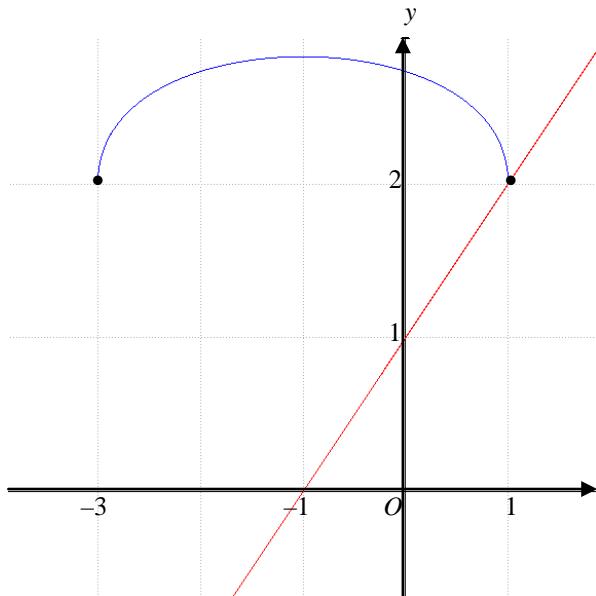


SII 2011 Q1



Domain $-3 \leq x \leq 1$ **B1**

(-3, 2) and (1, 2) on graph **B1**

y-intercept at $(0, 1 + \sqrt{3})$ **B1**

Symmetry in the line $x = -1$ **B1**

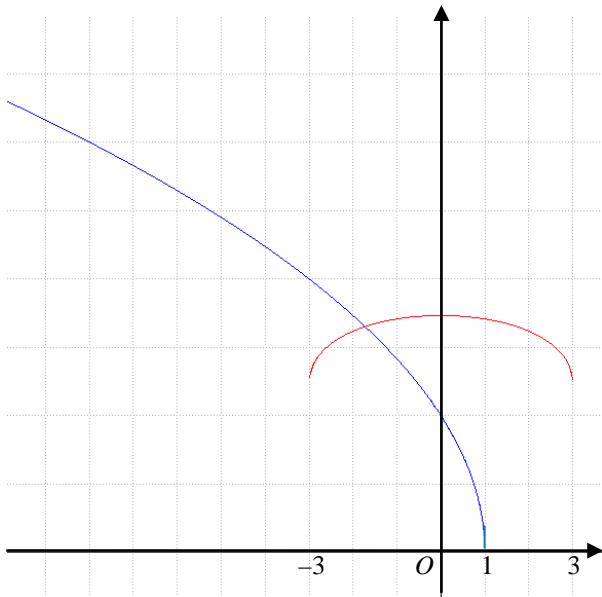
Correct shape of curve (ignore gradient at endpoints, but no vertex at peak) **B1**

Max. at $(-1, 2\sqrt{2})$ **B1**

$y = x + 1$ correctly drawn on diagram **B1**

Noting that curve & line cross only once so equation only has one root **E1**

$x = 1$ **B1**



Half-parabola $y = 2\sqrt{1-x}$ from (1, 0) **B1**

Dome-shaped curve for $y = \sqrt{3+x} + \sqrt{3-x}$ in $[-3, 3]$ **B1**

Noting that equation only has one root **E1**

Squaring both sides: **M1**

$$\begin{aligned} 4 - 4x &= 3 + x + 2\sqrt{9 - x^2} + 3 - x \\ \Rightarrow \sqrt{9 - x^2} &= -(2x + 1) \end{aligned} \quad \text{A1}$$

Squaring again after suitable rearrangement:

$$\begin{aligned} 9 - x^2 &= 4x^2 + 4x + 1 \\ \Rightarrow 0 &= 5x^2 + 4x - 8 \end{aligned} \quad \text{M1} \quad \text{A1}$$

Solving a quadratic equation **M1**

$$x = \frac{-4 \pm \sqrt{16 + 4 \cdot 5 \cdot 8}}{10} = -\frac{2}{5}(1 \pm \sqrt{11}) \quad \text{A1}$$

$$x = -\frac{2}{5}(1 + \sqrt{11}) \text{ or exact } \equiv \quad \text{A1}$$

Valid reason given for choosing this root; e.g. from graph, intersection is at negative x **E1**

SII 2011 Q2

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000 (Sorry, no marks)

$$x + y = k, \quad (x + y)(x^2 - xy + y^2) = kz^3$$

M1 Substituting in

$$\Rightarrow x^2 - (k-x)x + (k-x)^2 - z^3 = 0$$

$$\Rightarrow 3x^2 - 3kx + k^2 - z^3 = 0 \quad (*)$$

A1 Correct, identifiable quadratic

$$x = \frac{3k \pm \sqrt{9k^2 - 12(k^2 - z^3)}}{6} = \frac{1}{2} \left\{ k \pm \sqrt{\frac{4z^3 - k^2}{3}} \right\}$$

For x real, we need $4z^3 - k^2 \geq 0$ i.e. $z^3 \geq \frac{1}{4}k^2$ **M1** Considering discriminant; **A1 Given Answer**For x integer, we need $\frac{4z^3 - k^2}{3}$ a perfect square **E1 Given answer Explained**For distinct positive roots (N.B. one root is x , the other is y)

$$\text{we need } k - \sqrt{\frac{4z^3 - k^2}{3}} > 0 \quad \text{i.e. } 3k^2 > 4z^3 - k^2 \quad \text{i.e. } z^3 < k^2 \quad \mathbf{M1 A1 Given Answer}$$

Alternatively, $y = k - x$ in $(*) \Rightarrow z^3 = k^2 - 3xy < k^2$ (since $x, y > 0$) etc.When $k = 20$, $100 \leq z^3 < 400 \Rightarrow z = 5, 6, 7$ **M1** Using given results to get a suitable small set of values of z

$$\text{with } \frac{4z^3 - k^2}{3} = \frac{100}{3}, 88, 324 = 18^2$$

M1 Using other given condition to test these z 'sThus $z = 7$ gives $(x, y) = (1, 19)$

$$\mathbf{A1} \quad \text{i.e. } 20 = \left(\frac{1}{7}\right)^3 + \left(\frac{19}{7}\right)^3$$

$$\mathbf{NB} \quad \frac{4z^3 - k^2}{3} = \begin{cases} (k-2x)^2 \\ (y-x)^2 \\ < k^2 \end{cases} \quad \begin{array}{l} \text{gives 2 of the results directly (M1 A1 E1)} \\ \text{gives the 3rd (M1 A1)} \end{array}$$

$$\begin{aligned}
x + y &= z^2, \quad (x + y)(x^2 - xy + y^2) = kz.z^2 \\
\Rightarrow x^2 - (z^2 - x)x + (z^2 - x)^2 - kz &= 0 \\
\Rightarrow 3x^2 - 3z^2x + z^4 - kz &= 0 \quad (*) \\
x &= \frac{3z^2 \pm \sqrt{9z^4 - 12(z^4 - kz)}}{6} = \frac{1}{2} \left\{ z^2 \pm \sqrt{\frac{4kz - z^4}{3}} \right\}
\end{aligned}$$

For x real, we need $4kz - z^4 \geq 0$ i.e. $z^3 \leq 4k$

and $\frac{4kz - z^4}{3}$ a perfect square

and $z^2 - \sqrt{\frac{4kz - z^4}{3}} > 0$ i.e. $3z^4 > 4kz - z^4$ i.e. $z^3 > k$

M1 Substituting in

A1 Correct, identifiable quadratic

M1 Considering discriminant; **A1**

E1 Noted

B1

OR by alternate “squares” methods, as before

When $k = 19$, $19 < z^3 \leq 76 \Rightarrow z = 3$ or 4

with $\frac{z(76 - z^3)}{3} = 49 = 7^2$ or $16 = 4^2$

Thus $z = 3$ gives $(x, y) = (1, 8)$

$z = 4$ gives $(x, y) = (6, 10)$

M1 Using given results to get a suitable small set of values of z

M1 Using other given condition to test them

$$\mathbf{A1} \quad \text{i.e. } 19 = \left(\frac{1}{3}\right)^3 + \left(\frac{8}{3}\right)^3$$

$$\mathbf{A1} \quad \text{i.e. } 19 = \left(\frac{3}{2}\right)^3 + \left(\frac{5}{2}\right)^3$$

Additional Note

$$\begin{aligned}
x + y &= kz, \quad (x + y)(x^2 - xy + y^2) = kz.z^2 \Rightarrow x^2 - (kz - x)x + (kz - x)^2 - z^2 = 0 \\
\Rightarrow 3x^2 - 3kzx + z^2(k^2 - 1) &= 0
\end{aligned}$$

$$x = \frac{3kz \pm \sqrt{9k^2z^2 - 12z^2(k^2 - 1)}}{6} = \frac{1}{6}z \left\{ 3k \pm \sqrt{12 - 3k^2} \right\}$$

requiring $12 - 3k^2 \geq 0$ i.e. $k^2 \leq 4 \Rightarrow k = 1$ or 2

$k = 1$: $x^3 + y^3 = z^3$ has NO solutions by *Fermat's Last Theorem*

$k = 2$: $x^3 + y^3 = 2z^3$ has (trivially) infinitely many solutions $x = y = z$

SII 2011 Q3

$$f'(x) = \cos x - \{\sin x + \cos x\} = x \sin x \quad \mathbf{M1} \text{ Product Rule; A1}$$

$$\geq 0 \text{ for } x \in [0, \frac{1}{2}\pi]$$

B1 Noted that f is increasing (larger interval ok)

and since $f(0) = 0$ (and f increasing)

$$f(x) = \sin x - x \cos x \geq 0 \text{ for } 0 \leq x \leq \frac{1}{2}\pi$$

E1 Fully explained/noted

$$\text{For } 0 \leq x < 1, \frac{d}{dx}(\arcsin x) \geq \frac{d}{dx}(x)$$

M1 for using $1 = \frac{d}{dx}(x)$

$$\Rightarrow \frac{d}{dx}(\arcsin x - x) \geq 0$$

M1 Combining the two sides

$$\Rightarrow f(x) = \arcsin x - x \text{ an increasing fn.}$$

E1 Fully explained

and since $f(0) = 0$ (and f increasing)

$$f(x) = \arcsin x - x \geq 0 \text{ for } 0 \leq x < 1$$

M1

$$\text{i.e. } \arcsin x \geq x \text{ for } 0 \leq x < 1$$

E1 Fully explained

$$g(x) = \frac{x}{\sin x} \Rightarrow g'(x) = \frac{\sin x - x \cos x}{\sin^2 x}$$

B1

$$> 0 \text{ for } 0 < x < \frac{1}{2}\pi$$

E1 that g is an increasing fn. from (a)

$$\text{Let } u = \arcsin x. \text{ Then } u \geq x \text{ (for } 0 < x < 1)$$

M1 by (b)

$$\Rightarrow g(u) \geq g(x) \text{ since } g'(x) \geq 0$$

M1

$$\Rightarrow \frac{\arcsin x}{x} \geq \frac{x}{\sin x} \text{ (for } 0 < x < 1) \quad \mathbf{A1} \text{ Answer Given}$$

$$g(x) = \frac{\tan x}{x}, g'(x) = \frac{x \sec^2 x - \tan x}{x^2} = \frac{2x - \sin 2x}{2x^2 \cos^2 x}$$

M1

$$\text{and examine } f(x) = 2x - \sin 2x, f'(x) = 2 - 2\cos 2x \geq 0 \text{ in } [0, \frac{1}{2}\pi] \Rightarrow f \text{ increasing,}$$

M1

$$\Rightarrow \text{since } f(0) = 0 \text{ that } g'(x) \geq 0 \Rightarrow g \text{ (strictly) increasing [ignore } f(0) = 0]$$

E1

$$\text{Given } \frac{d}{dx}(\arctan x) \leq \frac{d}{dx}(x) \Rightarrow \frac{d}{dx}(x - \arctan x) \geq 0 \quad \mathbf{M1} \text{ for using } 1 = \frac{d}{dx}(x) \text{ etc.}$$

$$\text{Let } u = \arctan x. \text{ Then } x \geq u \text{ (for } 0 < x < \frac{1}{2}\pi)$$

$$\Rightarrow g(x) \geq g(u)$$

M1

$$\Rightarrow \frac{\tan x}{x} \geq \frac{x}{\arctan x} \text{ (for } 0 < x < \frac{1}{2}\pi) \quad \mathbf{A1} \text{ Answer Given}$$

SII 2011 Q4

Using $\sin A = \cos(90^\circ - A)$ to get $\theta = 360n \pm (90^\circ - 4\theta)$ **M1** (or \equiv Gen. Soln. for sine)

$$\Rightarrow 5\theta = 360n + 90^\circ \text{ or } 3\theta = 360n + 90^\circ$$

M1 either case

$$\Rightarrow \theta = 72n + 18^\circ \Rightarrow \theta = 18^\circ, 90^\circ, 162^\circ$$

A1

$$\text{or } \theta = 120n + 30^\circ \Rightarrow \theta = 30^\circ, 150^\circ$$

A1

$$c = 2.2sc.(1 - 2s^2)$$

M1 double-angle formulae, twice

$$(c \neq 0 \text{ for } \theta = 18^\circ) \Rightarrow 1 = 4s(1 - 2s^2) \text{ or } 8s^3 - 4s + 1 = 0$$

M1 cubic formed

$$\Rightarrow (2s - 1)(4s^2 + 2s - 1) = 0$$

M1 cubic factorisation attempted

$$(c \neq \frac{1}{2} \text{ for } \theta = 18^\circ) \Rightarrow s = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

M1 quadratic solution method

$$\theta \text{ acute} \Rightarrow s = \sin 18^\circ > 0 \Rightarrow \sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

A1 GIVEN ANSWER legitimately obtained

Explanation of $c \neq 0$, $s \neq \frac{1}{2}$ and $s > 0$

E1 All 3 must appear somewhere

$$4s^2 + 1 = 16s^2(1 - s^2) \Rightarrow 0 = 16s^4 - 12s^2 + 1$$

M1 double-angles; **A1** correct quartic

$$s^2 = \frac{12 \pm \sqrt{80}}{32} = \frac{3 \pm \sqrt{5}}{8}$$

M1

$$= \frac{6 \pm 2\sqrt{5}}{16} = \left(\frac{\sqrt{5} \pm 1}{4} \right)^2$$

M1 Method to find an exact square-root

$$\Rightarrow \sin x = \pm \left(\frac{\sqrt{5} \pm 1}{4} \right)$$

A1 Must be *four* answers

Noting $\sin^2 x + \frac{1}{4} = \sin^2 2x$ from (ii) with $x = 3\alpha = 18^\circ$

works provided $5\alpha = 30^\circ \Rightarrow \alpha = 6^\circ$

M1 A1

$$\text{Also, } \sin x = -\left(\frac{\sqrt{5} - 1}{4} \right) \Rightarrow 3\alpha = 180^\circ + 18^\circ = 198^\circ$$

M1

can also work, since $5\alpha = 330^\circ$ also has $\sin 5\alpha = -\frac{1}{2}$ **E1** Must be mentioned explicitly

$$\Rightarrow \alpha = 66^\circ$$

A1

NB: $\alpha = 45^\circ$ works also, but does not follow from a “hence” argument

SII 2011 Q5

$$\text{Let } M = OA \cap BC. \text{ Then } \mathbf{m} = m\mathbf{a}, \quad \mathbf{M1}$$

$$\text{and } (\mathbf{ma} - \mathbf{b}) \cdot \mathbf{a} = 0 \quad \mathbf{M1} \quad \Rightarrow m = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$$

$$\text{Then } \mathbf{c} = \overrightarrow{OB} + 2\overrightarrow{BM} = \mathbf{b} + 2m\mathbf{a} - 2\mathbf{b} \quad \mathbf{M1} \quad \Rightarrow \mathbf{c} = \lambda\mathbf{a} - \mathbf{b} \text{ where } \lambda = 2m = 2\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \quad \mathbf{A1}$$

ALT. OA is the bisector of $\angle BOC$ and $OB = OC$

$$\Rightarrow A \text{ is on the diagonal } OA' \text{ of } \text{gm } OBA'C \Rightarrow \mathbf{b} + \mathbf{c} = \lambda\mathbf{a} \quad \mathbf{M1} \quad \mathbf{M1} \quad \mathbf{A1}$$

$$BC \perp^r OA \Rightarrow (\mathbf{b} - \mathbf{c}) \cdot \mathbf{a} = 0 \quad \mathbf{M1} \Rightarrow (2\mathbf{b} - \lambda\mathbf{a}) \cdot \mathbf{a} = 0 \Rightarrow \lambda = 2\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \quad \mathbf{A1}$$

Similarly (replacing \mathbf{a} by \mathbf{b} and \mathbf{b} by \mathbf{c} in the above) **M1**

$$\begin{aligned} \mathbf{d} &= k\mathbf{b} - \mathbf{c} \text{ where } k = 2\left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{b} \cdot \mathbf{b}}\right) \quad \mathbf{M1} \\ &= 2\left(\frac{\mathbf{b} \cdot \lambda\mathbf{a} - \mathbf{b} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) = 2\lambda\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) - 2 \quad \mathbf{M1} \\ \Rightarrow \mathbf{d} &= \left(2\lambda\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) - 2\right)\mathbf{b} - (\lambda\mathbf{a} - \mathbf{b}) \quad \mathbf{M1} \\ &= \mu\mathbf{b} - \lambda\mathbf{a} \text{ where } \mu = 2\lambda\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right) - 1 \text{ or } 4\left(\frac{[\mathbf{a} \cdot \mathbf{b}]^2}{[\mathbf{a} \cdot \mathbf{a}][\mathbf{b} \cdot \mathbf{b}]}\right) - 1 \quad \mathbf{A1} \end{aligned}$$

$$\text{Now } \overrightarrow{AB} = \mathbf{b} - \mathbf{a} \parallel \overrightarrow{AD} = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$$

$$\Leftrightarrow t(\mathbf{b} - \mathbf{a}) = \mu\mathbf{b} - (\lambda + 1)\mathbf{a} \text{ for some } t (\neq 0) \quad \mathbf{M1}$$

$$\text{Comparing coeffts. of } \mathbf{a} \text{ and } \mathbf{b} \quad \mathbf{M1} \Rightarrow (t =) \mu = \lambda + 1 \quad \mathbf{A1}$$

$$\lambda = -\frac{1}{2} \Rightarrow \mu = \frac{1}{2} \quad \mathbf{B1} \Rightarrow D \text{ is the midpoint of } AB \quad \mathbf{B1}$$

$$\mu = \frac{1}{2} \Rightarrow \frac{1}{2} = 4\left(\frac{[\mathbf{a} \cdot \mathbf{b}]^2}{[\mathbf{a} \cdot \mathbf{a}][\mathbf{b} \cdot \mathbf{b}]}\right) - 1 = 4\left(\frac{\mathbf{a} \cdot \mathbf{b}}{ab}\right)^2 - 1 \quad \mathbf{M1} \quad \mathbf{M1} \quad \mathbf{A1}$$

$$\text{Use of } \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} \quad \mathbf{M1} \text{ to get } \cos \theta = -\sqrt{\frac{3}{8}} \quad \mathbf{A1}$$

[NB : $\mathbf{a} \cdot \mathbf{b}$ has the same sign as λ]

ALT

$$\mu = \frac{1}{2} \Rightarrow 4ab \cos \theta = -a^2 \text{ and } 2ab \cos \theta = -3a^2 \Rightarrow a = b\sqrt{6} \text{ and } a, b > 0 \Rightarrow \cos \theta = -\sqrt{\frac{3}{8}}$$

M1

A1

A1

M1

A1

SII 2011 Q6

$$I = \int [f'(x)]^2 [f(x)]^n dx = \int [f'(x)] \times \{ [f'(x)] [f(x)]^n \} dx \quad \text{M1 Splitting M1 Parts}$$

$$= f'(x) \times \frac{1}{n+1} [f(x)]^{n+1} - \int \left([f''(x)] \times \frac{1}{n+1} [f(x)]^{n+1} \right) dx \quad \text{A1}$$

$$\begin{aligned} \text{Using } f''(x) &= kf(x)f'(x), \quad I = f'(x) \times \frac{1}{n+1} [f(x)]^{n+1} - \int \left(kf'(x) \times \frac{1}{n+1} [f(x)]^{n+2} \right) dx \quad \text{M1} \\ &= f'(x) \times \frac{1}{n+1} [f(x)]^{n+1} - \frac{1}{(n+1)(n+3)} \times k [f(x)]^{n+3} (+ C) \quad \text{A1} \end{aligned}$$

$$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \text{ and } f''(x) = 2 \sec^2 x \tan x = kf(x)f'(x) \text{ with } k = 2 \quad \text{M1 A1 E1}$$

$$\Rightarrow I = \frac{\sec^2 x \tan^{n+1} x}{n+1} - \frac{2 \tan^{n+3} x}{(n+1)(n+3)} \quad \text{B1}$$

Differentiating this gives M1

$$\begin{aligned} \frac{dI}{dx} &= \frac{1}{n+1} \left(\sec^2 x (n+1) \tan^n x \sec^2 x + 2 \sec x \sec x \tan x \tan^{n+1} x \right) \\ &\quad - \frac{1}{(n+1)(n+3)} \left(2(n+3) \tan^{n+2} x \sec^2 x \right) \\ &= \sec^4 x \tan^n x = (f'(x))^2 \times (f(x))^n \text{ as required} \quad \text{A1 E1} \end{aligned}$$

NB This 4-mark chunk may be done in reverse as an integration

$$\int \frac{\sin^4 x}{\cos^8 x} dx = \int \sec^4 x \tan^4 x dx = \frac{\sec^2 x \tan^5 x}{5} - \frac{2 \tan^7 x}{35} + C \quad \text{M1 A1}$$

$$f(x) = \sec x + \tan x \Rightarrow f'(x) = \sec x \tan x + \sec^2 x = \sec x (\sec x + \tan x) \quad \text{M1}$$

$$\text{and } f''(x) = \sec^2 x (\sec x + \tan x) + \sec x \tan x (\sec x + \tan x) \quad \text{A1}$$

$$= \sec x (\sec x + \tan x)^2 = kf(x)f'(x) \text{ with } k = 1 \quad \text{E1}$$

$$\begin{aligned} \text{Then } \int \sec^2 x (\sec x + \tan x)^6 dx &= \int \{ \sec x (\sec x + \tan x) \}^2 \times (\sec x + \tan x)^4 dx \quad \text{M1} \\ &= \frac{\sec x (\sec x + \tan x)^6}{5} - \frac{(\sec x + \tan x)^7}{35} + C \quad \text{A1 A1} \end{aligned}$$

NB Lack of “+ C” not penalised throughout

SII 2011 Q7

$$\sum_{r=0}^n b_r = (1 + \lambda + \lambda^2 + \dots + \lambda^n) - (1 + \mu + \mu^2 + \dots + \mu^n) \quad \text{M1}$$

$$= \frac{\lambda^{n+1} - 1}{\lambda - 1} - \frac{\mu^{n+1} - 1}{\mu - 1}$$

M1 $S_\infty(\text{GP})$ used at least once

$$= \frac{1}{\sqrt{2}} (\lambda^{n+1} - 1 + \mu^{n+1} - 1) \quad \text{since } \lambda - 1 = \sqrt{2} \text{ and } \mu - 1 = -\sqrt{2} \quad \text{M1}$$

$$= \frac{1}{\sqrt{2}} a_{n+1} - \sqrt{2}$$

A1 Answer Given

$$\sum_{r=0}^n a_r = \frac{\lambda^{n+1} - 1}{\sqrt{2}} - \frac{\mu^{n+1} - 1}{\sqrt{2}} = \frac{1}{\sqrt{2}} b_{n+1}$$

B1 May be just stated/observed

$$\sum_{m=0}^{2n} \left(\sum_{r=0}^m a_r \right) = \sum_{m=0}^{2n} \left(\frac{1}{\sqrt{2}} b_{m+1} \right) = \frac{1}{\sqrt{2}} \sum_{m=0}^{2n+1} b_m \quad \text{since } b_0 = 0 \quad \text{M1 A1}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} a_{2n+2} - \sqrt{2} \right) \quad \text{from (i)}$$

M1 Use of earlier result

$$= \frac{1}{2} (\lambda^{2n+2} + \mu^{2n+2} - 2)$$

M1

$$= \frac{1}{2} \left([\lambda^{n+1}]^2 - 2[\lambda\mu]^{n+1} + [\mu^{n+1}]^2 \right) \quad \text{since } \lambda\mu = -1 \text{ and } n+1 \text{ is even when } n \text{ is odd E1}$$

$$= \frac{1}{2} (b_{n+1})^2 \quad \text{when } n \text{ is odd}$$

A1 Answer Given

$$\text{However, when } n \text{ is even, } n+1 \text{ is odd and } \sum_{m=0}^{2n} \left(\sum_{r=0}^m a_r \right) = \frac{1}{2} (b_{n+1})^2 - 2 \quad \text{or} \quad \frac{1}{2} (a_{n+1})^2 \quad \text{B1}$$

$$\left(\sum_{r=0}^n a_r \right)^2 = \frac{1}{2} (b_{n+1})^2$$

$$\text{and } \sum_{r=0}^n a_{2r+1} = (\lambda + \lambda^3 + \lambda^5 + \dots + \lambda^{2n+1}) + (\mu + \mu^3 + \mu^5 + \dots + \mu^{2n+1}) \quad \text{M1}$$

$$= \frac{\lambda(\lambda^{2n+2} - 1)}{\lambda^2 - 1} + \frac{\mu(\mu^{2n+2} - 1)}{\mu^2 - 1} \quad \text{M1 } S_\infty(\text{GP}) \text{ used at least once}$$

$$\text{Now } \lambda^2 - 1 = 3 + 2\sqrt{2} - 1 = 2(1 + \sqrt{2}) = 2\lambda \quad \text{and} \quad \mu^2 - 1 = 3 - 2\sqrt{2} - 1 = 2(1 - \sqrt{2}) = 2\mu \quad \text{M1 A1}$$

$$\text{so } \sum_{r=0}^n a_{2r+1} = \frac{1}{2} (\lambda^{2n+2} + \mu^{2n+2} - 2) = \frac{1}{2} (b_{n+1})^2 \quad \text{when } n \text{ is odd} \quad \text{M1}$$

$$\text{and } = \frac{1}{2} (b_{n+1})^2 - 2 \quad \text{when } n \text{ is even} \quad \text{M1}$$

$$\text{Thus } \left(\sum_{r=0}^n a_r \right)^2 - \sum_{r=0}^n a_{2r+1} = 0 \quad \text{when } n \text{ is odd} / = 2 \quad \text{when } n \text{ is even} \quad \text{A1 A1 2nd Answer Given}$$

SII 2011 Q8

The string leaves the circle at $C(-\cos\theta, \sin\theta)$

B1

Arc $AC = \pi - t = \theta$ (so $\cos\theta = -\cos t$ and $\sin\theta = \sin t$)

M1

Then $B = (-\cos\theta + t \sin\theta, \sin\theta + t \cos\theta) = (\cos t + t \sin t, \sin t - t \cos t)$

A1 Answer Given

$$\frac{dx}{dt} = -\sin t + t \cos t + \sin t = t \cos t \text{ by the Product Rule } \mathbf{M1} \mathbf{A1}$$

$$= 0 \quad \mathbf{M1} \text{ when } t = 0, (x, y) = (1, 0) \text{ or } t = \frac{1}{2}\pi, (x, y) = \left(\frac{1}{2}\pi, 1\right). \text{ This is } x_{\max} \text{ so } t_0 = \frac{1}{2}\pi \quad \mathbf{A1}$$

Area under curve and above x -axis is

$$\begin{aligned} A &= \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} y \frac{dx}{dt} dt \quad \mathbf{B1} \text{ Including limits} &= \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\sin t - t \cos t) t \cos t dt \quad \mathbf{M1} \\ &= \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} -\frac{1}{2}t \sin 2t dt + \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{2}t^2(1 + \cos 2t) dt \quad \mathbf{M1} \end{aligned}$$

$$\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} -\frac{1}{2}t \sin 2t dt = \left[\frac{1}{4}t \cos 2t \right]_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{4} \cos 2t dt = \left[\frac{1}{4}t \cos 2t + \frac{1}{8} \sin 2t \right]_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \frac{3\pi}{8} \quad \mathbf{M1} \mathbf{M1} \mathbf{A1}$$

and

$$\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{2}t^2 dt = \left[\frac{1}{6}t^3 \right]_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \frac{7\pi^3}{48} \quad \mathbf{B1}$$

and

$$\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{2}t^2 \cos 2t dt = \left[\frac{1}{4}t^2 \sin 2t \right]_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{2}t \sin 2t dt = 0 - -\frac{3\pi}{8} = \frac{3\pi}{8} \text{ using previous answer } \mathbf{M1} \mathbf{A1}$$

$$\text{Thus } A = \frac{7\pi^3}{48} + \frac{3\pi}{4} \quad \mathbf{A1}$$

$$\text{Using limits } \frac{1}{2}\pi \text{ and } 0 \text{ gives } -\frac{\pi^3}{48} + \frac{\pi}{4} \quad \mathbf{B1}$$

$$\begin{aligned} \text{Total area swept out by string is then } & \frac{7\pi^3}{48} + \frac{3\pi}{4} + -\frac{\pi^3}{48} + \frac{\pi}{4} - \frac{\pi}{2} \quad \text{(area inside semi-circle)} \quad \mathbf{M1} \\ &= \frac{\pi^3}{6} \quad \mathbf{A1} \end{aligned}$$

SII 2011 Q9

CLM $3mu = 2mV_A + mV_B$ **B1**

NEL $e.3u = V_B - V_A$ **B1**

Solving simultaneously for V_A and V_B **M1**

$V_A = u(1 - e)$, $V_B = u(1 + 2e)$ **A1 A1**

Vel. B after collision with wall is $|f V_B|$ **B1**

CLM (away from wall) $f m V_B - 2m V_A = 2m W_A - m W_B$ **M1 A1** (W_B -ve since towards wall)

NEL $W_A + W_B = e(V_A + f V_B)$ **B1**

Subst^{g.} for V_A & V_B from before in *both* equations **M1**

$2W_A - W_B = u\{f(1+2e) - 2(1-e)\}$ **A1**

$W_A + W_B = eu\{(1-e) + f(1+2e)\}$ **A1**

Solving simultaneously for W_A and W_B **M1**

$(W_A = \frac{1}{3}u\{f(1+2e)(1+e) - (2-e)(1-e)\}$ not required)

$W_B = \frac{1}{3}u\{2(1-e^2) - f(1-4e^2)\}$ **A1 Answer Given**

If $e = \frac{1}{2}$, $W_B = \frac{1}{3}u\{2(\frac{3}{4}) - f(0)\} = \frac{1}{2}u > 0$ (May be incorporated into one of the other cases)

If $\frac{1}{2} < e < 1$, $W_B = \frac{1}{3}u\{2(1-e^2) + f(4e^2 - 1)\} > 0$ for all e, f since each term in the bracket is +ve

If $0 < e < \frac{1}{2}$, $1 - e^2 > \frac{3}{4}$ and $W_B > \frac{1}{3}u\{\frac{3}{2} - f(1-4e^2)\} > \frac{1}{3}u\{\frac{3}{2} - 1 \times 1\} > 0$

Attempt to show $W_B > 0$ for some values of e, f **M1**

Splitting into suitable cases **M1**

Cases as above **B1 B1 M1 A1** However done

SII 2011 Q10

$$\dot{y} = u \sin \theta - gt = 0 \Rightarrow t = \frac{u \sin \theta}{g} \text{ subst. into } y = ut \sin \theta - \frac{1}{2}gt^2 \Rightarrow H = \frac{u^2 \sin^2 \theta}{2g} \quad \mathbf{M1 A1}$$

This may just be quoted

$$l = \frac{1}{2}H = \frac{u^2 \sin^2 \theta}{4g} \quad \mathbf{B1}$$

$$l = ut \sin \theta - \frac{1}{2}gt^2$$

M1 using vertical distance expression for l

$$gt^2 - (2u \sin \theta)t + H = 0$$

M1 forming a quadratic in t

$$t = \frac{2u \sin \theta \pm \sqrt{4u^2 \sin^2 \theta - 4gH}}{2g}$$

M1 good solving attempt

$$= \frac{2\sqrt{2gH} \pm \sqrt{8gH - 4gH}}{2g}$$

M1 using $u \sin \theta = \sqrt{2gH}$

$$= \frac{1}{g} (\sqrt{2gH} \pm \sqrt{gH}) = \sqrt{\frac{H}{g}} (\sqrt{2} - 1) \quad \mathbf{A1 Answer Given}$$

explained by observing that we want the first time when an unimpeded P is at this height **E1**

$$\text{For } P \text{ vertically, } v = \dot{y} = u \sin \theta - g \sqrt{\frac{H}{g}} (\sqrt{2} - 1)$$

$$= \sqrt{2gH} - \sqrt{gH} (\sqrt{2} - 1) = \sqrt{gH} \text{ or } \frac{u \sin \theta}{\sqrt{2}} \quad \mathbf{M1 A1}$$

Thus, common speed after string goes taut, by CLM, is

$$v = \frac{1}{2} \sqrt{gH} \text{ or } \frac{u \sin \theta}{2\sqrt{2}} \quad \mathbf{M1 A1}$$

Consider now the projectile R ,

$$\text{with initial velocity components } u \cos \theta \rightarrow \text{ and } \frac{u \sin \theta}{2\sqrt{2}} \uparrow \quad \mathbf{M1}$$

$$y = \frac{u \sin \theta}{2\sqrt{2}} t - \frac{g}{2} t^2 = 0 \text{ (} t \neq 0 \text{) at Range, when } t = \frac{u \sin \theta}{g\sqrt{2}} \quad \mathbf{M1 A1}$$

$$\text{Then } D = x_1 + x_2 \text{ where } x_1 = u \cos \theta \frac{u \sin \theta}{g\sqrt{2}} (\sqrt{2} - 1) \text{ and } x_2 = u \cos \theta \frac{u \sin \theta}{g\sqrt{2}} \quad \mathbf{M1 2 distances}$$

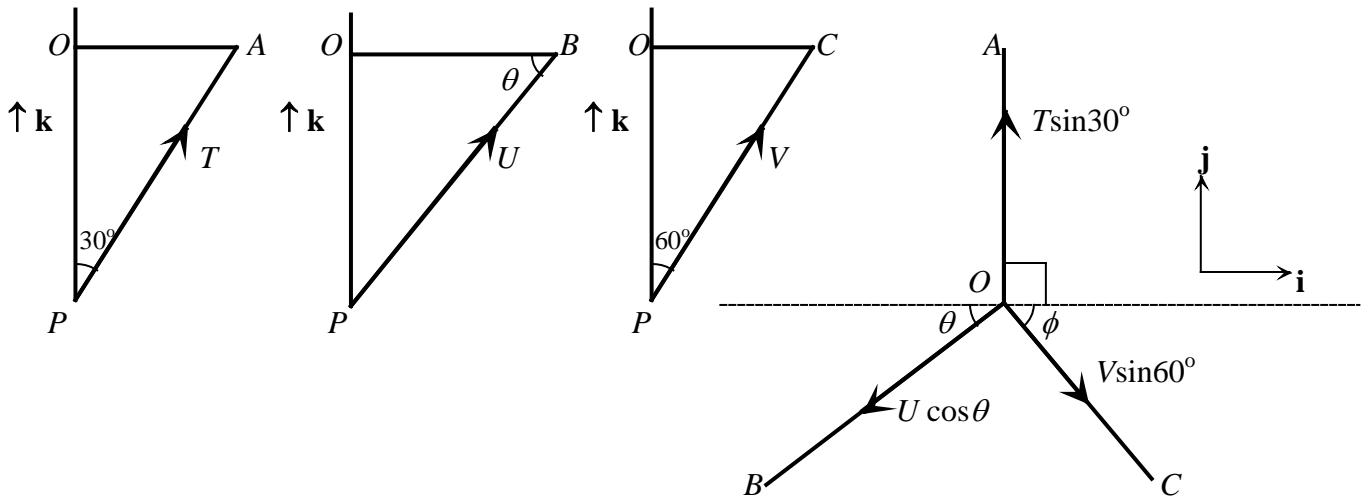
$$= \frac{u^2 \sin \theta \cos \theta}{g} \quad \mathbf{A1}$$

$$D = H \Rightarrow \tan \theta = 2$$

M1 Must involve cancelling trig. terms **A1 CAO**

NB Throughout, results may be in terms of g & H rather than u and θ .

SII 2011 Q11



$$\tan \theta = \sqrt{2} \Rightarrow \sin \theta = \frac{\sqrt{2}}{\sqrt{3}} \text{ and } \cos \theta = \frac{1}{\sqrt{3}} \quad \mathbf{B1}$$

$$\tan \phi = \frac{\sqrt{2}}{4} \Rightarrow \sin \phi = \frac{1}{3} \text{ and } \cos \phi = \frac{2\sqrt{2}}{3} \quad \mathbf{B1}$$

Vector in direction PB is $-(U \cos \theta) \cos \theta \mathbf{i} - (U \cos \theta) \sin \theta \mathbf{j} + U \sin \theta \mathbf{k}$ **M1 A1 A1 A1**

$$\underline{T}_B = \left(-\frac{1}{3} \mathbf{i} - \frac{\sqrt{2}}{3} \mathbf{j} + \frac{\sqrt{2}}{\sqrt{3}} \mathbf{k} \right) U \quad \text{and the bracketed vector has magnitude 1}$$

This must be explicitly verified **A1 Answer Given**

$$\underline{T}_A = T \sin 30^\circ \mathbf{j} + T \cos 30^\circ \mathbf{k} = \frac{1}{2} T (\mathbf{j} + \sqrt{3} \mathbf{k}) \quad \mathbf{B1}$$

$$\underline{T}_C = V \sin 60^\circ \cos \phi \mathbf{i} - V \sin 60^\circ \sin \phi \mathbf{j} + V \cos 60^\circ \mathbf{k} = \frac{1}{2} V \left(\frac{2\sqrt{2}}{\sqrt{3}} \mathbf{i} - \frac{1}{\sqrt{3}} \mathbf{j} + \mathbf{k} \right) \quad \mathbf{B1 B1}$$

$$\underline{W} = -W \mathbf{k} \quad \mathbf{B1}$$

$$\underline{T}_A + \underline{T}_B + \underline{T}_C + \underline{W} = \underline{0} \quad \mathbf{M1} \text{ May be implied by zero components}$$

$$\text{Comparing terms : (i)} \quad 0 - \frac{1}{3} U + \frac{\sqrt{6}}{3} V = 0 \quad \mathbf{M1}$$

$$\Rightarrow U = V\sqrt{6} \quad \mathbf{A1 Answer Given}$$

$$\text{(j)} \quad \frac{1}{2} T - \frac{\sqrt{2}}{3} U - \frac{\sqrt{3}}{6} V = 0 \quad \mathbf{A1}$$

$$\text{Use of } U = V\sqrt{6} \Rightarrow T = \frac{5\sqrt{3}}{3} V \quad \mathbf{A1}$$

$$\text{(k)} \quad \frac{\sqrt{3}}{2} T + \frac{\sqrt{6}}{3} U + \frac{1}{2} V = W \quad \mathbf{A1}$$

$$\text{Use of } U = V\sqrt{6} \text{ and } T = \frac{5\sqrt{3}}{3} V \Rightarrow T = \frac{W\sqrt{3}}{3}, \quad U = \frac{W\sqrt{6}}{5}, \quad V = \frac{W}{5} \quad \mathbf{A1 A1 A1}$$

SII 2011 Q12

$$P(\text{re-match}) = P(XYX) + P(YXY) = p(1-p)^2 + (1-p)^3 = (1-p)^2 \quad \mathbf{B1}$$

$$P(Y \text{ wins directly}) = P(YY) + P(XYY)$$

$$= (1-p)p + p(1-p)p = p(1-p)(1+p) \text{ or } p(1-p^2) \quad \mathbf{B1}$$

$$P(Y \text{ wins}) = w = p(1-p^2) + w(1-p)^2 \quad \mathbf{M1} \text{ recurrently defined or via } S_\infty(\text{GP})$$

$$\Rightarrow w = \frac{p(1-p^2)}{1-(1-p)^2} = \frac{p(1-p^2)}{(1-(1-p))(1+(1-p))} = \frac{p(1-p^2)}{p(2-p)} \quad \mathbf{M1} \text{ rearranging for } w$$

$$= \frac{1-p^2}{2-p} \text{ for } p \neq 0 \quad \mathbf{A1} \text{ GIVEN ANSWER legitimately obtained}$$

$$w - \frac{1}{2} = \frac{2(1-p^2) - (2-p)}{2(2-p)} = \frac{p(1-2p)}{2(2-p)} \quad \mathbf{M1} \text{ A1}$$

Since $2-p > 0$, $w - \frac{1}{2}$ has the same sign as $1-2p$ and hence as $\frac{1}{2}-p$ **M1**

Hence, $w > \frac{1}{2}$ if $p < \frac{1}{2}$ and $w < \frac{1}{2}$ if $p > \frac{1}{2}$ **E1** both correctly concluded

[May be done by calculus: **M1 A1** then **M1 E1** for the explanation]

$$\frac{dw}{dp} = \frac{(2-p)(-2p) - (1-p^2)(-1)}{(2-p)^2} \quad \mathbf{M1} \text{ A1} \text{ correct unsimplified}$$

$$= \frac{dw}{dp} = \frac{1}{(2-p)^2} (p^2 - 4p + 1) = \frac{1}{(2-p)^2} ([2-p]^2 - 3) \quad \mathbf{M1} \text{ A1}$$

$$\frac{dw}{dp} > 0 \text{ for } 0 < p < 2 - \sqrt{3} \text{ and } \frac{dw}{dp} < 0 \text{ for } 2 - \sqrt{3} < p \leq 1 \quad \mathbf{M1} \text{ A1} \text{ considering sign of } \frac{dw}{dp}$$

E1 Justification

$$p = \frac{2}{3} \Rightarrow w = \frac{5}{12}, 1-w = \frac{7}{12} \text{ and so } k = \frac{7}{5} \times £1 = £1.40 \text{ (to balance the game)}$$

or by “expected gain” approach **M1 A1** $k = 1.4$

When $p = 0$, results run $YXY \dots$ re-match ... $YXY \dots$ re-match **E1**

... and the match never ends **B1**

SII 2011 Q13

Skewness is a measure of a distribution's *lack of symmetry* **B1**

$$\begin{aligned} E[(X - \mu)^3] &= E[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3] \\ &= E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3 \\ &= E[X^3] - 3\mu(\sigma^2 + \mu^2) + 3\mu^2 \cdot \mu - \mu^3 \\ &= E[X^3] - 3\mu\sigma^2 - \mu^3 \end{aligned}$$

B1 Correct binomial expansion

M1 Use of distributivity

M1 Use of both $E[X] = \mu$ & $E[X^2] = \sigma^2 + \mu^2$

A1 Answer Given

$$E[X] = \int_0^1 2x^2 dx = \left[\frac{2}{3}x^3 \right]_0^1 = \frac{2}{3} = \mu$$

B1

$$E[X^2] = \int_0^1 2x^3 dx = \left[\frac{1}{2}x^4 \right]_0^1 = \frac{1}{2} \Rightarrow \sigma^2 = \frac{1}{18}$$

B1

$$E[X^3] = \int_0^1 2x^4 dx = \left[\frac{2}{5}x^5 \right]_0^1 = \frac{2}{5}$$

B1

$$\Rightarrow \gamma = \frac{\frac{2}{5} - 3 \cdot \frac{2}{3} \cdot \frac{1}{18} - \frac{8}{27}}{\frac{1}{18\sqrt{18}}} = -\frac{2\sqrt{2}}{5}$$

M1 A1 Answer Given

$$F(x) = \int_0^x 2x dx = x^2 \quad (0 \leq x \leq 1)$$

B1

$$\Rightarrow F^{-1}(x) = \sqrt{x} \quad (0 \leq x \leq 1)$$

B1 B1

$$\Rightarrow D = \frac{F^{-1}\left(\frac{9}{10}\right) - 2F^{-1}\left(\frac{1}{2}\right) + F^{-1}\left(\frac{1}{10}\right)}{F^{-1}\left(\frac{9}{10}\right) - F^{-1}\left(\frac{1}{10}\right)} = \frac{\frac{3}{\sqrt{10}} - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}}}$$

M1 All correctly substituted

$$= \frac{3 - 2\sqrt{5} + 1}{3 - 1} = \frac{4 - 2\sqrt{5}}{2} = 2 - \sqrt{5}$$

A1 Answer Given

$$M \text{ is given by } \int_0^M 2x dx = \frac{1}{2}$$

M1

$$\Rightarrow M^2 = \frac{1}{2} \Rightarrow M = \frac{1}{\sqrt{2}}$$

A1 OR by $M = F^{-1}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$

$$\text{Then } P = \frac{3\left(\frac{2}{3} - \frac{1}{\sqrt{2}}\right)}{\frac{1}{3\sqrt{2}}} = 6\sqrt{2} - 9$$

B1

$$\text{RTP } D > P > \gamma \text{ i.e. } 2 - \sqrt{5} > 6\sqrt{2} - 9 > -\frac{2}{5}\sqrt{2} \quad \mathbf{B1 B1}$$

1st B1 for $D > P$; 2nd for $P > \gamma$

May use inequality arguments (e.g. squaring) or use of approximations to $\sqrt{2}$, $\sqrt{5}$ – be strict on improper rounding