System of linear equations

- Why solve System of linear equations ?
- Geometric Interpretation
- Understanding Ax = b intuitively
- A step by step method to find Solution for Ax=b (Gaussian Elimination)
    - Rank
    - Parametric form of solution

Before That: A quick recap so far

(1) What is LD or LI

$v_1 \ldots v_k$ are LD iff there is

atleast one vector can be written in form

of others.

$$c_1 v_1 + c_2 v_2 + \cdots c_k v_k = 0$$

$\exists\, c_i \neq 0$?

Yes → LD

No → LI

Mathematically

given that set of vectors are LI and we have $c_1 v_1 + c_2 v_2 + \cdots c_k v_k = 0$ $\Rightarrow$ ALL c_i's are zero

(2) filling the space

$v_1, v_2 - - - v_n \in R^n$

$\underbrace{\{v_1, v_2 - - - v_n\}}$ will fill space R^n ?

LI then these vectors will fill the space.

$\underline{\underline{R^3}}$ 3 linearly Indep. vectors will fill the space.

③ How many max. LI vectors in R^n

$$\Rightarrow n \qquad \underline{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\6\\1\end{bmatrix}}$$

there are many sets in R^3 which are LI

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Convenient vectors}$$

$$\begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Matrix × vector

Ax $\Rightarrow$ this is a linear combination of columns of A

examples / questions

$Ax = 0$ is having nontrivial sol^n then what does this mean

if b is a LC of columns of A then does $Ax = b$ have sol^n

Suppose

$$\boxed{v = c_1 v_1 + c_2 v_2 + c_3 v_3}$$

v is LC of v_1, v_2, v_3

+

$\{v_1, v_2, v_3\}$ are LD $\Rightarrow$ there are many $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$\{v_1, v_2, v_3\}$ are LI $\Rightarrow$ there is exactly one combination

LI and LD vectors defination

multiply a matrix and vector

$Ax = b$

Step by Step . . .

Solving system of linear equations

Ax=0

Ax=b

Solving System of linear equations using high school way

$$x - 2y = 1$$
$$3x + 2y = 11$$

How would you **eliminate** x completely from 2nd equation ?

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

$$\begin{aligned} \cancel{3x} - 6y &= 3 \\ 3x + 2y &= 11 \\ \hline -8y &= -8 \end{aligned}$$

$$y = 1$$

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

$-8y = -8 \Rightarrow x, y =$

$$x - 2y = 1$$

$$x - 2y = 1$$
$$3x + 2y = 11$$

How would you **eliminate** x completely from 2nd equation ?

Before $\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$ **After** $\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$ $\begin{aligned} &(\textit{multiply equation } 1 \textit{ by } 3) \\ &(\textit{subtract to eliminate } 3x) \end{aligned}$

$$4x - 8y = 4$$
$$3x + 2y = 11$$

Eliminate x from 2nd equation

$$\cancel{12x} - 24y = 12$$
$$\underset{-}{\cancel{12x}} \underset{-}{+} 8y = \underset{-}{44}$$

$$-32y = -32 \rightarrow y = 1$$

$$\begin{aligned} 4x - 8y &= 4 \\ 3x + 2y &= 11 \end{aligned}$$

Eliminate x from 2nd equation

$$\begin{aligned} 4x - 8y &= 4 \\ 3x + 2y &= 11 \end{aligned}$$

Multiply equation 1 by $\frac{3}{4}$
Subtract from equation 2

$$\begin{aligned} 4x - 8y &= 4 \\ 8y &= 8. \end{aligned}$$

$y = 1$

$4x - 8 = 4 \Rightarrow x = 3$

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
4x + 9y - 3z &= 8 \\
-2x - 3y + 7z &= 10
\end{aligned}
$$

(Three Equations in Three Unknowns)

$$2x + 4y - 2z = 2$$

$$4x + 9y - 3z = 8$$

$$-2x - 3y + 7z = 10$$

$$0 \quad +y \quad +5z = 12$$

Step 1 Subtract 2 times equation 1 from equation 2.
Step 2 Add equation 1 and equation 3.

$$2x + 4y - 2z = 2$$

$$\mathbf{1}y + 1z = 4$$

$$1y + 5z = 12$$

Step 3: Subtract equation 2 from equation 3

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

$Ax = b$ has become

$$\begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ \mathbf{1}y + 1z &= 4 \\ \mathbf{4}z &= 8. \end{aligned}$$

Goal Achieved

What if there are 10 equations ?

What if there are 10 equations ?

What if there are 20 equations ?

What if there are 10 equations ?

What if there are 20 equations ?

What if there are 100 equations ?

What if there are 10 equations ?

What if there are 20 equations ?

What if there are 100 equations ?

What if there are 10000 equations ?

Gaussian Elimination

An algorithm for solving systems of linear equations.

(Wikipedia)

Gaussian Elimination

Matrix $\xrightarrow[\text{Elimination}]{\text{Gaussian}}$ Echelon form of Matrix

Gaussian Elimination

- Echelon form of Matrix
- Pivot and Free variables
- Elementary row operations
- Row Reduced Echelon form (Optional)

Echelon Form (or Row Echelon Form)

- All nonzero rows are above any rows of all zeros
- All entries in a column below a **leading entry** are zero
- The leading entry of any row occurs to the right of the leading entry of the row above it.

(This method can also be used to compute the rank of a matrix)

Examples of Echelon Form:

$$\begin{pmatrix} 5 & 1 & -6 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -6 \end{pmatrix} \qquad \begin{pmatrix} 0 & -3 & 3 & -4 & 5 & -2 \\ 0 & 0 & 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following examples are not in echelon form:

A
$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

B
$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

x

x

Non zero

Nonze

$$\begin{bmatrix} 0 & * & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & * & \bullet & \bullet & \bullet \\ \vdots & 0 & 0 & 0 & 0 & 0 & * & \bullet \\ \vdots & \vdots & \vdots & & \cdots & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix}
\square & * & * & * & * & * & * & * & * & * & * \\
 & \square & * & * & * & * & * & * & * & * & * \\
 & & & & \square & * & * & * & * & * & * \\
 & & & & & & \square & * & * & * & * \\
 & & & & & & & \square & * & * & * \\
 & & & & & & & & & & \square
\end{pmatrix}$$

SSES

Which matrices are in echelon form?

✓ $$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

✗ $$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * & * \end{bmatrix}$$ ✗

Pivot Pivot

✓ $$\begin{bmatrix} 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Which matrices are in echelon form?

$$\begin{bmatrix} 0 & 0 & 1 & -3 & 1 \\ 1 & 4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Pivot and Free variables

In Row echelon form -

Column

A variable whose coefficient is a leading nonzero is called a pivot variable. Otherwise, the variable is known as a free variable

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$2x + 3y = 5$

$3x - 2y = 9$

$$\begin{array}{cc|c} x & y & \\ A & & b \\ \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 3 & -2 & 9 \end{array}\right]$$

← Augmented matrix

first column
or first variable x

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$$

$$3x + 2y - z = 5$$

$$2x - 3y + 5z = 17$$

$$9x + 7y - 3z = 12$$

$$\underbrace{\begin{array}{cccc} x & & z & \\ \left[\begin{array}{ccc} 3 & 2 & -1 \\ 2 & -3 & 5 \\ 9 & 7 & -3 \end{array}\right. & & & \left|\begin{array}{c} 5 \\ 17 \\ 12 \end{array}\right] \end{array}}_{\text{Augmented matrix}}$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \Rightarrow \text{cofficient matrix}$$

$$\left[\begin{array}{c|c} A & b \\ & \end{array}\right] \Rightarrow \text{Augmented matrix}$$

(pivot)

which are basic variables and which are free variables in given augmented matrix?

pivot column

basic variable

$$1.\ \begin{bmatrix} 1 & 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad 2.\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(pivot)

which are basic variables and which are free variables in given augmented matrix?

$$1.\ \begin{array}{ccccc} P & & P & & \\ \left[\begin{array}{cccc|c} 1 & 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] & & & & \\ & f & & f & \end{array}$$

$$2.\ \begin{array}{cc} P & P \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] & \end{array}$$

pivot column

basic variable

which are basic variables and which are free variables in given augmented matrix?

3. $$\begin{bmatrix} 1 & 2 & 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which are basic variables and which are free variables in given augmented matrix?

$$\begin{array}{cc} & \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & & \end{array} \\ 3. & \left[\begin{array}{cccccc} 1 & 2 & 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ & \begin{array}{cccccc} P & P & F & P & & \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} P & F & P & \end{array} \\ 4. & \left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\boxed{c_3 = c_2 - 2c_1 + 0c_4}$$

free column is ALWAYS linearly dependent on pivot columns

pivot

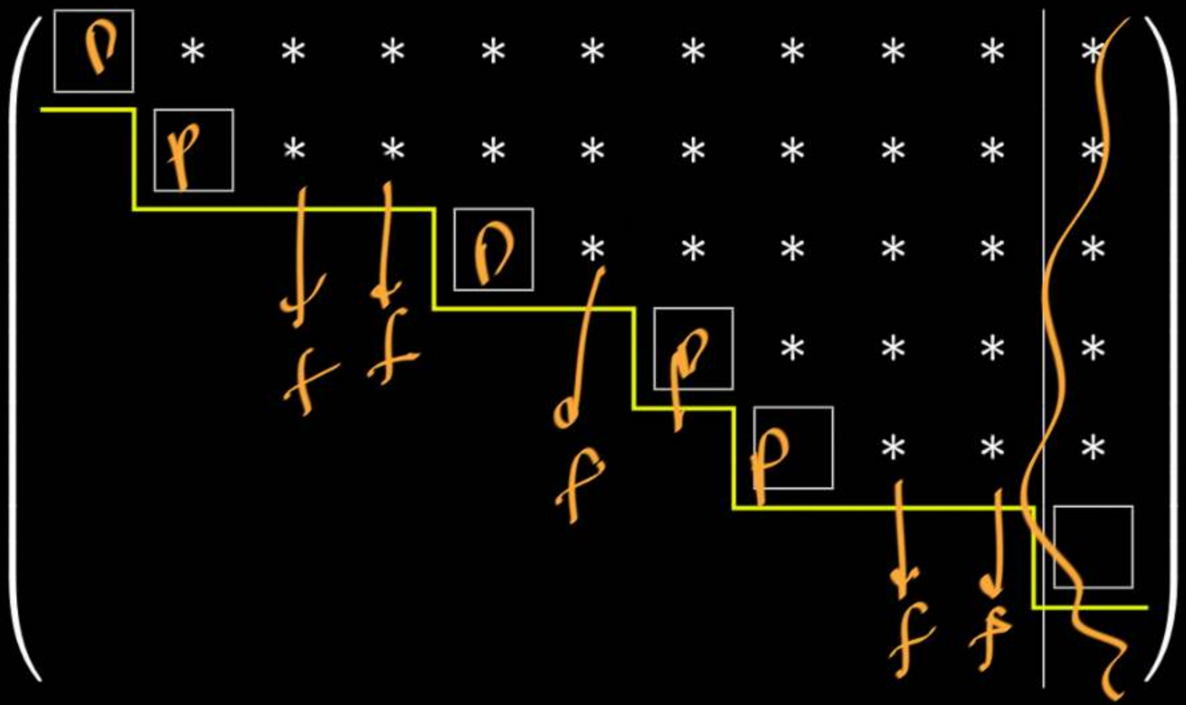
free coulmn = $\square + \square + \square$

Mark free and pivot columns in the given echelon form of some augmented matrix

$$\begin{array}{cccccccc} & P & & & P & & P & \end{array}$$

$$\begin{bmatrix} 0 & * & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & * & \bullet & \bullet & \bullet \\ \vdots & 0 & 0 & 0 & 0 & 0 & * & \bullet \\ \vdots & \vdots & \vdots & & \cdots & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Mark free and pivot columns in the given echelon form of some augmented matrix



Elementary row operations

There are three types of **elementary row operations** which may be performed on the rows of a matrix:

- Swap the positions of two rows. $R_i \leftrightarrow R_j$
- Multiply a row by a non-zero scalar. $R_i \to cR_i$
- Add to one row a scalar multiple of another.

$R_1 \to \underbrace{3R_1 + \alpha}_{R_1}$

$R_i \to R_i + cR_j$

$$3x + 2y - z = 5$$
$$2x - 3y + 5z = 17$$
$$9x + 7y - 3z = 12$$

$$\begin{array}{cccc} x & & z & \\ \left[\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 2 & -3 & 5 & 17 \\ 9 & 7 & -3 & 12 \end{array}\right] \end{array}$$

$$R_1 \longrightarrow R_1 + 2R_2$$

$$R_i \rightarrow R_i + cR_j$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 2 & -3 & 5 & 17 \\ 9 & 7 & -3 & 12 \end{array}\right] \xrightarrow{R_1 \rightarrow R_1 + 2R_2} \left[\begin{array}{ccc|c} 7 & -4 & 9 & 39 \\ 2 & -3 & 5 & 17 \\ 9 & 7 & -3 & 12 \end{array}\right]$$

$$\left.\begin{aligned} 3x + 2y - z &= 5 \\ 2x - 3y + 5z &= 17 \\ 9x + 7y - 3z &= 12 \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} 7x - 4y + 9z &= 39 \\ 2x - 3y + 5z &= 17 \\ 9x + 7y - 3z &= 12 \end{aligned}\right.$$

$$R_1 \longrightarrow R_1 + 2R_2$$

✓ $R_1 \rightarrow 2R_1$

✓ $R_1 \rightarrow R_1 + R_2 + R_2$

✓ $R_1 \leftrightarrow R_2$

✓ $R_1 \rightarrow R_1 + 2R_2$

✓ $R_1 \rightarrow 2R_1 + 3R_2$

$R_1 \rightarrow R_2 + R_3$ ✗

✓ $R_2 \rightarrow 3R_1 - 2R_2$

✗ $R_3 \rightarrow 3R_1 + 5R_2$ } ← this is NOT allowed because R_3 is not in RHS.

$R_1 \longleftrightarrow R_2$

$R_1 \longrightarrow R_1 + R_2 + R_3 + R_4 - SR_5$

$R_1 \longrightarrow$ — — — — — —

Here we need R_1

$R_1 \rightarrow 3R_1 + 2R_2$ ✓

$R_1 \rightarrow 3R_1 + 2R_2$ ← in case of finding the determinant, it will change value of determinant.

$R_1 \rightarrow R_1 + 2R_2$

$R_1 \rightarrow R_1 + 2R_2$ ← in case of finding the determinant, it will NOT change value of determinant.

Convert given matrix to Echelon Form

(another way of asking same question – Apply Gauss Elimination to given matrix)

$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix}$$

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Guass elemination

$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

Row echelon form

$$\begin{bmatrix} \boxed{1} & 3 & 1 & 9 \\ 0 & \boxed{-2} & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Convert given matrix to Echelon Form

$$\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \frac{3}{2} R_2$$

free

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank ← single number

Rank

- # Linearly independent rows
- # Linearly independent columns
- # Pivot elements in echelon form of matrix
- # non-zero rows of an echelon form of matrix

Find Rank

$$M = \begin{bmatrix} 1 & 5 & -1 & 7 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rank}(M) = 4$$

GATE CSE 1994

The rank of matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is:

A. 0
B. 1
C. 2
D. 3

GATE CSE 1994

The rank of matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is:

A. 0
B. 1
C. 2
D. 3

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 \to} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank is zero only for Zero matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 0$$

In Row Echelon Form

Number of Pivot Variables

||

Number of Linearly Independent Columns

||

Number of Linearly Independent Rows

||

Rank of Matrix

Number of Pivot Variables + Number of Free Variables

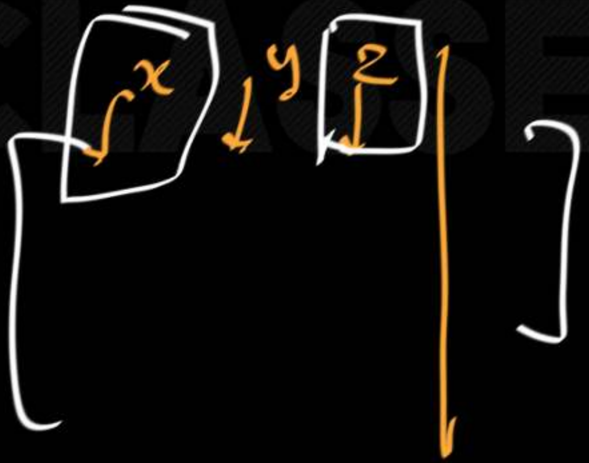
= Total number of variables

= Total number of Columns

3

Pivot 2

free 1



n columns in cofficient matrix

free

Pivot

$x \quad y \quad z \quad w \quad \ldots$

$n - r$ = nullity

r = rank

Nullity : Number of free variables

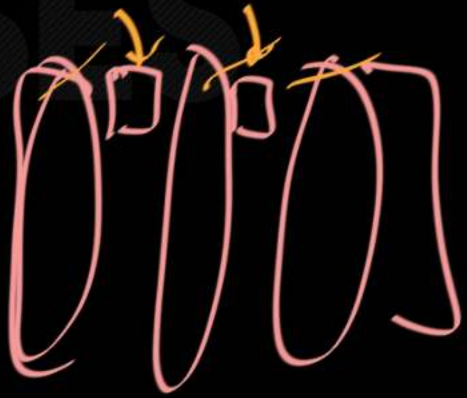
Rank : Number of pivot Columns (r)

Nullity : Number of free columns

n

Rank r

Nullity $n-r$



Rank-Nullity Theorem

Rank + Nullity = n

pivot

free

No. of columns in A

Types of System of Linear Equations

- Homogenous $Ax = 0$
- Heterogenous $Ax = b$

(Non homogenous)

Question 1

Solve the following system of equations using Gaussian reduction.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 2 \\ 2x_1 - x_2 - x_3 &= -3 \end{aligned}$$

https://math.okstate.edu/people/binegar/3013-S99/3013-l05.pdf

$$x_1 + x_2 - x_3 = 0$$
$$x_1 - x_2 + x_3 = 2$$
$$2x_1 - x_2 - x_3 = -3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & -1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & -3 & 1 & -3 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_3 \to -2R_3 + 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

$x_1 = 1$
$x_2 = 2$
$x_3 = 3$

$$\boxed{\begin{array}{l} x_1 + 2 - 3 = 0 \\ \Rightarrow x_1 = 1 \end{array}}$$

$-2x_2 + 2x_3 = 2 \Rightarrow -2x_2 + 6 = 2$

$\Rightarrow$

$-2x_2 = -4$

$x_2 = 2$

$4x_3 = 12 \Rightarrow x_3 = 3$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & -1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

First we write down the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & -1 & -3 \end{array}\right]$$

We now use elementary row operations to convert this matrix into one that is upper triangular.
Adding -1 times the first row to the second row produces

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 2 & -1 & -1 & -3 \end{array}\right]$$

Adding -2 times the first row to the second row produces

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & -3 & 1 & -3 \end{array}\right]$$

Adding $-\frac{3}{2}$ times the second row to the third row produces

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -2 & -6 \end{array}\right]$$

The augmented matrix is now in upper triangular form. It corresponds to the following system of equations

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_2 + 2x_3 &= 2 \\ -2x_3 &= -6 \end{aligned}$$

which can easily be solved via back-substitution:

$$\begin{aligned} &-2x_3 = -6 \quad \Rightarrow \quad x_3 = 3 \\ \Rightarrow \quad &-2x_2 + 6 = 2 \quad \Rightarrow \quad x_2 = 2 \\ \Rightarrow \quad &x_1 + 2 - 3 = 0 \quad \Rightarrow \quad x_1 = 1 \end{aligned}$$

Question 2

Solve the following system of equations.

$$5x_1 - 11x_2 = -2$$
$$-4x_1 + 9x_2 = 1$$
$$x_1 - 2x_2 = -1$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -4 & 9 & 1 \\ 5 & -11 & -2 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 4R_1$
$R_3 \rightarrow R_3 - 5R_1$

$$\begin{bmatrix} 1 & -2 & -1 \\ -4 & 9 & 1 \\ 5 & -11 & -2 \end{bmatrix}$$

$R_2 \to R_2 + 4R_1$

$R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -4 & 9 & 1 \\ 5 & -11 & -2 \end{bmatrix} \rightsquigarrow \begin{array}{ccc} x_1 & x_2 & \\ \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$x_2 = -3$$

$$x_1 - 2x_2 = -1$$

$$x_1 = -1 - 6$$
$$= -7$$

Question 3

Solve the following system of equations.

$$3x_1 = 6,$$

$$2x_2 = 5,$$

$$4x_3 = 7$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 4 & 7 \end{array}\right]$$

$3x_1 = 6$

$x_1 = 2$

$2x_2 = 5$

$\Rightarrow x_2 = 5/2$

$4x_3 = 7 \Rightarrow x_3 = 7/4$

Question 4

Solve the following system of equations.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_3$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \rightsquigarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Very first step:

identify the free variable and assign a constant parameter to that

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$x_3 = K$ (1st step)

$3x_2 = 0 \Rightarrow x_2 = 0$

$3x_1 + 5x_2 - 4x_3 = 0$

$3x_1 - 4K = 0 \Rightarrow x_1 = \frac{4}{3}K$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\underline{x_3 = K}$ (1^{st} step)

$3x_2 = 0 \Rightarrow \underline{x_2 = 0}$

$3x_1 + 5x_2 - 4x_3 = 0$

$3x_1 - 4K = 0 \Rightarrow \underline{x_1 = \frac{4}{3}K}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}K \\ 0 \\ K \end{bmatrix}$$

$$x = K \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$$x = K \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$K = -1$ $\begin{bmatrix} -4/3 \\ 0 \\ -1 \end{bmatrix}$

$K = 1$ $\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$

$K = 5$ $\begin{bmatrix} 20/3 \\ 0 \\ 5 \end{bmatrix}$

How may sol?

one possible = inf

How many LI sol?

one possible = 1

Question 5

Solve the following system of equations.

$$
\begin{aligned}
x - 3y + z &= 4 \\
-x + 2y - 5z &= 3 \\
5x - 13y + 13z &= 8
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
1 & -3 & 1 & 4 \\
-1 & 2 & -5 & 3 \\
5 & -13 & 13 & 8
\end{array}\right]
$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ -1 & 2 & -5 & 3 \\ 5 & -13 & 13 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 2 & 8 & -12 \end{array}\right]$$

$R_2 \to R_1 + R_2$

$R_3 \to R_3 - 5R_1$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ -1 & 2 & -5 & 3 \\ 5 & -13 & 13 & 8 \end{array}\right] \rightsquigarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 0 & 0 & 2 \end{array}\right] \end{array}$$

No soln

1 – 3 unique sol^n

4 inf sol^n

5 No sol^n

sol^n exist

Homework Question

Solve the following system of equations.

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 2x_1 - x_2 - 3x_3 &= 3 \\ 4x_1 + 5x_2 - x_3 &= 7 \end{aligned}$$

Why sometimes solution exist and sometimes not ?

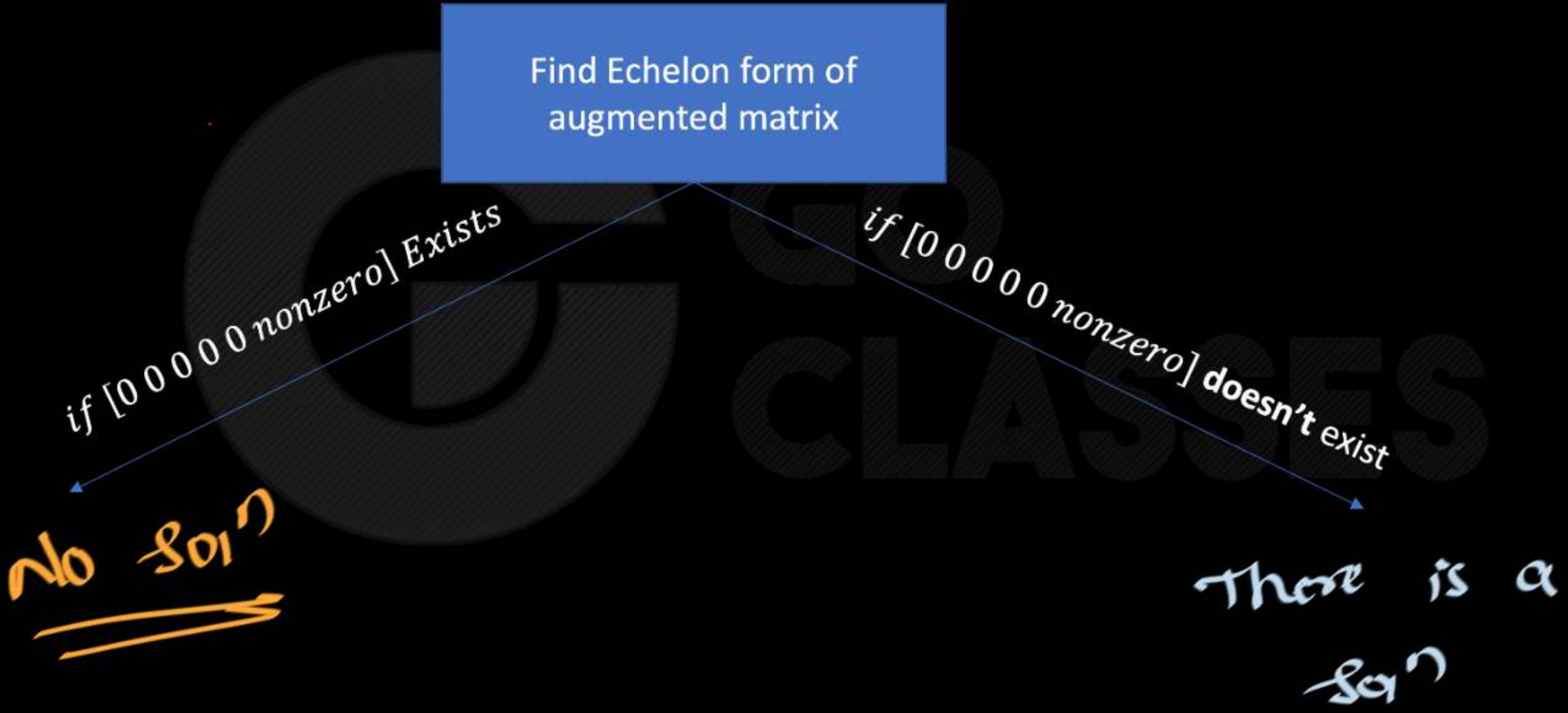
$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & b \end{array}\right]$$

b → nonzero scalar

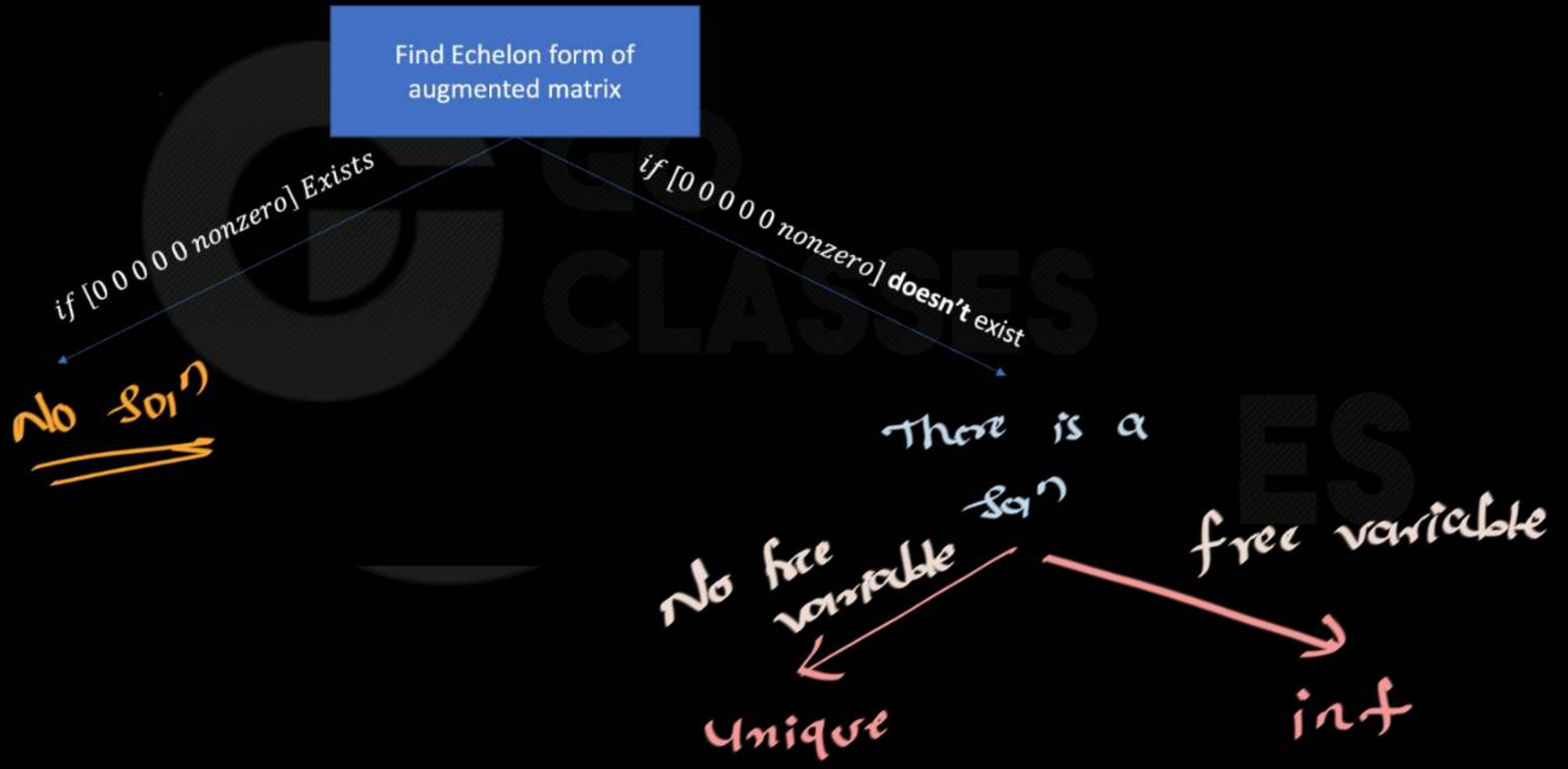
$$\left[\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{nonzero} \end{array}\right]$$

Solution does not exist

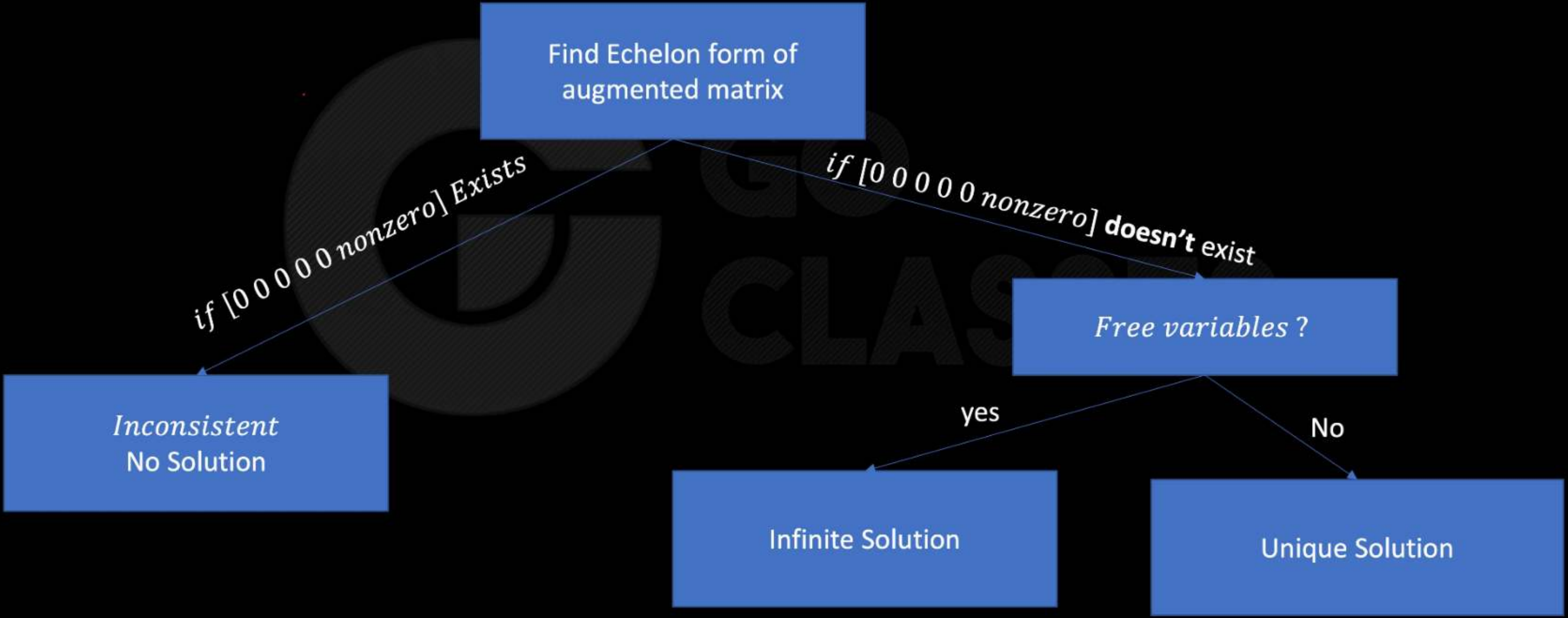
System of Linear Eqns flowchart (Ax = b)



System of Linear Eqns flowchart (Ax = b)



System of Linear Eqns flowchart (Ax = b)



optional read

Existence and Uniqueness

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

David clay

$$\left[\begin{array}{cccccc|c} & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{non zero} \end{array}\right] \longleftarrow$$

sol^n does not exist

$\underline{\underline{Ax=0}}$

Homogeneous always have trivial

Sol^n

we don't have $[0\ 0\ 0\ 0\ 0 \mid \text{Nonzero}]$ row there.

True/False

A homogeneous system with 5 equations and 5 unknowns always have unique solution?

True/False

A homogeneous system with 5 equations and 5 unknowns always have unique solution?

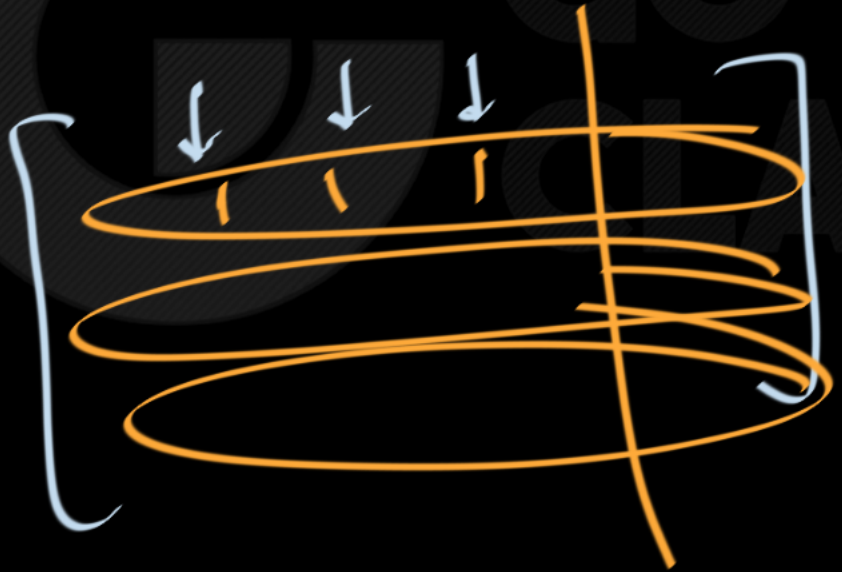
there can be inf sol's also

free

$$
\begin{cases}
x+y+z+w+v=0 \\
2x+2y+2z+2w+2v=0 \\
3x+3y+3z+3w+3v=0
\end{cases}
$$

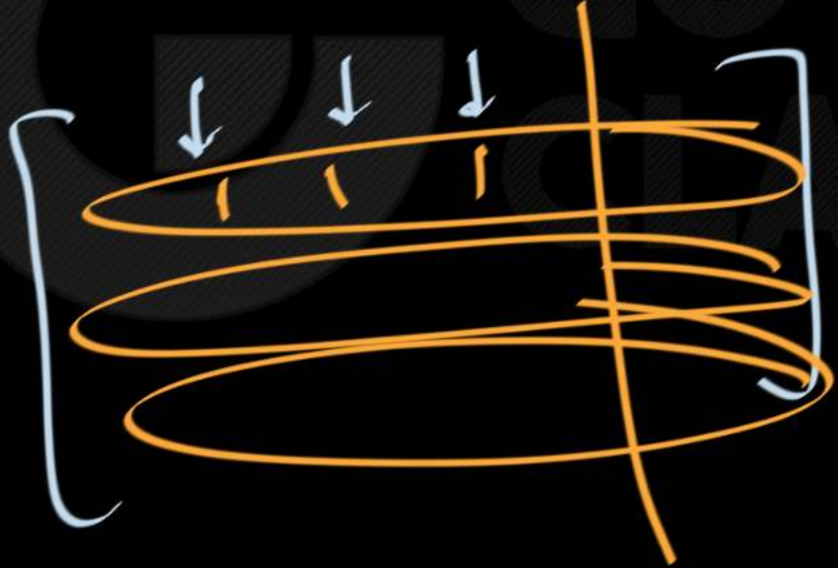
True/False

A homogeneous system with 10 equations and 3 unknowns always have unique solution?



True/False

A homogeneous system with 10 equations and 3 unknowns always have unique solution?



depends on whether we have free variable or not.

True/False

Non Homogeneous system does not have solution iff

$[\,0\ 0\ 0 \ldots 0 \mid Nonzero\,]$ exist

Question

Find numbers a, b, c, and d such that the linear system corresponding to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{array}\right)$$

has

a) no solutions, and

b) infinitely many solutions.

Question

Find numbers a, b, c, and d such that the linear system corresponding to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{array}\right)$$

has

$d = 0 \quad c \neq 0$

a) no solutions, and

b) infinitely many solutions.

$\underline{d = 0} \quad \underline{c = 0} \quad a \quad b$ could be anything

Solution.

a) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ has no solutions.

b) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ has infinitely many solutions.

Question

Example Let

$$A = \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \end{array}$$

Solve $A\mathbf{x} = 0$

$$\begin{array}{cccc} x & y & z & w \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 3 & 0 \end{array}\right]$$

$$R_2 \rightarrow 2R_2 - R_1$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 4 & 0 \end{array}\right]$$

$$\begin{array}{cccc|c} x & y & z & w & \\ \hline \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 4 & 0 \end{array}\right]$$

free variables :

z, w $\quad z = s$

$w = t$

$y + s + 4t = 0$

$y = -(s+4t)$

$2x - s - 4t - s + 2t = 0$

$2x = 2s + 2t$

$x = s + t$

$y = -(s+4t)$

$2x - s - 4t - s + 2t = 0$

$2x = 2s + 2t$

$x = s + t$

$$x = \begin{bmatrix} s+t \\ -(s+4t) \\ s \\ t \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} s+t \\ -s-4t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$s=1$ $\quad$ $t=1$

$t=0$ $\quad$ $s=0$

$$x = \begin{bmatrix} 4/3\,k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} s+t \\ -s-4t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

No. of Soln = inf.

No of linearly indep Soln = 2

imagine a scenerio where you have 3 free variables

$$x = \begin{bmatrix} \vdots \\ s+2t+u \\ s+t \\ s \\ t \\ u \end{bmatrix} = s \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$x = \begin{bmatrix} s-2t+u \\ s+t \\ s \\ t \\ u \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$s=1$
$t=0$
$u=0$

$t=1$
$s=0$
$u=0$

$u=1$
$t=0$
$s=0$

for inf sol^{n}s

No. of linearly indep. sol^{n}s =

No. of free variables

= Nullity

$$\mathbf{x} = \begin{bmatrix} s+t \\ -s-4t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} t$$

Question

Suppose a matrix $A_{20\times 50}$ has 10 pivot column, then –

- Number of Linearly independent solutions in $Ax\ =\ 0$?
- Number of Linearly independent columns in A ?

Question

Suppose the parametric form of the solutions to $Ax = 0$ looks like following –

Dimensions of A - $A_{20\times 30}$

$n = 30$ ← total variables

$$x = s\begin{bmatrix} \star \\ \star \\ \star \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} \star \\ \star \\ \star \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} + p\begin{bmatrix} \star \\ \star \\ \star \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Number of Linearly independent columns in A ?

Question

Suppose the parametric form of the solutions to $Ax = 0$ looks like following –

$\leftarrow$ total variables

Dimensions of A - $A_{20\times30}$

$n = 30$

$$x = s\begin{bmatrix}\star\\\star\\\star\\\vdots\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}\star\\\star\\\star\\\vdots\\0\\1\\0\end{bmatrix} + p\begin{bmatrix}\star\\\star\\\star\\\vdots\\0\\0\\1\end{bmatrix}$$

Number of Linearly independent columns in A ?

No. of free variables $= 3$

No. of free variables = 3

No. of pivot variables = 30 - 3

= 27

[No. of LI col^n = 27]

Homework Question

$$B = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 0 & 0 & 1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve $B\mathbf{x} = 0$

$$B = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 0 & 0 & 1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

► **Solution.** Putting B in reduced row-echelon form (it is already in row-echelon form) produces

$$R = \begin{bmatrix} 1 & -2 & 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2s+3t+9u \\ s \\ 2t+4u \\ t \\ -5u \\ u \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2 + u\mathbf{v}_3$$

GATE 2021

Suppose that P is a 4×5 matrix such that every solution of the equation $\mathrm{Px}=0$ is a scalar multiple of $\begin{bmatrix} 2 & 5 & 4 & 3 & 1 \end{bmatrix}^T$. The rank of P is ________