Homework 1

习题3

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> Note
>
> 设$X^{(1)}$和$X^{(2)}$为p维随机向量,且
>
> $$X=\begin{pmatrix}X^{(1)}\\X^{(2)}\end{pmatrix}\sim N_{2p}\left(\begin{pmatrix}\mu^{(1)}\\\mu^{(2)}\end{pmatrix},\begin{pmatrix}\mathbf{\Sigma}_1&\mathbf{\Sigma}_2\\\mathbf{\Sigma}_2&\mathbf{\Sigma}_1\end{pmatrix}\right)$$
>
> 其中$\mu^{(1)}$和 $\mu^{(2)}$为p维列向量,$\mathbf{\Sigma}_1$和$\mathbf{\Sigma}_2$为p阶正定矩阵,
> (1)试证$X^{(1)}+X^{(2)}$与$X^{(1)}-X^{(2)}$相互独立;
> (2)试求$X^{(1)}+X^{(2)}$与$X^{(1)}-X^{(2)}$的分布.

$$\begin{aligned}
&Proof.(1)\text{已知}\mathbb{E}\left[X^{(i)}\right]=\mu^{(i)},\operatorname{Cov}\left(X^{(i)},X^{(i)}\right)=\mathbb{E}\left[X^{(i)}\right]=\mathbf{\Sigma}_1,i=1,2,\\
&\qquad\operatorname{Cov}\left(X^{(1)},X^{(2)}\right)=\operatorname{Cov}\left(X^{(2)},X^{(1)}\right)=\mathbf{\Sigma}_2.\\
&\qquad\text{则}\operatorname{Cov}\left(X^{(1)}+X^{(2)},X^{(1)}-X^{(2)}\right)\\
&\qquad=\operatorname{Cov}\left(X^{(1)},X^{(1)}\right)-\operatorname{Cov}\left(X^{(2)},X^{(2)}\right)+\operatorname{Cov}\left(X^{(2)},X^{(1)}\right)-\operatorname{Cov}\left(X^{(1)},X^{(2)}\right)\\
&\qquad=\mathbf{\Sigma}_1-\mathbf{\Sigma}_1+\mathbf{\Sigma}_2-\mathbf{\Sigma}_2=0\\
&\qquad\text{又}X^{(1)}+X^{(2)}\text{与}X^{(1)}-X^{(2)}\text{都为}X\text{的一个线性组合,则服从正态分布,}\\
&\qquad\text{因此,}X^{(1)}+X^{(2)}\text{和}X^{(1)}-X^{(2)}\text{相互独立.}\\
&\qquad(2)\text{由已知,}\mathbb{E}\left[X^{(1)}+X^{(2)}\right]=\mu^{(1)}+\mu^{(2)},\mathbb{E}\left[X^{(1)}-X^{(2)}\right]=\mu^{(1)}-\mu^{(2)},\\
&\qquad\operatorname{Var}\left(X^{(1)}+X^{(2)}\right)=\operatorname{Var}\left(X^{(1)}\right)+\operatorname{Var}\left(X^{(2)}\right)+\operatorname{Cov}\left(X^{(1)},X^{(2)}\right)+\operatorname{Cov}\left(X^{(2)},X^{(1)}\right)\\
&\qquad=2\mathbf{\Sigma}_1+2\mathbf{\Sigma}_2,\text{同理,}\operatorname{Var}\left(X^{(1)}-X^{(2)}\right)=2\mathbf{\Sigma}_1-2\mathbf{\Sigma}_2\\
&\qquad X^{(1)}+X^{(2)}\sim N\left(\mu^{(1)}+\mu^{(2)},2\mathbf{\Sigma}_1+2\mathbf{\Sigma}_2\right),X^{(1)}-X^{(2)}\sim N\left(\mu^{(1)}-\mu^{(2)},2\mathbf{\Sigma}_1-2\mathbf{\Sigma}_2\right)
\end{aligned}$$

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> Note
>
> 设$X\sim N_p\left(\mu,\mathbf{\Sigma}\right),\mathbf{\Sigma}>0$,对$\mu$和$\mathbf{\Sigma}$作如下剖分:

$$\mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

其中$\mu^{(1)}$为r维列向量，$\boldsymbol{\Sigma}_{11}$为r维方阵，$1 \leqslant r < p$.

(1)试证明:$\mu' \boldsymbol{\Sigma}^{-1} \mu \geqslant \left(\mu^{(1)}\right)' \boldsymbol{\Sigma}_{11}^{-1} \mu^{(1)}$;

(2)试证明:$X^{(2)} | X^{(1)} = x^{(1)} \sim N_{p-r}(\mu_{2.1}, \boldsymbol{\Sigma}_{22.1}$,其中$\mu_{2.1} = \mu^{(2)} + \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \left(x^{(1)} - \mu^{(1)}\right)$和$\boldsymbol{\Sigma}_{22.1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12}$.

Proof.(1)已知$\boldsymbol{\Sigma}^{-1}=\begin{pmatrix}\boldsymbol{\Sigma}_{11}^{-1}+\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22.1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & -\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22.1}^{-1}\\ \boldsymbol{\Sigma}_{22.1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{\Sigma}_{22.1}^{-1}\end{pmatrix}$,要证$\mu'\boldsymbol{\Sigma}^{-1}\mu-\left(\mu^{(1)}\right)'\boldsymbol{\Sigma}_{1}$

即证$\mu'\boldsymbol{\Sigma}^{-1}\mu-\left(\mu^{(1)}\right)'\boldsymbol{\Sigma}_{11}^{-1}\mu^{(1)}$

$$=\begin{pmatrix}\mu^{(1)}\\ \mu^{(2)}\end{pmatrix}'\begin{pmatrix}\boldsymbol{\Sigma}_{11}^{-1}+\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22.1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & -\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22.1}^{-1}\\ \boldsymbol{\Sigma}_{22.1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{\Sigma}_{22.1}^{-1}\end{pmatrix}\begin{pmatrix}\mu^{(1)}\\ \mu^{(2)}\end{pmatrix}-\left(\mu^{(1)}\right)'\boldsymbol{\Sigma}_{11}^{-1}\mu^{(1)}$$

$$=\left(\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mu^{(1)}-\mu^{(2)}\right)'\boldsymbol{\Sigma}_{22.1}^{-1}\left(\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mu^{(1)}-\mu^{(2)}\right)\geqslant 0$$

因此我们只需说明$\boldsymbol{\Sigma}_{22.1}^{-1}=\boldsymbol{\Sigma}_{22}-\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$半正定即可. 事实上

$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix}\backsimeq\begin{pmatrix}\boldsymbol{\Sigma}_{11} & 0\\ 0 & \boldsymbol{\Sigma}_{22}-\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\end{pmatrix}$且$\boldsymbol{\Sigma}$正定, 从而$\boldsymbol{\Sigma}_{22.1}^{-1}=\boldsymbol{\Sigma}_{22}-\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$

故原命题得证.

(2)作变换$Y=\begin{pmatrix}Y^{(1)}\\ Y^{(2)}\end{pmatrix}=\begin{pmatrix}\mathbf{I}_r & 0\\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{I}_{P-r}\end{pmatrix}\begin{pmatrix}X^{(1)}\\ X^{(2)}\end{pmatrix}=\begin{pmatrix}X^{(1)}\\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}X^{(1)}+X^{(2)}\end{pmatrix}$

故$\mathbb{E}[Y]=\begin{pmatrix}\mu^{(1)}\\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mu^{(1)}+\mu^{(2)}\end{pmatrix}$, $\mathrm{Var}(Y)=\begin{pmatrix}\mathbf{I}_r & 0\\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{I}_{P-r}\end{pmatrix}\boldsymbol{\Sigma}\begin{pmatrix}\mathbf{I}_r & 0\\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \mathbf{I}_{P-r}\end{pmatrix}'$

$$=\begin{pmatrix}\boldsymbol{\Sigma}_{11} & 0\\ 0 & \boldsymbol{\Sigma}_{22}-\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\end{pmatrix}=\begin{pmatrix}\boldsymbol{\Sigma}_{11} & 0\\ 0 & \boldsymbol{\Sigma}_{22.1}\end{pmatrix},$$

从而$Y^{(1)}$与$Y^{(2)}$独立，Y的密度函数为$f(Y)=f(Y^{(1)})f(Y^{(2)})$

$$=\frac{1}{(2\pi)^{r/2}|\boldsymbol{\Sigma}_{11}|^{1/2}}\exp\left\{-\frac{1}{2}\left(Y^{(1)}-\mu^{(1)}\right)'\boldsymbol{\Sigma}_{11}^{-1}\left(Y^{(1)}-\mu^{(1)}\right)\right\}$$

$$\times\frac{1}{(2\pi)^{(p-r)/2}|\boldsymbol{\Sigma}_{22.1}|^{1/2}}\exp\left\{-\frac{1}{2}\left(Y^{(2)}-\mu^{(2)}+\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mu^{(1)}\right)'\boldsymbol{\Sigma}_{22.1}^{-1}\left(Y^{(2)}-\mu^{(2)}+\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mu\right.\right.$$

故$f(X)=f\left(X^{(1)},X^{(2)}\right)\cdot|\mathbf{J}|^{-1}=\dfrac{1}{(2\pi)^{r/2}|\boldsymbol{\Sigma}_{11}|^{1/2}}\exp\left\{-\dfrac{1}{2}\left(X^{(1)}-\mu^{(1)}\right)'\boldsymbol{\Sigma}_{11}^{-1}\left(X^{(1)}-\mu^{(\right.\right.$

$$\times\frac{1}{(2\pi)^{(p-r)/2}|\boldsymbol{\Sigma}_{22.1}|^{1/2}}$$

$$\cdot\exp\left\{-\frac{1}{2}\left(X^{(2)}-\mu^{(2)}-\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\left(X^{(1)}-\mu^{(1)}\right)\right)'\boldsymbol{\Sigma}_{22.1}^{-1}\left(X^{(2)}-\mu^{(2)}-\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\left(X^{(1)}-\mu^{(\right.\right.\right.$$

又$f(X^{(2)}|X^{(1)})=\dfrac{f\left(X^{(1)},X^{(2)}\right)}{f\left(X^{(1)}\right)}$, 注意到, $f\left(X^{(1)}\right)$为上式的前半部分, 后半部分即$f(X^{(2)}|X$

故$X^{(2)}|X^{(1)}=x^{(1)}\sim N_{p-r}\left(\mu^{(2)}+\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(x^{(1)}-\mu^{(1)}),\boldsymbol{\Sigma}_{22.1}\right)$.

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Note

令$X_1,\cdots,X_n$是相互独立的,且$X_i\sim N\left(\beta+\gamma z_i,\sigma^2\right)$,其中$z_i$是给定的常数,$i=1,\cdots,n$,且$\sum\limits_{i=1}^n z_i=0.$

(1)求$(X_1,\cdots,X_n)'$的分布;

(2)求$\overline{X}=\frac{1}{n}\sum\limits_{i=1}^n X_i$和$Y=\sum\limits_{i=1}^n z_iX_i\Big/\sum\limits_{i=1}^n z_i^2$的分布，其中$\sum\limits_{i=1}^n z_i^2>0.$

$Sol.$(1)由$X_1,\cdots,X_n$相互独立,$(X_1,\cdots,X_n)'\sim N_p\left(\beta\mathbb{1}_n+\gamma z,\sigma^2\mathbf{I}_n\right)$,其中$\mathbb{1}_n$表示元素全为1的$n\times$

$z=(z_1,\cdots,z_n)'.$

(2)由正态分布的可加性及$\sum\limits_{i=1}^n z_i=0,\sum\limits_{i=1}^n X_i\sim N\left(n\beta+\gamma\sum\limits_{i=1}^n z_i,n\sigma^2\right)$,则$\overline{X}=\dfrac{1}{n}\sum\limits_{i=1}^n X_i\sim N$

$\sum\limits_{i=1}^n z_iX_i\sim N\left(\beta\sum\limits_{i=1}^n z_i+\gamma\sum\limits_{i=1}^n z_i^2,\sigma^2\sum\limits_{i=1}^n z_i^2\right)$,则$Y=\sum\limits_{i=1}^n z_iX_i\Big/\sum\limits_{i=1}^n z_i^2\sim N\left(\gamma,\sigma^2\Big/\sum\limits_{i=1}^n z_i^2\right)$

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Note

令$a=(-4,3)'$和$b=(1,1)'$,以及$\mathbf{A}=\begin{pmatrix}2&-2\\-2&5\end{pmatrix}$,试验证推广的$\mathrm{Cauchy-Schwarz}$不等式$:(a'b)^2\leqslant(a'\mathbf{A}a)(b'\mathbf{A}^{-1}b).$

$Sol.(1)a'b=(-4,3)\begin{pmatrix}1\\1\end{pmatrix}=-1,(a'b)^2=1,\mathbf{A}^{-1}=\begin{pmatrix}\frac{5}{6}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}\end{pmatrix},$

$a'\mathbf{A}a=(-4,3)\begin{pmatrix}2&-2\\-2&5\end{pmatrix}\begin{pmatrix}-4\\3\end{pmatrix}=(-14,23)\begin{pmatrix}-4\\3\end{pmatrix}=125,b'\mathbf{A}^{-1}b=(1,1)\begin{pmatrix}\frac{5}{6}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=$

$(a'\mathbf{A}a)(b'\mathbf{A}^{-1}b)=\dfrac{1375}{6}>1=(a'b)^2.$

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Note

证明:

$$\mathbf{\Sigma}^{-1}=\begin{pmatrix}\mathbf{\Sigma}_{11.2}^{-1}&-\mathbf{\Sigma}_{11.2}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\\-\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11.2}^{-1}&\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11.2}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}+\mathbf{\Sigma}_{22}^{-1}\end{pmatrix}=\begin{pmatrix}0&0\\0&\mathbf{\Sigma}_{22}^{-1}\end{pmatrix}+\begin{pmatrix}\mathbf{I}\\\beta'\end{pmatrix}\mathbf{\Sigma}_{11.2}^{-1}(\mathbf{I},-\beta)$$

> 其中$\beta = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$.

Proof. 即证$\mathbf{\Sigma}^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{\Sigma}_{22}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{\Sigma}_{11.2}^{-1} & -\mathbf{\Sigma}_{11.2}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1} \\ -\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11.2}^{-1} & \mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11.2}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I} \\ \beta' \end{pmatrix} \mathbf{\Sigma}_{11.2}^{-1}(\mathbf{I}, -\beta)$,

$$\begin{pmatrix} \mathbf{I} \\ -\beta' \end{pmatrix} \mathbf{\Sigma}_{11.2}^{-1}(I, -\beta) = \begin{pmatrix} \mathbf{I} \\ -\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} \end{pmatrix} \mathbf{\Sigma}_{11.2}^{-1}(\mathbf{I}, -\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}) = \begin{pmatrix} \mathbf{\Sigma}_{11.2}^{-1} \\ -\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{11.2}^{-1} \end{pmatrix} (\mathbf{I}, -\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1})$$

$$= \begin{pmatrix} \mathbf{\Sigma}_{11.2}^{-1} & -\mathbf{\Sigma}_{11.2}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1} \\ -\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11.2}^{-1} & \mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11.2}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1} \end{pmatrix}.$$