

Single Qubit states and their visualization

Introduction to Quantum Computing

Jothishwaran C.A.

Department of Electronics and Communication Engineering
Indian Institute of Technology Roorkee

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Outline

Recap

Complex Vectors and Qubits

- Qubit states in \mathbb{C}^2

- Basis vectors and State representation

- Projections and Photons

Visualising the Qubit state

- Polar Coordinates

- The global phase

- States in Polar Coordinates

- Visualising the Qubit state

What we know so far

- ▶ In quantum computing, the fundamental unit of information is a quantum bit (qubit).
- ▶ A qubit can be used to represent the state of many physical objects.
- ▶ We considered photons, where the qubit was used to represent the polarization state of the photon.
- ▶ Vertical ($|0\rangle$) and horizontal ($|1\rangle$) polarization states were considered as the basis for these qubits.

What we know so far

- ▶ A real superposition (linear combination) of $|0\rangle$ and $|1\rangle$ formed oblique polarization states.
- ▶ Complex superposition also accounts for circular and elliptical polarization states.
- ▶ A vertically aligned polarizer can be used to “measure” the polarization state of these photons.
- ▶ If a photon passes through this polarizer its polarization is measured to be vertically aligned. If a photon is blocked (absorbed) by this polarizer, its polarization is measured to be horizontally aligned.

What we know so far

- ▶ Measuring a qubit state with respect to a basis will change the quantum state of the photon.
- ▶ After measurement, the state of the photon changes to one of the basis states.
- ▶ The outcome of a quantum measurement is in general represented by a real number.
- ▶ In the case of a qubit, the measurement outcome is represented by a single classical bit.

The vector space \mathbb{C}^2

- ▶ A vector in the space \mathbb{C}^2 is represented as follows:

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}; a, b \in \mathbb{C}$$

- ▶ If $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$, the inner products $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$ are defined as:

$$\langle\phi|\psi\rangle = (\bar{c} \quad \bar{d}) \begin{pmatrix} a \\ b \end{pmatrix} = a\bar{c} + b\bar{d}$$

and

$$\langle\psi|\phi\rangle = \left\{ (\bar{c} \quad \bar{d}) \begin{pmatrix} a \\ b \end{pmatrix} \right\}^\dagger = \bar{a}c + \bar{b}d = \overline{\langle\phi|\psi\rangle}$$

Normalization and the Qubit state

- ▶ Consider the quantity $\langle\psi|\psi\rangle$ which has a value

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2$$

- ▶ Therefore a vector $|\psi\rangle$ is a qubit state if $\langle\psi|\psi\rangle = 1$.
- ▶ A general vector $|\phi\rangle$ can be converted to a qubit state $|\tilde{\phi}\rangle$ as follows:

$$|\tilde{\phi}\rangle = \frac{1}{\langle\phi|\phi\rangle^{\frac{1}{2}}} |\phi\rangle$$

- ▶ The above process is referred to as normalization and the quantity $\langle\phi|\phi\rangle^{\frac{1}{2}}$ is called the norm of the vector $|\phi\rangle$.
- ▶ Qubit states can now be formally defined as vectors in \mathbb{C}^2 with a unit norm.

Basis States

- ▶ Defining the vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the vector $|\psi\rangle$ is now expressed as:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

It is possible to represent any vector in \mathbb{C}^2 in the above manner.

- ▶ The set $\{|0\rangle, |1\rangle\}$ is called the standard or the computational basis and is said to span \mathbb{C}^2 .
- ▶ The inner products have the values; $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.
- ▶ A basis satisfying the above property is known as an orthonormal basis. It is worth noting that orthonormal basis vectors are valid qubit states.

Coordinates and Projections

- ▶ Considering the vector $|\psi\rangle$ and the computational basis $\{|0\rangle, |1\rangle\}$, the following is true

$$\langle 0|\psi\rangle = (1 \ 0) \begin{pmatrix} a \\ b \end{pmatrix} = a; \quad \langle 1|\psi\rangle = (0 \ 1) \begin{pmatrix} a \\ b \end{pmatrix} = b$$

- ▶ The coordinates of the vector $|\psi\rangle$ can be defined in terms of the inner product with the basis vectors.
- ▶ The inner product of $|\psi\rangle$ with a basis vector is known as the projection of $|\psi\rangle$ along that basis vector.
- ▶ The vector $|\psi\rangle$ can now be represented in terms of the projections as follows:

$$|\psi\rangle = \langle 0|\psi\rangle |0\rangle + \langle 1|\psi\rangle |1\rangle$$

“Non-standard” Basis

- ▶ The idea of coordinates and projections is true for any orthonormal basis.
- ▶ Consider the basis orthonormal basis $\{|+\rangle, |-\rangle\}$, where
$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
- ▶ In this basis, the vector $|\psi\rangle$ is represented as:

$$|\psi\rangle = \langle +|\psi\rangle |+\rangle + \langle -|\psi\rangle |-\rangle$$

evaluating the inner products give the result

$$|\psi\rangle = \frac{a+b}{\sqrt{2}} |+\rangle + \frac{a-b}{\sqrt{2}} |-\rangle$$

Projections and Photons

- ▶ The state of an obliquely polarized photon $|\chi\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$, this state can be represented in the standard basis as:

$$|\chi\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle$$

where the basis vectors $|0\rangle$, $|1\rangle$ represent the vertical and horizontal polarization states.

- ▶ The probability that this photon is transmitted by a polarizer aligned along the $|0\rangle$ is given by $|\cos \theta|^2$.
- ▶ The transmission probability can now be correctly reinterpreted as $|\langle 0|\chi\rangle|^2$ and this result may be used to calculate the transmission probabilities for circularly polarized light as well.
- ▶ The above result can be generalised to transmission probabilities for a polarizer oriented along any direction using the same method as described before.

Polar Coordinates

- ▶ A complex number, $z = x + \mathbf{i}y$ can be represented in the polar form as, $z = re^{\mathbf{i}\phi}$ where

$$r = \sqrt{x^2 + y^2} ; \phi = \arctan\left(\frac{y}{x}\right)$$

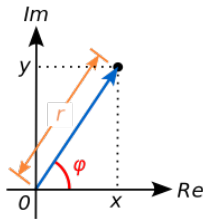


Figure 1: Figure showing the representations of a complex number. Source: Wikipedia²

²Complex_number_illustration.svg: The original uploader was Wolfkeeper at English Wikipedia. derivative work: Kan8eDie (talk)
https://commons.wikimedia.org/wiki/File:Complex_number_illustration_modarg.svg, “Complex number illustration modarg”,
<https://creativecommons.org/licenses/by-sa/3.0/legalcode>

The global phase

- ▶ The unit complex number $e^{i\phi}$ is also referred to as a phase factor.
- ▶ Multiplying a photon state $|\psi\rangle$ with a phase factor gives a state $e^{i\phi} |\psi\rangle$. The phase factor is now referred to as a global phase factor.
- ▶ The projection of this new vector with respect to a basis state (say $|0\rangle$) is given by $e^{i\phi} \langle 0|\psi\rangle$.
- ▶ While this is different from $\langle 0|\psi\rangle$, it should be noted that this new state will have the same transmission probabilities as that of $|\psi\rangle$.
- ▶ It is therefore not possible to distinguish $e^{i\phi} |\psi\rangle$ from $|\psi\rangle$ by performing polarization measurements.
- ▶ Therefore, states that differ from each other by only a global phase are considered to be equivalent.

Qubit state in polar coordinates

- ▶ Consider the state $|\psi\rangle$ in the standard basis with the coordinates represented in polar coordinates. $a = r_0 e^{i\phi_0}$ and $b = r_1 e^{i\phi_1}$
- ▶ The state may now be expressed as follows:

$$\begin{aligned} |\psi\rangle &= r_0 e^{i\phi_0} |0\rangle + r_1 e^{i\phi_1} |1\rangle \\ &= e^{i\phi_0} \left(r_0 |0\rangle + r_1 e^{i(\phi_1 - \phi_0)} |1\rangle \right) \\ &\equiv r_0 |0\rangle + r_1 e^{i(\phi_1 - \phi_0)} |1\rangle \end{aligned}$$

setting $\phi_1 - \phi_0 = \phi$,

$$|\psi\rangle = r_0 |0\rangle + r_1 e^{i\phi} |1\rangle$$

additionally,

$$\langle\psi|\psi\rangle = 1 \Rightarrow r_0^2 + r_1^2 = 1$$

Parameter Selection

- ▶ Since, $0 \leq r_0, r_1 \leq 1$ and $r_0^2 + r_1^2 = 1$ it is possible to represent $r_0 = \cos(\theta/2)$ and $r_1 = \sin(\theta/2)$ where, $\theta \in [0, \pi]$
- ▶ Since, $e^{i\phi} = \cos \phi + i \sin \phi \Rightarrow \phi \in [0, 2\pi)$. This angle is known as the relative phase.
- ▶ These parameters are identical to the angle variable in spherical polar coordinates. Therefore, each qubit state defined using these parameters corresponds to a point on a unit sphere.

- ▶ The qubit state in terms of this parameter is represented as

$$|\psi\rangle = \cos(\theta/2) |0\rangle + e^{i\phi} \sin(\theta/2) |1\rangle$$

- ▶ The sphere on which the point corresponding to the state is present is known as the *Bloch Sphere*.

Qubit states on the Bloch Sphere

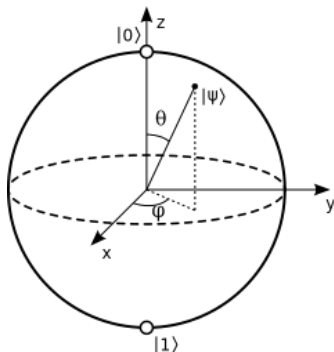


Figure 2: Figure showing a point on the Bloch Sphere. Source: Wikipedia³

³Smite-Meister

(https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg), "Bloch sphere",

(<https://creativecommons.org/licenses/by-sa/3.0/legalcode>)

Qubit states on the Bloch Sphere

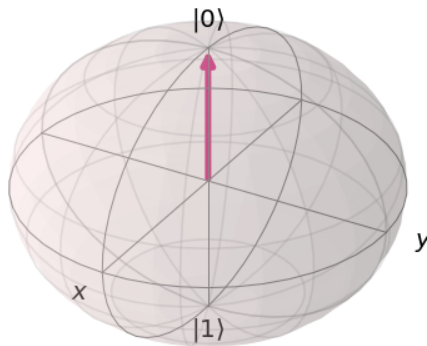


Figure 3: Figure showing the the point corresponding to $|0\rangle$

Qubit states on the Bloch Sphere

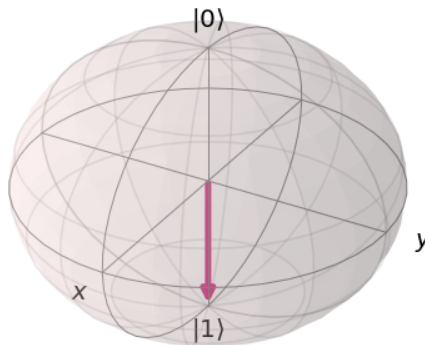


Figure 4: Figure showing the point corresponding to $|1\rangle$

Qubit states on the Bloch Sphere

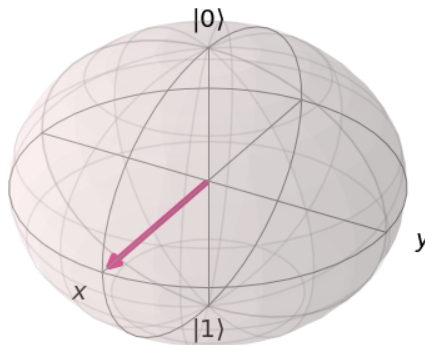


Figure 5: Figure showing the point corresponding to $|+\rangle$