1}{2\sqrt{5}}\left(\frac{1}{1-\frac{1}{4}\left(1-\sqrt{5}\right)}\right)$$ **M1** $S\infty$ GP used (at least once)

$$= \frac{1}{2\sqrt{5}}\left(\frac{4}{3-\sqrt{5}}\right) - \frac{1}{2\sqrt{5}}\left(\frac{4}{3+\sqrt{5}}\right)$$ **M1** Simplifying

$$= \frac{2}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{9-5}\right) - \frac{2}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{9-5}\right)$$ **M1** Rationalising denominators (or equivalent)

$$= \frac{2}{\sqrt{5}}\left(\frac{2\sqrt{5}}{4}\right)$$

$= 1$ **A1** ***cao*** ⑥

Note: **(ii)** F_6 can be found by $a\,\lambda^6 + b\,\mu^6 = a\left(\lambda^6 - \mu^6\right) = a\left(\lambda^3 - \mu^3\right)\left(\lambda^3 + \mu^3\right) = F_3\left(\lambda^3 + \mu^3\right)$ etc.

4(i) **M1** Using the substn. $y = a - x$ **M1** Full substn. involving $\mathrm{d}y = -\,\mathrm{d}x$ and $(0, a) \to (a, 0)$

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,\mathrm{d}x = \int_a^0 \frac{f(a-y)}{f(a-y)+f(y)}.-\mathrm{d}y$$

$$= \int_0^a \frac{f(a-y)}{f(a-y)+f(y)}\,\mathrm{d}y = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,\mathrm{d}x \quad \textbf{A1} \quad ③$$

Then $2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,\mathrm{d}x = \int_0^a 1.\,\mathrm{d}x = [x]_0^a = a \Rightarrow I = \tfrac{1}{2}a$ **M1 A1** ②

Let $f(x) = \ln(1 + x)$ **M1**

Then $\ln(2 + x - x^2) = \ln[(1 + x)(2 - x)]$ **M1** Factorisation

$= \ln(1 + x) + \ln(2 - x)$ **M1** Log. work

and $\ln(2 - x) = \ln(1 + [1 - x]) = f(a - x)$ with $a = 1$ **M1** Or shown via $x \to 1 - x$

so that $\int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,\mathrm{d}x = \tfrac{1}{2}$ **A1** ⑤

$$\int_0^{\pi/2} \frac{\sin x}{\sin\left(x+\frac{1}{4}\pi\right)}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\sin x}{\sin x.\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}}}\,\mathrm{d}x$$ **M1** $\sin(A + B)$ used; **A1** incl. the $\sqrt{2}$

$$= \sqrt{2}\int_0^{\pi/2} \frac{\sin x}{\sin x + \sin\left(\frac{1}{2}\pi - x\right)}\,\mathrm{d}x$$ **A1** **M1** $\cos = \sin(\tfrac{1}{2}\pi - \ )$

$= \tfrac{1}{4}\pi\sqrt{2}$ **A1** ④

(ii) **M1** for $u = \dfrac{1}{x}$ **M1** Full substn. involving $\mathrm{d}u = -\dfrac{1}{x^2}\,\mathrm{d}x$ and $\left(\tfrac{1}{2}, 2\right) \to \left(2, \tfrac{1}{2}\right)$

Then $$\int_{0.5}^2 \frac{1}{x}.\frac{\sin x}{\left(\sin x + \sin\left(\frac{1}{x}\right)\right)}\,\mathrm{d}x = \int_{0.5}^2 \frac{1}{x^2}.\frac{x\sin x}{\left(\sin x + \sin\left(\frac{1}{x}\right)\right)}\,\mathrm{d}x$$

$$= \int_2^{0.5} \frac{\frac{1}{u}.\sin\left(\frac{1}{u}\right)}{\left(\sin\left(\frac{1}{u}\right) + \sin u\right)}.-\mathrm{d}u$$ **M1**

$$= \int_{0.5}^2 \frac{1}{u}.\frac{\sin\left(\frac{1}{u}\right)}{\left(\sin u + \sin\left(\frac{1}{u}\right)\right)}\,\mathrm{d}u \quad \text{or} \quad \int_{0.5}^2 \frac{1}{x}.\frac{\sin\left(\frac{1}{x}\right)}{\left(\sin x + \sin\left(\frac{1}{x}\right)\right)}\,\mathrm{d}x$$ **A1**

Adding then gives $2I = \int_{0.5}^2 \frac{1}{x}\,\mathrm{d}x = [\ln x]_{0.5}^2 = 2\ln 2 \Rightarrow I = \ln 2$ **M1 A1** ⑥

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5 $\cos 2\alpha = \dfrac{(1,\ 1,\ 1)\bullet(5,\ -1,\ -1)}{\sqrt{3}.\sqrt{27}} = \dfrac{1}{3}$ **M1** Scalar product/product of moduli **A1** ②

(i) l_1 equally inclined to OA and OB iff

$$\frac{(m,\ n,\ p)\bullet(1,\ 1,\ 1)}{\sqrt{m^2+n^2+p^2}.\sqrt{3}} = \frac{(m,\ n,\ p)\bullet(5,\ -1,\ -1)}{\sqrt{m^2+n^2+p^2}.\sqrt{27}}$$ **M1** Two expressions of this form **A1** **A1**

i.e. $3(m + n + p) = 5m - n - p$ or $m = 2(n + p)$ **M1** equated **A1** relationship ⑤

For l_1 the angle bisector, we also require $\dfrac{m+n+p}{\sqrt{m^2+n^2+p^2}.\sqrt{3}} = \cos\alpha$ **M1**

Now $\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{1}{3} \Rightarrow \cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$ **M1** **A1**

so $m + n + p = \sqrt{m^2+n^2+p^2}.\sqrt{2}$

Squaring both sides: $m^2 + n^2 + p^2 + 2mn + 2np + 2pm = 2(m^2 + n^2 + p^2)$ **M1**

$\Rightarrow 2mn + 2np + 2pm = m^2 + n^2 + p^2$ **A1**

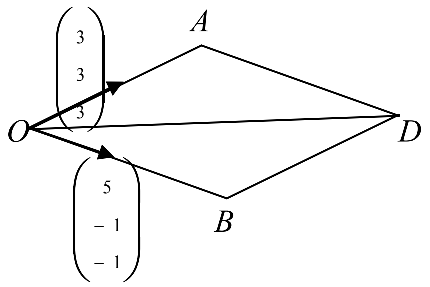
M1 Setting $m = 2n + 2p$ (or equivalent) then gives

$$2np + (2n + 2p)^2 = (2n + 2p)^2 + n^2 + p^2$$

which gives $(n - p)^2 = 0$ **M1** simplifying $\Rightarrow p = n$, $m = 4n$

and $\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ (or any non-zero multiple) **A1** ⑧

ALT.



Form rhombus $OADB$.
Then angle bisector is in
the direction $\overrightarrow{OD} = 8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ **M2** **A1** this way

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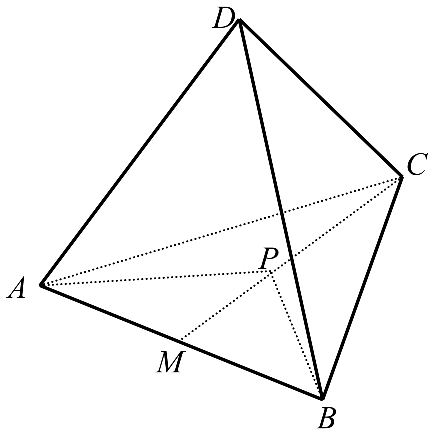
(ii) We already have this (if first method used above);
namely, $2uv + 2vw + 2wu = u^2 + v^2 + w^2$ **M1** **A1**

In this case, $2xy + 2yz + 2zx = x^2 + y^2 + z^2$ gives
M1 all lines inclined at an angle $\cos^{-1}\frac{\sqrt{2}}{\sqrt{3}}$ to OA
describing the surface which is a (double-) cone **M1** Ignore lack of "double" here
vertex at O, having central axis OA **A1** Must say this & "double" ⑤

6(i)



Take M = midpt.AB = origin, **M1** set-up
the x-axis along AB and
the y-axis along MC.

Then $A = \left(-\frac{1}{2},\ 0,\ 0\right),\ B = \left(\frac{1}{2},\ 0,\ 0\right)$ (**A1**)

$C = \left(0,\ \frac{\sqrt{3}}{2},\ 0\right)$ by trig. or Pythagoras **M1 A1**

$P = \left(0,\ \frac{\sqrt{3}}{6},\ 0\right)$ **A1**

PA (or PB) = $\frac{\sqrt{3}}{3}$ by Pythagoras

and $PD = \frac{\sqrt{6}}{3}$ or $\sqrt{\frac{2}{3}}$ by Pythagoras **A1** ⑥

i.e. $D = \left(0,\ \frac{\sqrt{3}}{6},\ \frac{\sqrt{6}}{3}\right)$

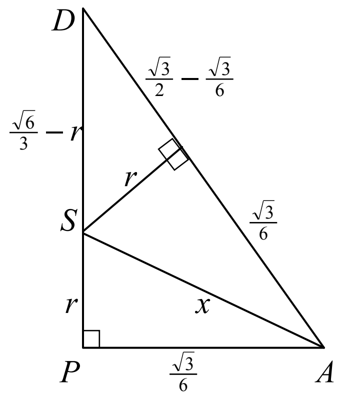
(ii) Angle betn. adjacent faces is $\angle DMP = \cos^{-1}\left(\dfrac{\frac{1}{6}\sqrt{3}}{\frac{1}{2}\sqrt{3}}\right)$ in Rt.∠d. ΔDMP

or $\angle DMC = \cos^{-1}\left(\dfrac{\frac{3}{4}+\frac{3}{4}-1}{2\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}}\right)$ by the Cosine Rule in ΔDMC

M1 Suitable Δ **M1** Appropriate method for chosen Δ **A1** correct unsimplified

$= \cos^{-1}\frac{1}{3}$ **A1** Legit. (**ANSWER GIVEN**) ④

(iii) Centre of sphere, S, is on PD **M1** equidistant from each vertex **M1**



M1 Valid Δ **A1 A1 A1** Correct relevant lengths

By Pythagoras, $x^2 = \frac{1}{12} + \left(\frac{6}{9} - 2\frac{\sqrt{6}}{3}x + x^2\right)$ **M1**

$\Rightarrow x = \frac{\sqrt{6}}{4}$ **A1**

Then $r = x\sin(90^\circ - \text{(ii)}) = \frac{1}{3}x = \frac{\sqrt{6}}{12}$ **M1 A1**

ALT.1: By similar Δs with same lengths.
ALT.2: By working with $\angle DAS = \angle PAS = \frac{1}{2}$ (answer to (ii)).

Then (e.g.) $\cos\theta = \frac{1}{3} \Rightarrow \tan\theta = 2\sqrt{2} \Rightarrow t = \tan\frac{1}{2}\theta$ g.b. $t^2\sqrt{2} + t - \sqrt{2} = 0$

and so $t = \frac{1}{\sqrt{2}}$ and $r = \frac{\sqrt{3}}{6}\tan\frac{1}{2}\theta = \frac{\sqrt{6}}{12}$

ALT.3: Of course, if they know that the sphere's centre is at the centre of mass of the tetrahedron $\left(\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})\right)$ then the answer is just $\frac{1}{4}DP = \frac{\sqrt{6}}{12}$ ⑩

7(i) $y = x^3 - 3qx - q(1+q) \Rightarrow \dfrac{dy}{dx} = 3(x^2 - q) = 0$ **M1** Diff^g.

M1 setting $\dfrac{dy}{dx} = 0$ for TPs

M1 Subst^g. either/both x's back

When $x = +\sqrt{q}$, $y = -q\left(\sqrt{q}+1\right)^2$

< 0 since $q > 0$ **E1** Explained (or via all terms < 0)

When $x = -\sqrt{q}$, $y = -q\left(\sqrt{q}-1\right)^2$ **M1** Compl^g. the sq. attempted (or ≡)

< 0 since $q > 0$ **and** $q \neq 1$ **E1** Both needed

Since both TPs below x-axis,

the curve crosses the x-axis once only **E1** explained (possibly with sketch) ⑦

(ii) $x = u + \dfrac{q}{u} \Rightarrow x^3 = u^3 + 3uq + 3\dfrac{q^2}{u} + \dfrac{q^3}{u^3}$ **B1**

$0 = x^3 - 3qx - q(1+q) = u^3 + 3uq + 3\dfrac{q^2}{u} + \dfrac{q^3}{u^3} - 3qu - 3\dfrac{q^2}{u} - q - q^2$ **M1** substn.

$\Rightarrow u^3 + \dfrac{q^3}{u^3} - q(1+q) = 0$ or $\left(u^3\right)^2 - q(1+q)\left(u^3\right) + q^3 = 0$ **M1** quadratic in u^3 **A1** ④

$u^3 = \dfrac{q(1+q) \pm \sqrt{q^2(1+q)^2 - 4q^3}}{2} = \dfrac{q}{2}\left\{1 + q \pm \sqrt{1 + 2q + q^2 - 4q}\right\}$ **M1** quadratic formula

$= \dfrac{q}{2}\left\{1 + q \pm \sqrt{(1-q)^2}\right\} = \dfrac{q}{2}\left\{1 + q \pm (1-q)\right\} = q$ or q^2 **M1** Compl^g. the sq.

giving $u = q^{\frac{1}{3}}$ or $q^{\frac{2}{3}}$ and $x = q^{\frac{1}{3}} + q^{\frac{2}{3}}$ **A1** ③

(iii) $\alpha + \beta = p$, $\alpha\beta = q \Rightarrow \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$ **M1**

$= p^3 - 3qp$ **A1** legit. (**ANSWER GIVEN**)

ALT. $\alpha = \frac{1}{2}\left\{p + \sqrt{p^2 - 4q}\right\}$, $\beta = \frac{1}{2}\left\{p - \sqrt{p^2 - 4q}\right\}$

Then $\alpha^3 + \beta^3 = \frac{1}{8}\left\{p^3 + 3p^2\sqrt{p^2-4q} + 3p\left(p^2-4q\right) + \left(p^2-4q\right)\sqrt{p^2-4q}\right\}$

$+ \frac{1}{8}\left\{p^3 - 3p^2\sqrt{p^2-4q} + 3p\left(p^2-4q\right) - \left(p^2-4q\right)\sqrt{p^2-4q}\right\} = p^3 - 3qp$ ②

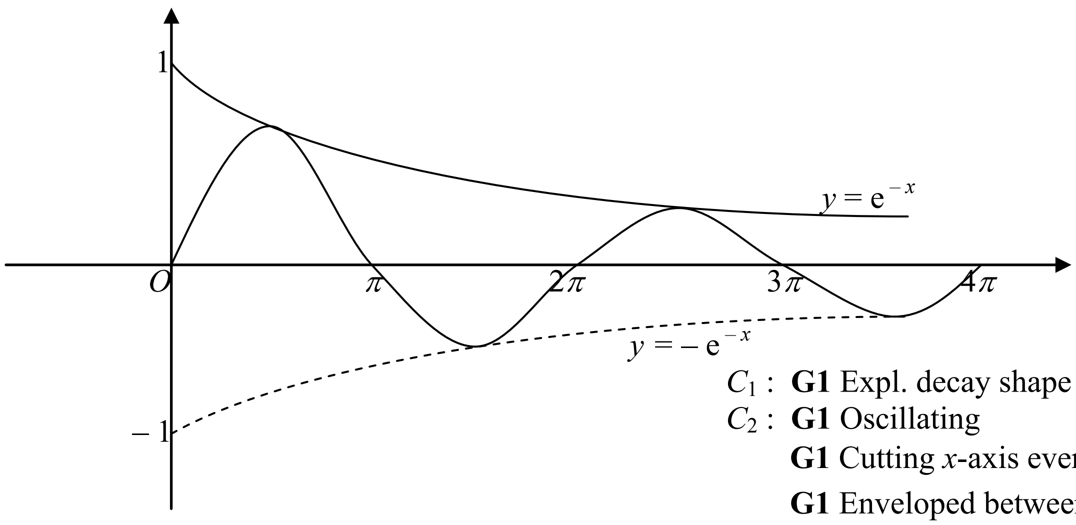
One root the square of the other $\Leftrightarrow \alpha = \beta^2$ or $\beta = \alpha^2 \Leftrightarrow 0 = \left(\alpha^2 - \beta\right)\left(\alpha - \beta^2\right)$ **E1**

$\left(\alpha^2 - \beta\right)\left(\alpha - \beta^2\right) = \alpha^3 + \beta^3 - \alpha\beta - (\alpha\beta)^2$ **M1** $= p^3 - 3qp - q(1+q)$ **A1**

$\Leftrightarrow p = q^{\frac{1}{3}} + q^{\frac{2}{3}}$ **A1** ***ft*** (ii)'s final answer only

ALT. Let roots be α and α^2. Then $p = \alpha + \alpha^2$ and $q = \alpha^3$; i.e. $p = q^{\frac{1}{3}} + q^{\frac{2}{3}}$ ④

8



C_1 : **G1** Expl. decay shape
C_2 : **G1** Oscillating
G1 Cutting x-axis every π
G1 Enveloped between C_1 and $-C_1$ ④

The curves meet each time $\sin x = 1$ **M1** when $x = 2n\pi + \frac{\pi}{2}$ ($n = 0, 1, 2, \ldots$) **M1**

(These two M's might be implicit) ③⇒ $x_n = \dfrac{(4n-3)\pi}{2}$, $x_{n+1} = \dfrac{(4n+1)\pi}{2}$ **A1 A1 Limits** ④

$\int \left(e^{-x} \sin x\right) dx$ **M1** attempted by parts

$$= -e^{-x}.\cos x - \int \left(e^{-x}.\cos x\right) dx \text{ or } -e^{-x}.\sin x - \int \left(e^{-x}.\sin x\right) dx \quad \textbf{A1}$$

$$= -e^{-x}.\cos x - \left\{ e^{-x}.\sin x + \int \left(e^{-x}.\sin x\right) dx \right\} \quad \textbf{M1 } 2^{\text{nd}} \text{ round of parts}$$

$\Rightarrow I = -e^{-x}(\cos x + \sin x) - I$ **M1** by "looping"

$= -\frac{1}{2} e^{-x}(\cos x + \sin x)$ **A1** **Anywhere it appears** ⑤

$A_n = \displaystyle\int_{x_n}^{x_{n+1}} \left(e^{-x} - e^{-x} \sin x\right) dx$ **M1** (ignore limits for now)

$A_n = \left[-e^{-x} + \frac{1}{2} e^{-x}\left(\cos x + \sin x\right)\right]_{x_n}^{x_{n+1}}$ or $\left[\frac{1}{2} e^{-x}\left(\cos x + \sin x - 2\right)\right]_{x_n}^{x_{n+1}}$ **M1** use of insert working

$= \frac{1}{2} e^{-\frac{1}{2}\pi(4n+1)}\left(0+1-2\right) - \frac{1}{2} e^{-\frac{1}{2}\pi(4n-3)}\left(0+1-2\right)$ **M1** use of limits

$= \frac{1}{2} e^{-\frac{1}{2}\pi(4n+1)}\left(-1 + e^{2\pi}\right)$ **A1 (ANSWER GIVEN)** ④

Note that $A_1 = \frac{1}{2} e^{-\frac{5}{2}\pi}\left(e^{2\pi} - 1\right)$ and $A_{n+1} = e^{-2\pi} A_n$ **M1**

so that $\displaystyle\sum_{n=1}^{\infty} A_n = A_1 \left\{1 + \left(e^{-2\pi}\right) + \left(e^{-2\pi}\right)^2 + \ldots\right\}$

$$= \tfrac{1}{2} e^{-\frac{5}{2}\pi}\left(e^{2\pi} - 1\right) \times \frac{1}{1 - e^{-2\pi}} = \tfrac{1}{2} e^{-\frac{5}{2}\pi}\left(e^{2\pi} - 1\right) \times \frac{e^{2\pi}}{e^{2\pi} - 1} \quad \textbf{M1 } S\infty \text{ GP used}$$

$= \frac{1}{2} e^{-\frac{1}{2}\pi}$ **A1** ③

9 For P_1, $\ddot{x}_1 = 0$, $\dot{x}_1 = u\cos\alpha$, $x_1 = ut\cos\alpha$, $\ddot{y}_1 = -g$, $\dot{y}_1 = u\sin\alpha - gt$, $y_1 = ut\sin\alpha - \frac{1}{2}gt^2$

P_2, $\ddot{x}_2 = 0$, $\dot{x}_2 = v\cos\beta$, $x_2 = vt\cos\beta$, $\ddot{y}_2 = -g$, $\dot{y}_2 = v\sin\beta - gt$, $y_2 = vt\sin\beta - \frac{1}{2}gt^2$

P_1 at greatest height when $\dot{y}_2 = 0$ **M1** $\Rightarrow t = \dfrac{u\sin\alpha}{g}$ **A1**

M1 Substd. into y_1 formula $\Rightarrow y_1 = h = \dfrac{u^2\sin^2\alpha}{2g}$ **A1**

$\Rightarrow u\sin\alpha = \sqrt{2gh}$ **A1** This may be implicit in following working ⑤

Note that if the two particles are at the same height at any two distinct times (one of which is $t = 0$ here), then their vertical speeds are the same throughout their motions. **E1**
Thus $u\sin\alpha = v\sin\beta$ **B1** **Somewhere**

ALT. P_1, P_2 at the same height at a common time $t = \tau \neq 0$, then
$u\tau\sin\alpha - \frac{1}{2}g\tau^2 = v\tau\sin\beta - \frac{1}{2}g\tau^2$ **E1** $\Rightarrow u\sin\alpha = v\sin\beta$ **B1** ②

$y_2 = 0$, $t \neq 0 \Rightarrow t = \dfrac{2v\sin\beta}{g}$ **M1 A1**

Collision at $x_2 = b \Rightarrow t = \dfrac{b}{v\cos\beta}$ **M1 A1**

Then $t\left(P_2\ \tfrac{1}{2}-\text{range}\right) < t(\text{collision}) < t\left(P_2\ \text{range}\right)$ **M1** or by distances

$\Rightarrow \dfrac{v\sin\beta}{g} < \dfrac{b}{v\cos\beta} < \dfrac{2v\sin\beta}{g}$ **A1** or $\frac{1}{2}.\dfrac{v^2\sin 2\beta}{g} < b < \dfrac{v^2\sin 2\beta}{g}$

$\Rightarrow \dfrac{v^2\sin\beta\cos\beta}{g} < b < \dfrac{2v^2\sin\beta\cos\beta}{g}$

$\Rightarrow \dfrac{(v\sin\beta)^2}{g}\cot\beta < b < \dfrac{2(v\sin\beta)^2}{g}\cot\beta$ **M1** relevant trig. work

M1 use of $u\sin\alpha = v\sin\beta$ **M1** use of $u\sin\alpha = \sqrt{2gh}$

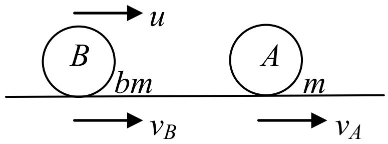
$\Rightarrow \dfrac{2gh}{g}\cot\beta < b < \dfrac{4gh}{g}\cot\beta \Rightarrow 2h\cot\beta < b < 4h\cot\beta$ **A1** legit. (**ANSWER GIVEN**) ⑩

Particles at max. ht. simultaneously (see above reasoning) **M1**
and would achieve max. ranges simultaneously also **M1**
$\Rightarrow 2h\cot\alpha < a < 4h\cot\alpha$ **A1** (**ANSWER GIVEN**) ③

Anyone who says "similarly" without explaining why … gets **0**

Those who do all the work again, give **M1** for clear intention to repeat it all, **M1** for *actually* doing it all again, and **A1** for legitimately obtaining given result.

10(i)

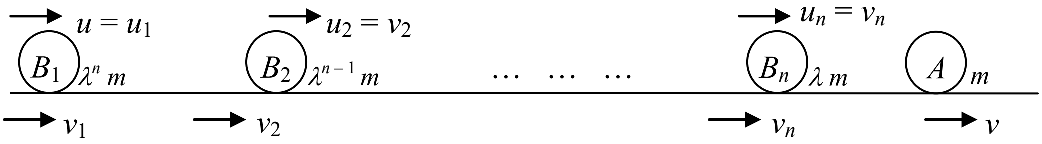


CLM→ $b\cancel{m}u = b\cancel{m}v_B + \cancel{m}v_A$ **M1 A1**

NEL $u = v_A - v_B$ **B1**

M1 Solving simultaneously: $v_A = \dfrac{2bu}{b+1}$ **A1** $v_B = \dfrac{(b-1)u}{b+1}$

Then $v_A = \left(\dfrac{2}{1+\frac{1}{b}}\right)u \to 2u-$ as $b \to \infty$, and $v_A < 2u$ always **E1** convincing ⑥



(ii) **M1** Using the results of (i), $v_2 = u_2 = \left(\dfrac{2\lambda}{\lambda+1}\right)u$

M1 repeatedly $u_3 = \left(\dfrac{2\lambda}{\lambda+1}\right)u_2 = \left(\dfrac{2\lambda}{\lambda+1}\right)^2 u$

… … … …

M1 all the way down to u $u_n = \left(\dfrac{2\lambda}{\lambda+1}\right)u_{n-1} = \left(\dfrac{2\lambda}{\lambda+1}\right)^{n-1} u$

and $v = \left(\dfrac{2\lambda}{\lambda+1}\right)u_n = \left(\dfrac{2\lambda}{\lambda+1}\right)^n u$ **A1 A1**

Since $u_n = \dfrac{2\lambda}{\lambda+1} > 1$, as $\lambda > 1$ **E1**

it follows that v can be made as large as possible **E1** ⑦

In the case when $\lambda = 4$, $v = \left(\frac{8}{5}\right)^n u > 20u$ requires $n \log\left(\frac{8}{5}\right) > \log 20 \Rightarrow n > \dfrac{\log 20}{\log\left(\frac{8}{5}\right)}$ **M1 A1**

Now $\log 2 = 0.30103 \Rightarrow \log 8 = 3\log 2 = 0.90309$ **M1**

$\log 5 = \log 10 - \log 2 = 1 - 0.30103 = 0.69897$ **M1**

so that $\log\left(\frac{8}{5}\right) = \log 8 - \log 5 = 0.20412$

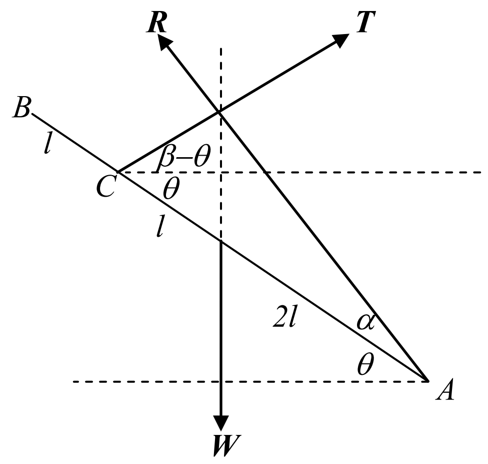
Also $\log 20 = \log 10 + \log 2 = 1 + 0.30103 = 1.30103$ **M1**

and $n > \dfrac{1.30103}{0.20412}$.

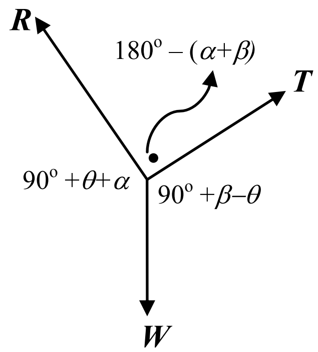
Since $6 \times 0.20412 = 1.22472$ and $7 \times 0.20412 = 1.42884$,

$n_{\min} = 7$ **A1** answer **E1** suitable justification ⑦

11



N.B. The three forces must be concurrent for equilibrium **M1**



Angles **A1**
A1

By *Lami's Theorem* (or a triangle of forces and the *Sine Rule*): **M3**

$$\frac{T}{\sin(90^\circ+\theta+\alpha)}=\frac{R}{\sin(90^\circ+\beta-\theta)}=\frac{W}{\sin(180^\circ-[\alpha+\beta])}$$

$$\Rightarrow\ \frac{T}{\cos(\theta+\alpha)}=\frac{R}{\cos(\beta-\theta)}=\frac{W}{\sin(\alpha+\beta)}$$ **A1**

<u>**A**↵</u> $W.2l\cos\theta = T.3l\sin\beta$ **M1 A1** ⑨

Then $T=\dfrac{2W\cos\theta}{3\sin\beta}=\dfrac{W\cos(\theta+\alpha)}{\sin(\alpha+\beta)}$ **M1**

$\Rightarrow\ 2\cos\theta\ \sin(\alpha+\beta) = 3\sin\beta\cos(\theta+\alpha)$ **M1**

$\Rightarrow\ 2\cos\theta\ \sin\alpha\ \cos\beta + 2\cos\theta\ \cos\alpha\ \sin\beta = 3\sin\beta\ \cos\theta\ \cos\alpha - 3\sin\beta\ \sin\theta\ \sin\alpha$ **M1**

Dividing by $\cos\theta\ \cos\alpha\ \cos\beta \Rightarrow\ 2\tan\alpha + 2\tan\beta = 3\tan\beta - 3\tan\beta\ \tan\theta\ \tan\alpha$ **M1**

$\Rightarrow\ 2\tan\alpha + 3\tan\beta\ \tan\theta\ \tan\alpha = \tan\beta$

Dividing by $\tan\alpha\ \tan\beta$ **M1**

$\Rightarrow\ 2\cot\beta + 3\tan\theta = \cot\alpha$ **A1 (ANSWER GIVEN)** ⑥

$\theta = 30^\circ,\ \beta = 45^\circ \Rightarrow \cot\alpha = 2.1+3.\dfrac{1}{\sqrt{3}} = 2+\sqrt{3}$ **B1**

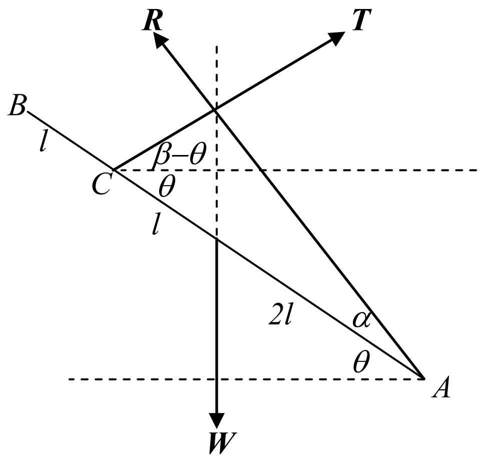
Now $\tan(2A) = \dfrac{2\tan A}{1-\tan^2 A} \Rightarrow \frac{1}{\sqrt{3}}\left(1-t^2\right)=2t \Rightarrow 0=t^2+2t\sqrt{3}-1$ **M1**

$\Rightarrow\ t = \tan 15^\circ = \dfrac{-2\sqrt{3}\pm\sqrt{12+4}}{2} = -\sqrt{3}\pm 2$ **M1**

However, $\tan 15^\circ > 0$ since 15° is acute, so $\tan 15^\circ = 2-\sqrt{3}$ and $\cot 15^\circ = 2+\sqrt{3}$ **M1 A1**

ALT. $\tan(60^\circ - 45^\circ) = \dfrac{\sqrt{3}-1}{1+\sqrt{3}} = \dfrac{\left(\sqrt{3}-1\right)^2}{3-1} = 2-\sqrt{3} = \dfrac{1}{2+\sqrt{3}}$ or verification ⑤

11 ALTERNATIVE



M1 A1 A1 for relevant, correct angles for their working

$(\beta-\theta)$ & $(\alpha+\theta)$ below

Res.↑ $T\sin(\beta-\theta)+R\sin(\alpha+\theta)=W$ **M1 A1**

Res.→ $T\cos(\beta-\theta)=R\cos(\alpha+\theta)$ **M1 A1**

A↲ $W.2l\cos\theta = T.3l\sin\beta$ **M1 A1** ⑨

Subst^{g}. to eliminate T's (for instance): **M1**

$$T\sin(\beta-\theta)+\frac{T\cos(\beta-\theta)}{\cos(\alpha+\theta)}\sin(\alpha+\theta)=\frac{3T\sin\beta}{2\cos\theta}$$

$$\Rightarrow\ 2\cos\theta\left(\cos\alpha.\cos\theta-\sin\alpha.\sin\theta\right)\left(\sin\beta.\cos\theta-\cos\beta.\sin\theta\right)$$
$$+\ 2\cos\theta\left(\cos\beta.\cos\theta+\sin\beta.\sin\theta\right)\left(\sin\alpha.\cos\theta+\cos\alpha.\sin\theta\right)$$
$$=3\sin\beta\left(\cos\alpha.\cos\theta-\sin\alpha.\sin\theta\right)$$

M1 Correct trig. expansions

Dividing by $\cos\theta\ \cos\alpha\ \cos\beta$ **M1**

$$\Rightarrow\ 2(\cos\theta-\tan\alpha.\sin\theta)(\tan\beta.\cos\theta-\sin\theta)+2(\cos\theta+\tan\beta.\sin\theta)(\tan\alpha.\cos\theta+\sin\theta)$$
$$=3\tan\beta(1-\tan\alpha.\tan\theta)$$

M1 Multiplying out, cancelling and collecting up terms

M1 Dividing by $\tan\alpha\ \tan\beta$

$\Rightarrow\ 2\cot\beta+3\tan\theta=\cot\alpha$ **A1 (ANSWER GIVEN)** ⑥

$\theta=30^{\circ},\ \beta=45^{\circ}\ \Rightarrow\ \cot\alpha=2.1+3.\frac{1}{\sqrt{3}}=2+\sqrt{3}$ **B1**

Now $\tan(2A)=\dfrac{2\tan A}{1-\tan^{2}A}\ \Rightarrow\ \frac{1}{\sqrt{3}}\left(1-t^{2}\right)=2t\ \Rightarrow\ 0=t^{2}+2t\sqrt{3}-1$ **M1**

$\Rightarrow\ t=\tan 15^{\circ}=\dfrac{-2\sqrt{3}\pm\sqrt{12+4}}{2}=-\sqrt{3}\pm2$ **M1**

However, $\tan 15^{\circ}>0$ since 15° is acute, so $\tan 15^{\circ}=2-\sqrt{3}$ and $\cot 15^{\circ}=2+\sqrt{3}$ **M1 A1**

ALT. $\tan(60^{\circ}-45^{\circ})=\dfrac{\sqrt{3}-1}{1+\sqrt{3}}=\dfrac{\left(\sqrt{3}-1\right)^{2}}{3-1}=2-\sqrt{3}=\dfrac{1}{2+\sqrt{3}}$ or verification ⑤

12	Since the pdf is only non-zero between 0 & 1 and the area under its graph = 1 if a, b both $< / > 1$ then total area will be $< / > 1$	**M1** considering graph or ≡ **M1** consideration of area … relative to 1	②
(i)	$1 = \int_0^1 \mathrm{f}(x)\,\mathrm{d}x = \int_0^k a\,\mathrm{d}x + \int_k^1 b\,\mathrm{d}x$	**M1** use of total prob. = 1	
	$= [ax]_0^k + [bx]_k^1 = ak + b - bk$	**M1** calculus used to find k	
	$\Rightarrow\ k = \dfrac{1-b}{a-b}$	**A1**	③
	$\mathrm{E}(X) = \int_0^1 x\mathrm{f}(x)\,\mathrm{d}x = \int_0^k ax\,\mathrm{d}x + \int_k^1 bx\,\mathrm{d}x$ **M1** $= \left[\dfrac{ax^2}{2}\right]_0^k + \left[\dfrac{bx^2}{2}\right]_k^1 = \dfrac{ak^2}{2} + \dfrac{b}{2} - \dfrac{bk^2}{2}$		
	M1 use of k in terms of a, b	$\mathrm{E}(X) = \dfrac{b}{2} + \dfrac{(a-b)}{2} \times \left(\dfrac{1-b}{a-b}\right)^2 = \dfrac{ba - b^2 + 1 - 2b + b^2}{2(a-b)}$ $= \dfrac{1-2b+ab}{2(a-b)}$ **A1 (ANSWER GIVEN)**	③
(ii)	If $ak \ge \frac{1}{2}$ (i.e. $M \in (0, k)$) **M1** recognition of this then $\dfrac{a-ab}{a-b} \ge \frac{1}{2} \Rightarrow 2a - 2ab \ge a - b \Rightarrow a + b \ge 2ab$ **B1** correct condition confirmed and $aM = \frac{1}{2}$ or $M = \dfrac{1}{2a}$ **A1 (ANSWER GIVEN)**		③
	If $ak \le \frac{1}{2}$ (i.e. $M \in (k, 1)$) **M1** recognition that this $\equiv a + b \le 2ab$ then $ak + (M-k)b = \frac{1}{2}$ or $(1-M)b = \frac{1}{2}$ **M1** $\Rightarrow M = 1 - \dfrac{1}{2b}$ **A1**		③
(iii)	If $a + b \ge 2ab$, then $\mu - M = \dfrac{1-2b+ab}{2(a-b)} - \dfrac{1}{2a}$ **M1** applying correct case $= \dfrac{a - 2ab + a^2b - a + b}{2a(a-b)} = \dfrac{b(1-a)^2}{2a(a-b)}$ **M1** single fraction, fact$^{\mathrm{g}}$. & compl$^{\mathrm{g}}$. the sq. or equivalent (inequalities) method > 0 **A1** correctly concluded		③
	If $a + b \le 2ab$, then $\mu - M = \dfrac{1-2b+ab}{2(a-b)} - 1 + \dfrac{1}{2b}$ **M1** applying correct case $= \dfrac{b - 2b^2 + ab^2 - 2ab + 2b^2 + a - b}{2b(a-b)} = \dfrac{a(1-b)^2}{2b(a-b)}$ **M1** sing. frac., fact$^{\mathrm{g}}$. & compl$^{\mathrm{g}}$. the sq. (or ≡) > 0 **A1** correctly concluded		③

13 (i) $P(W_{PPQ}) = P(W_P\ W_Q\ –) + P(L_P\ W_Q\ W_P)$ **M1** A sum of 2(3) probs or ≡ product

$= p\,.\,q.\ 1 + (1 – p)qp = pq(2 – p)$ **A1**

$P(W_{PQQ}) = pq(2 – q)$ similarly **B1** ***ft***

$P(W_{PPQ}) – P(W_{PQQ}) = pq(q – p)$ **M1** Or comparing two sides of a relevant inequality (Ditto throughout qn.)

> 0 since $q > p \Rightarrow P(W_{PPQ}) > P(W_{PQQ})$ for all p, q

and "Ros plays Pardeep twice" is always her best strategy **A1** ⑤

(ii) **SI:** $P(W_1) = P(W_Q\ W_P\ – –) + P(W_Q\ L_P\ W_P\ –) + P(W_Q\ L_P\ L_P\ W_P)$ **M1** cases

$= pq + pq(1 – p) + pq(1 – p)^2$ **A1** unsimplified $= pq(3 – 3p + p^2)$

SIII: $p(W_3) = pq(3 – 3q + q^2)$ similarly **B1** ***ft***

SII: $p(W_2) = p(W_P\ W_Q\ – –) + p(L_P\ W_P\ W_Q\ –) + p(W_P\ L_Q\ W_Q\ –) + p(L_P\ W_P\ L_Q\ W_Q)$ **M1** cases

$= pq + pq(1 – p) + pq(1 – q) + pq(1 – p)(1 – q)$ **A1** unsimplified

$= pq(4 – 2p – 2q + pq)$ or $pq(2 – p)(2 – q)$ ⑤

$P(W_1) – P(W_3) = pq(q – p)\left(3-[p+q]\right) > 0$ since $q > p$ and $p + q < 2 < 3$

so that **SI** is always better than **S3** **B1** ①

$P(W_1) – P(W_2) = pq\left(p^2 - p - 1 - pq + 2q\right)$ **M1**

$= pq\left((2-p)(q-p)-(1-p)\right)$ **M1**

> 0 whenever $q - p > \dfrac{1-p}{2-p} = 1 - \dfrac{1}{2-p}$ **A1** [arrangements with > one q term not helpful]

Now $p + \frac{1}{2} < q < 1 \Rightarrow 0 < p < \frac{1}{2} \Rightarrow \frac{1}{3} < 1 - \dfrac{1}{2-p} < \frac{1}{2}$,

so that **SI** always better than **SII** when $q - p > \frac{1}{2}$. **E1**

<u>**ALT.**</u> Setting $q = p + \frac{1}{2} + \varepsilon$ where $\varepsilon > 0$ gives

$P(W_1) – P(W_2) = p\left(p + \tfrac{1}{2} + \varepsilon\right)\left(p^2 - p - 1 - p^2 - \tfrac{1}{2}p - p\varepsilon + 2p + 1 + 2\varepsilon\right)$

$= p\left(p + \tfrac{1}{2} + \varepsilon\right)\left(\tfrac{1}{2}p + (2-p)\varepsilon\right) > 0$ since all terms positive ④

$\text{P}(W_1) - \text{P}(W_2) >< 0 \;\Leftrightarrow\; q - p >< \dfrac{1-p}{2-p}$ **M1** Some clear method for deciding

Take $p = \frac{1}{4}, q = \frac{1}{2} \;\Rightarrow\; q - p = \frac{1}{4} < \frac{1}{2}$ and $\dfrac{1-p}{2-p} = \frac{3}{7} > \frac{1}{4}$ so **SII** is better than **SI** **M1 A1**

Take $p = \frac{1}{4}, q = \frac{3}{4} - \varepsilon \;\Rightarrow\; q - p = \frac{1}{2} - \varepsilon < \frac{1}{2}$ and $\dfrac{1-p}{2-p} = \frac{3}{7}$

so choosing $\varepsilon < \frac{3}{7} - \frac{1}{2} = \frac{1}{14}$ (say $\frac{1}{16}$) will give **M1**

$p = \frac{1}{4}, q = \frac{11}{16}$ and $q - p = \frac{7}{16} > \dfrac{1-p}{2-p} = \frac{3}{7}$ so **SI** is better than **SII** **A1**

For the most part, candidates are just picking values of a and b and subst$^{\text{g}}$. into
SI : $pq(3 - 3p + p^2)$ and **SII** : $pq(2-p)(2-q)$

If they pick an a and a b and then do nothing with them, they score M0.
To score the M1, they must show that $q - p < \frac{1}{2}$ and attempt to work out the two probs.
To score the A1, *they* must demonstrate the result. Also, their numerical working must be both visible and correct

⑤

[I think that $q - p > k$ has $k = \frac{1}{2}$ as the least positive k which *always* gives **SI** better than **SII**]