

## NOTES

- 1) In the marking scheme there are three types of marks:

M marks are for method

A marks are for accuracy and are not given unless the relevant M mark has been given (M0 A1 is impossible!).

B marks are for a particular answer independent of method

- 2) Be especially careful where the answer is given in the question. \* by an answer indicates that it is given in the question.  
 3) Each question is followed by a summary showing for what each mark is awarded.

Q1

(i) Find the value of  $m$  for which the line  $y = mx$  touches the curve  $y = \ln x$ .

If instead the line intersects the curve when  $x = a$  and  $x = b$ , where  $a < b$ , show that  $a^b = b^a$ . Show by [7] means of a sketch that  $a < e < b$ .

The gradient of the line joining the point  $(t, \ln t)$  to the origin is  $\frac{\ln t}{t}$ . B1

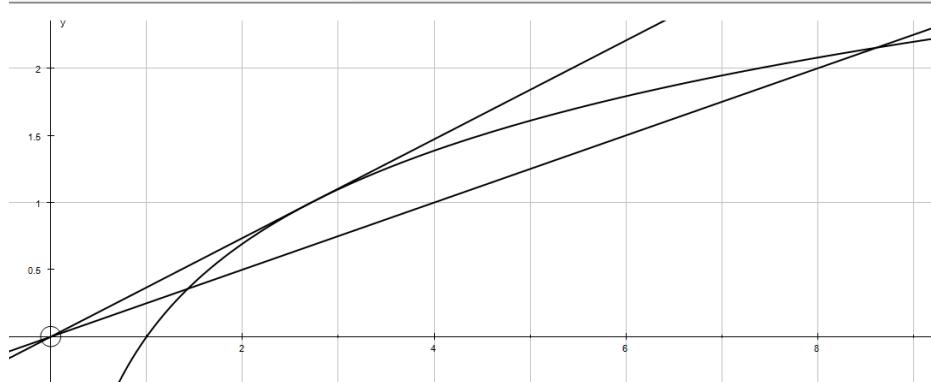
The gradient of the curve  $y = \ln x$  at the point  $(t, \ln t)$  is  $\frac{1}{t}$ . M1

Therefore  $t$  must satisfy  $\ln t = 1$  and so the point is  $(e, 1)$ . A1

The gradient is  $\frac{1}{e}$ . A1

$ma = \ln a$  and  $mb = \ln b$  M1

Therefore  $\frac{\ln a}{a} = \frac{\ln b}{b}$ , which leads to  $a^b = b^a$ . \* A1



The points of intersection are on different sides of the intersection with the tangent, so  $a < e < b$ . B1

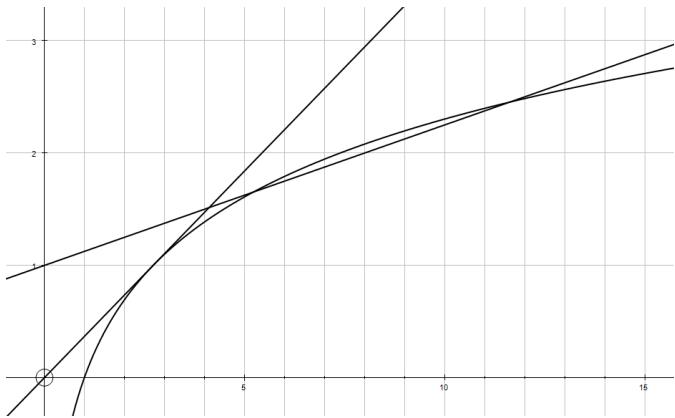
(ii) The line  $y = mx + c$ , where  $c > 0$ , intersects the curve  $y = \ln x$  when  $x = p$  and  $x = q$ , where  $p < q$ . Show by means of a sketch, or otherwise, that  $p^q > q^p$ . [3]

Considering gradients:

$$\frac{\ln p - c}{p} = \frac{\ln q - c}{q}$$

Therefore  $\ln p^q = \ln q^p + c(q - p)$  and so  $p^q > q^p$ . \* A1

(iii) Show by means of a sketch that the straight line through the points  $(p, \ln p)$  and  $(q, \ln q)$ , where  $e \leq p < q$ , intersects the y-axis at a positive value of  $y$ . Which is greater  $\pi^e$  or  $e^\pi$ ? [5]



B1  
B1

The gradient must be less than the gradient of the tangent.  
Therefore the point of intersection must be on the positive  $y$ -axis.

B1

So with  $p = e$  and  $q = \pi$  the conditions in part (ii) are satisfied  
Applying the result of part (ii)  $e^\pi > \pi^e$ .

B1  
B1

(iv) Show, using a sketch or otherwise, that if  $0 < p < q$  and  $\frac{\ln q - \ln p}{q-p} = e^{-1}$ , then  $q^p > p^q$ . [5]

$\frac{\ln q - \ln p}{q-p} = e^{-1}$  means that the line is parallel to the tangent through the origin.

Therefore the intersection is at a negative value of  $y$ .

Considering gradients as in part (ii) gives  $\ln p^q = \ln q^p + c(q - p)$ ,  
but  $c(q - p) < 0$ , so  $q^p > p^q$ . \*

[5]

B1

B1

M1

A1

A1

(i)	
B1	Identifying the gradients.
M1	Identification of the point at which the line meets the curve.
A1	Correct value for $m$ .
M1	Substitution into the equations.
A1	Correct solution ( <b>note that the answer is given in the question</b> ).
B1	Sketch showing the curve, the tangent through the origin and the other line which intersects the curve twice.
B1	Explanation of the required inequality.
(ii)	
M1	Consideration of gradients to establish equation.
A1	Correct equation.
A1	Justification that $p^q > q^p$ ( <b>note that the answer is given in the question</b> ).
(iii)	
B1	Sketch with an appropriate line and the tangent through the origin.
B1	Explanation that the gradient is less than that of the tangent.
B1	Justification that intersection is at a positive value of $y$ .
B1	Observation that the conditions in part (ii) are satisfied.
B1	Correct choice.
(iv)	
B1	Observation that this is the gradient of the line.
B1	Conclusion about position of intersection.
M1	Consideration of gradients to establish equation.
A1	Correct equation.
A1	Justification ( <b>note that the answer is given in the question</b> ).

Q2

(i) For  $n \geq 1$ , show by means of a substitution that  $\int_0^1 x^{n-1}(1-x)^n dx = \int_0^1 x^n(1-x)^{n-1} dx$  and deduce that  $2 \int_0^1 x^{n-1}(1-x)^n dx = I_{n-1}$ . [10]

Show also, for  $n \geq 1$ , that  $I_n = \frac{n}{n+1} \int_0^1 x^{n-1}(1-x)^{n+1} dx$  and hence that  $I_n = \frac{n}{2(2n+1)} I_{n-1}$ .

Let  $t = 1 - x$

$$\frac{dx}{dt} = -1$$

At  $x = 0, t = 1$ .

At  $x = 1, t = 0$ .

$$\text{Therefore } \int_0^1 x^{n-1}(1-x)^n dx = - \int_1^0 (1-t)^{n-1} t^n dt$$

$$\text{So } \int_0^1 x^{n-1}(1-x)^n dx = \int_0^1 x^n(1-x)^{n-1} dx *$$

$$\int_0^1 x^{n-1}(1-x)^n dx + \int_0^1 x^n(1-x)^{n-1} dx = \int_0^1 x^{n-1}(1-x)^{n-1}[(1-x) + x] dx,$$

$$\text{So } 2 \int_0^1 x^{n-1}(1-x)^n dx = I_{n-1}.$$

Integrating by parts:

$$u = x^n, \text{ so } \frac{du}{dx} = nx^{n-1}$$

$$\frac{dv}{dx} = (1-x)^n, \text{ so } v = -\frac{1}{n+1}(1-x)^{n+1}$$

$$\text{Therefore, } I_n = \left[ \frac{1}{n+1} x^n (1-x)^{n+1} \right]_0^1 + \frac{n}{n+1} \int_0^1 x^{n-1} (1-x)^{n+1} dx$$

$$I_n = \frac{n}{n+1} \int_0^1 x^{n-1} (1-x)^{n+1} dx *$$

$$\text{Therefore } I_n = \frac{n}{n+1} \left( \frac{1}{2} I_{n-1} - I_n \right)$$

$$\text{Which simplifies to } I_n = \frac{n}{2(2n+1)} I_{n-1}. *$$

(ii) When  $n$  is a positive integer, show that  $I_n = \frac{(n!)^2}{(2n+1)!}$  [4]

$$I_0 = 1$$

$$\text{Therefore } I_n = \frac{n}{2(2n+1)} \times \frac{n-1}{2(2n-1)} \times \frac{n-2}{2(2n-3)} \times \dots \times \frac{1}{2(3)}$$

Multiplying numerator and denominator by  $2^n n! = (2n)(2n-2)(2n-4)\dots(2)$  gives:

$$I_n = \frac{(n!)^2}{(2n+1)!} *$$

(iii) Use the substitution  $x = \sin^2 \theta$  to show that  $I_{\frac{1}{2}} = \frac{\pi}{8}$ , and evaluate  $I_{\frac{3}{2}}$  [6]

$$\frac{dx}{d\theta} = 2 \sin \theta \cos \theta$$

At  $x = 0, \theta = 0$ .

At  $x = 1, \theta = \frac{\pi}{2}$ .

$$\text{Therefore } I_{\frac{1}{2}} = \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta \times 2 \sin \theta \cos \theta d\theta$$

$$I_{\frac{1}{2}} = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta = \frac{1}{4} \int_0^{\frac{\pi}{2}} 1 - \cos 4\theta d\theta$$

$$\text{Which leads to } I_{\frac{1}{2}} = \frac{\pi}{8} *$$

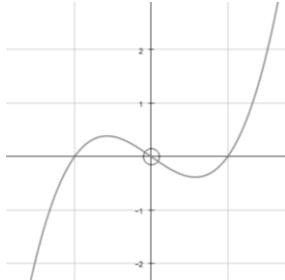
$$I_{\frac{3}{2}} = \frac{3/2}{2(4)} \times \frac{\pi}{8} = \frac{3\pi}{128}$$

(i)	
B1	Choice of an appropriate substitution.
A1	Differentiation.
A1	Correct substitution into the integral.
B1	A justification is needed as the answer is given in the question.
B1	Clear explanation as the answer is given in the question.
M1	Identification of split used for integration by parts.
A1	Differentiation and integration.
A1	Integration by parts ( <b>note that the answer is given in the question</b> ).
M1	Split the integral and write in terms of $I_n$ .
A1	Simplify ( <b>note that the answer is given in the question</b> ).
(ii)	
A1	Evaluation of $I_0$ .
M1	Writing down an expression for $I_n$ without reference to other cases of $I_k$ .
M1	Filling in the gaps in the product.
A1	Clear working to conclusion ( <b>note that the answer is given in the question</b> ).
(iii)	
B1	Differentiation.
A1	Correct substitution.
M1	Use of double angle formulae.
A1	Correct solution ( <b>note that the answer is given in the question</b> ).
M1	Substitution into formula from part (i).
A1	Correct answer.

Q3

(i) Given that the cubic equation  $x^3 + 3ax^2 + 3bx + c = 0$  has three distinct real roots and  $c < 0$ , show with the help of sketches that either exactly one of the roots is positive or all three of the roots are positive. [4]

To have three roots the shape of the graph must be (the axes can be translated):



B1

Since  $c < 0$  the curve intersects the  $y$ -axis below the  $x$ -axis,

So there are two possibilities for the position of the origin:

- The  $y$ -axis is to the left of the first root (from the left)
- The  $y$ -axis is between the second and third roots (from the left)

B1

B1

In the first case, all three roots are positive.

B1

In the second case, one root is positive and the other two are negative.

(ii) Given that the equation  $x^3 + 3ax^2 + 3bx + c = 0$  has three distinct real positive roots show that  $a^2 > b > 0$ ,  $a < 0$ ,  $c < 0$ . (\*) [6]

Differentiate to find the stationary points:

M1

$$3x^2 + 6ax + 3b = 0,$$

A1

There must be two distinct real solutions to this equation, so  $a^2 - b > 0$ , and so  $a^2 > b$ .

A1

If all roots are positive, then the turning points must both be at positive values of  $x$ :

M1

$$\text{Therefore } -a - \sqrt{a^2 - b} > 0.$$

$$\text{Therefore } a < 0,$$

A1

$$\sqrt{a^2 - b} < a, \text{ which means that } b > 0.$$

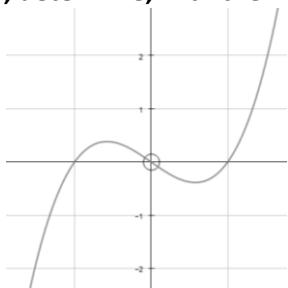
A1

$$\text{Since the product of the roots is positive, } c < 0.$$

A1

- (iii) Given that the equation  $x^3 + 3ax^2 + 3bx + c = 0$  has three distinct real roots and that  $ab < 0$ ,  $c > 0$ , determine, with the help of sketches, the signs of the roots.

[5]



If  $c > 0$ , then the  $y$ -axis must either pass between the first and second roots or to the right of the third root.

B1

From the reasoning in part (ii), the turning points will be on the same side of the  $y$ -axis if  $b > 0$  and on opposite sides if  $b < 0$ .

M1

A1

If the roots were all negative, then the turning points would both be at negative values of  $x$  and so both  $a$  and  $b$  would be positive, which would give  $ab > 0$

B1

Therefore there must be one negative and two positive roots.

A1

- (iv) Show, by means of an explicit example (giving values for  $a$ ,  $b$  and  $c$ ) that it is possible for the conditions of (\*) to be satisfied even though the corresponding cubic equation has only one real root.

[5]

Changing the value of  $c$  translates the graph vertically and if the graph is moves sufficiently far downwards there will only be one root.

B1

Choose values for  $a$  and  $b$  (for example,  $a = -2$  and  $b = 3$ ).

M1

Find the coordinates of the turning points:

A1

$-a \pm \sqrt{a^2 - b} = 2 \pm 1$ , so the turning points are at  $x = 1$  and  $x = 3$ .

A1

The first one must be the maximum and its  $y$ -coordinate is equal to  $4 + c$ .

A1

This point needs to be below the  $x$ -axis,  $c = -5$  will do.

A1

$a = -2, b = 3, c = -5$ .

(i)	<b>Note that the question requires the use of sketches, so no marks should be awarded to answers which are based on the product of the roots is negative.</b>
B1	Sketch of the shape of a cubic that would admit three distinct real roots with a positive coefficient of $x^3$ .
B1	Explanation that the $y$ -intercept must be below the $x$ -axis.
B1	Identification of the two cases possible.
B1	Matching of the two cases with the possibilities given in the question.
(ii)	
M1	Differentiation of the equation.
A1	Setting equal to 0 to find turning points.
A1	Explanation of $a^2 > b$ .
M1	Consideration that both turning points are at positive values of $x$ .
A1	Explanation of $a < 0$ and $b > 0$
A1	Explanation of $c < 0$ ( <b>note that some solutions may assume that this follows from part (i)</b> )
(iii)	
B1	Sketch of graph and explanation of where the $y$ -axis must be.
M1	Consideration of the positions of turning points.
A1	Explanation that the sign of $b$ determines whether the turning points are on the same side of the $y$ -axis or not.
B1	Consideration of the case with all negative roots.
A1	Correct identification of nature of roots.
(iv)	
B1	Appreciation that changing the value of $c$ can generate a solution from any case that satisfies (*)
M1	Use derivative to locate stationary points.
A1	Find values in chosen case.
A1	Find $y$ -coordinate of maximum point.
A1	Choose an appropriate value for $c$ .

Q4

- (i) Suppose  $b$  is fixed and positive. As  $a$  varies,  $M$  traces out a curve (the *locus* of  $M$ ). Show that  $x = -by$  on this curve. Given that  $a$  varies with  $-1 \leq a \leq 1$ , show that the locus is a line segment of length  $2b/(1+b^2)^{\frac{1}{2}}$ . Give a sketch showing the locus and the unit circle.

[9]

The equation of the line passing through the point  $(a, 0)$  with gradient  $b$  is  $y = bx - ab$ .

A1

The coordinates of  $P$  and  $Q$  therefore satisfy:

M1

$$x^2 + (bx - ab)^2 = 1,$$

A1

$$\text{which simplifies to } x^2(1+b^2) - 2ab^2x + a^2b^2 - 1 = 0.$$

The  $x$ -coordinate of  $P$  is therefore  $\frac{ab^2}{1+b^2}$

A1

And so (substituting into the equation of the line) the  $y$ -coordinate is  $-\frac{ab}{1+b^2}$

A1

Therefore  $x = -by$  on this curve. \*

A1

If  $b$  is fixed, the coordinates of  $P$  increase with the value of  $a$ , so the end-points of the line segment will be determined by the values when  $a = -1$  and  $a = 1$ .

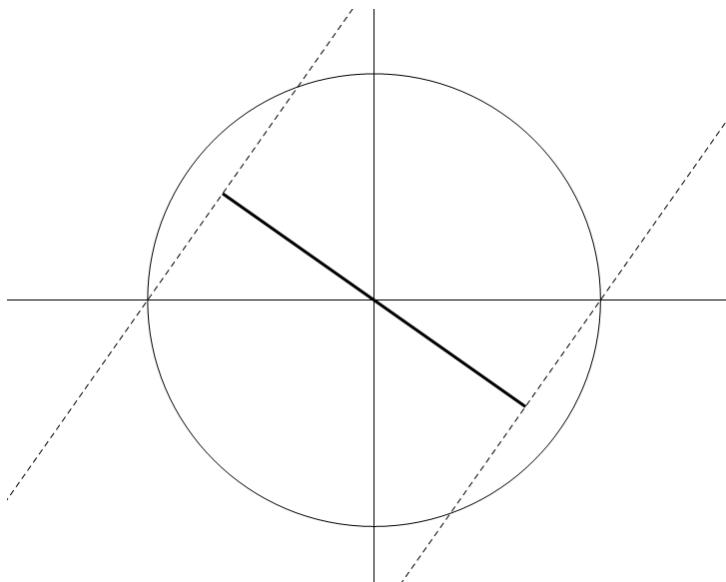
B1

The line segment goes from  $(\frac{-b^2}{1+b^2}, \frac{b}{1+b^2})$  to  $(\frac{b^2}{1+b^2}, \frac{-b}{1+b^2})$ .

M1

Therefore the length can be calculated as  $2b/(1+b^2)^{\frac{1}{2}}$ . \*

A1



B1

(ii) Find the locus of  $M$  in the following cases, giving in each case its cartesian equation, describing it geometrically and sketching it in relation to the unit circle:

(a)  $a$  is fixed with  $0 < a < 1$ , and  $b$  varies with  $-\infty < b < \infty$ ;

(b)  $ab = 1$ , and  $b$  varies with  $0 < b \leq 1$ .

[11]

If  $a$  is fixed then the coordinates of  $P$  satisfy  $x^2 + y^2 = ax$ ,

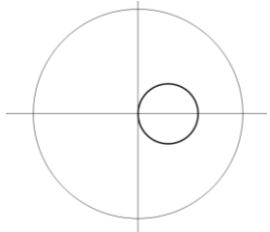
M1

which is the equation of a circle, centre  $(\frac{a}{2}, 0)$ , radius  $\frac{a}{2}$ .

A1

B1

B1



B1

If  $ab = 1$ , then the coordinates of  $P$  are  $(\frac{b}{1+b^2}, \frac{-1}{1+b^2})$

M1

Therefore the coordinates satisfy  $x^2 + y^2 = -y$ ,

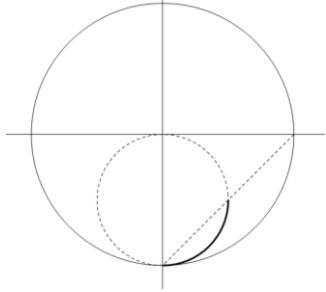
A1

which is the equation of a circle, centre  $(0, -\frac{1}{2})$ , radius  $\frac{1}{2}$ .

B1

Since  $0 < b \leq 1$ ,  $a \geq 1$  and so the locus is only the lower right quarter of the circle.

B1



B1

(i)	
A1	Equation of the straight line.
M1	Eliminating one variable from the equation of the circle.
A1	Simplifying to a quadratic equation.
A1	Finding the $x$ -coordinate of $P$ (or the $y$ -coordinate if $x$ was eliminated rather than $y$ ).
A1	Finding the $y$ -coordinate of $P$ and confirming the relationship ( <b>note that the relationship is given in the question.</b> )
B1	Finding the endpoints of the line segment.
M1	Using the formula for the length of a line segment.
A1	Simplifying to the final answer ( <b>note that the answer is given in the question</b> ).
B1	Parallel lines with locus perpendicular and passing through the centre of the circle.
(ii)	
M1	Attempt to eliminate $b$ from the parametric equations.
A1	Cartesian equation of circle.
B1	Description as circle.
B1	Correct centre and radius.
B1	Circle centred on positive $x$ -axis and passing through the origin.
M1	Attempt to eliminate $a$ and $b$ from the parametric equations.
A1	Cartesian equation of circle.
B1	Description as circle.
B1	Correct centre and radius.
B1	Explanation that the locus is only the lower right quarter of the circle.
B1	Correct circle and clear indication of the section that is the locus.

Q5

(i) A function  $f(x)$  satisfies  $f(x) = f(1-x)$  for all  $x$ . Show, by differentiating with respect to  $x$ , that $f'(\frac{1}{2}) = 0$ . If, in addition,  $f(x) = f(\frac{1}{x})$  for all (non-zero)  $x$ , show that  $f'(-1) = 0$  and  $f'(2) = 0$ .

[5]

Differentiating  $f(x) = f(1-x)$  with respect to  $x$ :

M1

$$f'(x) = -f'(1-x)$$

A1

$$\text{Substituting } x = \frac{1}{2} \text{ gives } f'(\frac{1}{2}) = 0. *$$

A1

Differentiating  $f(x) = f(\frac{1}{x})$  with respect to  $x$ :

M1

$$f'(x) = -\frac{1}{x^2} f'(\frac{1}{x})$$

A1

$$\text{Substituting } x = -1 \text{ gives } f'(-1) = 0. *$$

A1

$$\text{Substituting } x = 2 \text{ gives } f'(2) = \frac{1}{4} f'(\frac{1}{2}) = 0. *$$

A1

(ii) The function  $f$  is defined, for  $x \neq 0$  and  $x \neq 1$ , by  $f(x) = \frac{(x^2-x+1)^3}{(x^2-x)^2}$ .

[8]

Show that  $f(x) = f(\frac{1}{x})$  and  $f(x) = f(1-x)$ .Given that it has exactly three stationary points, sketch the curve  $y = f(x)$ .

M1

A1

$$f(\frac{1}{x}) = \frac{\left(\frac{1}{x^2}(1-x+x^2)\right)^3}{\left(\frac{1}{x^3}(x-x^2)\right)^2},$$

M1

A1

$$\text{so } f(\frac{1}{x}) = \frac{(1-x+x^2)^3}{(x-x^2)^2} = f(x) *$$

M1

A1

$$f(1-x) = \frac{((1-x)^2-(1-x)+1)^3}{((1-x)((1-x)-1))^2},$$

M1

A1

$$\text{so } f(1-x) = \frac{(x^2-x+1)^3}{(x^2-x)^2} = f(x) *$$

The three stationary points must be at  $x = -1$ ,  $x = \frac{1}{2}$  and  $x = 2$ .

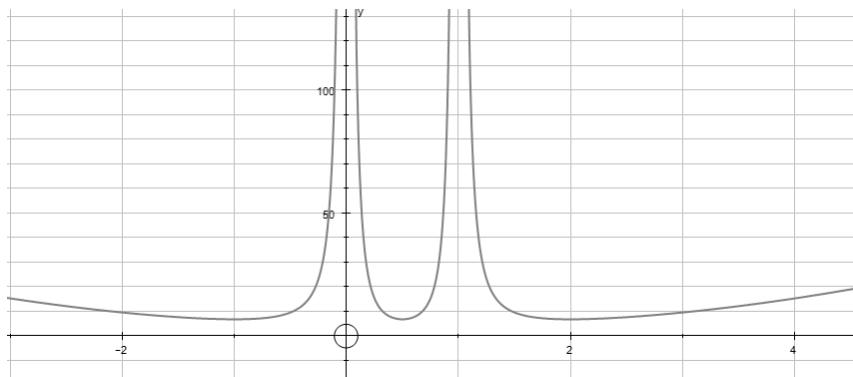
B1

At each of these points,  $f(x) = \frac{27}{4}$ .

A1

The curve has vertical asymptotes at  $x = 0$  and  $x = 1$ 

B1



B1

(iii) Hence, or otherwise, find all the roots of the equation  $f(x) = \frac{27}{4}$  and state the ranges of values for which  $f(x) > \frac{27}{4}$ .

Find also the roots of the equation  $f(x) = \frac{343}{36}$  and state the ranges of values for which  $f(x) > \frac{343}{36}$ .

	The roots are $x = -1$ , $x = \frac{1}{2}$ and $x = 2$ (as found above). It is clear from the graph that there are only three roots.	A1
	By considering the graph, the range of values is: $x < -1$ , $-1 < x < 0$ , $0 < x < \frac{1}{2}$ , $\frac{1}{2} < x < 1$ , $1 < x < 2$ and $x > 2$ .	A1
	By inspection $x = 3$ is a root of the equation $f(x) = \frac{343}{36}$ .	A1
	By considering the graph it is clear that there are six roots in total.	B1
	The other roots can be obtained by the properties of $f(x)$ proved in part (ii): $x = -2$ , $x = \frac{1}{3}$ .	A1
	$x = -\frac{1}{2}$ , $x = \frac{2}{3}$ , $x = \frac{3}{2}$ .	A1
	By considering the graph, the range of values is: $x < -2$ , $-\frac{1}{2} < x < 0$ , $0 < x < \frac{1}{3}$ , $\frac{2}{3} < x < 1$ , $1 < x < \frac{3}{2}$ and $x > 3$ .	A1

(i)	
M1	Application of the chain rule to differentiate $f(1 - x)$ .
A1	Substitution to obtain $f'(\frac{1}{2}) = 0$ ( <b>note that the answer is given in the question</b> ).
M1	Application of the chain rule to differentiate $f(\frac{1}{x})$ .
A1	Substitution to obtain $f'(-1) = 0$ ( <b>note that the answer is given in the question</b> ).
A1	Substitution and relation to first part to obtain $f'(2) = 0$ ( <b>note that the answer is given in the question</b> ).
(ii)	
M1	Substitution of $\frac{1}{x}$ for $x$ in the function $f$ .
A1	Clear simplification to obtain $f(\frac{1}{x}) = f(x)$ ( <b>note that the answer is given in the question</b> ).
M1	Substitution of $1 - x$ for $x$ in the function $f$ .
A1	Clear simplification to obtain $f(1 - x) = f(x)$ ( <b>note that the answer is given in the question</b> ).
B1	Correct stationary points on the graph.
A1	All three stationary points at the same height.
B1	Vertical asymptotes.
B1	Shape of the graph (line of symmetry at $x = \frac{1}{2}$ )
(iii)	
A1	The correct three roots.
A1	The correct set of ranges.
A1	Any one root found by inspection.
B1	Understanding that there will be six roots (award this mark if all six roots are found even if there is no explicit statement).
A1	The two roots found by applying the properties of $f$ directly to the first root found.
A1	The remaining three roots.
A1	The correct set of ranges found from the graph.

Q6

(i) Show that, for  $n \geq 3$ ,  $u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1+u_n)(1+u_{n-2})}$

$$u_n = 1 + \frac{1}{1 + \frac{1}{u_{n-2}}} = \frac{2u_{n-2} + 1}{1 + u_{n-2}} \text{ and similarly, } u_{n+2} = \frac{2u_n + 1}{1 + u_n}.$$

Therefore  $u_{n+2} - u_n = \frac{2u_n + 1}{1 + u_n} - \frac{2u_{n-2} + 1}{1 + u_{n-2}}$ , which simplifies to  $u_{n+2} - u_n = \frac{u_n - u_{n-2}}{(1+u_n)(1+u_{n-2})}$  \*

as required.

(ii) Prove, by induction or otherwise, that  $1 \leq u_n \leq 2$  for all  $n$ .

$u_1 = 1$ , so  $1 \leq u_1 \leq 2$ .

Assume that  $1 \leq u_k \leq 2$ :

$$\text{Then } u_{k+1} = 1 + \frac{1}{u_k}$$

$$\frac{1}{2} \leq \frac{1}{u_k} \leq 1, \text{ so } \frac{3}{2} \leq u_{k+1} \leq 2.$$

Therefore if  $1 \leq u_k \leq 2$ , then  $1 \leq u_{k+1} \leq 2$ .

By induction,  $1 \leq u_n \leq 2$  for all  $n$ .

(iii) Show that the sequence  $u_1, u_3, u_5, \dots$  tends to a limit, and that the sequence  $u_2, u_4, u_6, \dots$  tends to a limit. Find these limits and deduce that the sequence  $u_1, u_2, u_3, \dots$  tends to a limit.

Would this conclusion change if the sequence were defined by (\*) and  $u_1 = 3$ ?

$u_3 - u_1 > 0$  and so the result in part (i) shows (inductively) that

$u_{2k+1} - u_{2k-1} > 0$  for all values of  $k$ .

Therefore  $u_1, u_3, u_5, \dots$  is an increasing sequence.

Part (ii) shows that the sequence is bounded above.

Therefore the sequence tends to a limit.

Similarly:

$u_4 - u_2 < 0$  and so the result in part (i) shows (inductively) that

$u_{2k+2} - u_{2k} < 0$  for all values of  $k$ .

Therefore  $-u_2, -u_4, -u_6, \dots$  is an increasing sequence and is bounded above (by -1)

Therefore the sequence tends to a limit.

If the limit is  $L$  (for either sequence), then  $L = \frac{2L+1}{1+L}$

$$\text{Therefore } L = \frac{1+\sqrt{5}}{2}.$$

Both sequences converge to the same limit and therefore  $u_1, u_2, u_3, \dots$  also converges to this limit.

If  $u_1 = 3$ , then  $u_2 = \frac{4}{3}$ , which is between 1 and 2, so the arguments given will still hold.

[5]

M1

A1

A1

M1

A1

[4]

B1

M1

M1

A1

A1

[11]

A1

A1

B1

B1

B1

B1

M1

A1

B1

B1

B1

(i)	
M1	Expression of $u_n$ in terms of $u_{n-2}$ .
A1	Correct simplification.
A1	Equivalent expression found for $u_{n+2}$ .
M1	Expression for difference.
A1	Simplification to the required result ( <b>note that the answer is given in the question</b> ).
(ii)	
B1	Verify the first case.
M1	Find $u_{k+1}$ in terms of $u_k$ .
M1	Justify that $1 \leq u_{k+1} \leq 2$ .
A1	Conclusion.
(iii)	
M1	Considering the sign of $u_{2k+1} - u_{2k-1}$ .
A1	Concluding that the sequence is increasing.
B1	Observing that the sequence is bounded above.
B1	Concluding that the sequence converges to a limit.
A1	Concluding that it is a decreasing sequence.
B1	Concluding that the sequence converges to a limit.
M1	Appropriate method to find the limit.
A1	Correct value (justifying choice of positive root).
B1	Conclusion that sequence converges to a limit.
B1	Calculation of $u_2$ .
B1	Observation that all reasoning will still hold from this point.

Q7

- (i) Write down a solution of the equation  $x^2 - 2y^2 = 1$  (\*), for which  $x$  and  $y$  are non-negative integers. Show that, if  $x = p, y = q$  is a solution of (\*), then so also is  $x = 3p + 4q, y = 2p + 3q$ . Hence find two solutions of (\*) for which  $x$  is a positive odd integer and  $y$  is a positive even integer.

[7]

Any valid solution. e.g.  $x = 1, y = 0$ .

B1

$$(3p + 4q)^2 - 2(2p + 3q)^2 \equiv (9p^2 + 24pq + 16q^2) - 2(4p^2 + 12pq + 9q^2)$$

M1

M1

A1

Which simplifies to  $p^2 - 2q^2$ .

B1

$$\text{Therefore, if } p^2 - 2q^2 = 1, (3p + 4q)^2 - 2(2p + 3q)^2 = 1$$

So if  $x = p, y = q$  is a solution of (\*), then so also is  $x = 3p + 4q, y = 2p + 3q$ .

B1

Applying this to the initial solution gives the further solutions:

B1

$$x = 3, y = 2$$

B1

$$x = 17, y = 12$$

- (ii) Show that, if  $x$  is an odd integer and  $y$  is an even integer, (\*) can be written in the form

[3]

$$n^2 = \frac{1}{2}m(m+1), \text{ where } m \text{ and } n \text{ are integers.}$$

If  $x$  is written as  $2m + 1$  and  $y$  is written as  $2n$ , then (\*) becomes:

B1

$$(2m+1)^2 - 2(2n)^2 = 1$$

M1

$$8n^2 = (2m+1)^2 - 1 \text{ which simplifies to } n^2 = \frac{1}{2}m(m+1).$$

A1

- (iii) The positive integers  $a, b$  and  $c$  satisfy  $b^3 = c^4 - a^4$ , where  $b$  is a prime number. Express  $a$  and  $c^2$  in terms of  $b$  in the two cases that arise. Find a solution of  $b^3 = c^4 - a^4$ , where  $a, b$  and  $c$  are positive integers but  $b$  is not prime.

[10]

$$b^3 = (c^2 - a)(c^2 + a).$$

A1

Since  $b$  is prime,  $b^3$  can only be written as a product of 2 factors in 2 ways:  $(1 \times b^3)$  and  $(b \times b^2)$

B1

B1

In the first case:

M1

$$a = \frac{b^3 - 1}{2}, c^2 = \frac{b^3 + 1}{2}$$

A1

In the second case:

M1

$$a = \frac{b(b-1)}{2}, c^2 = \frac{b(b+1)}{2}$$

A1

From the solution  $x = 17, y = 12$ , the solution  $a = 28, b = 8, c = 6$  can be obtained.

M1

M1

A1

(i)	
B1	Any solution of the equation with non-negative integers.
M1	Substitution of $x = 3p + 4q$ , $y = 2p + 3q$ into the equation.
M1	Correct expansions.
A1	Simplification.
B1	Convincing explanation of the result.
B1	Solution found by applying the result just shown to initial solution (or the initial solution itself if it satisfied the conditions of this part of the question).
B1	Solution found by applying the result just shown to the solution above.
(ii)	
B1	Expression of $x$ as $2m + 1$ and $y$ as $2n$ .
M1	Substitution into the equation.
A1	Simplification ( <b>note that the final equation is given in the question</b> ).
(iii)	
A1	For factorising the RHS.
B1	For recognising one of the possible pairs.
B1	For recognising the other pair.
M1	Generating the simultaneous equations in the first case.
A1	Solving for $a$ and $c^2$
M1	Generating the simultaneous equations in the second case.
A1	Solving for $a$ and $c^2$
M1	For recognition that the solution to part (ii) provides possible values of $b$ and $c$ .
M1	For applying the solution to part (i) to find values for $b$ and $c$ .
A1	For finding the value of $a$ . ( <b>note that a solution that satisfies the criteria should be awarded all three marks regardless of whether it follows from earlier answers to the question</b> ).

Q8

- (i) For  $t \geq 0$ , let  $A_0(t)$  be the area of the largest rectangle with sides parallel to the coordinate axes that can fit in the region bounded by the curve  $y = f(x)$ , the  $y$ -axis and the line  $y = f(t)$ . Show that  $A_0(t)$  can be written in the form  $A_0(t) = x_0(f(x_0) - f(t))$ , where  $x_0$  satisfies  $x_0f'(x_0) + f(x_0) = f(t)$ .

[4]

For a given value of  $t$ , the largest rectangle with a width of  $x$  will have a height of  $f(x) - f(t)$  and so its area will be  $x(f(x) - f(t))$ .

B1

Differentiate to find maximum:

M1

$$f(x) - f(t) + xf'(x) = 0$$

A1

Therefore  $A_0(t)$  can be written in the form  $A_0(t) = x_0(f(x_0) - f(t))$ , where  $x_0$  satisfies  $x_0f'(x_0) + f(x_0) = f(t)$ . \*

A1

- (ii) The function  $g$  is defined, for  $t > 0$ , by  $g(t) = \frac{1}{t} \int_0^t f(x) dx$ .

Show that  $tg'(t) = f(t) - g(t)$ .

[7]

Making use of a sketch show that, for  $t > 0$ ,  $\int_0^t (f(x) - f(t)) dx > A_0(t)$

and deduce that  $-t^2 g'(t) > A_0(t)$ .

$$tg(t) = \int_0^t f(x) dx.$$

M1

Therefore (by differentiating),  $tg'(t) + g(t) = f(t)$ .

A1

$$\text{So } tg'(t) = f(t) - g(t).$$

Sketch showing the area represented by  $\int_0^t (f(x) - f(t)) dx$  – the region bounded by the curve

B1

$y = f(x)$ , the  $y$ -axis and the line  $y = f(t)$ .

An indication that the rectangle with area  $A_0(t)$  is contained within this region.

B1

Evaluation of the LHS of the inequality as  $tg(t) - tf(t)$ .

A1

Factorisation as  $-t(f(t) - g(t))$  and substitution of the first result from this section to obtain  
 $-t^2 g'(t) > A_0(t)$ . \*

M1

A1

- (iii) In the case  $f(x) = \frac{1}{1+x}$ , use the above to establish the inequality  $\ln \sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}}$  for  $t > 0$ .

[9]

$$g(t) = \frac{1}{t} \int_0^t \frac{1}{1+x} dx = \frac{1}{t} [\ln|1+x|]_0^t,$$

M1

$$\text{So } g(t) = \frac{1}{t} \ln(1+t).$$

A1

$$g'(t) = \frac{1}{t(1+t)} - \frac{1}{t^2} \ln(1+t).$$

M1

A1

$$x_0 \text{ satisfies } -\frac{x_0}{(1+x_0)^2} + \frac{1}{1+x_0} = \frac{1}{1+t},$$

M1

$$\text{So } x_0 = -1 + \sqrt{1+t}. \text{ (since } x_0 > 0\text{)}$$

A1

$$\text{And } A_0(t) = (-1 + \sqrt{1+t}) \left( \frac{1}{\sqrt{1+t}} - \frac{1}{1+t} \right) = 1 - \frac{2}{\sqrt{1+t}} + \frac{1}{1+t}$$

M1

A1

$$\text{Therefore } -\frac{t}{(1+t)} + \ln(1+t) > 1 - \frac{2}{\sqrt{1+t}} + \frac{1}{1+t}$$

A1

$$\ln \sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}} *$$

(i)	
B1	For finding the area of the largest rectangle for a particular width.
M1	Differentiation of the formula for the area.
A1	Correct derivative.
A1	Convincing explanation of how this leads to the required result ( <b>note that the result is given in the question</b> ).
(ii)	
M1	Differentiation of a formula including $g(t)$ .
A1	Correct rearrangement ( <b>note that the answer is given in the question</b> ).
B1	Graph showing the required region.
B1	Clear indication that the rectangle with area $A_0(t)$ is contained in this region.
A1	Correct evaluation of the integration.
M1	Simplification of the resulting inequality.
A1	Clear justification of the inequality ( <b>note that the inequality is given in the question</b> ).
(iii)	
M1	Integration of $f(x)$ .
A1	Expression for $g(t)$ .
M1	Differentiation of $g(t)$ .
A1	Correct expression.
M1	Process to find $A_0(t)$ .
A1	Correct value of $x_0$
M1	Evaluation of $A_0(t)$
A1	Simplification of expression found.
A1	Substitution into inequality and simplification to give required result ( <b>note that the final inequality is given in the question</b> ).

Q9

- (i) Show that the normal reaction between the horizontal surface and a disc in contact with the surface is  $\frac{3}{2}W$ . [3]

The reaction force between a disc and the horizontal surface is  $R_S$ .

By symmetry, the reactions for the two points of contact with the horizontal surface are equal.

Resolving vertically:

$$2R_S = 3W, \text{ so } R_S = \frac{3}{2}W. *$$

- (ii) Find the normal reaction between two discs in contact and show that the magnitude of the frictional force between two discs in contact is  $\frac{W \sin \theta}{2(1+\cos \theta)}$ . [10]

The frictional force between a disc and the horizontal surface is  $F_S$ . The frictional force between two discs is  $F_D$ .

Taking moments about the centre of a lower disc:

$$F_S = F_D.$$

The reaction force between two discs is  $R_D$ .

Resolving horizontally for a lower disc:

$$F_S + F_D \cos \theta = R_D \sin \theta.$$

Resolving vertically for the upper disc:

$$W = 2R_D \cos \theta + 2F_D \sin \theta.$$

Eliminating the frictional forces:

$$W = 2R_D \cos \theta + \frac{2R_D \sin^2 \theta}{1+\cos \theta}.$$

$$R_D = \frac{1}{2}W.$$

Substituting into one of the equations gives:

$$F_S = \frac{W \sin \theta}{2(1+\cos \theta)}. *$$

- (iii) Show that if  $\mu < 2 - \sqrt{3}$  there is no value of  $\theta$  for which equilibrium is possible. [7]

Since all frictional forces are equal and  $R_D < R_S$ , slipping will occur first between the two discs.

Therefore equilibrium is only possible if  $\frac{W \sin \theta}{2(1+\cos \theta)} \leq \frac{1}{2}\mu W$ .

$$\frac{\sin \theta}{1+\cos \theta} \leq \mu.$$

As  $\theta$  increases,  $\sin \theta$  increases and  $1 + \cos \theta$  decreases, so  $\frac{\sin \theta}{1+\cos \theta}$  increases.

Since the lower two discs cannot overlap,  $\theta \geq \frac{\mu}{6}$ .

The smallest possible value of  $\frac{\sin \theta}{1+\cos \theta}$  is therefore  $\frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}$ .

There will be no value of  $\theta$  for which equilibrium is possible if  $\mu < 2 - \sqrt{3}$ . \*

B1

M1

A1

M1

A1

M1

A1

M1

A1

M1

A1

M1

A1

[7]

B1

M1

A1

B1

M1

A1

B1

B1

(i)	
B1	Use of symmetry to identify that the two reactions are equal.
M1	Resolving vertically.
A1	Establishing that $R_S = \frac{3}{2}W$ ( <b>note that the answer is given in the question</b> ).
(ii)	
M1	Applying moments about the centre of a lower disc.
A1	Establishing that all frictional forces are equal.
M1	Resolving forces horizontally for the lower disc.
A1	Correct equation.
M1	Resolving forces vertically for the upper disc.
A1	Correct equation.
M1	Elimination of the frictional forces from the equations.
A1	Correct value for $R_D$ .
M1	Substitution into one of the equations.
A1	Correct working to the value of the frictional force ( <b>note that the answer is given in the question</b> ).
(iii)	
B1	Observation that slipping will occur first between the discs.
M1	Using the limiting value of friction to establish an inequality / critical value.
A1	Correct inequality.
B1	Establishing that $\frac{\sin \theta}{1+\cos \theta}$ is an increasing function.
M1	Identifying that there is a minimum value of $\theta$ defined by the geometry of the situation.
A1	Correct evaluation of this value of $\theta$ .
B1	Calculation of minimum value for $\theta$ and statement of the conclusion ( <b>note that the critical value is given in the question</b> ).

Q10

(i) Show that, at a general point on the trajectory,  $2\tan\theta = \tan\alpha + \tan\phi$ .

[8]

If the point of projection has coordinates  $(0, 0)$  and the general point has coordinates  $(x, y)$ , then:  
 $x = k\cos\theta$  and  $y = k\sin\theta$ , for some positive value of  $k$ .

B1

Assume that the initial velocity was  $u$  and the velocity at the general point is  $v$ .

Vertically:

M1

$$k\sin\theta = utsin\alpha - \frac{1}{2}gt^2$$

A1

$$vsin\phi = usin\alpha - gt$$

A1

Horizontally:

B1

$$k\cos\theta = ut\cos\alpha$$

B1

$$vcos\phi = u\cos\alpha$$

Eliminating  $t$ :

M1

$$k\sin\theta = u\left(\frac{k\cos\theta}{u\cos\alpha}\right)\sin\alpha - \frac{1}{2}\left(\frac{k\cos\theta}{u\cos\alpha}\right)(usin\alpha - vsin\phi)$$

Simplifying:

A1

$$\tan\theta = \tan\alpha - \frac{1}{2}(\tan\alpha - \tan\phi)$$

A1

$$2\tan\theta = \tan\alpha + \tan\phi *$$

(ii) Show that, if  $B$  and  $C$  are the same point, then  $\alpha = 60^\circ$ .

[4]

If  $\theta = \frac{1}{2}\alpha$  and  $\phi = -\frac{1}{2}\alpha$  then:

A1

$$2\tan\left(\frac{1}{2}\alpha\right) = \tan\alpha - \tan\left(\frac{1}{2}\alpha\right), \text{ so } \tan\alpha = 3\tan\left(\frac{1}{2}\alpha\right).$$

M1

$$\text{Therefore } \frac{2\tan\left(\frac{1}{2}\alpha\right)}{1-\tan^2\left(\frac{1}{2}\alpha\right)} = 3\tan\left(\frac{1}{2}\alpha\right)$$

A1

$$\text{Since } \tan\left(\frac{1}{2}\alpha\right) \neq 0, 3\tan^2\left(\frac{1}{2}\alpha\right) = 1$$

M1

$$\text{Therefore } \tan\left(\frac{1}{2}\alpha\right) = \frac{\sqrt{3}}{3}, \text{ which leads to } \alpha = 60^\circ. *$$

A1

(iii) Given that  $\alpha < 60^\circ$ , determine whether the particle reaches the point  $B$  first or the point  $C$  first.

If  $t_B$  is the time that the particle reaches  $B$ :

M1

$$\tan\left(\frac{1}{2}\alpha\right) = \frac{u\sin\alpha - \frac{1}{2}gt_B}{u\cos\alpha} = \tan\alpha - \frac{gt_B}{2u\cos\alpha} \text{ (using the equations from part (i))}$$

A1

If  $t_C$  is the time that the particle reaches  $C$ :

M1

$$-\tan\left(\frac{1}{2}\alpha\right) = \frac{u\sin\alpha - gt_C}{u\cos\alpha} = \tan\alpha - \frac{gt_C}{u\cos\alpha} \text{ (using the equations from part (i))}$$

A1

Therefore, subtracting the second equation from twice the first gives:

M1

$$3\tan\left(\frac{1}{2}\alpha\right) = \tan\alpha + \frac{g}{u\cos\alpha}(t_C - t_B)$$

A1

$$\frac{g}{u\cos\alpha} > 0, \text{ so } t_C > t_B \text{ when } 3\tan\left(\frac{1}{2}\alpha\right) > \tan\alpha \text{ and } t_C < t_B \text{ when } 3\tan\left(\frac{1}{2}\alpha\right) < \tan\alpha$$

M1

If  $\alpha < 60^\circ$ ,  $3\tan\left(\frac{1}{2}\alpha\right) > \tan\alpha$  so  $t_C > t_B$ , meaning that  $B$  is reached first.

A1

(i)	
B1	For identifying the different variables that need to be considered (the position of the particle and its velocity at the general point).
M1	Application of the formulae for uniform motion in the vertical direction.
A1	Correct formula relating to the displacement.
A1	Correct formula relating to the velocity.
B1	Correct statement of horizontal displacement.
B1	Correct statement of horizontal velocity.
M1	Elimination of $t$ from the equations.
A1	Correct justification ( <b>note that the answer is given in the question</b> ).
(ii)	
A1	Simplification of the result from part (i) in this case.
M1	Use of formula for $\tan 2A$ .
A1	Simplification of equation.
A1	Solution leading to $\alpha = 60^\circ$ ( <b>note that the answer is given in the question</b> ).
(iii)	
M1	Use of the equations from part (i) to find an expression for $t_B$ .
A1	Simplification of equation.
M1	Use of the equations from part (i) to find an expression for $t_C$ .
A1	Simplification of equation.
M1	Linear combination of simultaneous equations to find $t_C - t_B$ (or attempt to solve the simultaneous equations).
A1	Correct expression for $t_C - t_B$ .
M1	Consideration of the value of $t_C - t_B$ in the case $\alpha < 60^\circ$ .
A1	Conclusion that $B$ is reached first.

Q11

- (i) Show that, after the second collision, the speeds of the particles are  $\frac{1}{2}u(1-e)$ ,  $\frac{1}{4}u(1-e^2)$  and  $\frac{1}{4}u(1+e)^2$ . Deduce that there will be a third collision whatever the value of  $e$ .

[11]

First collision:

$$\text{Conservation of momentum: } mu = mv_1 + mv_2 \quad \text{B1}$$

$$\text{Law of Restitution: } v_2 - v_1 = e(u - 0) \quad \text{B1}$$

Solving simultaneously:

$$v_1 = \frac{1}{2}u(1-e) * \quad \text{M1}$$

$$v_2 = \frac{1}{2}u(1+e) \quad \text{A1}$$

Second collision:

$$\text{Conservation of momentum: } \frac{1}{2}mu(1+e) = mv'_2 + mv'_3 \quad \text{B1}$$

$$\text{Law of Restitution: } v'_3 - v'_2 = e(\frac{1}{2}u(1+e) - 0) \quad \text{B1}$$

Solving simultaneously:

$$v'_2 = \frac{1}{4}u(1-e^2) * \quad \text{M1}$$

$$v'_3 = \frac{1}{4}u(1+e)^2 * \quad \text{A1}$$

A third collision will occur if  $v_1 > v'_2$ .

$$v_1 - v'_2 = \frac{1}{4}u(2 - 2e - (1 - e^2)) \quad \text{M1}$$

$$v_1 - v'_2 = \frac{1}{4}u(1 - e)^2 > 0 \text{ for all values of } e \quad \text{A1}$$

- (ii) Show that there will be a fourth collision if and only if  $e$  is less than a particular value which you should determine.

[9]

Third collision:

$$\text{Conservation of momentum: } \frac{1}{2}mu(1-e) + \frac{1}{4}mu(1-e^2) = mv''_1 + mv''_2 \quad \text{B1}$$

$$\text{Law of Restitution: } v''_2 - v''_1 = e(\frac{1}{2}u(1-e) - \frac{1}{4}u(1-e^2)) = \frac{1}{4}eu(1-e)^2 \quad \text{B1}$$

Solving simultaneously:

$$v''_2 = \frac{1}{8}u(3-e)(1-e)(1+e) \quad \text{B1}$$

B1

For a fourth collision, we require  $v''_2 > v'_3$

$$\frac{1}{8}u(3-e)(1-e)(1+e) - \frac{1}{4}u(1+e)^2 = \frac{1}{8}u(1+e)(3-4e+e^2-2-2e) \quad \text{M1}$$

$$\frac{1}{8}u(1+e)(e^2-6e+1) > 0, \text{ so since } e > 0 \text{ we require } e^2-6e+1 > 0 \quad \text{A1}$$

$$e < 3 - 2\sqrt{2} \quad \text{M1}$$

A1

(i)	
B1	For correct statement of momentum before the first collision. (Accept $u = v_1 + v_2$ )
B1	For applying law of restitution.
M1	For attempting to solve the equations.
A1	For correctly finding $v_1$ and $v_2$ . ( <b>note that the value of <math>v_1</math> is given in the question</b> )
B1	For correct statement of momentum before the second collision. (Accept $\frac{1}{2}u(1 + e) = v'_2 + v'_3$ )
B1	For applying law of restitution.
M1	For attempting to solve the equations.
A1	For correctly finding $v'_2$ and $v'_3$ . ( <b>note that these values are given in the question</b> )
B1	For observing that a third collision will require $v_1 > v'_2$ .
M1	For simplifying the expression for $v_1 - v'_2$ .
A1	For a justification that $v_1 > v'_2$
(ii)	
B1	For correct statement of momentum before the third collision.
B1	For applying law of restitution.
A1	For finding a correct expression for $v''_2$ (note that it is not necessary to find the value of $v''_1$ ).
B1	For factorising part of the expression for $v''_2$ (for example leaving it as a difference of two factorised parts).
B1	For fully factorising the expression for $v''_2$ .
	Note that these two marks are to simplify the next part of the calculation – a correct final answer should be awarded these marks if the factorisation was not done and the more complicated algebra was followed through.
M1	For establishing the correct inequality.
A1	For simplifying it to the quadratic inequality to be solved.
M1	For applying an appropriate method to solving their inequality.
A1	For the correct solution.

Q12

(i) Find  $E(X)$  and  $E(Y)$  in terms of  $\lambda$ ,  $\alpha$  and  $\beta$  where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots \text{ and } \beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots$$

[5]

$$E(X) = \sum_{r=1}^n (2r-1) \frac{e^{-\lambda} \lambda^{2r-1}}{(2r-1)!}$$

M1

$$E(X) = \lambda e^{-\lambda} \sum_{r=1}^n \frac{\lambda^{2r-2}}{(2r-2)!} = \alpha \lambda e^{-\lambda}$$

M1

$$E(Y) = \sum_{r=1}^n (2r) \frac{e^{-\lambda} \lambda^{2r}}{(2r)!}$$

M1

$$E(Y) = \lambda e^{-\lambda} \sum_{r=1}^n \frac{\lambda^{2r-1}}{(2r-1)!} = \beta \lambda e^{-\lambda}$$

A1

(ii) Show that  $Var(X) = \frac{\lambda\alpha+\lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$  and obtain the corresponding expression for  $Var(Y)$ . Are there any non-zero values of  $\lambda$  for which  $Var(X) + Var(Y) = Var(X + Y)$ ?

[15]

$$e^{-\lambda} = \frac{1}{\alpha+\beta}$$

B1

$$E(X^2) = \sum_{r=1}^n (2r-1)^2 \frac{e^{-\lambda} \lambda^{2r-1}}{(2r-1)!}$$

M1

$$E(X^2) = \lambda e^{-\lambda} \sum_{r=1}^n (2r-1) \frac{\lambda^{2r-2}}{(2r-2)!}$$

M1

$$E(X^2) = \lambda^2 e^{-\lambda} \sum_{r=2}^n \frac{\lambda^{2r-3}}{(2r-3)!} + \lambda e^{-\lambda} \sum_{r=1}^n \frac{\lambda^{2r-2}}{(2r-2)!}$$

M1

$$E(X^2) = \beta \lambda^2 e^{-\lambda} + \alpha \lambda e^{-\lambda}$$

A1

$$Var(X) = \beta \lambda^2 e^{-\lambda} + \alpha \lambda e^{-\lambda} - (\alpha \lambda e^{-\lambda})^2$$

M1

$$Var(X) = \frac{\lambda\alpha+\lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} *$$

A1

$$E(Y^2) = \sum_{r=1}^n (2r)^2 \frac{e^{-\lambda} \lambda^{2r}}{(2r)!}$$

M1

$$E(Y^2) = \lambda e^{-\lambda} \sum_{r=1}^n (2r) \frac{\lambda^{2r-1}}{(2r-1)!}$$

M1

$$E(Y^2) = \lambda^2 e^{-\lambda} \sum_{r=1}^n \frac{\lambda^{2r-2}}{(2r-2)!} + \lambda e^{-\lambda} \sum_{r=1}^n \frac{\lambda^{2r-1}}{(2r-1)!}$$

M1

$$E(Y^2) = \alpha \lambda^2 e^{-\lambda} + \beta \lambda e^{-\lambda}$$

A1

$$Var(Y) = \alpha \lambda^2 e^{-\lambda} + \beta \lambda e^{-\lambda} - (\beta \lambda e^{-\lambda})^2$$

M1

$$Var(Y) = \frac{\lambda\beta+\lambda^2\alpha}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2}$$

A1

 $Var(X + Y)$  will be the same as  $Var(U) = \lambda$ 

B1

$$Var(X) + Var(Y) = \frac{\lambda\alpha+\lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} + \frac{\lambda\beta+\lambda^2\alpha}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2}$$

M1

$$Var(X) + Var(Y) = \lambda + \lambda^2 \left( 1 - \frac{\alpha^2 + \beta^2}{(\alpha+\beta)^2} \right)$$

A1

These can only be equal if one of  $\alpha$  and  $\beta$  is zero. Since  $\alpha > 0$  and  $\beta > 0$ ,  $Var(X) + Var(Y)$  cannot equal  $Var(X + Y)$ 

B1

(i)	
M1	For a correct statement of the sum required to calculate either $E(X)$ or $E(Y)$
M1	For extracting the factor of $\lambda e^{-\lambda}$ from the sum in the calculation for $E(X)$ .
A1	For the value of $E(X)$ . It is not necessary to write $e^{-\lambda}$ as $\frac{1}{\alpha+\beta}$ for this mark.
M1	For extracting the factor of $\lambda e^{-\lambda}$ from the sum in the calculation for $E(Y)$ .
A1	For the value of $E(Y)$ . It is not necessary to write $e^{-\lambda}$ as $\frac{1}{\alpha+\beta}$ for this mark.
(ii)	
B1	For recognising that $e^{-\lambda} = \frac{1}{\alpha+\beta}$ . This mark could be awarded at any point in the question (including in part (a)).
M1	For a correct statement of the sum required to calculate $E(X^2)$ .
M1	For separating out into two sums that can be expressed in terms of $\alpha$ and $\beta$ .
A1	For the correct value of $E(X^2)$ .
M1	For substituting their values of $E(X)$ and $E(X^2)$ into the formula for $Var(X)$ .
A1	For obtaining the value of $Var(X)$ ( <b>note that the answer is given in the question</b> ).
M1	For a correct statement of the sum required to calculate $E(Y^2)$ .
M1	For separating out into two sums that can be expressed in terms of $\alpha$ and $\beta$ .
A1	For the correct value of $E(Y^2)$ .
M1	For substituting their values of $E(Y)$ and $E(Y^2)$ into the formula for $Var(Y)$ .
A1	For obtaining the value of $Var(Y)$ .
B1	For realising that the variance of $X + Y$ will be the same as the variance of $U$ .
M1	For calculating $Var(X) + Var(Y)$ in terms of $\lambda$ , $\alpha$ and $\beta$ .
A1	For simplifying the expression for $Var(X) + Var(Y)$ .
B1	For explaining why $Var(X) + Var(Y)$ cannot equal $Var(X + Y)$ .

Q13

(i) Explain why  $P(A = 1) = p^2 + q^2$  and find  $P(S = 1)$ . Show that  $P(S = 1) < P(A = 1)$ . [5]

For  $A = 1$  the first two tosses must have the same result (HH or TT) B1

$$\text{Therefore } P(A = 1) = p^2 + q^2 *$$

For  $S = 1$  the first two tosses must have different results (HT or TH) M1

$$\text{Therefore } P(S = 1) = pq + qp = 2pq \text{ A1}$$

Since  $p \neq q$ ,  $(p - q)^2 > 0$ . M1

$$p^2 - 2pq + q^2 > 0$$

$$p^2 + q^2 > 2pq$$

$P(A = 1) > P(S = 1)$  as required. A1

(ii) Show that  $P(S = 2) = P(A = 2)$  and determine the relationship between  $P(S = 3)$  and  $P(A = 3)$ . [6]

For  $S = 2$  the two possibilities are HHT and TTH. B1

For  $A = 2$  the two possibilities are HTT and THH. B1

Since  $P(HHT) = P(THH)$  and  $P(TTH) = P(HTT)$ ,  $P(S = 2) = P(A = 2)$ . B1

For  $S = 3$  the two possibilities are HHHT and TTTT. B1

For  $A = 3$  the two possibilities are HTHH and THTT. B1

Since  $P(HHHT) = P(HTHH)$  and  $P(TTTT) = P(THTT)$ ,  $P(S = 3) = P(A = 3)$ . B1

(iii) Show that, for  $n > 1$ ,  $P(S = 2n) > P(A = 2n)$  and determine the corresponding relationship between  $P(S = 2n + 1)$  and  $P(A = 2n + 1)$ . [9]

For  $S = 2n$  the two possibilities are HH...HT and TT...TH.

For  $A = 2n$  the two possibilities are (HT)(HT)...(HT)T and (TH)(TH)...(TH)H.

$$P(S = 2n) = p^{2n}q + pq^{2n}$$

$$P(A = 2n) = p^{n+1}q^n + p^nq^{n+1}$$

$$\text{Therefore } P(S = 2n) - P(A = 2n) = pq(p^n - q^n)(p^{n-1} - q^{n-1})$$

Either  $(p^n - q^n) > 0$  and  $(p^{n-1} - q^{n-1}) > 0$

or  $(p^n - q^n) < 0$  and  $(p^{n-1} - q^{n-1}) < 0$ .

Therefore  $P(S = 2n) - P(A = 2n) > 0$  and so  $P(S = 2n) > P(A = 2n)$  A1

For  $S = 2n + 1$  the two possibilities are HH...HT and TT...TH.

For  $A = 2n + 1$  the two possibilities are (HT)(HT)...(HT)HH and (TH)(TH)...(TH)TT.

$$P(S = 2n + 1) = p^{2n+1}q + pq^{2n+1}$$

$$P(A = 2n + 1) = p^{n+2}q^n + p^nq^{n+2}$$

$$\text{Therefore } P(S = 2n + 1) - P(A = 2n + 1) = pq(p^{n+1} - q^{n+1})(p^{n-1} - q^{n-1})$$

Either  $(p^{n+1} - q^{n+1}) > 0$  and  $(p^{n-1} - q^{n-1}) > 0$

or  $(p^{n+1} - q^{n+1}) < 0$  and  $(p^{n-1} - q^{n-1}) < 0$ .

Therefore  $P(S = 2n + 1) - P(A = 2n + 1) > 0$  and so  $P(S = 2n + 1) > P(A = 2n + 1)$  A1

(i)	
B1	For identifying the possible sequences that result in $A = 1$ and confirming that $P(A = 1) = p^2 + q^2$ .
M1	For identifying the possible sequences that result in $S = 1$ .
A1	For correctly calculating $P(S = 1)$ .
M1	For selecting an appropriate approach to showing that $P(A = 1) > P(S = 1)$ .
A1	For a clear justification that $P(A = 1) > P(S = 1)$ .
(ii)	
B1	For identifying the possible sequences that result in $S = 2$ .
B1	For identifying the possible sequences that result in $A = 2$ .
B1	For explaining that the probabilities are equal (calculation of the probabilities is not required)
B1	For identifying the possible sequences that result in $S = 3$ .
B1	For identifying the possible sequences that result in $A = 3$ .
B1	For explaining that the probabilities are equal (calculation of the probabilities is not required)
M1	For identifying the sequence resulting in $S = 2n$ or $A = 2n$ .
A1	For correct calculation of $P(S = 2n)$ .
A1	For correct calculation of $P(A = 2n)$ .
M1	For considering the difference of their $P(S = 2n)$ and their $P(A = 2n)$ .
A1	For justifying the inequality.
M1	For identifying the sequence resulting in $S = 2n + 1$ or $A = 2n + 1$ .
A1	For correct calculation of $P(S = 2n + 1)$ .
A1	For correct calculation of $P(A = 2n + 1)$ .
A1	For justifying the inequality.