NP-Complete Problems: Detailed Proofs

1. Independent Set Problem (NP-Complete)

Proof Strategy: Show Vertex Cover $\leq_P$ Independent Set.

Definition:
Independent Set: Given graph $G=(V,E)$ and integer k, determine if there exists a set $S \subseteq V$ of size $\geq k$ such that no two vertices in S are adjacent.

Reduction:
Given Vertex Cover instance <G,k>, convert to Independent Set instance <G, |V|-k>.
A set C is a vertex cover iff V–C is an independent set.

Therefore, VC polynomially reduces to Independent Set and Independent Set $\in$ NP.
Hence Independent Set is NP-Complete.

2. Hamiltonian Path

Definition: Given graph G, determine whether there exists a simple path visiting every vertex exactly once.

Proof Outline:
Reduce from NP-complete problem Hamiltonian Cycle.
Given graph G, create graph G' by splitting a node and adding edges.

G has a Hamiltonian cycle $\Leftrightarrow$ G' has a Hamiltonian path.

Thus Hamiltonian Path is NP-Complete.

3. Knapsack Problem

Definition:
Given items (w1,v1),...,(wn,vn) and capacity W, value V, does there exist a subset with total weight $\leq$ W and value $\geq$ V?

Proof: Reduce Subset-Sum $\leq_P$ Knapsack.

Subset-Sum instance: {a1,...,an} and target T.
Convert to knapsack instance where wi = vi = ai and W=V=T.
Solution exists $\Leftrightarrow$ subset sum exists.
Knapsack $\in$ NP, so NP-Complete.

4. Bin Packing

Definition: Given n items with sizes s(i) (0<s(i)<=1) and k bins (capacity 1), check if items can fit.

Proof:
Reduce Partition problem to Bin Packing.
Partition $\Rightarrow$ can pack 2 equal halves in 2 bins.
Bin Packing $\in$ NP and NP-Hard $\Rightarrow$ NP-Complete.

5. Set Cover

Definition: Given universe U and collection of subsets S, determine if k subsets cover U.

Reduction: