STEP Mathematics III 2010: Markscheme

Section A: Pure Mathematics

1. (i)

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k$$

$$= \frac{1}{n+1}\left(\sum_{k=1}^{n} x_k + x_{n+1}\right)$$

M1

$$= \frac{1}{n+1}(nA + x_{n+1})$$

A1

2 marks

(ii)

$$B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2$$

$$= \frac{1}{n}\sum_{k=1}^{n} {x_k}^2 - \frac{1}{n}\sum_{k=1}^{n} 2Ax_k + \frac{1}{n}\sum_{k=1}^{n} A^2$$

M1 expanding

$$= \frac{1}{n}\sum_{k=1}^{n} {x_k}^2 - \frac{2A}{n}\sum_{k=1}^{n} x_k + \frac{1}{n}nA^2$$

$$= \frac{1}{n}\sum_{k=1}^{n} {x_k}^2 - 2A^2 + A^2$$

M1 manipulating 2nd & 3rd sums

$$= \frac{1}{n}\sum_{k=1}^{n} {x_k}^2 - A^2$$

***A1**

3 marks

(iii)

$$D = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - C)^2$$

$$= \frac{1}{n+1}\sum_{k=1}^{n+1} {x_k}^2 - \frac{1}{n+1}\sum_{k=1}^{n+1} 2C\,x_k + \frac{1}{n+1}\sum_{k=1}^{n+1} C^2$$

M1 expanding

$$= \frac{1}{n+1}\left[n(B + A^2) + {x_{n+1}}^2\right] - 2C^2 + C^2$$

M1 use of (ii), **M1** manipulating 2nd & 3rd sums

$$= \frac{1}{n+1}\left[n(B + A^2) + {x_{n+1}}^2\right] - C^2$$

A1

$$= \frac{1}{n+1}\left[n(B + A^2) + {x_{n+1}}^2\right] - \left[\frac{1}{n+1}(nA + x_{n+1})\right]^2$$

M1 use of (i)

$$= \frac{nB}{n+1} + \frac{1}{(n+1)^2}\left[n(n+1)A^2 - n^2A^2 - 2nAx_{n+1} + (n+1){x_{n+1}}^2 - {x_{n+1}}^2\right]$$

M1 expanding

$$= \frac{nB}{n+1} + \frac{n}{(n+1)^2}\left[A^2 - 2Ax_{n+1} + {x_{n+1}}^2\right]$$

M1 collection of terms

$$= \frac{n}{(n+1)^2}\left[(n+1)B + (A - x_{n+1})^2\right]$$

o.e. A1

8 marks

Hence, as

$$(n+1)D = nB + \frac{n}{n+1}(A - x_{n+1})^2$$

and as

$$\frac{n}{n+1}(A - x_{n+1})^2 \geq 0 \ \forall\, x_{n+1}$$

M1 completion of square and use

$$(n+1)D \geq nB \ \forall\, x_{n+1}$$

***A1**

2 marks

$$D - B = \frac{nB}{n+1} + \frac{n}{(n+1)^2}(A - x_{n+1})^2 - B$$

$$= \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{1}{n+1}B$$

M1

$$D < B \ \Leftrightarrow D - B < 0$$

$$\Leftrightarrow \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{1}{n+1}B < 0$$

M1

$$\Leftrightarrow (A - x_{n+1})^2 < \frac{n+1}{n}B$$

A1

$$\Leftrightarrow -\sqrt{\frac{n+1}{n}B} < x_{n+1} - A < \sqrt{\frac{n+1}{n}B}$$

M1

$$\Leftrightarrow A - \sqrt{\frac{n+1}{n}B} < x_{n+1} < A + \sqrt{\frac{n+1}{n}B}$$

***A1**

5 marks

Withhold final A mark if ⇔ not used.

2. (i)

$$\cosh a = \frac{e^a + e^{-a}}{2}$$

B1

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,dx = \int_0^1 \frac{1}{x^2 + x(e^a + e^{-a}) + 1}\,dx$$

$$= \int_0^1 \frac{1}{(x + e^a)(x + e^{-a})}\,dx$$

or alternatively

$$= \int_0^1 \frac{1}{(x + \cosh a + \sinh a)(x + \cosh a - \sinh a)}\,dx$$

M1 factorising

$$= \int_0^1 \frac{1}{(e^a - e^{-a})}\frac{1}{(x + e^{-a})} - \frac{1}{(e^a - e^{-a})}\frac{1}{(x + e^a)}\,dx$$

M1 partial fractions

$$= \frac{1}{(e^a - e^{-a})}\left[ln\left(\frac{x + e^{-a}}{x + e^a}\right)\right]_0^1$$

M1 integrating

$$= \frac{1}{2\sinh a}\left(ln\left(\frac{1 + e^{-a}}{1 + e^a}\right) - ln\left(\frac{e^{-a}}{e^a}\right)\right)$$

$$= \frac{1}{2\sinh a}\left(ln\left(\frac{1 + e^{-a}}{1 + e^a}\right) + ln(e^{2a})\right)$$

$$= \frac{1}{2\sinh a}\left(ln\left(e^a\frac{1 + e^a}{1 + e^a}\right)\right)$$

M1 handling limits and lns

$$= \frac{a}{2\sinh a}$$

***A1**

6 marks

(ii)

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,dx = \int_1^\infty \frac{1}{x^2 + x(e^a - e^{-a}) - 1}\,dx$$

B1

or alternatively

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,dx = \int_1^\infty \frac{1}{x^2 + 2x\sinh a + \sinh^2 a - \cosh^2 a}\,dx$$

$$= \int_1^\infty \frac{1}{(x+e^a)(x-e^{-a})} dx$$

M1 factorising

$$= \int_1^\infty \frac{1}{(e^a+e^{-a})} \frac{1}{(x-e^{-a})} - \frac{1}{(e^a+e^{-a})} \frac{1}{(x+e^a)} dx$$

M1 partial fractions

$$= \frac{1}{(e^a+e^{-a})} \left[ln\left(\frac{x-e^{-a}}{x+e^a}\right)\right]_1^\infty$$

M1

$$= \frac{1}{(e^a+e^{-a})} \left[ln\left(\frac{1-\frac{e^{-a}}{x}}{1+\frac{e^a}{x}}\right)\right]_1^\infty$$

$$= \frac{1}{(e^a+e^{-a})} \left(0 - ln\left(\frac{1-e^{-a}}{1+e^a}\right)\right)$$

M1 A1

$$= \frac{1}{2\cosh a} \left(ln\left(e^a \frac{1+e^a}{e^a-1}\right)\right)$$

$$= \frac{1}{2\cosh a} \left(a + ln\left(\coth \frac{a}{2}\right)\right)$$

M1 A1

8 marks

$$\int_0^\infty \frac{1}{x^4+2x^2 \cosh a + 1} dx = \int_0^\infty \frac{1}{(x^2+e^a)(x^2+e^{-a})} dx$$

$$= \frac{1}{(e^a-e^{-a})} \int_0^\infty \frac{1}{(x^2+e^{-a})} - \frac{1}{(x^2+e^a)} dx$$

M1 A1

$$= \frac{1}{(e^a-e^{-a})} \left[\frac{1}{e^{-\frac{a}{2}}} \tan^{-1}\left(\frac{x}{e^{-\frac{a}{2}}}\right) - \frac{1}{e^{\frac{a}{2}}} \tan^{-1}\left(\frac{x}{e^{\frac{a}{2}}}\right)\right]_0^\infty$$

M1 A1

$$= \frac{1}{2\sinh a}\left(\frac{\pi}{2} 2\sinh\frac{a}{2}\right)$$

$$= \frac{\pi}{4\cosh\frac{a}{2}}$$

M1 A1 any correct equivalent in hyperbolic functions

6 marks

3. The two primitive 4th roots of unity are $\pm i$

B1

So $C_4(x) = (x - i)(x + i) = x^2 + 1$

M1 *A1

3 marks

(i)

$C_1(x) = x - 1$

B1

$x^2 - 1 = (x - 1)(x + 1)$ so $C_2(x) = x + 1$

B1

$x^3 - 1 = (x - 1)(x^2 + x + 1)$

M1

so $C_3(x) = x^2 + x + 1$

A1

(or correct answer without working **B2**)

$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$ so $C_5(x) = x^4 + x^3 + x^2 + x + 1$

M1 A1

(or correct answer without working **B2**)

$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x^3 - 1)(x + 1)(x^2 - x + 1)$
so $C_6(x) = x^2 - x + 1$

M1 (must remove all 4 non-primitives) **A1**

(or correct answer without working **B2**)

8 marks

(ii) $C_n(x) = 0 \Rightarrow x^4 = -1$

$\Rightarrow x^8 = 1$ so n is a multiple of 8,

M1

and as there are 4 primitive 8th roots of unity,

M1

n must be 8

A1

3 marks

(iii) $x^p = 1 \Rightarrow x^p - 1 = 0 \Rightarrow (x - 1)(x^{p-1} + x^{p-2} + x^{p-3} + \cdots + 1)$

M1

1 is the only non-primitive root as no power of any other root less than the pth equals unity, because p is prime.

E1

So $C_p(x) = x^{p-1} + x^{p-2} + x^{p-3} + \cdots + 1$

A1

3 marks

(iv) No root of $C_n(x) = 0$ is a root of $C_t(x) = 0$ for any $t \neq n$.
(For if $t < n$, by the definition of $C_n(x)$, there is no integer t such that $a^t = 1$ when $a^n = 1$. Similarly, if $t > n$.)

E1

Thus if $C_q(x) \equiv C_r(x)C_s(x)$, and if $C_q(x) = 0$, then $C_r(x) = 0$ or $C_s(x) = 0$, so $q = r$ or $q = s$.

M1

If $q = r$, then $C_q(x) \equiv C_r(x)$, and so $C_s(x) \equiv 1$ which is not possible for positive s, and likewise in the alternative case.

E1

3 marks

4. (i) $\alpha^2 + a\alpha + b = 0$

$\alpha^2 + c\alpha + d = 0$

Subtracting gives $(a - c)\alpha + (b - d) = 0$

So $(a - c)\alpha = -(b - d)$ and as $a \neq c$

$\alpha = -\frac{(b-d)}{(a-c)}$

M1

1 mark

So if there is a common root ($a \neq c$), then $\alpha = -\frac{(b-d)}{(a-c)}$ is it, and it satisfies

$x^2 + ax + b = 0$,

so $\left(\frac{(b-d)}{(a-c)}\right)^2 - a\frac{(b-d)}{(a-c)} + b = 0$ i.e. $(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0$

M1 A1

If $(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0$ and $a \neq c$,

Then $\left(\frac{(b-d)}{(a-c)}\right)^2 + a\left(-\frac{(b-d)}{(a-c)}\right) + b = 0$, and so $-\frac{(b-d)}{(a-c)}$ satisfies $x^2 + ax + b = 0$

M1

Also, $\left(\frac{(b-d)}{(a-c)}\right)^2 + c\left(-\frac{(b-d)}{(a-c)}\right) + d$

M1

$= \left(\frac{(b-d)}{(a-c)}\right)^2 + a\left(-\frac{(b-d)}{(a-c)}\right) + b + (c - a)\left(-\frac{(b-d)}{(a-c)}\right) + (d - b)$

M1

$= 0 + (c - a)\left(-\frac{(b-d)}{(a-c)}\right) + (d - b) = 0 + (b - d) + (d - b) = 0$

so $-\frac{(b-d)}{(a-c)}$ satisfies $x^2 + cx + d = 0$ as well, so there is a common root.

A1

6 marks

Alternatively, if there is a common root and $a = c$, then initial subtraction yields $b = d$, and so result is trivially true.

B1

If $(b - d)^2 - a(b - d)(a - c) + b(a - c)^2 = 0$ and $a = c$, then $b = d$, so the two equations are one and the same, and they have common roots.

A1

2 marks

(ii) If α is the common root, $\alpha^2 + a\alpha + b = 0$,

and $\alpha^3 + (a + 1)\alpha^2 + q\alpha + r = 0$.

$\alpha(\alpha^2 + a\alpha + b) = 0$

M1

Subtracting gives $\alpha^2 + (q - b)\alpha + r = 0$

M1

Thus, using the result from part (i),

M1

$(b - r)^2 - a(b - r)\big(a - (q - b)\big) + b\big(a - (q - b)\big)^2 = 0$

i.e. $(b - r)^2 - a(b - r)(a + b - q) + b(a + b - q)^2 = 0$

***A1**

If $(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$, then
$x^2 + ax + b = 0$ and $x^2 + (q-b)x + r = 0$ have a common root from (i)

M1

So $x(x^2 + ax + b) = 0$ and $x^2 + (q-b)x + r = 0$ have that common root

M1

and thus, $x(x^2 + ax + b) + x^2 + (q-b)x + r = 0$ and $x^2 + ax + b = 0$ have that common root as required.

A1

7 marks

Using $= \frac{5}{2}$, $q = \frac{5}{2}$, $r = \frac{1}{2}$,
so $\left(b - \frac{1}{2}\right)^2 - \frac{5}{2}\left(b - \frac{1}{2}\right)b + b^3 = 0$

M1

$b^3 - \frac{3}{2}b^2 + \frac{1}{4}b + \frac{1}{4} = 0$

A1

$(b-1)\left(b^2 - \frac{1}{2}b - \frac{1}{4}\right) = 0$

So $= 1$, or $b = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4}+1}}{2} = \frac{1 \pm \sqrt{5}}{4}$

M1 A1

4 marks

5. P is $(an, 0)$, Q is $(0, am)$ $m, n \neq 0, or\ 1$

CP is the line $\frac{y}{x-an} = \frac{a}{a-an}$
i.e. $(1-n)y = x - an$

M1 A1

At R, $= 0$, so $y = \frac{an}{n-1}$ i.e. R is $\left(0, \frac{an}{n-1}\right)$

M1 A1

S is $\left(\frac{am}{m-1}, 0\right)$

B2

(If S not found correctly, allow **B1** for CQ is $(1-m)x = y - am$)

Thus RS is the line $\frac{m-1}{am}x + \frac{n-1}{an}y = 1$

M1 A1

and PQ is the line $\frac{1}{an}x + \frac{1}{am}y = 1$

M1 A1

i.e. $n(m-1)x + m(n-1)y = amn$ and $mx + ny = amn$

Subtracting, for the point of intersection T,
$(mn - n - m)x + (mn - m - n)y = 0$

M1

However, as RS and PQ intersect, $\frac{n}{m} \neq \frac{m(n-1)}{n(m-1)}$,

M1

this condition is $m^2n - mn^2 - m^2 + n^2 \neq 0$,
$(m-n)(mn - m - n) \neq 0$

M1 A1

So as $(mn - n - m)x + (mn - m - n)y = 0$, $x + y = 0$

A1

(Alternatively, if $mn - m - n = 0 \Leftrightarrow n = \frac{m}{m-1} < 0$, is a contradiction **M1A1A1**)
Thus TA has gradient -1 and as AC has gradient 1, TA & AC are perpendicular.

E1

16 marks

Labelling the square ABCD anti-clockwise, choose points on AB and AD different distances from A, label them P and Q, construct CP and CQ, and find their intersections with AD and AB, R and S, respectively, and find the intersection of PQ and RS, label it T, then TA is perpendicular to AC.

E2

Rotating through right angles and repeating three more times gives sides of a square of area $2a^2$.

E2

4 marks

6. (i) P_1 is $(\cos\varphi, \sin\varphi, 0)$

B1, B1
2 marks

(ii) P_2 is $(\cos\varphi\cos\lambda, \sin\varphi\cos\lambda, \sin\lambda)$

B1, B1, B1
3 marks

Q_1 is $(-\sin\varphi, \cos\varphi, 0)$

B1

Q_2 is $(-\sin\varphi, \cos\varphi, 0)$

B1

R_1 is $(0,0,1)$

B1

R_2 is $(-\cos\varphi\sin\lambda, -\sin\varphi\sin\lambda, \cos\lambda)$

B1, B1, B1
6 marks

Q_1 & R_1 need not be quoted and can be implied by correct Q_2 & R_2

(iii) $OP_2 \cdot OP_0 = \begin{pmatrix}\cos\varphi\cos\lambda\\ \sin\varphi\cos\lambda\\ \sin\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$

M1 A1

$= 1 \times 1 \times \cos\theta$

M1

so $\cos\theta = \cos\varphi\cos\lambda$

***A1**
4 marks

Alternatively, by use of cosine rule,
$(1-\cos\varphi\cos\lambda)^2 + (\sin\varphi\cos\lambda)^2 + (\sin\lambda)^2 = 1 + 1 - 2\cos\theta$

M1 A1

and correct simplification yields $\cos\theta = \cos\varphi\cos\lambda$

M1 *A1
4 marks

Direction of axis is $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}\cos\varphi\cos\lambda\\ \sin\varphi\cos\lambda\\ \sin\lambda\end{pmatrix}$

M1 A1ft

$$= \begin{pmatrix}0\\ -\sin\lambda\\ \sin\varphi\cos\lambda\end{pmatrix}$$

A1 A1 A1ft
5 marks

7. $y = \cos(m \sin^{-1} x)$

$\cos^{-1} y = m \sin^{-1} x$

$-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}$

M1

$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$

M1

$2(1-x^2)\frac{d^2y}{dx^2}\frac{dy}{dx} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2y\frac{dy}{dx}$

M1

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$

M1

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

***A1**

5 marks

Alternatively, $y = \cos(m \sin^{-1} x)$

$\frac{dy}{dx} = -\sin(m \sin^{-1} x)\frac{m}{\sqrt{1-x^2}}$

M1

$\frac{d^2y}{dx^2} = -\cos(m \sin^{-1} x)\frac{m^2}{1-x^2} - \sin(m \sin^{-1} x)\frac{mx}{(1-x^2)^{\frac{3}{2}}}$

M1

$(1-x^2)\frac{d^2y}{dx^2} = -m^2\cos(m \sin^{-1} x) - \sin(m \sin^{-1} x)\frac{mx}{(1-x^2)^{\frac{1}{2}}}$

M1

$(1-x^2)\frac{d^2y}{dx^2} = -m^2y + x\frac{dy}{dx}$

M1

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

***A1**

5 marks

Thus similarly, $(1-x^2)\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} + m^2\frac{dy}{dx} = 0$

$(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} + (m^2-1)\frac{dy}{dx} = 0$

B1

and $(1-x^2)\frac{d^4y}{dx^4} - 2x\frac{d^3y}{dx^3} - 3x\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + (m^2-1)\frac{d^2y}{dx^2} = 0$

$(1-x^2)\frac{d^4y}{dx^4} - 5x\frac{d^3y}{dx^3} + (m^2-4)\frac{d^2y}{dx^2} = 0$

B1

2 marks

Conjecture $(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2-n^2)\frac{d^ny}{dx^n} = 0$

B1

Assume true for $n = k$

Differentiating gives

$(1-x^2)\frac{d^{k+3}y}{dx^{k+3}} - 2x\frac{d^{k+2}y}{dx^{k+2}} - (2k+1)x\frac{d^{k+2}y}{dx^{k+2}} - (2k+1)\frac{d^{k+1}y}{dx^{k+1}} + (m^2-k^2)\frac{d^{k+1}y}{dx^{k+1}} = 0$

$(1-x^2)\frac{d^{k+3}y}{dx^{k+3}} - (2(k+1)+1)x\frac{d^{k+2}y}{dx^{k+2}} + (m^2-(k+1)^2)\frac{d^{k+1}y}{dx^{k+1}} = 0$

M1

which is the required result for $k+1$

A1

Result is true for $n = 0$,

B1

so true for all *n* by PMI.

E1

5 marks

$x = 0$, $y = 1$,

B1

$\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = -m^2$,

B1

$\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = m^2(m^2-4)$

B1

$y = 1 - \frac{m^2}{2!}x^2 + \frac{m^2(m^2-2^2)}{4!}x^4 + \cdots$

B1

4 marks

If $\theta = \sin^{-1}x, x = \sin\theta$,

$\cos m\theta = 1 - \frac{m^2}{2!}x^2 + \frac{m^2(m^2-2^2)}{4!}x^4 + \cdots = 1 - \frac{m^2}{2!}\sin^2\theta + \frac{m^2(m^2-2^2)}{4!}\sin^4\theta + \cdots$

***B1**

All odd differentials are zero,

and even ones are $(-1)^{k+1}m^2(m^2-2^2)\ldots(m^2-(2k)^2)$

M1

Thus if m is even, the terms are zero from a certain point and therefore the Maclaurin series terminates and is thus a polynomial.

E1

The polynomial is of degree *m*

B1

4 marks

8. $\int \frac{P(x)}{\left(Q(x)\right)^2} dx = \int \frac{Q(x)R'(x)-Q'(x)R(x)}{\left(Q(x)\right)^2} dx = \frac{R(x)}{Q(x)} (+k)$

B2

2 marks

(i) $R(x) = a + bx + cx^2 \Rightarrow R'(x) = b + 2cx$

$Q(x) = 1 + 2x + 3x^2 \Rightarrow Q'(x) = 2 + 6x$

$5x^2 - 4x - 3 = (1 + 2x + 3x^2)(b + 2cx) - (2 + 6x\ )(a + bx + cx^2)$

M1

So equating coefficients,
$5 = 3b + 4c - 6b - 2c$, $-4 = 2b + 2c - 2b - 6a$, $-3 = b - 2a$
that is
$5 = -3b + 2c$, $-2 = -3a + c$, $-3 = -2a + b$

M1 A1, A1, A1

Thus, $\int \frac{5x^2-4x-3}{(1+2x+3x^2)^2} dx = \frac{1-x+x^2}{1+2x+3x^2} (+c) = \frac{-3x-2x^2}{1+2x+3x^2} (+c)$

A1

$5 = -3b + 2c$, $-2 = -3a + c$, $-3 = -2a + b$
are three linearly dependent equations, so a, b, and c are not uniquely defined.

E1

Choosing $a = 0$, $= -3$, $c = -2$
or $a = 1$, $b = -1$, $c = 1$

B1

$\frac{1-x+x^2}{1+2x+3x^2} = \frac{1+2x+3x^2-3x-2x^2}{1+2x+3x^2} = 1 + \frac{-3x-2x^2}{1+2x+3x^2}$ so both integrals are the same bar the arbitrary constant

E1

9 marks

(ii) $(1 + \cos x + 2\sin x)\frac{dy}{dx} + (\sin x - 2\cos x)y = 5 - 3\cos x + 4\sin x$

$\frac{dy}{dx} + \frac{(\sin x - 2\cos x)}{(1+\cos x+2\sin x)} y = \frac{(5-3\cos x+4\sin x)}{(1+\cos x+2\sin x)}$

So the integrating factor is $e^{\int \frac{(\sin x-2\cos x)}{(1+\cos x+2\sin x)} dx} = e^{-\ln(1+\cos x+2\sin x)} = \frac{1}{1+\cos x+2\sin x}$

M1 A1

$\frac{1}{1+\cos x+2\sin x} \frac{dy}{dx} + \frac{(\sin x-2\cos x)}{(1+\cos x+2\sin x)^2} y = \frac{(5-3\cos x+4\sin x)}{(1+\cos x+2\sin x)^2}$

$Q(x) = 1 + \cos x + 2\sin x \Rightarrow Q'(x) = -\sin x + 2\cos x$
Suppose $R(x) = a + b\sin x + c\cos x \Rightarrow R'(x) = b\cos x - c\sin x$

M1

Therefore, $5 - 3\cos x + 4\sin x = (1 + \cos x + 2\sin x)(b\cos x - c\sin x) - (-\sin x + 2\cos x)(a + b\sin x + c\cos x)$

M1

$(1+\cos x+2\sin x)(b\cos x-c\sin x)-(-\sin x+2\cos x)(a+b\sin x+c\cos x)$
$= (b-2a)\cos x-(c-a)\sin x$
$+ (b-2c)\cos^2 x+(2b-c-2b+c)\cos x\sin x+(b-2c)\sin^2 x$

A1

$5=b-2c$, $-3=b-2a$, $4=a-c$

M1

Solving/choosing $a=4$, $=5$, $c=0$

M1

Thus $\frac{1}{1+\cos x+2\sin x}y=\int\frac{(5-3\cos x+4\sin x)}{(1+\cos x+2\sin x)^2}dx=\frac{4+5\sin x}{1+\cos x+2\sin x}+k$

A1cso

$y=4+5\sin x+k(1+\cos x+2\sin x)$
$=-4\cos x-3\sin x+k(1+\cos x+2\sin x)$

A1ft

9 marks

Section B: Mechanics

9. Resolving in the direction *PO* for the mass *P*, we have

$$mg\sin\theta - R = \frac{mv^2}{a},$$

M1 A1, A1, A1

where *R* is the normal reaction of the block on P, and *v* is the (common) speed of the masses when OP makes an angle θ with the table.

(Thus $R = mg\sin\theta - \frac{mv^2}{a}$)

Conserving energy,

$$\frac{1}{2}mv^2 + \frac{1}{2}Mv^2 + mga\sin\theta - Mga\theta = 0$$

M1 A1, A1, A1, A1

Hence, $v^2 = \frac{2ga(M\theta - m\sin\theta)}{m+M}$

M1 A1

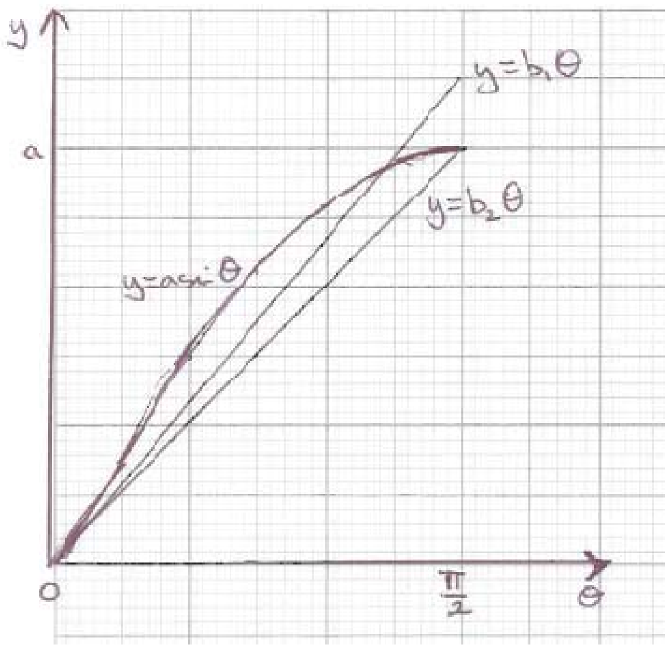
and so $R = mg\sin\theta - \frac{2mg(M\theta - m\sin\theta)}{m+M}$

M1 A1

$= \frac{mg\left((3m+M)\sin\theta - 2M\theta\right)}{m+M}$

A1

14 marks



Considering the graphs of $y = a\sin\theta$, and $y = b\theta$ for $0 \le \theta \le \frac{\pi}{2}$,

M1 G1

$a\sin\theta - b\theta \ge 0, \forall\theta, 0 \le \theta \le \frac{\pi}{2}$ if and only if $a\sin\theta - b\theta \ge 0$ for $\theta = \frac{\pi}{2}$

A1

so $R \ge 0$ for all $\theta, 0 \le \theta \le \frac{\pi}{2}$ if and only if $(3m + M)\sin\frac{\pi}{2} - 2M\frac{\pi}{2} \ge 0$

M1

i.e. $(3m + M) - \pi M \ge 0$

A1

which can be written $\frac{m}{M} \ge \frac{\pi - 1}{3}$

***A1**

6 marks

Alternatively, in place of conserving energy,

$Mg - T = Ma\ddot{\theta}$

M1 A1

$T - mg\cos\theta = ma\ddot{\theta}$

A1

Thus adding $Mg - mg\cos\theta = (M + m)a\ddot{\theta}$, and integrating, with initial conditions $= 0, \dot{\theta} = 0$, $Mg\theta - mg\sin\theta = \frac{1}{2}(M + m)a\dot{\theta}^2 = \frac{1}{2}(M + m)\frac{v^2}{a}$ yielding

M1 A1

$v^2 = \frac{2ga(M\theta - m\sin\theta)}{m+M}$

M1 A1

10. Conserving energy,

$$\frac{1}{2}mv^2 + \frac{1}{2}\lambda\frac{\left((a^2+b^2-2ab\cos\phi)^{\frac{1}{2}}-c\right)^2}{c} = A$$

M1 (energy) **M1** (cosine rule) **A1**

Differentiating,

$$mv\dot{v} + \frac{\lambda}{c}\left((a^2+b^2-2ab\cos\phi)^{\frac{1}{2}}-c\right)(a^2+b^2-2ab\cos\phi)^{-\frac{1}{2}}ab\sin\phi\,\dot{\phi} = 0$$

$$ma\dot{\phi}a\ddot{\phi} + \frac{\lambda}{c}ab\sin\phi\,\dot{\phi}\left[1-\frac{c}{(a^2+b^2-2ab\cos\phi)^{\frac{1}{2}}}\right] = 0$$

M1 A1

Thus,

$$ma\ddot{\phi} = -\frac{\lambda}{c}b\sin\phi\left[1-\frac{c}{(a^2+b^2-2ab\cos\phi)^{\frac{1}{2}}}\right]$$

so $ma\ddot{\phi} = -\lambda\left[\frac{b\sin\phi}{c}-\frac{b\sin\phi}{(a^2+b^2-2ab\cos\phi)^{\frac{1}{2}}}\right]$

A1

By the sine rule, $\frac{b}{\sin(\pi-(\theta+\phi))} = \frac{b}{\sin(\theta+\phi)} = \frac{a}{\sin\theta} = \frac{(a^2+b^2-2ab\cos\phi)^{\frac{1}{2}}}{\sin\phi}$

M1 A1 A1

so $ma\ddot{\phi} = -\lambda\left[\frac{a\sin(\theta+\phi)\sin\phi}{c\sin\theta}-\sin(\theta+\phi)\right] = -\lambda\left[\frac{a\sin\phi}{c\sin\theta}-1\right]\sin(\theta+\phi)$

M1 *A1

11 marks

Alternatively, resolving perpendicularly to OB,

$$ma\ddot{\phi} = -T\cos\left(\pi-\frac{\pi}{2}-\theta-\phi\right)$$

M1 A1 A1

where $T = \lambda\frac{PB-c}{c}$

M1 A1 A1

so $ma\ddot{\phi} = -\lambda\frac{PB-c}{c}\sin(\theta+\phi)$

M1 A1

$= -\lambda\left(\frac{a\sin\phi}{c\sin\theta}-1\right)\sin(\theta+\phi)$

M1 A1 *A1

11 marks

For ϕ and θ small, as $\frac{b}{\sin(\theta+\phi)} = \frac{a}{\sin\theta}$, $\frac{b}{\theta+\phi} \approx \frac{a}{\theta}$

M1 A1 A1

and so $a(\theta+\phi) \approx b\theta$

***A1**

4 marks

Therefore, further, $\theta \approx \frac{a\phi}{b-a}$

B1

Thus $ma\ddot{\phi} = -\lambda\left(\frac{a\sin\phi}{c\sin\theta} - 1\right)\sin(\theta+\phi) \approx -\lambda\left(\frac{a\phi}{c\theta} - 1\right)(\theta+\phi)$

i.e. $ma\ddot{\phi} \approx -\lambda\left(\frac{b-a}{c} - 1\right)\phi\left(\frac{a}{b-a} + 1\right)$

M1 A1

in other words, $\ddot{\phi} \approx -\frac{\lambda}{ma}\left(\frac{b-a-c}{c}\right)\left(\frac{b}{b-a}\right)\phi$

and so $\tau \approx 2\pi\sqrt{\frac{mac(b-a)}{\lambda b(b-a-c)}}$

M1 A1

5 marks

11.

(i) If the acceleration of the block is a' , then

$R = m(a - a')$ and

M1 A1

$R - \mu(M+m)g = Ma'$

A1

So $a = \frac{R}{m} + a' = \frac{R}{m} + \frac{R-\mu(M+m)g}{M}$

M1 *A1

5 marks

Alternatively, if the acceleration of the block is a' , and the acceleration of the bullet is a'',

$-R = ma''$ and

M1 A1

$R - \mu(M+m)g = Ma'$

A1

So relative acceleration $a = a' - a'' = \frac{R}{m} + \frac{R-\mu(M+m)g}{M}$

M1 *A1

5 marks

(ii) The initial velocity of the bullet relative to the block is $-u$

The final velocity of the bullet relative to the block is 0

If the time between the bullet entering the block and stopping moving through the block is T, then using $= u + at$, $0 = -u + \left(\frac{R}{m} + \frac{R-\mu(M+m)g}{M}\right)T$

M1 *A1

For the block, initial velocity is 0 , final velocity is v , and again using $v = u + at$,

$v = a'T = \frac{R-\mu(M+m)g}{M} \frac{u}{\left(\frac{R}{m}+\frac{R-\mu(M+m)g}{M}\right)}$

So $av = \left(\frac{R}{m} + \frac{R-\mu(M+m)g}{M}\right) \frac{R-\mu(M+m)g}{M} \frac{u}{\left(\frac{R}{m}+\frac{R-\mu(M+m)g}{M}\right)} = \frac{Ru-\mu(M+m)gu}{M}$

M1 A1

4 marks

(iii) For the block, initial velocity is 0 , final velocity is v , and if the distance moved by the block whilst the bullet is moving through the block is s,

using $v^2 = u^2 + 2as$, $v^2 = 2a's$

M1

so $s = \frac{v^2}{2a'} = \frac{Mv^2}{2(R-\mu(M+m)g)} = \frac{Mv^2}{2\frac{Mav}{u}} = \frac{uv}{2a}$

M1 A1

3 marks

(iv) Once the bullet stops moving through the block, initial velocity of block/bullet is v, final velocity is 0, acceleration is $-\mu g$, so distance moved s' using $v^2 = u^2 + 2as$ is given by $0 = v^2 - 2\mu g s'$ i.e. $s' = \frac{v^2}{2\mu g}$

M1 A1

2 marks

Thus total distance moved is $\frac{uv}{2a} + \frac{v^2}{2\mu g} = \frac{v}{2\mu g a}[\mu g u + a v]$

M1

$= \frac{v}{2\mu g a}\left[\mu g u + \frac{Ru - \mu(M+m)gu}{M}\right]$

M1

$= \frac{uv}{2\mu g a}\left[\frac{\mu M g + R - \mu(M+m)g}{M}\right]$

$= \frac{uv}{2\mu g}\left[\frac{R - \mu m g}{Ma}\right]$

$= \frac{uv}{2\mu g}\left[\frac{R - \mu m g}{M}\right]\frac{1}{\frac{R}{m} + \frac{R - \mu(M+m)g}{M}}$

M1

$= \frac{uv}{2\mu g}\left[\frac{R - \mu m g}{M}\right]\frac{Mm}{RM + Rm - \mu(M+m)gm}$

$= \frac{uv}{2\mu g}\left[\frac{R - \mu m g}{M}\right]\frac{Mm}{(M+m)(R - \mu m g)} = \frac{muv}{2(M+m)\mu g}$

***A1**

4 marks

If $R < (M+m)\mu g$, then the block does not move,

B1

and the bullet penetrates to a depth $\frac{mu^2}{2R}$.

B1

2 marks

Section C: Probability and Statistics

12. $S = 1 + (1+d)r + (1+2d)r^2 + \cdots + (1+nd)r^n + \cdots$

$S - rS = 1 + (1+d)r + (1+2d)r^2 + \cdots + (1+nd)r^n + \cdots$
$-r - (1+d)r^2 - \cdots - (1+(n-1)d)r^n - \cdots$

M1

$= 1 + dr + dr^2 + \cdots + dr^n + \cdots$

A1

$= 1 + \frac{dr}{1-r}$

So $S = \frac{1}{1-r} + \frac{rd}{(1-r)^2}$

***A1**

3 marks

$E(A) = 1a + 2(1-a)a + 3(1-a)^2a + \cdots + n(1-a)^{n-1}a + \cdots$

M1 A1

$= a\{1 + 2(1-a) + 3(1-a)^2 + \cdots + n(1-a)^{n-1} + \cdots\}$

The bracketed expression is S as above with $d = 1, r = (1-a)$

so $E(A) = a\left\{\frac{1}{1-(1-a)} + \frac{(1-a)}{\left(1-(1-a)\right)^2}\right\} = a\left\{\frac{1}{a} + \frac{1-a}{a^2}\right\}$

$= a\frac{1}{a^2} = \frac{1}{a}$

M1 A1

4 marks

$\alpha = a + (1-a)(1-b)\alpha = a + a'b'\alpha$

M1 A1

Thus $\alpha = \frac{a}{1-a'b'}$

A1

3 marks

Alternatively,

$\alpha = a + a'b'a + {a'}^2{b'}^2a + \cdots$

M1 A1

$= \frac{a}{1-a'b'}$

A1

3 marks

$\beta = 1 - \alpha = 1 - \frac{a}{1-a'b'} = \frac{1-a'b'-a}{1-a'b'} = \frac{a'-a'b'}{1-a'b'} = \frac{a'(1-b')}{1-a'b'} = \frac{a'b}{1-a'b'}$

M1 A1

2 marks

Alternatively,

$\beta = a'b + a'^2b'b + a'^3b'^2b + \cdots$

$= \frac{a'b}{1-a'b'}$

M1 A1

2 marks

$E(S) = 1a + 2a'b + 3a'b'a + 4a'^2b'b + 5a'^2b'^2a + \cdots$

M1 A1

$= a\{1 + 3a'b' + 5a'^2b'^2 + \cdots\} + 2a'b\{1 + 2a'b' + \cdots\}$

M1

which using the initial result of the question

$= a\left[\frac{1}{1-a'b'} + \frac{2a'b'}{(1-a'b')^2}\right] + 2a'b\left[\frac{1}{1-a'b'} + \frac{a'b'}{(1-a'b')^2}\right]$

M1 A1

$= \frac{1}{1-a'b'}\left\{\frac{a(1-a'b')+2aa'b'+2a'b(1-a'b')+2a'^2b'b}{1-a'b'}\right\}$

$= \frac{1}{1-a'b'}\left\{\frac{a(1+a'b')+2a'b}{1-a'b'}\right\}$

$= \frac{1}{1-a'b'}\left\{\frac{a(1+a'b')+2a'b}{1-a'b'}\right\}$

$= \frac{1}{1-a'b'}\left\{\frac{a(1+(1-a)(1-b))+2(1-a)b}{1-a'b'}\right\}$

$= \frac{1}{1-a'b'}\left\{\frac{2a+2b-2ab-a^2-ab+a^2b}{1-a'b'}\right\}$

$= \frac{1}{1-a'b'}\left\{\frac{(2-a)(a+b-ab)}{1-a'b'}\right\}$

$= \frac{1}{1-a'b'}\left\{\frac{(2-a)(a+b-ab)}{(a+b-ab)}\right\}$

$= \frac{(2-a)}{1-a'b'}$

M1 A1

$= \frac{1}{1-a'b'} + \frac{1-a}{1-a'b'}$

$= \frac{\alpha}{a} + \frac{\beta}{b}$

***A1**

8 marks

13. $Corr(Z_1, Z_2) = 0$

B1

1 mark

$E(Y_2) = E\left(\rho_{12}Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_2\right)$
$= \rho_{12}E(Z_1) + (1-\rho_{12}^2)^{\frac{1}{2}}E(Z_2)$
$= \rho_{12} \times 0 + (1-\rho_{12}^2)^{\frac{1}{2}} \times 0 = 0$

M1 A1

$Var(Y_2) = Var\left(\rho_{12}Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_2\right)$

$= \rho_{12}^2 Var(Z_1) + (1-\rho_{12}^2)Var(Z_2)$

$= \rho_{12}^2 + (1-\rho_{12}^2) = 1$

M1 A1

$Corr(Y_1, Y_2) = \frac{Cov(Y_1,Y_2)}{\sqrt{Var(Y_1)Var(Y_2)}} = Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$

M1

$= E\left(\rho_{12}Z_1^2 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_1Z_2\right)$
$= \rho_{12}Var(Z_1) + (1-\rho_{12}^2)^{\frac{1}{2}}E(Z_1)E(Z_2)$

$= \rho_{12}$

M1 A1

7 marks

$E(Y_3) = E(aZ_1 + bZ_2 + cZ_3) = aE(Z_1) + bE(Z_2) + cE(Z_3) = 0$ as given

$Var(Y_3) = Var(aZ_1 + bZ_2 + cZ_3) = a^2Var(Z_1) + b^2Var(Z_2) + c^2Var(Z_3)$

$= a^2 + b^2 + c^2 = 1$

M1 A1

$Corr(Y_1, Y_3) = E(aZ_1^2 + bZ_1Z_2 + cZ_1Z_3) = a = \rho_{13}$

M1 A1

$Corr(Y_2, Y_3) = E(Y_2Y_3) - E(Y_2)E(Y_3)$
$= E\left(\left(\rho_{12}Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_2\right)(aZ_1 + bZ_2 + cZ_3)\right)$

M1

$= \rho_{12}aVar(Z_1) + (1-\rho_{12}^2)^{\frac{1}{2}}bVar(Z_2)$
$= \rho_{12}a + (1-\rho_{12}^2)^{\frac{1}{2}}b = \rho_{23}$

M1 A1

Hence $a = \rho_{13},\ b = \frac{\rho_{23}-\rho_{12}\rho_{13}}{(1-\rho_{12}^2)^{\frac{1}{2}}}$

M1 A1

and $c = \sqrt{1-\rho_{13}^2 - \frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{(1-\rho_{12}^2)}}$

A1

10 marks

$X_i = \mu_i + \sigma_i Y_i$ for $i = 1,2,3$

B1

as $E(X_i) = E(\mu_i + \sigma_i Y_i) = E(\mu_i) + E(\sigma_i Y_i) = \mu_i + \sigma_i E(Y_i) = \mu_i$
$Var(X_i) = Var(\mu_i + \sigma_i Y_i) = Var(\sigma_i Y_i) = \sigma_i^2 Var(Y_i) = \sigma_i^2$
and $Corr(X_i, X_j) = Corr(Y_i, Y_j) = \rho_{ij}$ as a linear transformation will not affect correlation.

E1

2 marks