1 (i) $(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + 1,\ 2x_n y_n + 1)$

M1 for setting $x^2 - y^2 + 1 = x$ and $2xy + 1 = y$

M1 A1 for identifying y's in each case: $y^2 = x^2 - x + 1$ and $y = \dfrac{1}{1-2x}$

M1 for eliminating y's

M1 for creating a polynomial in x

A1 for correct quartic $4x^4 - 8x^3 + 9x^2 - 5x$

M1 for attempt to factorise (e.g. by factor theorem or long-division etc.) to at least quadratic stage

A1 for $x(x - 1)(4x^2 - 4x + 5) = 0$

B1 for convincing demonstration that the quadratic factor here has no real roots e.g. by $\Delta = 4^2 - 4.4.5 = -64 < 0$ or $4x^2 - 4x + 5 \equiv (2x - 1)^2 + 4 > 0\ \forall x$

A1 A1 for each solution-pair: $(x, y) = (0, 1)$ and $(1, -1)$
[N.B. A1 A0 if extras appear] **11**

ALTERNATIVE

$(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + 1,\ 2x_n y_n + 1)$

M1 for setting $x^2 - y^2 + 1 = x$ and $2xy + 1 = y$

M1 A1 for eliminating $y - 1 = 2xy$ and $x^2 - x = (y + 1)(y - 1)$

to get $x^2 - x = 2xy(y + 1)$

M1 A1 for 1st solution-pair: $(x, y) = (0, 1)$

M1 for other case $x = 1 + 2y + 2y^2$ with x eliminated to give a cubic eqn. in y

A1 for correct cubic eqn. $4y^3 + 4y^2 + y + 1 = 0$

M1 for attempt to factorise

A1 for $(y + 1)(4y^2 + 1) = 0$

B1 for convincing demonstration that the quadratic factor here has no real roots e.g. by $\Delta = 0^2 - 4.4.1 = -16 < 0$ or observing that $4y^2 + 1 > 0$ (or ≥ 1) $\forall x$

A1 for 2nd solution-pair: $(x, y) = (1, -1)$
[N.B. A1 A0 if extras appear] **11**

(ii) $(x_1 , y_1) = (-1 , 1) \quad \Rightarrow \quad (x_2 , y_2) = (a , b)$ **B1**

$\Rightarrow \quad (x_3 , y_3) = (a^2 - b^2 + a , 2ab + b + 2)$ **B1**

M1 for setting both $a^2 - b^2 + a = -1$ and $2ab + b + 2 = 1$

M1 A1 for identifying b's in each case: $b^2 = a^2 + a + 1$ and $b = \dfrac{-1}{1+2a}$

M1 for noting that the algebra is the same as the above, with $a = -x$ and $b = -y$
or via longer approach

A1 A1 for each solution-pair: $(a , b) = (0 , -1)$ and $(-1 , 1)$

B1 for rejecting, with reasoning, $(-1 , 1)$ since this gives a constant sequence. **9**

2 **M1** for use of correct PF form: $\dfrac{1+x}{(1-x)^2\left(1+x^2\right)} \equiv \dfrac{A}{1-x} + \dfrac{B}{(1-x)^2} + \dfrac{Cx+D}{1+x^2}$

M1 for $1 + x \equiv A(1 - x)(1 + x^2) + B(1 + x^2) + (Cx + D)(1 - x)^2$ and use of comparing coeffts. and/or susbtn.

A1 A1 A1 A1 for each of $A = \frac{1}{2}$, $B = 1$, $C = \frac{1}{2}$, $D = -\frac{1}{2}$ **6**

M1 for use of $\dfrac{1+x}{(1-x)^2\left(1+x^2\right)} \equiv A(1-x)^{-1} + B(1-x)^{-2} + Cx\left(1+x^2\right)^{-1} + D\left(1+x^2\right)^{-1}$ with attempt at binomial series and numerical A, B, C, D (ft from above work)

$$\equiv \tfrac{1}{2}\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(n+1)x^n + \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n+1} - \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

A1 A1 A1 for each series expansion correct (may be in explicit power series form) **4**

M1 for examining cases for n (mod 4)

A1 for $n \equiv 0$ (mod 4), coefft. of x^n is $\frac{1}{2} + n + 1 + 0 - \frac{1}{2} = n + 1$

A1 for $n \equiv 1$ (mod 4), coefft. of x^n is $\frac{1}{2} + n + 1 + \frac{1}{2} - 0 = n + 2$

A1 for $n \equiv 2$ (mod 4), coefft. of x^n is $\frac{1}{2} + n + 1 + 0 + \frac{1}{2} = n + 2$

A1 for $n \equiv 3$ (mod 4), coefft. of x^n is $\frac{1}{2} + n + 1 - \frac{1}{2} + 0 = n + 1$

Withhold the last A mark if these are merely stated without justification **5**

M1 A1 for $\dfrac{11000}{8181} = \dfrac{1.1}{0.9^2 \times 1.01}$ i.e. $x = 0.1$

M1 for use of series $1 + 3x + 4x^2 + 4x^3 + 5x^4 + 7x^5 + 8x^6 + 8x^7 + 9x^8 + \ldots$ with some suitable value of x with $|x| < 1$

A1 for 1.344 578 90 correct to first 6dp **A1** for all 8dp correct **5**

3 (i) **M1** for finding $\frac{dy}{dx}$ $(= 81x^2 - 54x)$ **A1 A1** for TPs at $(0, 4)$ and $\left(\frac{2}{3}, 0\right)$

(Give **M1 A1 A0** if both x-coords correct but y's omitted or one/both incorrect)

B1 Sketch of a cubic

B1 for TPs at $x = 0$ and $x = \frac{2}{3}$ (ft)

B1 for observation that, for all $x \ge 0$, $y \ge 0 \Rightarrow x^2(1 - x) \le \frac{4}{27}$ *clearly* shown **6**

M1 for contrary assumption that all three numbers exceed $\frac{4}{27}$.

M1 for use of their product $bc(1 - a)ca(1 - b)ab(1 - c)$

M1 for re-arranging this into the form $a^2(1 - a).\ b^2(1 - b.)\ c^2(1 - c)$ at some stage

A1 for consequence of assumption that $a^2(1 - a).\ b^2(1 - b.)\ c^2(1 - c) > (\frac{4}{27})^3$.

M1 for use of previous result $x^2(1 - x) \le \frac{4}{27}$ for each of a, b, c to deduce that

$$a^2(1 - a).\ b^2(1 - b.)\ c^2(1 - c) \le (\tfrac{4}{27})^3$$

A1 and, hence, by contradiction, at least one of $bc(1 - a)$, $ca(1 - b)$, $ab(1 - c) \le \frac{4}{27}$. **6**

ALTERNATIVELY

Assume w.l.o.g. that $(0 <)\ a \le b \le c\ (< 1)$, for instance.
Then $ab(1 - c) \le c^2(1 - c) \le \frac{4}{27}$, as required.
NOT a proof by contradiction, but pretty good mathematics.
Give **5 / 6** . (Similarly for other alternative approaches.)

(ii) **M1** for use of the graph of $y = x - x^2$ or another suitable choice: e.g. $y = (2x - 1)^2$

M1 A1 A1 for diff^g. (or $\equiv$) and showing max. at $(\frac{1}{2}, \frac{1}{4})$ so that $x(1 - x) \le \frac{1}{4}$.

[Ignore which x's here, as Qn. restricts later so that x and $1 - x$ both ≥ 0.]

M1 A1 for assumption that $p(1 - q)$, $q(1 - p) > \frac{1}{4} \Rightarrow p(1 - p).q(1 - q) > (\frac{1}{4})^2$

M1 for use of previous result $x(1 - x) \le \frac{1}{4}$ for each of p, q to deduce that

$$p(1 - p).q(1 - q) \le (\tfrac{1}{4})^2$$

A1 and, hence, by contradiction, at least one of $p(1 - q)$, $q(1 - p) \le \frac{1}{4}$. **8**

4 **M1** for use of implicit diff^n. including the *Product Rule* on the xy term

A1 for $\frac{dy}{dx} = -\frac{x+ay}{ax+y}$ legit. (GIVEN ANSWER) from $2\left(x + y\frac{dy}{dx} + ax\frac{dy}{dx} + ay\right) = 0$ **2**

B1 for grad. nml. $\frac{dy}{dx} = \frac{ax+y}{x+ay}$

M1 for use of $\tan(A - B)$ on this and $\frac{y}{x}$: $\tan\theta = \left|\frac{\frac{y}{x} - \frac{ax+y}{x+ay}}{1 + \frac{y}{x} \times \frac{ax+y}{x+ay}}\right|$

A1 correct unsimplified

M1 for mult^g. throughout by $x(x + ay)$: $= \left|\frac{xy + ay^2 - ax^2 - xy}{x^2 + axy + axy + y^2}\right|$

M1 for use of $x^2 + y^2 + 2axy = 1$ from the curve's eqn.

A1 for $\tan\theta = a\left|y^2 - x^2\right|$ [Ignore modulus signs until the end] **6**

(i) **M1** for diff^g. wrt x : $\sec^2\theta\,\frac{d\theta}{dx} = a\left(2y\frac{dy}{dx} - 2x\right)$

M1 for equating this to 0 and using $\frac{dy}{dx} = -\frac{x+ay}{ax+y}$ from earlier

$$y\frac{x+ay}{ax+y} + x = 0 \Rightarrow xy + ay^2 + ax^2 + xy = 0$$

A1 for correctly deducing GIVEN ANSWER $a(x^2 + y^2) + 2xy = 0$ **3**

(ii) **M1** for adding $x^2 + y^2 + 2axy = 1$ and $a(x^2 + y^2) + 2xy = 0$

A1 for $(1 + a)(x + y)^2 = 1$ **2**

(iii) **M1** for subtracting these two eqns.

A1 for $(1 - a)(y - x)^2 = 1$

M1 for mult[g]. these two results together

A1 for $(1 - a^2)\,(y^2 - x^2)^2 = 1$

M1 for use of $\tan\theta = a\left|y^2 - x^2\right| \Rightarrow (y^2 - x^2)^2 = \frac{1}{a^2}\tan^2\theta$ subst[d]. in this to get

A1 for GIVEN ANSWER $\tan\theta = \frac{a}{\sqrt{1-a^2}}$

B1 for explaining that +ve sq.rt. taken since $\tan\theta$ is | something |, which is ≥ 0. **7**

5 **B1** for $\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,dx = \int_0^{\pi/2} \frac{2\sin x\cos x}{1+\sin^2 x}\,dx$

M1 for use of substn. $s = \sin x$ **M1** for $ds = \cos x\,dx$ used to eliminate all x's for s's

A1 for $\int_0^1 \frac{2s}{1+s^2}\,ds$ [Limits may be dealt with later, so ignore for now]

A1 for $\ln(1 + s^2)$ **ft** on constant errors **A1** for $\ln 2$ **6**

M1 for use of substn. $c = \cos x$ in $\int_0^{\pi/2} \frac{\sin x}{1+\sin^2 x}\,dx$

M1 for $dc = -\sin x\,dx$ used to eliminate all x's for c's

A1 for $\int_0^1 \frac{1}{2-c^2}\,dc$ [Limits may be dealt with later, so ignore for now]

M1 for use of $k\ln\left|\frac{\sqrt{2}+c}{\sqrt{2}-c}\right|$ form from F.Bks. $k = \frac{1}{2\sqrt{2}}$

B1 for sorting out limits correctly at some stage **A1** for $\frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$ **6**

M1 A1 for binomial expansion on $(1 + \sqrt{2})^5 = 1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2}$

B1 for correct sensible line of reasoning:
$41 + 29\sqrt{2} < 99 \Leftrightarrow 29\sqrt{2} < 58 \Leftrightarrow \sqrt{2} < 2$ (which I am happy to allow as obvious)

B1 for $1.96 < 2 \Rightarrow 1.4 < \sqrt{2}$

M1 for approach such as $2^{1.4} > 1 + \sqrt{2} \Leftrightarrow 2^7 > (1 + \sqrt{2})^5$
$128 > 41 + 29\sqrt{2} \Leftrightarrow 87 > 29\sqrt{2} \Leftrightarrow 3 > \sqrt{2}$

A1 for completely correct reasoning: $2^{\sqrt{2}} > 2^{7/5} > 1+\sqrt{2}$ **6**

M1 for taking logs : $2^{\sqrt{2}} > 1+\sqrt{2} \Rightarrow \sqrt{2}\ln 2 > \ln(1 + \sqrt{2})$

$$\Rightarrow \ln 2 > \frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$$

A1 for correct conclusion legitimately obtained; i.e. $\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,dx > \int_0^{\pi/2} \frac{\sin x}{1+\sin^2 x}\,dx$ **2**

6 (i) $\cos x$ has period $2\pi \Rightarrow \cos(2x)$ repeats after $\pi, 2\pi, 3\pi, 4\pi, \ldots$ (i.e. period π)

$\sin x$ has period $2\pi \Rightarrow \sin\left(\frac{3x}{2}\right)$ repeats after $\frac{4\pi}{3}, \frac{8\pi}{3}, \frac{12\pi}{3}, \ldots$ (i.e. period $\frac{4}{3}\pi$)

Thus $f(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right)$ has period 4π .

B1 for correct answer in either case
M1 for method; *lcm*(π , $\frac{4}{3}\pi$) – ft their answers
A1 for correct answer of 4π **3**

(ii) **M1** for use of $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ or equivalent to get

A1 $\cos\left(2x + \frac{\pi}{3}\right) = -\sin\left(\frac{3x}{2} - \frac{\pi}{4}\right) = \cos\left(\frac{3x}{2} + \frac{\pi}{4}\right)$

M1 A1 for $2x + \frac{\pi}{3} = \frac{3x}{2} + \frac{\pi}{4} \Rightarrow x = -\frac{\pi}{6}$

from $2x + \frac{\pi}{3} = 2n\pi + \left(\frac{3x}{2} + \frac{\pi}{4}\right)$, $n = 0$ only

M1 for approach at other solutions, i.e. from $2x + \frac{\pi}{3} = 2n\pi - \left(\frac{3x}{2} + \frac{\pi}{4}\right)$

A1 for any one correct answer

A1 A1 for second/third correct answers

+ A1 for all four and no extras (ignore correct answers outside range [– π , π]

$x = -\frac{31\pi}{42}$ (from $n = -1$), $x = -\frac{\pi}{6}$ ($n = 0$), $x = \frac{17\pi}{42}$ ($n = 1$), $x = -\frac{41\pi}{42}$ ($n = 2$) **9**

B1 for $x = -\frac{\pi}{6}$

B1 for explanation : it is a double root (i.e. repeated root, order 2) **2**

(iii) **M1** for $y = 2$ if and only if **both** $\cos\left(2x + \frac{\pi}{3}\right) = 1$ and $\sin\left(\frac{3x}{2} - \frac{\pi}{4}\right) = 1$

M1 for solving $\cos\left(2x + \frac{\pi}{3}\right) = 1 \Rightarrow 2x + \frac{\pi}{3} = 0, 2\pi, 4\pi, \ldots..$

A1 for $x = \frac{5\pi}{6}, \frac{11\pi}{6}, \ldots..$

M1 for solving $\sin\left(\frac{3x}{2} - \frac{\pi}{4}\right) = 1 \Rightarrow \frac{3x}{2} - \frac{\boldsymbol{\pi}}{4} = \frac{\boldsymbol{\pi}}{2}, \frac{5\boldsymbol{\pi}}{2}, \ldots..$

A1 for $x = \frac{\pi}{2}, \frac{11\pi}{6}, \ldots..$

A1 for $x = \frac{11\pi}{6}$ **6**

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6 **ALTERNATIVE to (ii)**

(ii) **B1** for use of $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ or equivalent to get

$$\cos\left(2x + \frac{\pi}{3}\right) + \cos\left(\frac{3\pi}{4} - \frac{3x}{2}\right) = 0$$

B1 for $2\cos\left(\frac{x}{4} + \frac{13\pi}{24}\right)\cos\left(\frac{7x}{4} - \frac{5\pi}{24}\right) = 0$

M1 A1 for $\frac{x}{4} + \frac{13\pi}{24} = \frac{\pi}{2} \Rightarrow x = -\frac{\pi}{6}$ from $\cos\left(\frac{x}{4} + \frac{13\pi}{24}\right) = 0$

M1 for approach at other solutions, i.e. from $\cos\left(\frac{7x}{4} - \frac{5\pi}{24}\right) = 0$

$$\frac{7x}{4} - \frac{5\pi}{24} = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots..$$

$$\Rightarrow x = -\frac{31\pi}{42}, x = -\frac{\pi}{6}, x = \frac{17\pi}{42}, x = -\frac{41\pi}{42}$$

A1 for at any one correct answer

A1 A1 for second/third correct answers

+ A1 for all four and no extras (ignore correct answers outside range [– π , π] **9**

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7 (i) **M1 A1** for $y = u\sqrt{1+x^2} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = u.\dfrac{x}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\dfrac{\mathrm{d}u}{\mathrm{d}x}$

Then $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy + \dfrac{x}{1+x^2}$ becomes

M1 for eliminating y from the given diff. eqn.

$$\frac{1}{u\sqrt{1+x^2}}\left\{\frac{ux}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\frac{\mathrm{d}u}{\mathrm{d}x}\right\} = xu\sqrt{1+x^2} + \frac{x}{1+x^2}$$

dM1 for simplifying and cancelling one term $\Rightarrow \dfrac{x}{1+x^2} + \dfrac{1}{u}.\dfrac{\mathrm{d}u}{\mathrm{d}x} = xy + \dfrac{x}{1+x^2}$

A1 for correct diff. eqn. in x and u : $\dfrac{1}{u}.\dfrac{\mathrm{d}u}{\mathrm{d}x} = xu\sqrt{1+x^2}$

M1 for sep^{g}, variables and integrating $\displaystyle\int \frac{1}{u^2}.d\mathrm{u} = \int x\sqrt{1+x^2}\,\mathrm{d}x$

A1 ft for $-\dfrac{1}{u} = \dfrac{1}{3}\left(1+x^2\right)^{3/2}$ $(+\ C)$

M1 for use of $x = 0$, $y = 1$ ($u = 1$) to find C

M1 for getting y explicitly in terms of x **A1** for $y = \dfrac{3\sqrt{1+x^2}}{4-\left(1+x^2\right)^{3/2}}$ **10**

ALTERNATIVE

B1 for $y = u\sqrt{1+x^2} \Rightarrow \ln y = \dfrac{1}{2}\ln\left(1+x^2\right) + \ln u$

M1 A1 for diff^{g}. implicitly $\dfrac{1}{y}.\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2}.\dfrac{2x}{1+x^2} + \dfrac{1}{u}.\dfrac{\mathrm{d}u}{\mathrm{d}x}$

M1 A1 for $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy + \dfrac{x}{1+x^2}$ becomes $\dfrac{1}{u}.\dfrac{\mathrm{d}u}{\mathrm{d}x} = xu\left(1+x^2\right)^{\frac{1}{2}}$

M1 for sep^{g}, variables and integrating $\displaystyle\int \frac{1}{u^2}.d\mathrm{u} = \int x\sqrt{1+x^2}\,\mathrm{d}x$

A1 ft for $-\dfrac{1}{u} = \dfrac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}$ $(+\ C)$

M1 for use of $x = 0$, $y = 1$ ($u = 1$) to find C

M1 for getting y explicitly in terms of x **A1** for $y = \dfrac{3\sqrt{1+x^2}}{4-\left(1+x^2\right)^{\frac{3}{2}}}$ **10**

(ii) **M1** for choosing $y = u\left(1+x^3\right)^{1/3}$

M1 A1 for $\frac{\mathrm{d}y}{\mathrm{d}x} = u.\, x^2\left(1+x^3\right)^{-2/3} + \left(1+x^3\right)^{1/3}\frac{\mathrm{d}u}{\mathrm{d}x}$

Then $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y + \frac{x^2}{1+x^3}$ becomes

M1 for eliminating y from the given diff. eqn.

A1 for correct diff. eqn. in x and u : $\frac{1}{u}.\frac{\mathrm{d}u}{\mathrm{d}x} = x^2 u\left(1+x^3\right)^{1/3}$

M1 for sep$^{\text{g}}$, variables and integrating $\int \frac{1}{u^2}.\,d\mathrm{u} = \int x^2\left(1+x^3\right)^{1/3}\,\mathrm{d}x$

A1 ft for $-\frac{1}{u} = \frac{1}{4}\left(1+x^3\right)^{4/3}$ (+ C)

M1 for use of $x = 0$, $y = 1$ ($u = 1$) to find C and for getting y explicitly in terms of x

A1 for $y = \frac{4\left(1+x^3\right)^{1/3}}{5-\left(1+x^3\right)^{4/3}}$ **9**

ALTERNATIVE

M1 for choosing $y = u\left(1+x^3\right)^{\frac{1}{3}}$ **B1** for $\ln y = \frac{1}{3}\ln\left(1+x^3\right) + \ln u$

M1 for diff$^{\text{g}}$. implicitly $\frac{1}{y}.\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}.\frac{3x^2}{1+x^3} + \frac{1}{u}.\frac{\mathrm{d}u}{\mathrm{d}x}$

M1 A1 for $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y + \frac{x^2}{1+x^3}$ becomes $\frac{1}{u}.\frac{\mathrm{d}u}{\mathrm{d}x} = x^2 u\left(1+x^3\right)^{\frac{1}{3}}$

M1 for sep$^{\text{g}}$, variables and integrating $\int \frac{1}{u^2}.\,d\mathrm{u} = \int x^2\left(1+x^3\right)^{\frac{1}{3}}\,\mathrm{d}x$

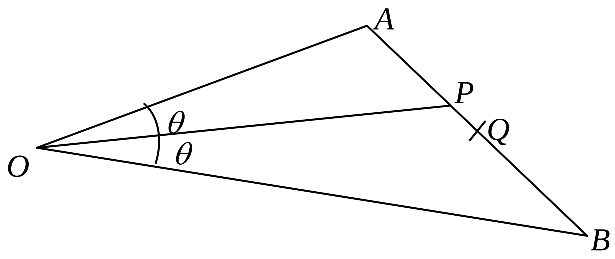
A1 ft for $-\frac{1}{u} = \frac{1}{4}\left(1+x^3\right)^{\frac{4}{3}}$ (+ C)

M1 for use of $x = 0$, $y = 1$ ($u = 1$) to find C and for getting y explicitly in terms of x

A1 for $y = \frac{4\left(1+x^3\right)^{\frac{1}{3}}}{5-\left(1+x^3\right)^{\frac{4}{3}}}$ **9**

(iii) **B1** for $y = \frac{(n+1)\left(1+x^n\right)^{1/n}}{(n+2)-\left(1+x^n\right)^{1+1/n}}$ **1**

8 **M1 A1** for $AP : PB = 1 - \lambda : \lambda \Rightarrow \mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ **2**



M1 for use of the scalar product

M1 A1 A1 for $\mathbf{a} \bullet \mathbf{p} = \lambda a^2 + (1-\lambda)(\mathbf{a} \bullet \mathbf{b})$ and $\mathbf{b} \bullet \mathbf{p} = \lambda(\mathbf{a} \bullet \mathbf{b}) + (1-\lambda)\, b^2$

M1 for equating these two expressions for $\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{p}}{ap} = \dfrac{\mathbf{b} \bullet \mathbf{p}}{bp}$

A1 for $\lambda a^2 b + (1-\lambda)b\,(\mathbf{a} \bullet \mathbf{b}) = \lambda a\,(\mathbf{a} \bullet \mathbf{b}) + (1-\lambda)\, ab^2$

M1 A1 for factorising: $ab\{\lambda(a+b) - b\} = \mathbf{a} \bullet \mathbf{b}\,\{\lambda(a+b) - b\}$

B1 for eliminating the possibility $ab = \mathbf{a} \bullet \mathbf{b}$ since this gives $\cos 2\theta = 1$, $\theta = 0$, $A = B$ (which violates the non-collinearity of O, A & B, for instance) – as opposed to ignoring or "cancelling" it

A1 for $\lambda = \dfrac{b}{a+b}$. **10**

B1 for $AP : PB = 1 - \lambda : \lambda \Rightarrow \mathbf{q} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$

M1 A1 for $OQ^2 = \mathbf{q} \bullet \mathbf{q} = (1-\lambda)^2\, a^2 + \lambda^2\, b^2 + 2\lambda(1-\lambda)\, \mathbf{a} \bullet \mathbf{b}$

A1 for $OP^2 = \mathbf{p} \bullet \mathbf{p} = (1-\lambda)^2\, b^2 + \lambda^2\, a^2 + 2\lambda(1-\lambda)\, \mathbf{a} \bullet \mathbf{b}$

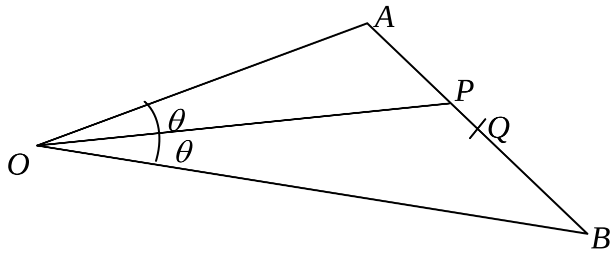
M1 for subtracting:
$OQ^2 - OP^2 = (b^2 - a^2)\,[\lambda^2 - (1-\lambda)^2] = (b^2 - a^2)\,(2\lambda - 1)$

M1 for substn. of their λ in terms of a and b into this expression

$= (b-a)(b+a) \times \dfrac{b-a}{b+a}$ **B1 ft** for $2\lambda - 1$ correct

A1 for $= (b-a)^2$ GIVEN ANSWER **8**

8 **M1 A1** for $AP : PB = 1 - \lambda : \lambda \Rightarrow \mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ **2**



ALTERNATIVE I

The *Angle Bisector Theorem* gives

$$\frac{AP}{PB} = \frac{OA}{OB} \Rightarrow \frac{(1-\lambda)(AB)}{\lambda(AB)} = \frac{a}{b} \Rightarrow b - b\lambda = a\lambda \Rightarrow \lambda = \frac{b}{a+b}$$

ALTERNATIVE II

OP is bisector of $\angle AOB$ if $\mathbf{p} = k\left(\frac{\mathbf{a}}{a} + \frac{\mathbf{b}}{b}\right)$

$$\Rightarrow \lambda = \frac{k}{a} \text{ and } 1 - \lambda = \frac{k}{b} \Rightarrow \lambda a = b - \lambda b \Rightarrow \lambda = \frac{b}{a+b}$$ **10**

M1 for repeated use of the Cosine Rule

A1 for $OQ^2 = OB^2 + BQ^2 - 2(OB)(BQ)\cos B$

A1 for $OP^2 = OA^2 + AP^2 - 2(OA)(AP)\cos A$

M1 for subtracting **dM1** for use of $AP = BQ$

$$OQ^2 - OP^2 = b^2 - a^2 + 2(AP)(a\cos A - b\cos B)$$

M1 for substn. of their λ in terms of a and b into this expression

$$= b^2 - a^2 + 2\frac{a}{a+b}c(a\cos A - b\cos B) \qquad \text{where } c = AB$$

M1 for use of $2ac\cos A = a^2 + c^2 - b^2$ and $2bc\cos B = b^2 + c^2 - a^2$

$$OQ^2 - OP^2 = b^2 - a^2 + \frac{a}{a+b}(a^2 + c^2 - b^2 - [b^2 + c^2 - a^2])$$

$$= b^2 - a^2 + \frac{a}{a+b}(2a^2 - 2b^2) = b^2 - a^2 + \frac{2a}{a+b}(a-b)(a+b)$$

A1 for $= (b - a)^2$ GIVEN ANSWER **8**

9 (i) **M1** for use of the (modified) trajectory eqn. $y = (h) + x\tan\alpha - \dfrac{gx^2}{2u^2}\sec^2\alpha$

M1 A1 for subst^{g}. in $g = 10$ and $u = 40$ to get $y = (h) + x\tan\alpha - \dfrac{gx^2}{320}\sec^2\alpha$

M1 for setting $x = 20$ and $y = 0\ (-h)$ in their trajectory eqn.

B1 for use of $\sec^2\alpha = 1 + \tan^2\alpha$ at some stage

A1 for $0 = h + 20t - \frac{5}{4}(1 + t^2)$

M1 for treating as a quadratic in $t = \tan\alpha$: $5t^2 - 80t - (4h - 5) = 0$

M1 for solving using the quadratic formula: $\tan\alpha = \dfrac{80 \pm \sqrt{6400 + 80h - 100}}{10}$

A1 for $\tan\alpha = 8 \pm \sqrt{63 + \frac{4}{5}h}$ **9**

B1 for rejecting $\tan\alpha = 8 + \sqrt{63 + \frac{4}{5}h}$
(gives a very high angle of projection/greater time for ball to arrive)

M1 A1 for Time of flight = $\dfrac{x}{u\cos\alpha} = \dfrac{1}{2\cos\alpha} \approx \dfrac{1}{2}$ since α small, $\cos\alpha \approx 1$ **3**

(ii) **B1** for $h > \frac{5}{4}$ for $\tan\alpha < 0$ $(= 8 - \sqrt{64 + \varepsilon})$ **1**

(iii) **M1** for re-writing into usable form: $h = 2.5 \Rightarrow \tan\alpha = 8 - \sqrt{64 + 1} = 8 - 8\left(1 + \frac{1}{64}\right)^{\frac{1}{2}}$

M1 for use of binomial series expansion (1st 2 terms): $\tan\alpha = 8 - 8\left(1 + \frac{1}{128} + ...\right)$

A1 for $\tan\alpha \approx -\frac{1}{16}$ [ignore sign]

M1 for small-angle use of $\tan\alpha \approx \alpha$

M1 for converting from radians into degrees

B1 for conversion factor $180/\pi \approx 57.3$

A1 for 3.6° **7**

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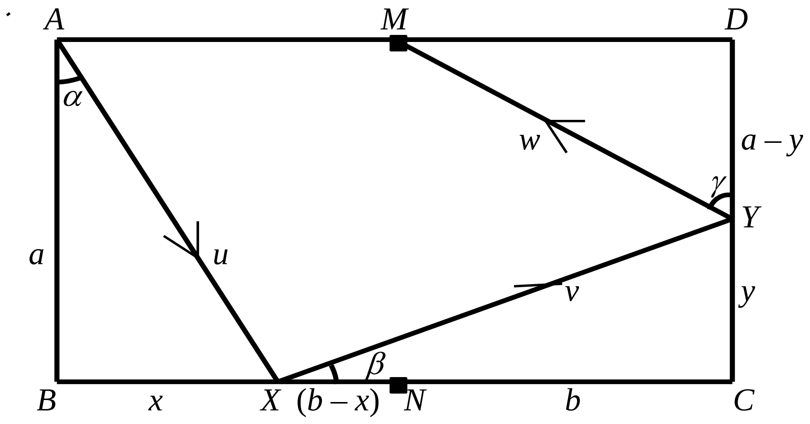

At X : $CLM \parallel BC$ $mu \sin\alpha = mv \cos\beta$ **B1**

NEL $eu \cos\alpha = v \sin\beta$ **B1**

Dividing ↑ $\Rightarrow$ $\tan\beta = e\cot\alpha$ or $\tan\alpha\tan\beta = e$ **B1**

At Y : $CLM \parallel BC$ $mv \sin\beta = mw \cos\gamma$ NEL $ev\cos\beta = w\sin\gamma$
Dividing ↓ $\Rightarrow$ $\tan\beta = e\cot\gamma$
OR "Similarly" $\tan\beta\tan\gamma = e$ **B1**

Hence $\alpha = \gamma$ (since all angles acute). **B1** **5**

(ii) **M1** for use of similar Δs (or ≡): Let $BX = x$ ($XN = b - x$) and $CY = y$ ($DY = a - y$)

B1 B1 B1 for $\tan\alpha = \dfrac{x}{a}$, $\tan\beta = \dfrac{y}{2b - x}$, $\tan\gamma = \dfrac{b}{a - y}$

M1 for use of $\alpha = \gamma$ to find (e.g) y in terms of a, b, x

$$\Rightarrow\ ax - xy = ab \ \Rightarrow\ y = \frac{a(x-b)}{x}$$

M1 for using $\tan\alpha\tan\beta = e$ to get x in terms of a and b

$$\Rightarrow\ \frac{x}{a}\times\frac{a(x-b)/x}{2b-x} = e \ \Rightarrow\ x - b = 2be - ex \ \Rightarrow\ x = \frac{b(1+2e)}{1+e}$$

A1 for $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)}$ GIVEN ANSWER **7**

M1 for argument such as

$\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{b}{a} + \dfrac{be}{a(1+e)} > \dfrac{b}{a}$ and $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{2b}{a} - \dfrac{be}{a(1+e)} < \dfrac{2b}{a}$

A1 so that $\dfrac{b}{a} < \tan\alpha < \dfrac{2b}{a}$ and shot is possible,

with ball striking *BC* between *N* and *C*, whatever the value of *e*

OR

M1 for as $e \to 0,\ \tan\alpha \to \dfrac{b}{a}+$ and as $e \to 1,\ \tan\alpha \to \dfrac{3b}{2a}-$

A1 so that $\dfrac{b}{a} < \tan\alpha < \dfrac{3b}{2a}$ and shot is possible, with ball striking *BC* between *N* and the midpoint of *NC*, whatever the value of *e* **2**

(iii) **SHORT VERSION**

At *X*, $\uparrow$-component of ball's velocity becomes $e \times$ initial $\uparrow$-component **B1**

and at *Y*, $\rightarrow$-component of ball's velocity becomes $e \times$ initial $\rightarrow$-component **B1**

Hence final velocity is eu **M2** and fraction of KE lost is

$$\frac{\frac{1}{2}mu^2 - \frac{1}{2}me^2u^2}{\frac{1}{2}mu^2} = 1 - e^2 \qquad \textbf{M1 A1}$$

LONG VERSION

Squaring and adding eqns. for collision at *X* $\Rightarrow$ $v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$ **B1**

Squaring and adding eqns. for collision at *Y* $\Rightarrow$ $w^2 = v^2(\sin^2\beta + e^2\cos^2\beta)$ **B1**

Initial KE = $\frac{1}{2}mu^2$ and final KE = $\frac{1}{2}mw^2$

Fraction of KE lost is $\dfrac{\frac{1}{2}mu^2 - \frac{1}{2}mw^2}{\frac{1}{2}mu^2} = 1 - \dfrac{w^2}{u^2}$ **M1**

$= 1 - (\sin^2\alpha + e^2\cos^2\alpha)(\sin^2\beta + e^2\cos^2\beta)$

$= 1 - \dfrac{\tan^2\alpha + e^2}{\sec^2\alpha} \times \dfrac{\tan^2\beta + e^2}{\sec^2\beta}$ **dM1**

M1 for use of $\tan\alpha\tan\beta = e$ $\quad = 1 - \dfrac{t^2 + e^2}{1 + t^2} \times \dfrac{e^2/t^2 + e^2}{1 + e^2/t^2}$

$= 1 - \dfrac{t^2 + e^2}{1 + t^2} \times \dfrac{e^2(1 + t^2)/t^2}{(t^2 + e^2)/t^2}$

A1 for $\quad = 1 - e^2$ **6**

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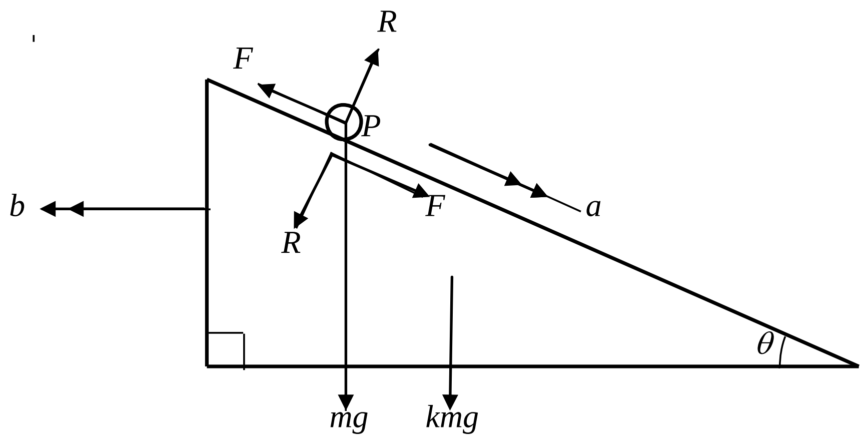

(i) **B1 B1** for the acceleration components of P : $a\cos\theta - b$ $(\rightarrow)$ and $a\sin\theta$ $(\downarrow)$

B1 for <u>*N2L → for P*</u> $m(a\cos\theta - b) = R\sin\theta - F\cos\theta$

B1 for <u>*N2L ↓ for P*</u> $ma\sin\theta = mg - F\sin\theta - R\cos\theta$

B1 for <u>*N2L ← for* wedge</u> $kmb = R\sin\theta - F\cos\theta$

M1 A1 for $a\cos\theta - b = kb \Rightarrow b = \dfrac{a\cos\theta}{k+1}$.

ALTERNATIVELY

B1 for P's → accln. component **B1** for wedge's accln. ←

M2 A1 for <u>*CLM* ↔</u> $kmbt = m(a\cos\theta - b)t$ t = time from release

M1 A1 for $a\cos\theta - b = kb \Rightarrow b = \dfrac{a\cos\theta}{k+1}$. **7**

M1 for noting that for P to move at 45° to the horizontal, $a\cos\theta - b = a\sin\theta$

A1 for $b = a(\cos\theta - \sin\theta)$

M1 for $(k+1)(\cos\theta - \sin\theta) = \cos\theta \Rightarrow k + 1 - (k+1)\tan\theta = 1$

A1 for $\Rightarrow \tan\theta = \dfrac{k}{k+1}$.

ALTERNATIVE

M1 for $\dfrac{a}{\sin 45^\circ} = \dfrac{b}{\sin(45^\circ - \theta)}$ (by the Sine Rule)

given that P moves at 45° to the horizontal

[Ignore other possibility involving $\sin(135^\circ - \theta)$]

A1 for $a(k+1)[\sin 45^\circ\cos\theta - \cos 45^\circ\sin\theta] = a\cos\theta\sin 45^\circ$

M1 for dividing by $\cos\theta$ and identifying $\tan\theta$

A1 for legitimately obtaining $\tan\theta = \dfrac{k}{k+1}$. **4**

B1 for $k = 3 \Rightarrow \tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$ noted or used.

B1 for $b = \frac{1}{5}a$.

M1 for 1st eqn. of motion

M1 for eliminating b

$\Rightarrow m(\frac{4}{5}a - b) = \frac{3}{5}R - \frac{4}{5}F$ or $3R - 4F = m(4a - 5b) = 3ma$

B1 for use of <u>*Friction Law (in motion)*</u> : $F = \mu R$ at any stage to eliminate F

$\Rightarrow R(3 - 4\mu) = 3ma$

M1 for 2nd eqn. of motion

$\Rightarrow \frac{3}{5}ma = mg - \frac{3}{5}F - \frac{4}{5}R$ or $4R + 3F = m(5g - 3a)$

$\Rightarrow R(4 + 3\mu) = 5mg - 3ma$

M1 for dividing/equating for R :

$$\frac{4+3\mu}{3-4\mu} = \frac{5g-3a}{3a} \Rightarrow (12 + 9\mu)a = 5(3 - 4\mu)g - (9 - 12\mu)a$$

A1 for $a = \dfrac{5(3-4\mu)g}{3(7-\mu)}$. **8**

(ii) **B1** for ….. If $\tan\theta \le \mu$, then both P and the wedge remain stationary. **1**

12 **B1** for $X \in \{0 , 1 , 2 , 3\}$ recognised somewhere

B1 for $p(X = 0) = (1 - p)(1 - \frac{1}{3}p)\,(1 - p^2)$ or any equivalent form

B1 for $p(X = 1) = p(1 - \frac{1}{3}p)(1 - p^2) + (1 - p)\frac{1}{3}p(1 - p^2) + (1 - p)(1 - \frac{1}{3}p)p^2$
$= p(1 - p)(\frac{4}{3} + \frac{5}{3}p - p^2)$ or aef

B1 for $p(X = 2) = p.\frac{1}{3}p(1 - p^2) + p(1 - \frac{1}{3}p)p^2 + (1 - p)\frac{1}{3}p.p^2$
$= \frac{1}{3}p^2(1 + 4p - 3p^2)$ or aef

B1 for $p(X = 3) = \frac{1}{3}p^4$ or aef

N.B. This work may be done later, numerically. **5**

M1 A1 for $E(X) = \sum x.p(x) = 0 + p(1 - p)(\frac{4}{3} + \frac{5}{3}p - p^2) + \frac{2}{3}p^2(1 + 4p - 3p^2) + p^4$

A1 $= \frac{4}{3}p + p^2$

ALTERNATIVELY
$E(X) = \sum E(X_i) = p + \frac{1}{3}p + p^2 = \frac{4}{3}p + p^2$ [**M2 A1**] **3**

M1 A1 for **e**quating this to $\frac{4}{3} \Rightarrow 0 = 3p^2 + 4p - 4$

M1 A1 for factorising/solving attempt at their quadratic $0 = (3p - 2)(p + 2)$

A1 for $0 < p < 1 \Rightarrow p = \frac{2}{3}$ **5**

Now, either p_0 and p_1 or p_2 and p_3 needed here:

M1 A1 for either $p_0 = \frac{35}{243}$ and $p_1 = \frac{108}{243}$ or $p_2 = \frac{84}{243}$ and $p_3 = \frac{16}{243}$

M1 A1 for careful statement of cases
p(correct pronouncement) = p(G and ≥ 2 judges say G) + p(NG and ≤ 1 judges say G)

A1 for correct (unsimplified) $= t\,.\,\frac{100}{243} + (1-t)\,.\,\frac{143}{243} = \frac{143-43t}{243}$

M1 for equating this to $\frac{1}{2}$ and solving for $t \Rightarrow 243 = 286 - 86t \Rightarrow 86t = 43$

A1 for $t = \frac{1}{2}$.

ALTERNATIVE
Let p(King pronounces guilty) = q .

Then "King correct" = "King pronounces guilty and defendant *is* guilty"
or "King pronounces not guilty and defendant *is* not guilty"
so that p(King correct) = $qt + (1-q)(1-t)$

Setting $qt + (1-q)(1-t) = \frac{1}{2} \Leftrightarrow (2q-1)(2t-1) = 0$

Since $q \neq \frac{1}{2}$, $t = \frac{1}{2}$. **7**

13 (i) **M1** for correct statement of cases

p(B in bag P) = p(B not chosen draw 1) + p(B chosen draw 1 and draw 2)

B1 for $\frac{k}{n}$ used; **B1 ft** for 1 – this; **B1** for $\frac{k}{n+k}$

$$= \left(1 - \frac{k}{n}\right) + \frac{k}{n} \times \frac{k}{n+k} = \frac{1}{n(n+k)}\left((n-k)(n+k) + k^2\right)$$

M1 for mult^{g}. the probs of 2 independent events

A1 for $= \frac{n}{n+k}$ **6**

B1 for $k = 0$

B1 for explanation that there are no others (e.g. since $p = 1 - \frac{k}{n+k} \le 1$ and for $k = 0$, $p = 1$ but $k > 0$, $p < 1$) **2**

(ii) **M1** for a correct listing of all cases

p(Bs in same bag) = p(B_1 chosen on D_1 and neither chosen on D_2)
+ p(B_1 chosen on D_1 and both chosen on D_2)
+ p(B_1 not chosen on D_1 and B_2 chosen on D_2)

$$= \frac{k}{n} \times \frac{{}^{n+k-2}C_k}{{}^{n+k}C_k} + \frac{k}{n} \times \frac{{}^{n+k-2}C_{k-2}}{{}^{n+k}C_k} + \left(1 - \frac{k}{n}\right) \times \frac{k}{n+k}$$

$$= \frac{k}{n} \times \frac{n(n-1)}{(n+k)(n+k-1)} + \frac{k}{n} \times \frac{k(k-1)}{(n+k)(n+k-1)} + \frac{k(n-k)}{n(n+k)}$$

$$= \frac{k}{n}\left\{\frac{n^2 - n + k^2 - k + (n^2 + nk - n - nk - k^2 + k}{(n+k)(n+k-1)}\right\}$$

$$= \frac{2k(n-1)}{(n+k)(n+k-1)}$$ **6**

$$\frac{\mathrm{d}p}{\mathrm{d}k} = \frac{(n^2 + 2nk + k^2 - n - k) \times 2(n-1) - 2k(n-1) \times (2n + 2k - 1)}{\left[(n+k)(n+k-1)\right]^2}$$

$= 0$ when $n^2 + 2nk + k^2 - n - k = 2nk + 2k^2 - k$ since $n > 2$, $n - 1 \neq 0$

$\Rightarrow k^2 = n(n-1)$

Allow $k = \left\lfloor \sqrt{n(n-1)} \right\rfloor$ or $k = \left\lfloor \sqrt{n(n-1)} \right\rfloor + 1$ or both, but must be an integer.

In fact, since $n^2 - n = (n - ½)^2 - ¼$, $\left\lfloor \sqrt{(n^2 - n)} \right\rfloor = n - 1$ and we find that,

when $k = n - 1$, $p = \dfrac{2(n-1)^2}{(2n-1)2(n-1)} = \dfrac{n-1}{2n-1}$

and when $k = n$, $p = \dfrac{2n(n-1)}{(2n)(2n-1)} = \dfrac{n-1}{2n-1}$ also **6**