STEP Mathematics III 2009: Markscheme

Section A: Pure Mathematics

1. Gradient $SV = \dfrac{ms-nv}{s-v}$, **B1** alternative $= \dfrac{nv-ms}{v-s}$

and so the equation of the line *SV* is $y - ms = \dfrac{ms-nv}{s-v}(x-s)$. **M1 A1** for full legitimate attempt to find line equation. If they use $y = mx + c$, they must get to evaluating +c to earn method mark. Alternatives $y - ms = \dfrac{nv-ms}{v-s}(x-s)$, $y - nv = \dfrac{ms-nv}{s-v}(x-v)$,

$y - nv = \dfrac{nv-ms}{v-s}(x-v)$

Hence, $-ms = \dfrac{ms-nv}{s-v}(p-s)$,

and so $p - s = \dfrac{-ms(s-v)}{ms-nv}$, (i). see below

$p = s - \dfrac{ms(s-v)}{ms-nv} = \dfrac{s}{ms-nv}(ms-nv-m(s-v))$ **M1** for substitution and attempting to make p the subject

$p = \dfrac{(m-n)sv}{ms-nv}$ as required. **(*)** **A1**

5 marks

Similarly $q = \dfrac{(m-n)tu}{mt-nu}$ **B2**

2 marks

As *S* and *T* lie on the circle, *s* and *t* are solutions of the equation

$\lambda^2 + (m\lambda - c)^2 = r^2$

i.e. $(1+m^2)\lambda^2 - 2mc\lambda + (c^2 - r^2) = 0$ **M1 A1**

Thus $st = \dfrac{c^2-r^2}{1+m^2}$, and $s + t = \dfrac{2mc}{1+m^2}$ **M1 A1, A1** sum, product of roots

5 marks

As *U* and *V* also lie on the circle,

similarly $uv = \dfrac{c^2-r^2}{1+n^2}$, and $u + v = \dfrac{2nc}{1+n^2}$ **M1 A1, A1** by interchange of letters

$p + q = \dfrac{(m-n)sv}{ms-nv} + \dfrac{(m-n)tu}{mt-nu}$ **M1** substitution

$= \dfrac{(m-n)}{(ms-nv)(mt-nu)}(sv(mt-nu) + tu(ms-nv))$ **M1** common denominator & factor

$$= \frac{(m-n)}{(ms-nv)(mt-nu)}\left(stm(u+v)-nuv(s+t)\right)$$ **M1** regroup of terms to enable next line

$$= \frac{(m-n)}{(ms-nv)(mt-nu)}\left(\frac{c^2-r^2}{1+m^2}m\frac{2nc}{1+n^2}-n\frac{c^2-r^2}{1+n^2}\frac{2mc}{1+m^2}\right)$$ **M1** substitution

$$= \frac{(m-n)}{(ms-nv)(mt-nu)}\frac{2mnc\left(c^2-r^2\right)}{\left(1+m^2\right)\left(1+n^2\right)}(1-1) = 0$$ as required. **(*)** **A1**

8 marks

(i) can be achieved directly by considering gradients or similar triangles, and hence

$\frac{-ms}{p-s} = \frac{ms-nv}{s-v}$ or equivalent

may be written straight down earning **M1 A1** (lhs) **A1** (rhs)

2. (i) $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$ **B1**

$y'' = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ either **B1**

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots..$ **B1**

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots..$ **B1**

$y'' = 2a_2 + 6a_3 x + \ldots..$ **B1**

5 marks

(ii) $xy'' - y' + 4x^3 y = 0$

$x(2a_2 + 6a_3 x + \ldots..) - (a_1 + 2a_2 x + 3a_3 x^2 + \ldots..) + 4x^3(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots..) = 0$

M1 A1 substituting series

$-a_1 + 3a_3 x^2 + (8a_4 + 4a_0)x^3 + \ldots.. = 0$ **M1 A1** simplification

$\therefore a_1 = 0$ comparing constants (and also $a_3 = 0$ comparing x^2 coefficients)

(*) **B1**

5 marks

Comparing coefficients of x^{n-1}, for $n \geq 4$, $n(n-1)a_n - na_n + 4a_{n-4} = 0$ **M1**

$n(n-2)a_n + 4a_{n-4} = 0$ and so $a_n = \frac{-4}{n(n-2)} a_{n-4}$ **M1 A1** rearrangement

If $a_0 = 1$, and $a_2 = 0$, and as $a_1 = 0$, and $a_3 = 0$, **B1** for a_3

$a_4 = \frac{-4}{4 \times 2} = \frac{-1}{2!}$, $a_5 = 0$, $a_6 = 0$, $a_7 = 0$, $a_8 = \frac{-4}{8 \times 6} \times \frac{-1}{2!} = \frac{1}{4!}$, etc.

Thus $y = 1 - \frac{1}{2!}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \ldots..$ **M1** substitution

$= 1 - \frac{1}{2!}(x^2)^2 + \frac{1}{4!}(x^2)^4 - \frac{1}{6!}(x^2)^6 + \ldots..$ **M1** making next step clear

$= \cos(x^2)$ **(*)** **A1**

7 marks

If $a_0 = 0$, and $a_2 = 1$, and as $a_1 = 0$, and $a_3 = 0$,

$a_4 = 0$, $a_5 = 0$, $a_6 = \frac{-4}{6 \times 4} = \frac{-1}{3!}$, $a_7 = 0$, $a_8 = 0$, $a_9 = 0$,

$a_{10} = \frac{-4}{10 \times 8} \times \frac{-1}{3!} = \frac{1}{5!}$ etc.

Thus $y = x^2 - \frac{-1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \ldots..$ **M1** substitution

$= (x^2) - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \frac{1}{7!}(x^2)^7 + \ldots..$ **M1** making next step clear

$= \sin(x^2)$ **A1**

3 marks

3. (i) $f(t)=\dfrac{t}{e^t-1}=\dfrac{t}{1+t+\dfrac{t^2}{2!}+.....-1}$

$=\dfrac{t}{t\left(1+\dfrac{t}{2!}+.....\right)}$

M1 sub power series and re-arrange

$=\dfrac{1}{\left(1+\dfrac{t}{2!}+.....\right)}$

So $\lim_{t\to 0} f(t)=1$ **(*)** **A1**

$f'(t)=\dfrac{\left(e^t-1\right)-te^t}{\left(e^t-1\right)^2}$ (which may be written $\dfrac{(1-t)e^t-1}{\left(e^t-1\right)^2}$) **M1** legitimate attempt at differentiation **A1**

$$f'(t)=\frac{(1-t)e^t-1}{\left(e^t-1\right)^2}=\frac{(1-t)\left(1+t+\dfrac{t^2}{2!}+....\right)-1}{\left(1+t+\dfrac{t^2}{2!}+....-1\right)^2}=\frac{-t^2+\dfrac{t^2}{2!}-\dfrac{t^3}{2!}+\dfrac{t^3}{3!}-.....}{t^2\left(1+\dfrac{t}{2!}+.....\right)^2}$$

$$=\frac{-\dfrac{1}{2}-t\left(\dfrac{1}{2!}-\dfrac{1}{3!}\right)-.....}{\left(1+\dfrac{t}{2!}+.....\right)^2}$$

M1 sub power series and re-arrange

So $\lim_{t\to 0} f'(t)=\dfrac{-1}{2}$

A1

6 marks

(ii) Let $g(t)=f(t)+\dfrac{1}{2}t$

$g(t)=f(t)+\dfrac{1}{2}t=\dfrac{t}{e^t-1}+\dfrac{1}{2}t$

$=\dfrac{2t+t\left(e^t-1\right)}{2\left(e^t-1\right)}$

M1 correct attempt to simplify algebra

$=\dfrac{te^t+t}{2\left(e^t-1\right)}$

$=\dfrac{t\left(e^t+1\right)}{2\left(e^t-1\right)}$

A1

$g(-t)=\dfrac{-t\left(e^{-t}+1\right)}{2\left(e^{-t}-1\right)}=\dfrac{-t\left(1+e^t\right)}{2\left(1-e^t\right)}=\dfrac{t\left(e^t+1\right)}{2\left(e^t-1\right)}=g(t)$ as required.

M1 correct algebra **A1**

(iii) Let $h(t) = e^t(1-t)$

Then $h'(t) = e^t(1-t) - e^t = -te^t$

So $h'(t) = 0 \Longrightarrow t = 0$ **M1** for diff'n and solving =0

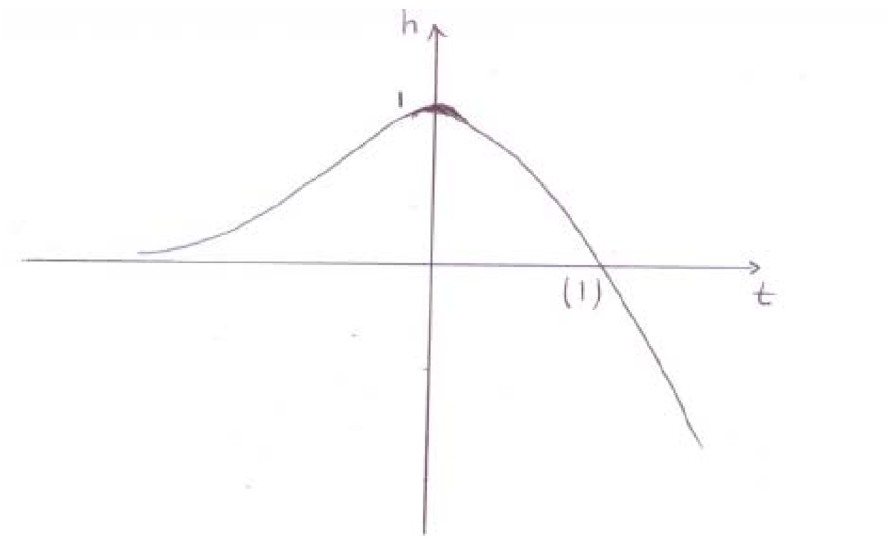
$t > 0 \Longrightarrow h'(t) < 0, t < 0 \Longrightarrow h'(t) > 0$

Thus $t = 0, h(t) = 1$ is a maximum and is the only turning point. **M1 A1**

Alternatively,

$h''(t) = -e^t - te^t = -(1+t)e^t$, so $h''(0) = -1$

Thus $t = 0, h(t) = 1$ is a maximum and is the only turning point. **M1 A1**



A1 *for correct shape and maximum correctly indicated (x intercept 1 optional)*

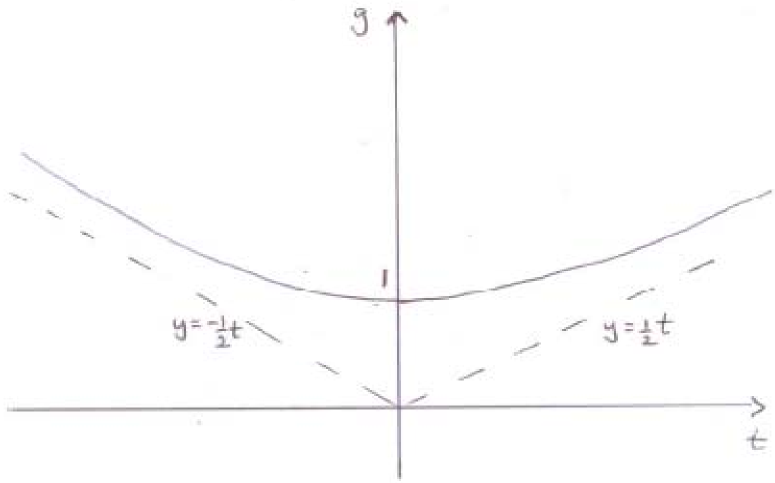
Hence $e^t(1-t) \leq 1$ and $e^t(1-t) - 1 \leq 0$

So $f'(t) = \frac{(1-t)e^t - 1}{(e^t-1)^2} \leq 0$, with equality only possible for $t = 0$ **E1**

5 marks

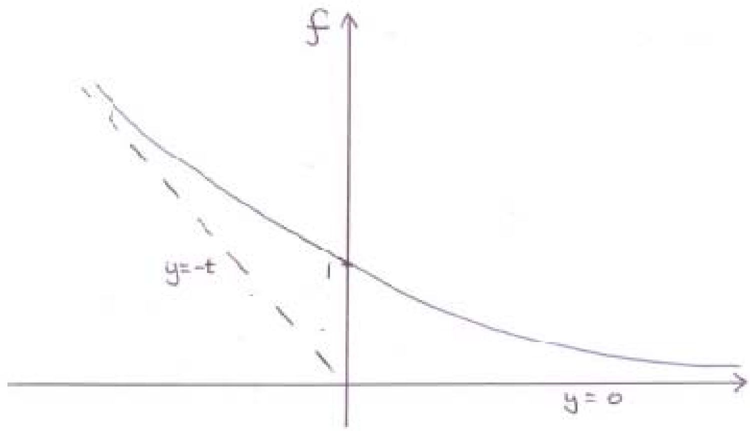
But we know $\lim_{t \to 0} f'(t) = \frac{-1}{2}$ and so, in fact, *f(t)* is always decreasing i.e. has no turning points. **E1**

The graph of $g(t) = f(t) + \frac{1}{2}t$ passes through (0,1), is symmetrical and approaches $y = \frac{1}{2}t$ as $t \to \infty$ and thus is



(optional but if drawn **B1** shape, **B1** asymptotes, **B1** maximum/intercept)

Therefore the graph of $f(t) = g(t) - \frac{1}{2}t$ also passes through (0,1), and has asymptotes $y = 0$ and $y = -t$ and thus is



B1 shape, **B1** $y = 0$, **B1** $y = -t$, **B1** y intercept and "correct" gradient. (If any optional marks earned for graph of g, award these marks instead if greater.)

5 marks

4. (i) $\int_0^\infty e^{-st}e^{-bt}f(t)dt = \int_0^\infty e^{-t(s+b)}f(t)dt = F(s+b)$ **(*)**

M1 **M1** **A1** 3 marks

(ii) $\int_0^\infty e^{-st}f(at)dt = \int_0^\infty e^{-s\frac{u}{a}}f(u)\frac{1}{a}du = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u}f(u)du = a^{-1}F\left(\frac{s}{a}\right)$ **(*)**

M1 change variable **M1** **A1**
3 marks

(iii) $\int_0^\infty e^{-st}f'(t)dt = \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty -se^{-st}f(t)dt$ **M1 A1**

$= -f(0) + sF(s)$

(*) **A1**

3 marks

(iv) $\int_0^\infty e^{-st}\sin t dt = \left[-e^{-st}\cos t\right]_0^\infty - \int_0^\infty se^{-st}\cos t dt$ **M1 A1**

$= 1 - \left(\left[se^{-st}\sin t\right]_0^\infty - \int_0^\infty -s^2e^{-st}\sin t dt\right)$ **M1 A1**

$= 1 - s^2\int_0^\infty e^{-st}\sin t dt$

Therefore $F(s) = 1 - s^2F(s)$

M1 for subbing F(s)

and so $(s^2+1)F(s) = 1$, i.e. $F(s) = \dfrac{1}{s^2+1}$ **(*)** **A1**

6 marks

So transform of $\cos t$ is $sF(s) - f(0) = \dfrac{s}{s^2+1} - \sin 0 = \dfrac{s}{s^2+1}$, **M1 A1**

transform of $\cos qt$ is $q^{-1}\left(\dfrac{s/q}{s^2/q^2+1}\right) = \dfrac{s}{s^2+q^2}$, **M1**

and so transform of $e^{-pt}\cos qt$ is $\dfrac{(s+p)}{(s+p)^2+q^2}$ **M1 A1**

5 marks

5. $\left(x+y+z\right)^2-\left(x^2+y^2+z^2\right)=2\left(yz+zx+xy\right)$ **M1 A1**

So $yz+zx+xy=\frac{1}{2}\left(S_1{}^2-S_2\right)=\frac{1}{2}\left(1^2-2\right)=-\frac{1}{2}$ **M1 A1**

(*)

4 marks

$\left(x^2+y^2+z^2\right)\left(x+y+z\right)=x^3+y^3+z^3+\left(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y\right)$ **M1 A1**

So $x^2y+x^2z+y^2z+y^2x+z^2x+z^2y=S_2S_1-S_3=2\times 1-3=-1$ **M1 A1**

(*)

4 marks

$\left(x+y+z\right)^3=\left(x^3+y^3+z^3\right)+3\left(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y\right)+6xyz$ **M1 A1**

So $xyz=\frac{1}{6}\left(S_1{}^3-S_3-3(ii)\right)=\frac{1}{6}\left(1^3-3+3\right)=\frac{1}{6}$ **(*)** **M1 A1**

4 marks

$x^{n+1}+y^{n+1}+z^{n+1}=(x+y+z)(x^n+y^n+z^n)-(xy^n+xz^n+yx^n+yz^n+zx^n+zy^n)$ **M1 A1**

$=1.S_n-(xy+yz+zx)(x^{n-1}+y^{n-1}+z^{n-1})+xyz(x^{n-2}+y^{n-2}+z^{n-2})$

M1 A1 **M1 A1**

So $S_{n+1}=S_n+\frac{1}{2}S_{n-1}+\frac{1}{6}S_{n-2}$ **M1 A1**

8 marks

Alternatively,

from 3rd result, $yz=\frac{1}{6x}$, and so 1st result becomes $\frac{1}{6x}+x\left(y+z\right)=-\frac{1}{2}$ **M1 A1**

and thus, $y+z=-\frac{1}{2x}-\frac{1}{6x^2}$. **M1 A1**

2nd result is therefore $x^2\left(-\frac{1}{2x}-\frac{1}{6x^2}\right)+\frac{1}{6x}\left(-\frac{1}{2x}-\frac{1}{6x^2}\right)+x\left(-\frac{1}{2x}-\frac{1}{6x^2}\right)^2-\frac{1}{3}=0$

This simplifies to $\frac{-x}{2}-\frac{1}{6}-\frac{1}{12x^2}-\frac{1}{36x^3}+\frac{1}{4x}+\frac{1}{6x^2}+\frac{1}{36x^3}-\frac{1}{3}=-1$

$\frac{-x}{2}+\frac{1}{2}+\frac{1}{4x}+\frac{1}{12x^2}=0$ **M1**

and so $6x^3-6x^2-3x-1=0$ **A1**

Hence $x^3=x^2+\frac{1}{2}x+\frac{1}{6}$, $x^{n+1}=x^n+\frac{1}{2}x^{n-1}+\frac{1}{6}x^{n-2}$, by symmetry, the same is true for *y* and *z*, and so $S_{n+1}=S_n+\frac{1}{2}S_{n-1}+\frac{1}{6}S_{n-2}$ **M1 A1**

8 marks

6. $e^{i\beta} - e^{i\alpha} = (\cos\beta - \cos\alpha) + i(\sin\beta - \sin\alpha)$

$\left|e^{i\beta} - e^{i\alpha}\right|^2 = (\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2$ **M1** for mod squared

$= \cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta + \sin^2\beta$

M1 for cos sq'd plus sin squ'd

$= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= 2 - 2\cos(\beta - \alpha)$ **M1** for compound angle (or factor formula)

$= 2 - 2\left(1 - 2\sin^2\frac{1}{2}(\beta - \alpha)\right)$ **M1** for half angle formula

$= 4\sin^2\frac{1}{2}(\beta - \alpha)$ **M1** for algebra throughout

$\therefore \left|e^{i\beta} - e^{i\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ as both are positive. **E1** for arguing + root

6 marks

Alternatively, $e^{i\beta} - e^{i\alpha} = (\cos\beta - \cos\alpha) + i(\sin\beta - \sin\alpha)$

$= -2\sin\left(\frac{1}{2}(\beta + \alpha)\right)\sin\left(\frac{1}{2}(\beta - \alpha)\right) + i2\cos\left(\frac{1}{2}(\beta + \alpha)\right)\sin\left(\frac{1}{2}(\beta - \alpha)\right)$

$\left|e^{i\beta} - e^{i\alpha}\right|^2 = 2^2\sin^2\left(\frac{1}{2}(\beta - \alpha)\right)\left(\sin^2\left(\frac{1}{2}(\beta + \alpha)\right) + \cos^2\left(\frac{1}{2}(\beta + \alpha)\right)\right)$

$\therefore \left|e^{i\beta} - e^{i\alpha}\right|^2 = 2^2\sin^2\left(\frac{1}{2}(\beta - \alpha)\right)$ and hence result as before.

same distribution of marks

$\left|e^{i\alpha} - e^{i\beta}\right|\left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right|\left|e^{i\alpha} - e^{i\delta}\right|$

$= 2\sin\left(\frac{1}{2}(\alpha - \beta)\right)2\sin\left(\frac{1}{2}(\gamma - \delta)\right) + 2\sin\left(\frac{1}{2}(\beta - \gamma)\right)2\sin\left(\frac{1}{2}(\alpha - \delta)\right)$ **M1 A1 A1**

$= -2\left(\cos\left(\frac{1}{2}(\alpha - \beta + \gamma - \delta)\right) - \cos\left(\frac{1}{2}(\alpha - \beta - \gamma + \delta)\right)\right)$

$-2\left(\cos\left(\frac{1}{2}(\beta - \gamma + \alpha - \delta)\right) - \cos\left(\frac{1}{2}(\beta - \gamma - \alpha + \delta)\right)\right)$ **M1 A1 A1**

$= 2\left(\cos\left(\frac{1}{2}(\alpha - \beta - \gamma + \delta)\right) - \cos\left(\frac{1}{2}(\beta - \gamma + \alpha - \delta)\right)\right)$

A1

$= -4\sin\left(\frac{1}{2}(\alpha - \gamma)\right)\sin\left(\frac{1}{2}(\delta - \beta)\right)$

M1 A1

$= 2\sin\left(\frac{1}{2}(\alpha - \gamma)\right)2\sin\left(\frac{1}{2}(\beta - \delta)\right)$

$= \left|e^{i\alpha} - e^{i\gamma}\right|\left|e^{i\beta} - e^{i\delta}\right|$ as required.

A1
10 marks

$AC.BD = AB.CD + BC.AD$ (this only **SC2**)
i.e. the product of the diagonals **E1** of a cyclic quadrilateral **E1** is equal to the sum **E1** of the products of the opposite pairs of sides **E1**.

(Merely stating "Ptolemy's Theorem" **SC4**)

4 marks

7. (i) Assume $\left(1+x^2\right)f_{k+1}(x)+2(k+1)xf_k(x)+k(k+1)f_{k-1}(x)=0$ for some k

B1

Then

$$\left(1+x^2\right)f_{k+2}(x)+2xf_{k+1}(x)+2(k+1)xf_{k+1}(x)+2(k+1)f_k(x)+k(k+1)f_k(x)=0$$

M1 diff'n

i.e. $\left(1+x^2\right)f_{k+2}(x)+2(k+2)xf_{k+1}(x)+(k+1)(k+2)f_k(x)=0$ which is the required result for k+1. **A1**

For $k=1$, $\left(1+x^2\right)f_{k+1}(x)+2(k+1)xf_k(x)+k(k+1)f_{k-1}(x)$

$$=\left(1+x^2\right)f_2(x)+4xf_1(x)+2f_0(x)=\left(1+x^2\right)\frac{6x^2-2}{\left(1+x^2\right)^3}+4x\frac{-2x}{\left(1+x^2\right)^2}+2\frac{1}{1+x^2}$$

M1 diff'n

B1 2nd diff. **B1** 1st diff.

$$=\frac{1}{\left(1+x^2\right)^2}\left(6x^2-2-8x^2+2\left(1+x^2\right)\right)=0$$

A1

Hence result true for $k=1$, and by principle of mathematical induction true for all n. **E1**

8 marks

(ii)

$$P_0(x)=\left(1+x^2\right)\frac{1}{1+x^2}=1$$

M1 A1

$$P_1(x)=\left(1+x^2\right)^2\frac{-2x}{\left(1+x^2\right)^2}=-2x$$

M1 A1

$$P_2(x)=\left(1+x^2\right)^3\frac{6x^2-2}{\left(1+x^2\right)^3}=6x^2-2$$

M1 A1

6 marks

$$P_{n+1}(x)-\left(1+x^2\right)\frac{dP_n(x)}{dx}+2(n+1)xP_n(x)$$

$$=\left(1+x^2\right)^{n+2}f_{n+1}(x)-\left(1+x^2\right)\left(\left(1+x^2\right)^{n+1}f_{n+1}(x)+(n+1)2x\left(1+x^2\right)^n f_n(x)\right)+2(n+1)x\left(1+x^2\right)^{n+1}f_n(x)$$

$=0$ as required.

M1 A1

2 marks

Assume $P_k(x)$ is a polynomial of degree *k*, then $\frac{dP_k(x)}{dx}$ is a polynomial of degree $k-1$, and so $\left(1+x^2\right)\frac{dP_k(x)}{dx}$ is a polynomial of degree $k+1$, and $2(k+1)xP_k(x)$ is also a polynomial of degree $k+1$. Thus $P_{k+1}(x)$ is a polynomial of degree <u>not greater than</u> $k+1$. **E1**

Further, assume that $P_k(x)$ has term of highest degree, **M1**

$(-1)^k(k+1)!x^k$, then as

$P_{n+1}(x)-\left(1+x^2\right)\frac{dP_n(x)}{dx}+2(n+1)xP_n(x)=0$, the term of highest degree of $P_{k+1}(x)$ is $(-1)^k(k+1)!kx^{k-1}x^2-2(k+1)x(-1)^k(k+1)!x^k$

$=(-1)^k(k+1)!x^{k+1}(k-2k-2)=(-1)^k(k+1)!x^{k+1}(-k-2)$ **A1**

$=(-1)^{k+1}(k+2)!x^{k+1}$ as required.

Result is true for $P_0(x)=1$, hence true for all *n* by PMI. **E1**

4 marks

8. (i) Let $x = e^{-t}$, **M1**

Then

$$\lim_{x\to 0}[x^m(\ln x)^n] = \lim_{t\to\infty}[(e^{-t})^m(-t)^n] = (-1)^n \lim_{x\to 0}[e^{-mt}t^n] = 0$$

M1 **M1** **A1**

4 marks

Let $m = n = 1$, then **M1**

$$\lim_{x\to 0}[x \ln x] = 0$$

M1

so

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} x\ln x} = e^0 = 1$$

M1 **(*)** **A1**

4 marks

(ii)

$$I_{n+1} = \int_0^1 x^m (\ln x)^{n+1} dx = \left[\frac{x^{m+1}(\ln x)^{n+1}}{m+1}\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\frac{(n+1)(\ln x)^n}{x} dx$$

M1 A1

$$= 0 - 0\ (using\ first\ result) - \int_0^1 \frac{n+1}{m+1} x^m(\ln x)^n dx = -\frac{n+1}{m+1} I_n$$

M1 **(*) A1**

4 marks

and so

$$I_n = \frac{-n}{m+1} \times \frac{-(n-1)}{m+1} \times \frac{-(n-2)}{m+1} \times \dots \times \frac{-1}{m+1} I_0 = \frac{(-1)^n n!}{(m+1)^n} \int_0^1 x^m dx$$

M1 **A1**

$$= \frac{(-1)^n n!}{(m+1)^n}\left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

A1

3 marks

(iii) $\int_0^1 x^x dx = \int_0^1 e^{x\ln x} dx = \int_0^1 1 + x\ln x + \frac{x^2(\ln x)^2}{2!} + \ldots\ldots dx$

M1 **M1 A1**

$= 1 + I_1 + \frac{1}{2!} I_2 + \ldots\ldots$ **A1**

$= 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \ldots\ldots = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \ldots\ldots$ as required. **(*) A1**

5 marks

9. (i) If V is the speed of projection from P, x and y are the horizontal and vertical displacements from P at a time t after projection, and T is the time of flight from P to Q, then

$x = Vt\cos\theta$, $y = Vt\sin\theta - \frac{1}{2}gt^2$, $\dot{x} = V\cos\theta$, and $\dot{y} = V\sin\theta - gt$

B1 any pair, **B1** other pair

So $\tan\alpha = \frac{VT\tan\theta - \frac{1}{2}gT^2}{VT\cos\theta} = \tan\theta - \frac{gT}{2V\cos\theta}$, **M1 A1**

and $\tan\varphi = \frac{V\sin\theta - gT}{V\cos\theta} = \tan\theta - \frac{gT}{V\cos\theta}$ **M1 A1**

Thus $\tan\theta + \tan\varphi = 2\tan\theta - \frac{gT}{V\cos\theta} = 2\left(\tan\theta - \frac{gT}{2V\cos\theta}\right) = 2\tan\alpha$ **(*) M1 A1**

8 marks

(ii) $t = \frac{x}{V\cos\theta}$, and so $y = V\frac{x}{V\cos\theta}\sin\theta - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$ **M1**

i.e. $y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta = x\tan\theta - \frac{gx^2}{2V^2}(1+\tan^2\theta)$ **M1**

As a quadratic equation in $\tan\theta$,

$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(\frac{gx^2}{2V^2} + y\right) = 0$ **A1**

Thus $\tan\theta + \tan\theta' = \frac{x}{\left(\frac{gx^2}{2V^2}\right)} = \frac{2V^2}{gx}$, **B1**

and $\tan\theta\tan\theta' = \frac{\left(\frac{gx^2}{2V^2}+y\right)}{\left(\frac{gx^2}{2V^2}\right)} = 1 + \frac{2V^2y}{gx^2} = 1 + \frac{2V^2}{gx}\tan\alpha$ **B1**

Thus $\tan(\theta+\theta') = \frac{(\tan\theta + \tan\theta')}{(1-\tan\theta\tan\theta')} = \frac{\left(\frac{2V^2}{gx}\right)}{1-\left(1+\frac{2V^2}{gx}\tan\alpha\right)} = \frac{-1}{\tan\alpha} = -\cot\alpha$ **(*) M1 A1**

7 marks

$\tan(\theta+\theta') = -\cot\alpha = \cot(-\alpha) = \tan\left(\frac{\pi}{2} - (-\alpha)\right) = \tan\left(\frac{\pi}{2} + \alpha + n\pi\right)$

Therefore, $+\theta' = \frac{\pi}{2} + \alpha + n\pi$, and as $0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $0 < \alpha < \frac{\pi}{2}$,

$\theta + \theta' = \frac{\pi}{2} + \alpha$ **M1 A1**

Reversing the motion we have, $(-\varphi) + (-\varphi') = \frac{\pi}{2} + (-\alpha) + n'\pi$, and therefore,

$\varphi + \varphi' = \alpha + \left(-n' - \frac{1}{2}\right)\pi = \theta + \theta' - n''\pi$ **M1 A1**

$0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi' < \frac{\pi}{2}$, and $\varphi < \theta$, $\varphi' < \theta'$

so $\varphi + \varphi' = \theta + \theta' - \pi$, or as required $\theta + \theta' = \varphi + \varphi' + \pi$ **(*)** **E1**

5 marks

10. Supposing that the particle P has mass m, that the spring has natural length l, and modulus of elasticity λ, $mg = \frac{\lambda d}{l}$ **M1 A1**

Conserving energy, if the speed of P when it hits the top of the spring is v,
$mgh = \frac{1}{2}mv^2$, and so $v = \sqrt{2gh}$ **M1 A1**

Using Newton's second law, the second-order differential equation is thus **M1**
$m\ddot{x} = mg - \frac{\lambda x}{l} = mg - \frac{mgx}{d}$ and so $\ddot{x} = g - \frac{gx}{d}$ **A1**
with initial conditions that $x = 0$, $\dot{x} = \sqrt{2gh}$, when $t = 0$. **B1**

7 marks

$\ddot{x} = g - \frac{gx}{d}$, i.e. $\ddot{x} + \frac{gx}{d} = g$ has complementary function $x = B\cos\omega t + C\sin\omega t$

M1

where $\omega = \sqrt{\frac{g}{d}}$, and particular integral $x = A$, where $\frac{gA}{d} = g$, i.e. $A = d$.

A1 **M1** **A1**

Using the initial conditions, $0 = d + B$, i.e. $B = -d$ and $\sqrt{2gh} = C\omega$,
i.e. $C = \sqrt{2dh}$ **B1** for B, **B1** for C
So $x = d - d\cos\sqrt{\frac{g}{d}}t + \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$

6 marks

$d\cos\sqrt{\frac{g}{d}}t - \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$ may be expressed in the form $R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$ **M1**

where $R^2 = d^2 + 2dh$, and $\tan\alpha = \frac{\sqrt{2dh}}{d} = \sqrt{\frac{2h}{d}}$ **A1, A1**

Thus $x = d - R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$ and $x = 0$ next when $= T$. **M1**

That is when $2\pi - \left(\sqrt{\frac{g}{d}}T + \alpha\right) = \alpha$ **M1**

So $\sqrt{\frac{g}{d}}T = 2\pi - 2\alpha = 2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}$, **M1**

and so $T = \sqrt{\frac{d}{g}}\left(2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}\right)$ as required. **(*)** **A1**

7 marks

11.

(i) Conserving momentum $MV = M(1+bx)v$ and so $V = (1+bx)v$ **M1A1**

$V = (1+bx)\frac{dx}{dt}$

$\int V\,dt = \int(1+bx)\,dx$ **M1**

$Vt + c = x + \frac{1}{2}bx^2$, **A1**

and as $= 0$, when $t = 0$, $c = 0$

So $\frac{1}{2}bx^2 + x - Vt = 0$, and so $x = \frac{-1\pm\sqrt{1+2bVt}}{b}$, **M1**

except $x > 0$, and thus $x = \frac{-1+\sqrt{1+2bVt}}{b}$ **M1 A1**

7 marks

(ii) $Mf = \frac{d}{dt}(mv) = \frac{d}{dt}(M(1+bx)v)$ **B1**

Therefore, $ft + c = (1+bx)v$ **M1 A1**

When $= 0$, $x = 0$, and $v = V$ so $c = V$ **M1**

Thus $v = \frac{ft+V}{1+bx}$ as required (*) **A1**

5 marks

$ft + V = (1+bx)\frac{dx}{dt}$

$\int ft + V\,dt = \int(1+bx)\,dx$ **M1**

$\frac{1}{2}ft^2 + Vt + c' = x + \frac{1}{2}bx^2$ and as $x = 0$, when $t = 0$, $c' = 0$ **M1**

So $\frac{1}{2}bx^2 + x - \frac{1}{2}ft^2 - Vt = 0$, and so $= \frac{-1\pm\sqrt{1+fbt^2+2bVt}}{b}$, except $x > 0$,

and thus $x = \frac{-1+\sqrt{1+fbt^2+2bVt}}{b}$ **M1 A1**

4 marks

If $1 + fbt^2 + 2bVt$ is a perfect square, **M1**

then *x* will be linear in *t* and $\frac{dx}{dt}$ will be constant, i.e. if $4b^2V^2 - 4fb = 0$, that is

$bV^2 = f$ **A1**

(in which case $x = \frac{-1+\sqrt{1+b^2V^2t^2+2bVt}}{b} = \frac{-1+(1+bVt)}{b} = Vt$, and $v = V$ as expected.)

Otherwise, $= \frac{ft+V}{1+bx} = \frac{ft+V}{\sqrt{1+fbt^2+2bVt}} = \frac{f+\frac{V}{t}}{\sqrt{fb+\frac{2bV}{t}+\frac{1}{t^2}}}$, and as $t \to \infty$, $v \to \frac{f}{\sqrt{fb}} = \sqrt{\frac{f}{b}}$ a

constant, as required. **M1 A1**

4 marks

12. (i) $E(X_1) = \frac{1}{2}k$ **B1**

$E(X_2|X_1 = x_1) = \frac{1}{2}x_1$, **B1**

and so $E(X_2) = \sum \frac{1}{2}x_1 P(X_1 = x_1) = \frac{1}{2}E(X_1) = \frac{1}{4}k$ **M1 A1**

$\sum_{i=1}^{\infty} E(X_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i k = k$ **M1 A1**

6 marks

(ii) $G_Y(t) = E(t^Y) = E\left(t^{\sum_{i=1}^{k} Y_i}\right) = \prod_{i=1}^{k} E(t^{Y_i})$

M1 **M1**

$P(Y_i = 0) = \frac{1}{2}$, $(Y_i = 1) = \frac{1}{4}, \dots, P(Y_i = r) = \left(\frac{1}{2}\right)^{r-1}$ **B1**

and so $E(t^{Y_i}) = \frac{1}{2} + \frac{1}{4}t + \dots + \left(\frac{1}{2}\right)^{r-1} t^r + \ ... = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}t\right)} = \frac{1}{2-t}$ **M1 A1**

Thus $G_Y(t) = \prod_{i=1}^{k} \frac{1}{2-t} = \left(\frac{1}{2-t}\right)^k$ **M1 A1**

7 marks

$G'_Y(t) = \frac{k}{(2-t)^{k+1}}$, $G''_Y(t) = \frac{k(k+1)}{(2-t)^{k+2}}$, and $G^{(r)}{}_Y(t) = \frac{k(k+1)(k+2)\dots(k+r-1)}{(2-t)^{k+r}}$ **B1**

So $(Y) = G'_Y(1) = k$, **M1 A1**

$Var(Y) = G''_Y(1) + G'_Y(1) - \left(G'_Y(1)\right)^2 = k(k+1) + k - k^2 = 2k$ **M1 A1**

and $P(Y = r) = \frac{G^{(r)}{}_Y(0)}{r!} = \frac{k(k+1)(k+2)\dots(k+r-1)}{2^{k+r}r!} = {}^{k+r-1}C_r \left(\frac{1}{2}\right)^{k+r}$ for $r = 0, 1, 2, \dots$

M1 A1

Alternatively, $P(Y = r)$ is coefficient of t^r in $G_Y(t)$

$G_Y(t) = \left(\frac{1}{2-t}\right)^k = \left(\frac{1}{2}\right)^k \left(1 - \frac{t}{2}\right)^{-k}$

$= \left(\frac{1}{2}\right)^k \left(1 + k\left(\frac{t}{2}\right) + \frac{k(k+1)}{2!}\left(\frac{t}{2}\right)^2 + \ \dots + \frac{k(k+1)\dots(k+r-1)}{r!}\left(\frac{t}{2}\right)^r + \ \dots\right)$

and so $P(Y = r) = \left(\frac{1}{2}\right)^k \frac{k(k+1)\dots(k+r-1)}{r!}\left(\frac{1}{2}\right)^r = {}^{k+r-1}C_r \left(\frac{1}{2}\right)^{k+r}$ same marks

7 marks

13.

(i) $F(x) = P(X < x) = P(\cos\theta < x) = P(\cos^{-1} x < \theta < 2\pi - \cos^{-1} x)$ **M1**

Therefore, $F(x) = \frac{2\pi - 2\cos^{-1} x}{2\pi}$ **A1**

So $(x) = \frac{dF}{dx} = \frac{1}{\pi\sqrt{1-x^2}}$, for $-1 \le x \le 1$ **M1 A1, B1**(domain)

5 marks

$E(X) = 0$ (trivially) **B1**

$E(X^2) = \int_{-1}^{1} x^2 \frac{1}{\pi\sqrt{1-x^2}} \quad dx = \int_{-1}^{1} x \frac{x}{\pi\sqrt{1-x^2}}\, dx$ **M1**

$= \left[x \frac{-1}{\pi}\sqrt{1-x^2}\right]_{-1}^{1} - \int_{-1}^{1} \frac{-1}{\pi}\sqrt{1-x^2}\, dx$, by parts

$= \int_{-1}^{1} \frac{1}{\pi}\sqrt{1-x^2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi}\sqrt{1-\sin^2 u}\, \cos u\, du$, by substitution

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi} \cos^2 u\, du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2\pi} (\cos 2u + 1) du = \left[\frac{1}{2\pi}\left(\frac{1}{2}\sin 2u + u\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}$

M1 (full integration method inc. double angle) **A1**

Alternatively,

$E(X^2) = \int_{-1}^{1} x^2 \frac{1}{\pi\sqrt{1-x^2}} \quad dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 u}{\pi} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos 2u}{2\pi}\, du$, by substitution

$= \left[\frac{u - \frac{1}{2}\sin 2u}{2\pi}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}$ same marks

So $Var(X) = \frac{1}{2} - 0^2 = \frac{1}{2}$ **A1**

5 marks

If $X = x$, $Y = \pm\sqrt{1-x^2}$ equiprobably, so $E(XY) = 0$
$E(Y) = 0$ (trivially) and thus $Cov(X, Y) = 0 - 0^2 = 0$, and hence $Corr(X, Y) = 0$

M1A1

X and *Y* are not independent for if $X = x$, $Y = \pm\sqrt{1-x^2}$ only, whereas without the restriction, *Y* can take all values in $[-1,1]$. **E1**

3 marks

(ii)

$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = 0$, and $E(\bar{Y}) = 0$ similarly. **B1**

$E(\overline{XY}) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i \sum_{j=i}^{n} Y_j\right) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i\, Y_i\right)$ as X_i and Y_j are independent and each have expectation zero, and $E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i\, Y_i\right) = 0$ from part (i), **M1**

and so $E(\overline{XY}) = 0$ **A1**

Thus $Cov(\bar{X}, \bar{Y}) = 0 - 0^2 = 0$, and hence $Corr(\bar{X}, \bar{Y}) = 0$ as required. **A1**

4 marks

For large n , $\bar{X} \sim N\left(0, \frac{1}{2n}\right)$ approximately, by Central Limit Theorem. **E1**

Therefore,

$$P\left(|\bar{X}| \leq \sqrt{\frac{2}{n}}\right) \approx P\left(|z| \leq \frac{\sqrt{\frac{2}{n}}}{\frac{1}{\sqrt{2n}}}\right) = P(|z| \leq 2) \approx P(|z| \leq 1 \cdot 960) \approx 0 \cdot 95$$

M1 **A1**

3 marks