NP-Completeness Proofs for Selected Problems

Independent Set Problem

Theorem: Independent Set is NP-complete.

Proof: We show Vertex Cover $\leq_P$ Independent Set.

Reduction: Given a graph $G = (V, E)$ and integer k for Vertex Cover:

- Construct the same graph $G' = G$
- Set $k' = |V| - k$

Correctness: Let $S \subseteq V$ be a vertex cover of size k in G. Then $V - S$ is an independent set because:

- For any edge $(u, v) \in E$, at least one of u or v is in S
- Therefore, no edge has both endpoints in $V - S$
- $|V - S| = |V| - k = k'$

Conversely, if I is an independent set of size k' in G, then $V - I$ is a vertex cover because:

- For any edge $(u, v) \in E$, u and v cannot both be in I
- So at least one of u or v is in $V - I$
- $|V - I| = |V| - k' = k$

Therefore, Independent Set is NP-complete.

Hamiltonian Path Problem

Theorem: Hamiltonian Path is NP-complete.

Proof: We show Hamiltonian Cycle $\leq_P$ Hamiltonian Path.

Reduction: Given graph $G = (V, E)$ for Hamiltonian Cycle:

1. Pick any vertex $v \in V$
2. Create new vertices v_{start} and v_{end}
3. Add edges from v_{start} to all neighbors of v
4. Add edges from v_{end} to all neighbors of v
5. Remove vertex v from the graph

Correctness: If G has a Hamiltonian cycle $(v, v_1, v_2, \ldots, v_n, v)$, then:

- The path $(v_{\text{start}}, v_1, v_2, \ldots, v_n, v_{\text{end}})$ is Hamiltonian in G'

If G' has a Hamiltonian path $(v_{\text{start}}, \ldots, v_{\text{end}})$, then:

- Replacing v_{start} and v_{end} with v gives a Hamiltonian cycle in G

Therefore, Hamiltonian Path is NP-complete.

Knapsack Problem

Theorem: Subset Sum $\leq_P$ Knapsack
Reduction: Given Subset Sum instance $(S = \{s_1, \ldots, s_n\}, t)$:

- Create Knapsack instance with n items
- $\text{Weight}_i = \text{Value}_i = s_i$
- Capacity $W = t$
- Target value $V = t$

Correctness: A subset $S' \subseteq S$ sums to t iff:

- The corresponding items have total weight $\leq W = t$
- And total value $= t = V$

Therefore, Knapsack is NP-complete.

Bin Packing Problem

Theorem: Partition $\leq_P$ Bin Packing
Reduction: Given Partition instance $(S = \{a_1, \ldots, a_n\})$:

- Let $T = \sum a_i/2$
- Create Bin Packing instance:
  - n items with sizes $a_1, \ldots, a_n$
  - Bin capacity $= T$
  - Number of bins $= 2$

Correctness: S can be partitioned into two equal-sum subsets iff the items can be packed into 2 bins of capacity T.
Therefore, Bin Packing is NP-complete.

Set Cover Problem

Theorem: Vertex Cover $\leq_P$ Set Cover
Reduction: Given graph $G = (V, E)$ for Vertex Cover:

- Universe $U = E$ (all edges)
- For each vertex $v \in V$, create set $S_i = \{e \in E : e \text{ incident to } v\}$
- k remains the same

Correctness: A vertex cover of size k corresponds to selecting k sets that cover all edges.
Therefore, Set Cover is NP-complete.

Multiprocessor Scheduling

Theorem: Partition $\leq_P$ Multiprocessor Scheduling

Reduction: Given Partition instance ($S = \{a_1, \ldots, a_n\}$):

- Create scheduling instance:
  - n jobs with processing times $a_1, \ldots, a_n$
  - 2 identical machines
  - Deadline $D = \sum a_i/2$

Correctness: A partition exists iff there's a schedule where all jobs finish by time D.

Therefore, Multiprocessor Scheduling is NP-complete.

Longest Path Problem

Theorem: Hamiltonian Path $\leq_P$ Longest Path

Reduction: Given graph $G = (V, E)$ with n vertices for Hamiltonian Path:

- Use the same graph G
- Set $k = n - 1$

Correctness: G has a Hamiltonian path iff G has a simple path of length $\geq n - 1$.

Therefore, Longest Path is NP-complete.

Summary of Reductions

Problem	**Reduced From**	**Key Idea**
Independent Set	Vertex Cover	Complement set
Hamiltonian Path	Hamiltonian Cycle	Split a vertex
Knapsack	Subset Sum	Weight = Value
Bin Packing	Partition	2 bins, capacity = half sum
Set Cover	Vertex Cover	Edges as universe
Multiprocessor Scheduling	Partition	2 machines
Longest Path	Hamiltonian Path	Path length = $n - 1$