Homework 2

习题4

4

> Note
>
> 设$\mathbf{W} \sim W_p(n,\mathbf{\Sigma})$,并令$\mathbf{W}=(w_{ij})$和$\mathbf{\Sigma}=(\mathbf{\Sigma}_{ij})$,其中$i,j=1,\cdots,p$.
> (1)试证明:$w_{ii} \sim \mathbf{\Sigma}_{ii}\chi^2_n$,$i=1,\cdots,p$;
> (2)试计算:$E(w_{ij})$和$\text{Cov}(w_{ij},w_{kl})$.提示:$\text{Cov}(w_{ij},w_{kl})=n(\mathbf{\Sigma}_{ik}\mathbf{\Sigma}_{jl}+\mathbf{\Sigma}_{il}\mathbf{\Sigma}_{jk})$.

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\begin{aligned}
&(1)\textit{Proof.} \text{ 令}\mathbf{I}_n=(e_1,\cdots,e_n),\text{由性质4.2.2可知，}e_i'\mathbf{W}e_i \sim W_1(n,e_i'\mathbf{\Sigma}e_i),\\
&\quad\text{又}w_{ii}=e_i'\mathbf{W}e_i,\mathbf{\Sigma}_{ii}=e_i'\mathbf{\Sigma}e_i,\text{则}w_{ii}\sim W_1(n,\mathbf{\Sigma}_{ii})\sim\mathbf{\Sigma}_{ii}\chi^2_n.\\
&(2)\textit{Sol.}\ E(w_{ij})=E(\mathbf{W})_{ij}=E(\textstyle\sum_{l=1}^n X_lX_l')_{ij}=n\cdot E(X_iX_i')_{ij}=n\cdot\text{Cov}(X_i)_{ij}=n\cdot\mathbf{\Sigma}_{ij}=n\cdot\mathbf{\Sigma}_{ij}.\\
&\quad\text{注意到}\text{Cov}(w_{ij},w_{kl})=\text{Cov}(\text{Vec}(\mathbf{W}))_{i+p(j-1),k+p(l-1)},(\mathbf{\Sigma}\otimes\mathbf{\Sigma})_{i+p(j-1),k+p(l-1)}\\
&\quad=(\mathbf{\Sigma}_{jl}\mathbf{\Sigma})_{ik}=\mathbf{\Sigma}_{ik}\mathbf{\Sigma}_{jl},(\mathbf{K}_p(\mathbf{\Sigma}\otimes\mathbf{\Sigma}))_{i+p(j-1),k+p(l-1)}=(\textstyle\sum_{t=1}^p\mathbf{E}_{tj}\cdot\mathbf{\Sigma}_{tl}\mathbf{\Sigma})_{ik}(\text{注意}t=j\text{时不为}0)\\
&\quad=\mathbf{\Sigma}_{il}(\mathbf{E}_{ij}\mathbf{\Sigma})_{ik}=\mathbf{\Sigma}_{il}\mathbf{\Sigma}_{jk},\text{其中}\mathbf{E}_{ij}=e_ie_j'.\\
&\quad\text{因此只需证明}\text{Cov}(\text{Vec}(\mathbf{W}))=n(\mathbf{I}_{p^2}+\mathbf{K}_p)(\mathbf{\Sigma}\otimes\mathbf{\Sigma})\text{即可,其中}\mathbf{K}_p=\textstyle\sum_{i=1}^p\sum_{j=1}^p(\mathbf{E}_{ij}\otimes\mathbf{E}_{ij}')\\
&\quad\textit{Lemma}1:\text{令}u\sim N_p(0,\mathbf{I}_p)\text{，则}E(uu'\otimes uu')=\mathbf{I}_{p^2}+\mathbf{K}_p+\text{Vec}(\mathbf{I})\text{Vec}(\mathbf{I})',E(u\otimes u)=\text{Vec}(\mathbf{I}_p),\\
&\qquad\qquad\text{Cov}(u\otimes u)=\mathbf{I}_{p^2}+\mathbf{K}_p.\\
&\quad\textit{Proof}:\text{令}\mathbf{T}_{ij}=\mathbf{E}_{ij}+\mathbf{E}_{ji},\text{已知}E(u_i)=E(u_i^3)=E(u_i^5)=0,E(u_i^2)=1\text{，}E(u_i^4)=3\text{，}E(u_i^6)=15.\\
&\qquad\text{则}E(u_iu_juu')=\mathbf{T}_ij+\delta_{ij}\mathbf{I}_p,\text{其中}\delta_{ij}\text{为}\mathbf{Kronecker}\text{记号}.\\
&\qquad\text{则}E(uu'\otimes uu')=\textstyle\sum_{ij}(\mathbf{E}_{ij}\otimes(t_{ij}+\delta_{ij}\mathbf{I}_p))=\sum_{ij}(\mathbf{E}_{ij}\otimes\mathbf{T}_{ij})+\sum_{ij}(\delta_{ij}\mathbf{E}_{ij}\otimes\mathbf{I}_p)\\
&\qquad=\textstyle\sum_{ij}(\mathbf{E}_{ij}\otimes\mathbf{E}_{ij})+\sum_{ij}(\mathbf{E}_{ij}\otimes\mathbf{E}_{ji})+\sum_{ii}(\delta_{ii}\mathbf{E}_{ii}\otimes\mathbf{I}_p).\\
&\qquad\text{其中},\textstyle\sum_{ij}(\mathbf{E}_{ij}\otimes\mathbf{E}_{ij})=\sum_{ij}(\text{Vec}(\mathbf{E}_{ii})\text{Vec}(\mathbf{E}_{jj})')=\text{Vec}(\mathbf{I}_p)\text{Vec}(\mathbf{I}_p)',\\
&\qquad\text{由定义},\textstyle\sum_{ij}(\mathbf{E}_{ij}\otimes\mathbf{E}_{ji})=\mathbf{K}_p,\sum_{ii}(\delta_{ii}\mathbf{E}_{ii}\otimes\mathbf{I}_p)=\mathbf{I}_p\otimes\mathbf{I}_p=\mathbf{I}_{p^2};\\
&\qquad E(u\otimes u)=E(\text{Vec}(uu\prime))=\text{Vec}(E(uu\prime))=\text{Vec}(\mathbf{I}_p);\\
&\qquad\text{Cov}(u\otimes u)=E(uu'\otimes uu')-\text{Vec}(\mathbf{I}_p)\text{Vec}(\mathbf{I}_p)'\\
&\qquad=\mathbf{I}_{p^2}+\mathbf{K}_p+\text{Vec}(\mathbf{I}_p)\text{Vec}(\mathbf{I}_p)'-\text{Vec}(\mathbf{I}_p)\text{Vec}(\mathbf{I}_p)'=\mathbf{I}_{p^2}+\mathbf{K}_p.\\
&\quad\text{Cov}(\text{Vec}(\mathbf{W}))=\text{Cov}(\text{Vec}(\textstyle\sum_{i=1}^n X_iX_i'))=\sum_{i=1}^n\text{Cov}(\text{Vec}(X_iX_i'))=\sum_{i=1}^n\text{Cov}(X_i\otimes X_i)\\
&\quad=n\text{Cov}((\mathbf{\Sigma}^{\frac12}u)\otimes(\mathbf{\Sigma}^{\frac12}u))=n\text{Cov}((\mathbf{\Sigma}^{\frac12}\otimes\mathbf{\Sigma}^{\frac12})(u\otimes u))\\
&\quad=n(\mathbf{\Sigma}^{\frac12}\otimes\mathbf{\Sigma}^{\frac12})\text{Cov}(u\otimes u)(\mathbf{\Sigma}^{\frac12}\otimes\mathbf{\Sigma}^{\frac12})'=n(\mathbf{\Sigma}^{\frac12}\otimes\mathbf{\Sigma}^{\frac12})(\mathbf{I}_{p^2}+\mathbf{K}_p)(\mathbf{\Sigma}^{\frac12}\otimes\mathbf{\Sigma}^{\frac12})'\\
&\quad=n(\mathbf{I}_{p^2}+\mathbf{K}_p)(\mathbf{\Sigma}^{\frac12}\otimes\mathbf{\Sigma}^{\frac12})(\mathbf{\Sigma}^{\frac12}\otimes\mathbf{\Sigma}^{\frac12})=n(\mathbf{I}_{p^2}+\mathbf{K}_p)(\mathbf{\Sigma}\otimes\mathbf{\Sigma}).
\end{aligned}
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5

> Note
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> (1)设$\mathbf{W},x_1,\cdots,x_m$相互独立,$\mathbf{W}\sim W_p(n,\mathbf{I}_p),x_i\sim N_p(0,\mathbf{I}_p)$,其中$i=1,\cdots,m.n\geqslant p$.试求$\mathbf{W}^{-1/2}\mathbf{X}'$的密度函数,其中$\mathbf{X}'=(x_1,\cdots,x_m)$;
> (2)设$\mathbf{W}_1$和$\mathbf{W}_2$相互独立,且$\mathbf{W}_1\sim W_p(n,\mathbf{I}_p)$和$\mathbf{W}_2\sim W_p(m,\mathbf{I}_p)$,其中$n,m\geqslant p$.试求$\mathbf{W}_1^{-1/2}\mathbf{W}_2\mathbf{W}_1^{-1/2}$的密度函数.

$Sol.(1)\mathbf{X}'\sim N_{p\times m}(0,\mathbf{I}_p\otimes\mathbf{I}_m)$, 令$\mathbf{Y}=\mathbf{W}^{-1/2}\mathbf{X}'$, 由$\mathbf{X}'$与$\mathbf{W}$独立, $F(\mathbf{X}',\mathbf{W})=f(\mathbf{X}')g(\mathbf{W})$, 其中

$$f(\mathbf{X}')=\frac{1}{(2\pi)^{mp/2}}\exp\left\{-\frac{1}{2}\mathrm{tr}(\mathbf{X}\mathbf{X}')\right\},g(\mathbf{W})=\frac{|\mathbf{W}|^{(n-p-1)/2}}{2^{np/2}\Gamma_p(\frac{n}{2})}\exp\left\{-\frac{1}{2}\mathrm{tr}(\mathbf{W})\right\}.$$

则$F(\mathbf{X}',\mathbf{W})=\dfrac{|\mathbf{W}|^{(n-p-1)/2}}{(2\pi)^{mp/2}2^{np/2}\Gamma_p(\frac{n}{2})}\exp\left\{-\dfrac{1}{2}\mathrm{tr}(\mathbf{W}+\mathbf{X}\mathbf{X}')\right\}.$

作变换$\mathbf{Y}=\mathbf{W}^{-1/2}\mathbf{X}'$, 则$\mathbf{J}(\mathbf{X}'\to\mathbf{Y})=|\mathbf{W}|^{\frac{m}{2}}$,

$$G(\mathbf{Y},\mathbf{W})=F(\mathbf{X}',\mathbf{W})\mathbf{J}(\mathbf{X}',\mathbf{W}\to\mathbf{Y},\mathbf{W})=F(\mathbf{X}',\mathbf{W})\mathbf{J}(\mathbf{X}'\to\mathbf{Y})$$

$$=\frac{|\mathbf{W}|^{(m+n-p-1)/2}}{\pi^{mp/2}2^{(m+n)p/2}\Gamma_p(\frac{n}{2})}\exp\left\{-\frac{1}{2}\mathrm{tr}(\mathbf{W}(\mathbf{I}_p+\mathbf{Y}'\mathbf{Y}))\right\},$$

$$h(\mathbf{Y})=\int_{\mathbf{W}>0}G(\mathbf{Y},\mathbf{W})\mathrm{d}\mathbf{W}=\frac{1}{\pi^{mp/2}2^{(m+n)p/2}\Gamma_p(\frac{n}{2})}\int_{\mathbf{W}>0}|\mathbf{W}|^{\frac{m+n-p-1}{2}}\mathrm{etr}\left\{-\mathbf{W}\frac{\mathbf{I}_p+\mathbf{Y}\mathbf{Y}'}{2}\right\}\mathrm{d}\mathbf{W}$$

(注 : $\mathrm{etr}(\mathbf{A})=\exp\{\mathrm{tr}(\mathbf{A})\}$, 多元$\Gamma$函数定义为$\Gamma_p(\alpha)=\displaystyle\int_{\mathbf{A}>0}|\mathbf{A}|^{\alpha-\frac{p+1}{2}}\mathrm{etr}(-\mathbf{A})\mathrm{d}\mathbf{A}$,

令$\mathbf{A}=\left(\dfrac{\mathbf{I}_p+\mathbf{Y}\mathbf{Y}'}{2}\right)^{'\frac{1}{2}}\mathbf{W}\left(\dfrac{\mathbf{I}_p+\mathbf{Y}\mathbf{Y}'}{2}\right)^{\frac{1}{2}}$, 则$\mathbf{J}(\mathbf{W}\to\mathbf{A})=|\dfrac{\mathbf{I}_p+\mathbf{Y}\mathbf{Y}'}{2}|^{-\frac{p+1}{2}})$

$$=\frac{1}{\pi^{\frac{mp}{2}}2^{\frac{(m+n)p}{2}}\Gamma_p(\frac{n}{2})}|\frac{\mathbf{I}_p+\mathbf{Y}\mathbf{Y}'}{2}|^{-\frac{m+n-p-1}{2}}\cdot|\frac{\mathbf{I}_p+\mathbf{Y}\mathbf{Y}'}{2}|^{-\frac{p+1}{2}}\int_{\mathbf{A}>0}|\mathbf{A}|^{\frac{m+n}{2}-\frac{p+1}{2}}\mathrm{etr}(-\mathbf{A})\mathrm{d}\mathbf{A}$$

$$=\frac{\Gamma_p(\frac{m+n}{2})}{\pi^{\frac{mp}{2}}\Gamma_p(\frac{n}{2})}\cdot|\mathbf{I}_p+\mathbf{Y}\mathbf{Y}'|^{-\frac{m+n}{2}}.\ This\ is\ a\ Matrix\ t-distribution.$$

$$(2)H(\mathbf{W}_1,\mathbf{W}_2)=g(\mathbf{W}_1)g(\mathbf{W}_2)=\frac{|\mathbf{W}_1|^{(n-p-1)/2}|\mathbf{W}_2|^{(m-p-1)/2}}{2^{(m+n)p/2}\Gamma_p(\frac{n}{2})\Gamma_p(\frac{m}{2})}\mathrm{etr}\left\{-\frac{1}{2}(\mathbf{W}_1+\mathbf{W}_2)\right\},$$

作变换$\mathbf{Z}=\mathbf{W}_1^{-1/2}\mathbf{W}_2\mathbf{W}_1^{-1/2}$, 则$\mathbf{J}(\mathbf{W}_2\to\mathbf{Z})=|\mathbf{W}_1|^{p+1/2}$,

$$\mathbf{Q}(\mathbf{W}_1,\mathbf{Z})=\frac{|\mathbf{W}_1|^{(n-p-1)/2}\cdot|\mathbf{W}_1^{1/2}\mathbf{Z}\mathbf{W}_1^{1/2}|^{(m-p-1)/2}}{2^{(m+n)p/2}\Gamma_p(\frac{n}{2})\Gamma_p(\frac{m}{2})}\mathrm{etr}\left\{-\frac{1}{2}(\mathbf{W}_1+\mathbf{W}_1^{1/2}\mathbf{Z}\mathbf{W}_1^{1/2})\right\}|\mathbf{W}_1|^{p+1/2}$$

$$=\frac{|\mathbf{W}_1|^{(m+n-p-1)/2}\cdot|\mathbf{Z}|^{(m-p-1)/2}}{2^{(m+n)p/2}\Gamma_p(\frac{n}{2})\Gamma_p(\frac{m}{2})}\mathrm{etr}\left\{-\mathbf{W}_1\frac{(\mathbf{I}_p+\mathbf{Z})}{2}\right\},$$

故$q(\mathbf{Z})=\displaystyle\int_{W_1>0}\mathbf{Q}(\mathbf{W}_1,\mathbf{Z})\mathrm{d}\mathbf{W}_1=\frac{\Gamma_p(\frac{m+n}{2})}{\Gamma_p(\frac{n}{2})\Gamma_p(\frac{m}{2})}\cdot\frac{|\mathbf{Z}|^{(m-p-1)/2}}{|\mathbf{I}_p+\mathbf{Z}|^{\frac{m+n}{2}}}$(积分形式同(1)).

This is a Matrix F − distribution.

6

Note

设$\mathbf{W}\sim W_p(n,\mathbf{I}_p)$, $\mathbf{W}=(w_{ij}),r_{ij}=w_{ij}/\sqrt{w_{ii}w_{jj}},\mathbf{R}=(r_{ij})_{p\times p}$.

(1)试证明:$w_{11},\cdots,w_{pp},\mathbf{R}$相互独立;

(2)试证明:$w_{11},\cdots,w_{pp}$相互独立同χ^2_n分布;

(3)试求$\mathbf{R}$的分布.

$Proof.$已知$f(\mathbf{W})=\frac{1}{2^{np/2}\Gamma_p(n/2)}|\mathbf{W}|^{\frac{n-p-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}(\mathbf{W})\right\}$, 令$\mathbf{W}$的对角元$u=(w_{11},\cdots,w_{pp})'$,

$\mathbf{W}$的上三角元排列为$w=(w_{12},\cdots,w_{1p},w_{23},\cdots,w_{2p},\cdots,w_{p-1,p})'$, $\mathbf{U}=\mathrm{diag}(w_{11},\cdots,w_{pp})$.

则$g(\mathbf{R},\mathbf{D})=f(\mathbf{D}^{\frac{1}{2}}\mathbf{R}\mathbf{D}^{\frac{1}{2}})\cdot\mathbf{J}(\mathbf{W}\to\mathbf{R},\mathbf{D})$, 其中

$$\mathbf{J}(\mathbf{W}\to\mathbf{R},\mathbf{D})=\mathbf{J}(\mathbf{W},\mathbf{D}\to\mathbf{R},\mathbf{D})=\mathbf{J}(w,u\to r,u)=\frac{\partial(w,u)'}{\partial(r',u')'}=\begin{vmatrix}\frac{\partial w'}{\partial r} & \frac{\partial u'}{\partial r}\\ \frac{\partial w}{\partial u} & \frac{\partial u}{\partial u}\end{vmatrix}$$

$=\begin{vmatrix}\frac{\partial w'}{\partial r} & 0\\ 0 & \mathbf{I}_p\end{vmatrix}$, 注意其中$w_{ij}=\sqrt{w_{ii}w_{jj}}r_{ij}$, 当$i<j$,

则$|\mathbf{J}(\mathbf{W}\to\mathbf{R},\mathbf{D})|=\prod_{i<j}\sqrt{w_{ii}w_{jj}}=\prod_{i=1}^{p}w_{ii}^{\frac{p-1}{2}}=|\mathbf{D}|^{\frac{p-1}{2}}$, 从而

$$g(\mathbf{R},\mathbf{D})=\frac{1}{2^{np/2}\Gamma_p(n/2)}|\mathbf{D}^{\frac{1}{2}}\mathbf{R}\mathbf{D}^{\frac{1}{2}}|^{\frac{n-p-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}(\mathbf{D}^{\frac{1}{2}}\mathbf{R}\mathbf{D}^{\frac{1}{2}})\right\}\cdot|\mathbf{D}|^{\frac{p-1}{2}}$$

$$=\frac{1}{2^{np/2}\Gamma_p(n/2)}|\mathbf{D}|^{\frac{n}{2}-1}|\mathbf{R}|^{\frac{n-p-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}(\mathbf{D})\right\}(\text{注意}r_{ii}=1,i=1,\cdots,p)$$

$$=\frac{1}{2^{np/2}\Gamma_p(n/2)}|\mathbf{R}|^{\frac{n-p-1}{2}}\prod_{i=1}^{p}w_{ii}^{n/2-1}\exp\left\{-\frac{1}{2}\sum_{i=1}^{p}w_{ii}\right\}$$

$$=\frac{(\Gamma(\frac{n}{2}))^p}{\Gamma_p(n/2)}|\mathbf{R}|^{\frac{n-p-1}{2}}\cdot\prod_{i=1}^{p}\left(\frac{w_{ii}^{n/2-1}}{2^{n/2}}\exp\left\{-\frac{1}{2}w_{ii}\right\}\right)$$

(1)(2)(3)由联合分布函数可知, $w_{11},\cdots,w_{pp},\mathbf{R}$相互独立,

且$w_{ii}\sim\frac{w_{ii}^{n/2-1}}{2^{n/2}}\exp\left\{-\frac{1}{2}w_{ii}\right\}\sim\chi^2_n, i=1,\cdots,p, \mathbf{R}\sim\frac{(\Gamma(\frac{n}{2}))^p}{\Gamma_p(n/2)}|\mathbf{R}|^{\frac{n-p-1}{2}}$.

10

Note

设$\mathbf{W}_i$相互独立,且$\mathbf{W}_i\sim W_p(n_i,\mathbf{I}_p)$,其中$n_i\geqslant p, i=0,1,\cdots,k$.

(1)令$\mathbf{M}_j=\left(\sum_{i=1}^{k}\mathbf{W}_i\right)^{-1/2}\mathbf{W}_j\left(\sum_{i=1}^{k}\mathbf{W}_i\right)^{-1/2}, j=1,\cdots,k$.试求$(\mathbf{M}_1,\cdots,\mathbf{M}_k)$的联合密度函数;

(2)令$\mathbf{V}_j=\mathbf{W}_0^{-1/2}\mathbf{W}_j\mathbf{W}_0^{-1/2}, j=1,\cdots,k$.试求$(\mathbf{V}_1,\cdots,\mathbf{V}_k)$的联合密度函数.

$$Sol.\ (1)\text{由独立性}, f(\mathbf{W}_1,\cdots,\mathbf{W}_k)=\frac{\prod\limits_{i=1}^{k}|\mathbf{W}_i|^{(n_i-p-1)/2}}{2^{\frac{p}{2}\sum\limits_{i=1}^{k}n_i}\cdot\prod\limits_{i=1}^{k}\Gamma_p(\frac{n_i}{2})}\mathrm{etr}\left\{-\frac{1}{2}\sum_{i=1}^{k}\mathbf{W}_i\right\},$$

$$\text{作变换}, \mathbf{W}=\sum_{i=1}^{k}\mathbf{W}_i, \mathbf{W}_i=\mathbf{W}^{\frac{1}{2}}\mathbf{M}_i\mathbf{W}^{\frac{1}{2}}, i=1,\cdots,k-1.$$

$$\mathbf{J}(\mathbf{W}_1,\cdots,\mathbf{W}_k\to\mathbf{M}_1,\cdots,\mathbf{M}_{k-1},\mathbf{W})=|\mathbf{W}|^{(p+1)(k-1)/2}$$

$$g(\mathbf{M}_1,\cdots,\mathbf{M}_{k-1},\mathbf{W})=f(\mathbf{W}_1,\cdots,\mathbf{W}_k)\mathbf{J}(\mathbf{W}_1,\cdots,\mathbf{W}_k\to\mathbf{M}_1,\cdots,\mathbf{M}_{k-1},\mathbf{W})$$

$$=\frac{\prod\limits_{i=1}^{k-1}|\mathbf{W}^{\frac{1}{2}}\mathbf{M}_i\mathbf{W}^{\frac{1}{2}}|^{(n_i-p-1)/2}\cdot|\mathbf{W}-\sum\limits_{i=1}^{k-1}\mathbf{W}_i|^{\frac{n_k-p-1}{2}}\cdot|\mathbf{W}|^{\frac{(p+1)(k-1)}{2}}}{2^{\frac{p}{2}\sum\limits_{i=1}^{k}n_i}\cdot\prod\limits_{i=1}^{k}\Gamma_p(\frac{n_i}{2})}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{W}\right\}$$

$$=\frac{\prod\limits_{i=1}^{k-1}|\mathbf{M}_i|^{(n_i-p-1)/2}|\mathbf{I}_p-\sum\limits_{i=1}^{k-1}\mathbf{M}_i|^{(n_k-p-1)/2}|\mathbf{W}|^{\frac{\sum_{i=1}^{k}n_i-(p+1)}{2}}}{2^{\frac{p}{2}\sum\limits_{i=1}^{k}n_i}\cdot\prod\limits_{i=1}^{k}\Gamma_p(\frac{n_i}{2})}\mathrm{etr}\left\{-\frac{1}{2}\mathbf{W}\right\}$$

$$h(\mathbf{M}_1,\cdots,\mathbf{M}_{k-1})=\int\limits_{\mathbf{W}>0}g(\mathbf{M}_1,\cdots,\mathbf{M}_{k-1},\mathbf{W})\mathrm{d}\mathbf{W}$$

$$=\frac{\prod\limits_{i=1}^{k-1}|\mathbf{M}_i|^{(n_i-p-1)/2}|\mathbf{I}_p-\sum\limits_{i=1}^{k-1}\mathbf{M}_i|^{(n_k-p-1)/2}\Gamma_p(\sum\limits_{i=1}^{k}\frac{n_i}{2})}{\prod\limits_{i=1}^{k}\Gamma_p(\frac{n_i}{2})}(\text{注意}\sum_{i=1}^{k}\mathbf{M}_i=\mathbf{I}_p)$$

$$\text{从而}l(\mathbf{M}_1,\cdots,\mathbf{M}_k)=\frac{\prod\limits_{i=1}^{k}|\mathbf{M}_i|^{(n_i-p-1)/2}\Gamma_p(\sum\limits_{i=1}^{k}\frac{n_i}{2})}{\prod\limits_{i=1}^{k}\Gamma_p(\frac{n_i}{2})}.\textit{This is a Dirichlet type }\mathbf{I}\textit{ distribution.}$$

$$(2)F(\mathbf{W}_0,\cdots,\mathbf{W}_k)=\frac{\prod\limits_{i=0}^{k}|\mathbf{W}_i|^{(n_i-p-1)/2}}{2^{\frac{p}{2}\sum\limits_{i=0}^{k}n_i}\cdot\prod\limits_{i=0}^{k}\Gamma_p(\frac{n_i}{2})}\mathrm{etr}\left\{-\frac{1}{2}\sum_{i=0}^{k}\mathbf{W}_i\right\}$$

$$\text{作变换}\mathbf{W}_j=\mathbf{W}_0^{\frac{1}{2}}\mathbf{V}_j\mathbf{W}_0^{\frac{1}{2}},\text{则}\mathbf{J}(\mathbf{W}_0,\cdots,\mathbf{W}_k\to\mathbf{W}_0,\mathbf{V}_1,\cdots,\mathbf{V}_k)=|\mathbf{W}_0|^{\frac{k(p+1)}{2}}.$$

$$G(\mathbf{W}_0,\mathbf{V}_1,\cdots,\mathbf{V}_k)=\frac{\prod\limits_{i=0}^{k}|\mathbf{W}_0^{\frac{1}{2}}\mathbf{V}_j\mathbf{W}_0^{\frac{1}{2}}|^{\frac{n_i-p-1}{2}}|\mathbf{W}_0|^{\frac{k(p+1)}{2}}}{2^{\frac{p}{2}\sum\limits_{i=0}^{k}n_i}\cdot\prod\limits_{i=0}^{k}\Gamma_p(\frac{n_i}{2})}\mathrm{etr}\left\{-\frac{1}{2}\left(\sum_{i=1}^{k}\mathbf{W}_0^{\frac{1}{2}}\mathbf{V}_i\mathbf{W}_0^{\frac{1}{2}}+\mathbf{W}_0\right)\right\}$$

$$=\frac{\prod\limits_{i=0}^{k}|\mathbf{V}_j|^{(n_i-p-1)/2}|\mathbf{W}_0|^{\frac{\sum_{i=0}^{k}n_i-(p+1)(k+1)}{2}}}{2^{\frac{p}{2}\sum\limits_{i=0}^{k}n_i}\cdot\prod\limits_{i=0}^{k}\Gamma_p(\frac{n_i}{2})}\mathrm{etr}\left\{-\frac{1}{2}(\mathbf{I}_p+\sum_{i=0}^{k}\mathbf{V}_i)\mathbf{W}_0\right\}$$

$$H(\mathbf{V}_1,\cdots,\mathbf{V}_k)=\int\limits_{\mathbf{W}_0>0}G(\mathbf{W}_0,\mathbf{V}_1,\cdots,\mathbf{V}_k)\mathrm{d}\mathbf{W}(\text{法同}5(1))$$

$$=\frac{\prod\limits_{i=1}^{k}|\mathbf{V}_i|^{\frac{n_i-p-1}{2}}\Gamma_p\left(\sum\limits_{i=0}^{k}n_i\right)}{\prod\limits_{i=0}^{k}\Gamma_p(\frac{n_i}{2})}|\mathbf{I}_p+\sum_{i=1}^{k}\mathbf{V}_i|^{-\frac{\sum_{i=0}^{k}n_i}{2}}\textit{This is a Dirichlet type }\mathbf{II}\textit{ distribution.}$$

11

Note

设$\mathbf{W}_1$和$\mathbf{W}_2$相互独立,且$\mathbf{W}_1 \sim W_p(n, \mathbf{\Sigma})$和$\mathbf{W}_2 \sim W_p(m, \mathbf{\Sigma})$,其中$\mathbf{\Sigma} > 0, n \geqslant p$和$m \geqslant p$.
(1)试证明:$\mathbf{W}_1 + \mathbf{W}_2$与$(\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}\mathbf{W}_1(\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}$相互独立;
(2)试由$(\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}\mathbf{W}_1(\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}$的密度函数,计算$\Lambda(p, n, m)$的矩;
(3)令$\mathbf{W}_1 + \mathbf{W}_2 = \mathbf{U}\mathbf{U}'$,其中$\mathbf{U}$为对角线元素为正的下三角矩阵.试证明$\mathbf{W}_1 + \mathbf{W}_2$与$\mathbf{U}^{-1}\mathbf{W}_1\mathbf{U}^{-1}$相互独立.
(4)试证明:$\mathbf{W}_1 + \mathbf{W}_2$与$\mathbf{C}\mathbf{W}_1\mathbf{C}'$相互独立,其中$\mathbf{C}$满足条件$\mathbf{C}(\mathbf{W}_1 + \mathbf{W}_2)\mathbf{C}' = \mathbf{I}_p$.

(1)$Proof.$ 令$\mathbf{T} = (\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}\mathbf{W}_1(\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}, \mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$.

$$f(\mathbf{W}_1, \mathbf{W}_2) = \frac{|\mathbf{W}_1|^{\frac{n-p-1}{2}}|\mathbf{W}_2|^{\frac{m-p-1}{2}}}{2^{\frac{(m+n)p}{2}}|\Sigma|^{\frac{m+n}{2}}\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})}\mathrm{etr}\left\{-\frac{1}{2}\Sigma^{-1}(\mathbf{W}_1 + \mathbf{W}_2)\right\},$$

有$\mathbf{J}(\mathbf{W}_1, \mathbf{W}_2 \to \mathbf{W}, \mathbf{W}_2) = 1$

$$\text{则}g(\mathbf{W}_1, \mathbf{W}) = \frac{|\mathbf{W}_1|^{\frac{n-p-1}{2}}|\mathbf{W} - \mathbf{W}_1|^{\frac{m-p-1}{2}}}{2^{\frac{(m+n)p}{2}}|\Sigma|^{\frac{m+n}{2}}\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})}\mathrm{etr}\left\{-\frac{1}{2}\Sigma^{-1}\mathbf{W}\right\},$$

记$\mathbf{W} = \mathbf{P}'\mathbf{P}$且满足$\mathbf{W}_1 = \mathbf{P}'\mathbf{T}\mathbf{P}$,

有$\mathrm{d}\mathbf{W}_1 \wedge \mathrm{d}\mathbf{W} = (\mathrm{d}\mathbf{P}' \cdot \mathbf{T}\mathbf{P} + \mathbf{P}'\mathrm{d}\mathbf{T} \cdot \mathbf{P} + \mathbf{P}'\mathbf{T} \cdot \mathrm{d}\mathbf{P}) \wedge \mathrm{d}(\mathbf{P}'\mathbf{P}) = (\mathbf{P}'\mathrm{d}\mathbf{T} \cdot \mathbf{P}) \wedge \mathrm{d}(\mathbf{P}'\mathbf{P})$

$= |\mathbf{P}|^{p+1}\mathrm{d}\mathbf{T} \wedge \mathrm{d}(\mathbf{P}'\mathbf{P}) = |\mathbf{P}'\mathbf{P}|^{\frac{p+1}{2}}\mathrm{d}\mathbf{T} \wedge \mathrm{d}(\mathbf{P}'\mathbf{P})$则$\mathbf{J}(\mathbf{A}, \mathbf{C} \to \mathbf{T}, \mathbf{P}'\mathbf{P}) = |\mathbf{P}'\mathbf{P}|^{\frac{p+1}{2}}$.

$$h(\mathbf{T}, \mathbf{P}'\mathbf{P}) = \frac{|\mathbf{P}'\mathbf{T}\mathbf{P}|^{\frac{n-p-1}{2}}|\mathbf{P}'\mathbf{P} - \mathbf{P}'\mathbf{T}\mathbf{P}|^{\frac{m-p-1}{2}}|\mathbf{P}'\mathbf{P}|^{\frac{p+1}{2}}}{2^{\frac{(m+n)p}{2}}|\Sigma|^{\frac{m+n}{2}}\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})}\mathrm{etr}\left\{-\frac{1}{2}\Sigma^{-1}\mathbf{P}'\mathbf{P}\right\}$$

$$= \frac{|\mathbf{P}'\mathbf{P}|^{\frac{m+n-p-1}{2}}|\mathbf{I}_p - \mathbf{T}|^{\frac{m-p-1}{2}}|\mathbf{T}|^{\frac{n-p-1}{2}}}{2^{\frac{(m+n)p}{2}}|\Sigma|^{\frac{m+n}{2}}\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})}\mathrm{etr}\left\{-\frac{1}{2}\Sigma^{-1}\mathbf{P}'\mathbf{P}\right\}$$

$$= \frac{|\mathbf{P}'\mathbf{P}|^{\frac{m+n-p-1}{2}}}{2^{\frac{(m+n)p}{2}}|\Sigma|^{\frac{m+n}{2}}\Gamma_p(\frac{m+n}{2})}\mathrm{etr}\left\{-\frac{1}{2}\Sigma^{-1}\mathbf{P}'\mathbf{P}\right\} \cdot \frac{\Gamma_p(\frac{m+n}{2})}{\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})}|\mathbf{I}_p - \mathbf{T}|^{\frac{m-p-1}{2}}|\mathbf{T}|^{\frac{n-p-1}{2}}$$

由联合分布函数可以看出,$\mathbf{W}_1 + \mathbf{W}_2$与$(\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}\mathbf{W}_1(\mathbf{W}_1 + \mathbf{W}_2)^{-1/2}$相互独立

(2)$Sol.$ $\mathbf{T} = (\mathbf{W}_1 + \mathbf{W}_2)^{-\frac{1}{2}}\mathbf{W}_1(\mathbf{W}_1 + \mathbf{W}_2)^{-\frac{1}{2}} \sim \dfrac{\Gamma_p(\frac{m+n}{2})}{\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})}|\mathbf{I}_p - \mathbf{T}|^{\frac{m-p-1}{2}}|\mathbf{T}|^{\frac{n-p-1}{2}} \sim BE_p(n, m)$.

$$E(\Lambda^k) = E(|\mathbf{T}|^k) = \int\limits_{T>0}\frac{\Gamma_p(\frac{m+n}{2})}{\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})}|\mathbf{I}_p - \mathbf{T}|^{\frac{m-p-1}{2}}|\mathbf{T}|^{\frac{n-p-1}{2}}|\mathbf{T}|^k\mathrm{d}\mathbf{T}$$

$$= \frac{\Gamma_p(\frac{m+n}{2})\Gamma_p(\frac{n}{2}+k)}{\Gamma_p(\frac{m+n}{2}+k)\Gamma_p(\frac{n}{2})}\int\limits_{T>0}\frac{\Gamma_p(\frac{m+n}{2}+k)}{\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2}+k)}|\mathbf{I}_p - \mathbf{T}|^{\frac{m-p-1}{2}}|\mathbf{T}|^{\frac{n-p-1}{2}+k}\mathrm{d}\mathbf{T}$$

$$= \frac{\Gamma_p(\frac{m+n}{2})\Gamma_p(\frac{n}{2}+k)}{\Gamma_p(\frac{m+n}{2}+k)\Gamma_p(\frac{n}{2})}.$$

当$k=1$时，$E(\Lambda) = \dfrac{\Gamma_p(\frac{m+n}{2})\Gamma_p(\frac{n}{2}+1)}{\Gamma_p(\frac{m+n}{2}+1)\Gamma_p(\frac{n}{2})} = \dfrac{\Gamma(\frac{m+n+1-p}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+n+1}{2})\Gamma(\frac{n+1-p}{2})}$.

((3)(4))少条件?下列证明在满足$\mathbf{W}_1 = \mathbf{U}'\mathbf{T}\mathbf{U}$或$\mathbf{W}_1 = \mathbf{C}'\mathbf{T}\mathbf{C}$时进行.)

(3)$Proof.$ 令(1)中$\mathbf{P} = \mathbf{U}$即可,由Bartlett分解可知这样的$\mathbf{U}$存在.

(4)$Proof.$ 令(1)中$\mathbf{P}^{-1} = \mathbf{C}$即可.