New Topic -

System of linear equations

$$\left.\begin{aligned} x - 3y &= -5 \\ 2x + 4y &= 10 \end{aligned}\right\} \leftarrow \text{system of linear equations}$$

$$\left.\begin{array}{l} x^2 + y = 5 \\[2ex] x - xy = 4 \\[2ex] x_1^2 + x_2^2 = 4 \\[2ex] -\sqrt{x} + y = 2 \end{array}\right\} \leftarrow \text{Non linear equations}$$

variables

$$
\left.
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\right\}
$$

System of linear equations

cofficient

$$
\begin{aligned}
x + y - z &= 8 \\
x - 2y + z &= 3 \\
x + 3y + 2z &= 7
\end{aligned}
$$

Example 1

$$
\begin{aligned}
2x - y + z &= -4 \\
-x + y + 3z &= -7 \\
x + 3y - 4z &= 22
\end{aligned}
$$

Example 2

$x, y, z = \; ?$

Why we want to solve

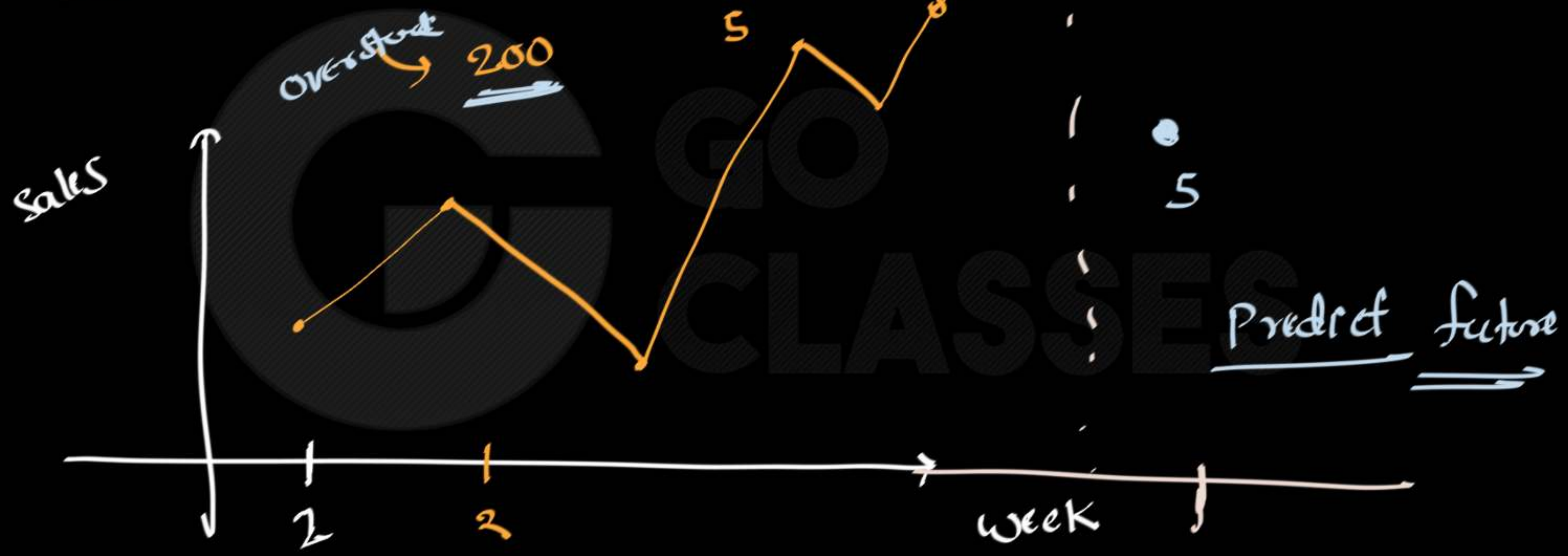
System of linear equations

?

Amazon

estimate real

out of stock → 5 200



Application 1: Price prediction

	Square feet	BHK	Number of bedrooms	Number of washrooms	Area of dinning room	Year built	Price
	x_1	x_2	x_3	x_4	x_5	x_6	
House1	1200	2	1	1	300	1990	35
House2	1500	3	2	3	400	2000	40
House3	1100	2	2	2	350	2020	45
House100							

Application 1: Price prediction

	x_1 Square feet	x_2 BHK	x_3 Number of bedrooms	x_4 Number of washrooms	x_5 Area of dinning room	x_6 Year built	Price
House1	1200	2	1	1	300	1990	35
House2	1500	3	2	3	400	2000	40
House3	1100	2	2	2	350	2020	45
⋮							
House100							

$x_1, x_2 \ldots x_6 =$

$$\begin{cases} 1200x_1 + 2x_2 + 1x_3 + 1x_4 + 300x_5 + 1990x_6 = 35 \\ 1500x_1 + 3x_2 + 2x_3 + \ldots = 40 \\ = 45 \end{cases}$$

Application 1: Price prediction

	x_1 Square feet	x_2 BHK	x_3 Number of bedrooms	x_4 Number of washrooms	x_5 Area of dinning room	x_6 Year built	Price
House1	1200	2	1	1	300	1990	35
House2	1500	3	2	3	400	2000	40
House3	1100	2	2	2	350	2020	45
⋮							
House100							30

$x_1, x_2 \ldots x_6$

House 101 1550 3 2 3 500 2022 ??

$x_1, x_2 \ldots x_6 =$

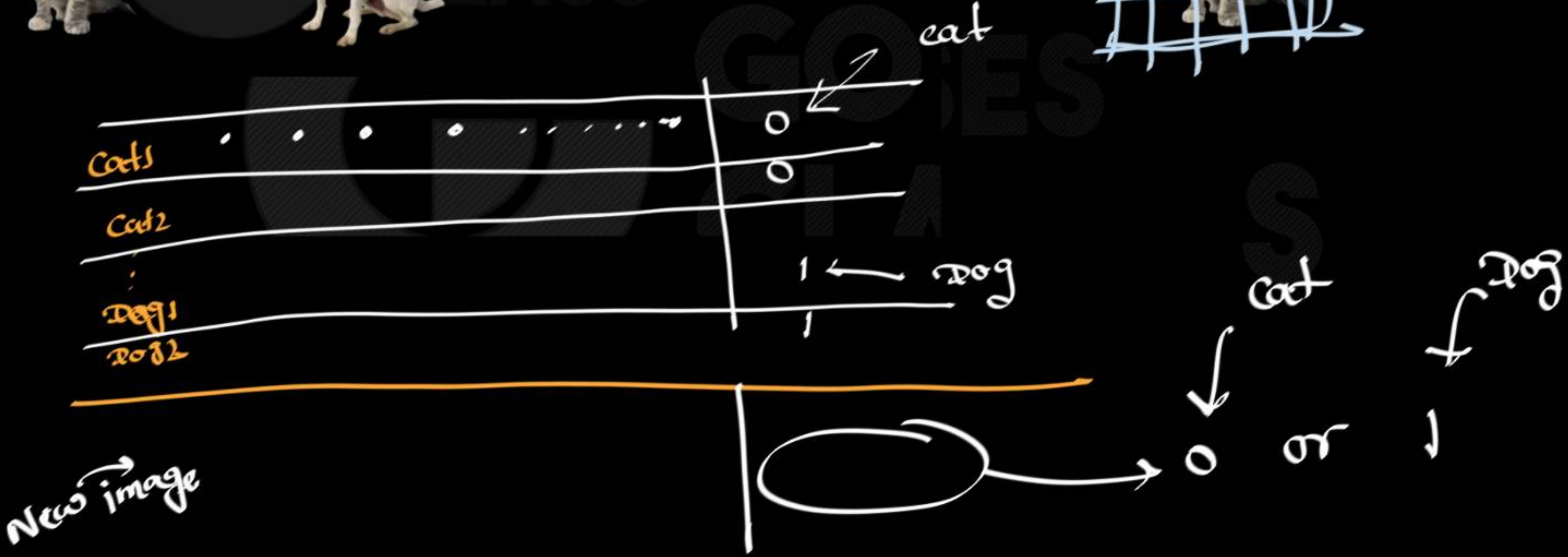
$1550x_1 + 3x_2 + 2x_3 + 3x_4 + 500x_5 + 2022x_6 =$

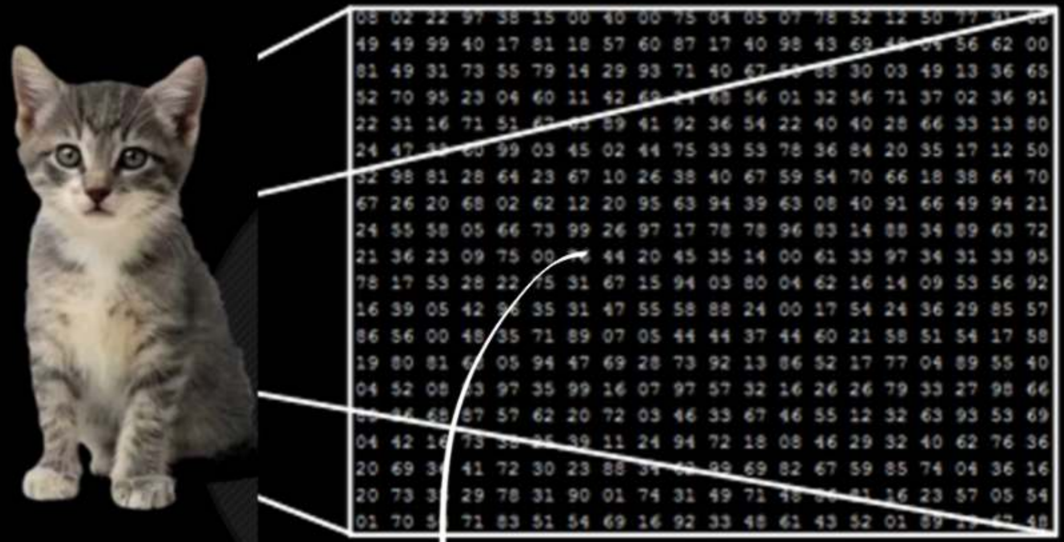
Application 2: Image Classification

Or

?

Application 2: Image Classification





What the computer sees

x_1 - - - - x_{400}

0

0

1

	x_1	x_2 ----	x_{100}	
image1				0
				1
				0
				0
				1

2 30 80 100

$x_1, x_2 \ldots x_{100}$

$2x_1 + 30x_2 + \cdots x_{100}$

dog 1

cat 0

System of linear equations

- Why solve System of linear equations ?
- Geometric Interpretation
- Understanding Ax = b intuitively and finding the number of solution
- A step by step method to find Solution for Ax=b (Gaussian Elimination)
    - Rank
    - Parametric form of solution

The Matrix form of equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 6 \\ 2x & + & 5y & + & 2z & = & 4 \\ 6x & - & 3y & + & z & = & 2 \end{array}$$

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \textit{2 times column 3} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Two equations
Two unknowns

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

Matrix equation
$\underline{Ax = b}$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

for our understanding

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$\begin{aligned} x + 2y + 3z &= 4 \\ 3x + 4y + z &= 5 \\ 2x + y + 3z &= 6 \end{aligned}$$

can be written as

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}}_{\text{This is called coefficient matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Geometric interpretation of $Ax = b$

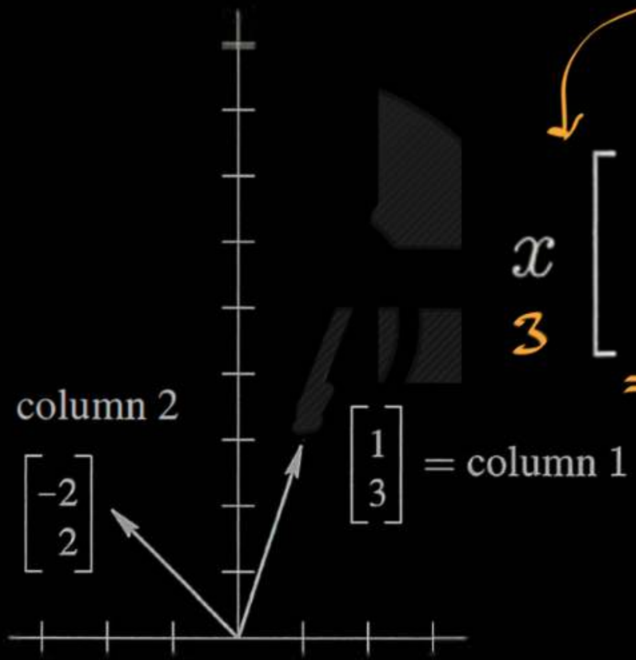
Geometric interpretation(1)

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$Ax = b$$

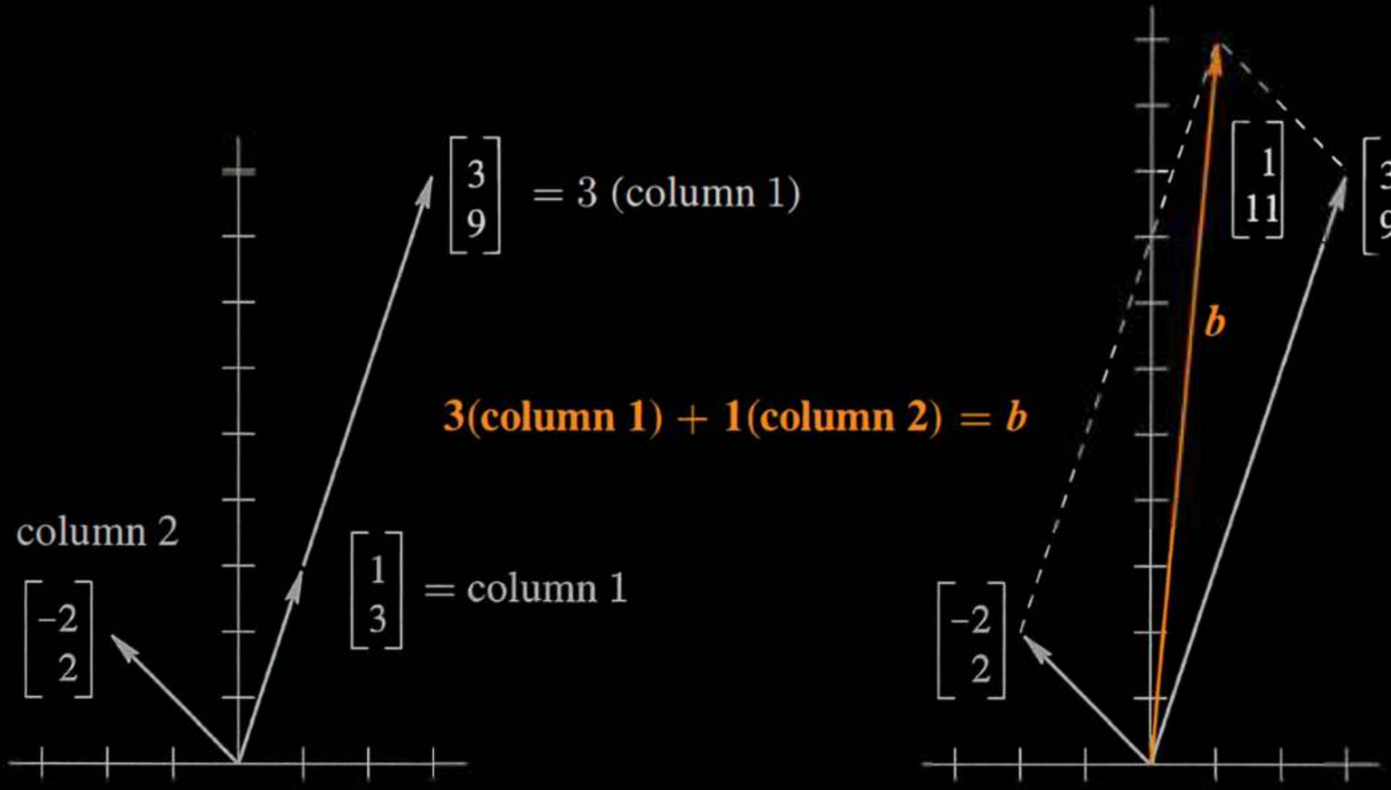
$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$



$$\underset{3}{x}\begin{bmatrix}1\\3\end{bmatrix}+\underset{1}{y}\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

This interpretation is called **Column Picture**



This interpretation is called **Column Picture**

Geometric interpretation(2)

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$\underline{\underline{Ax = b}}$

$$\boxed{\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}}$$

Column picture ✓

$$\boxed{x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}}$$

Geometric interpretation(2)

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

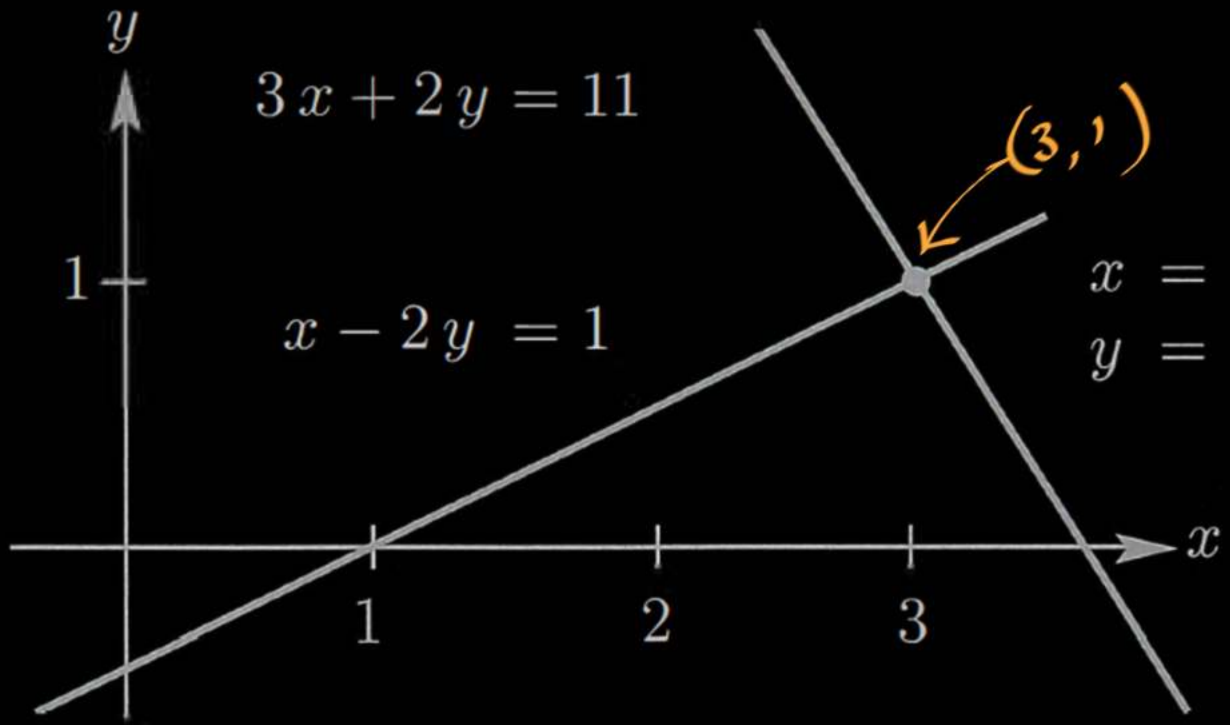
$$\underline{\underline{Ax = b}}$$

$$\left\{ \boxed{\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}} \right.$$

Row picture

Two equations
Two unknowns

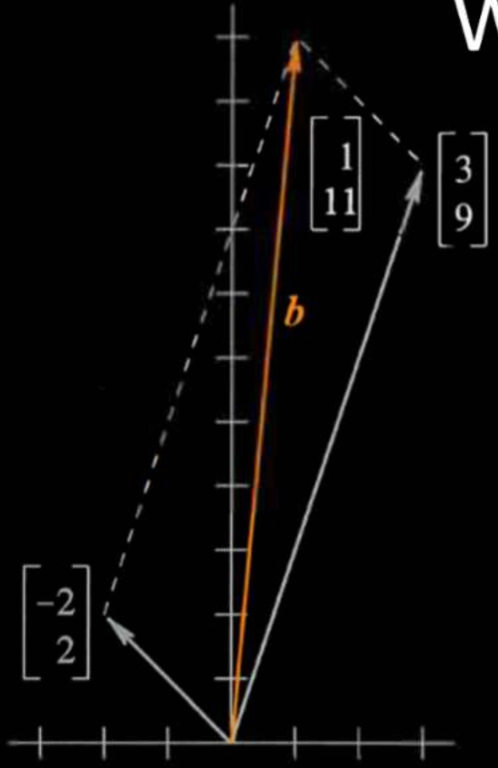
$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$



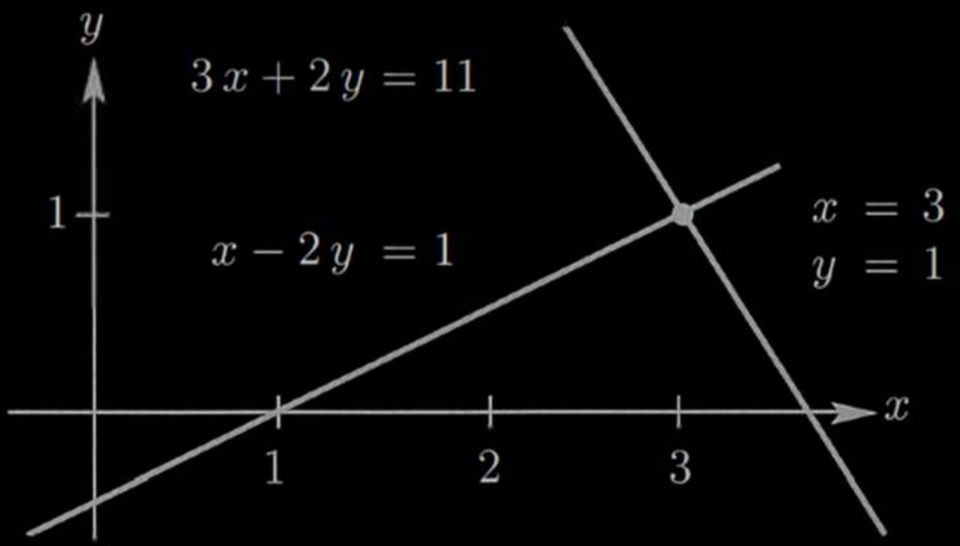
This interpretation is called **Row Picture**

Which Interpretation you like ?

Which Interpretation you like ?



Column Picture

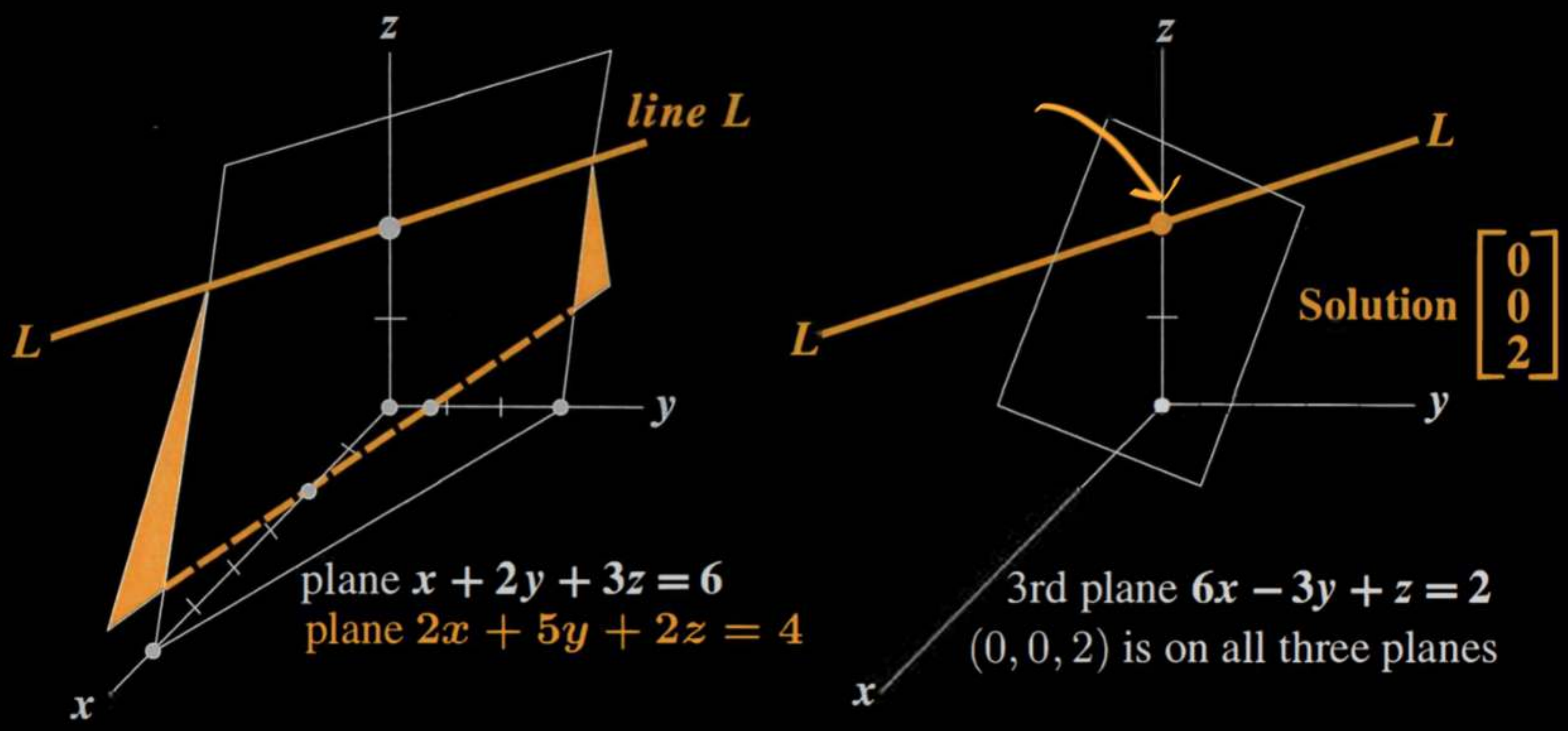


Row Picture

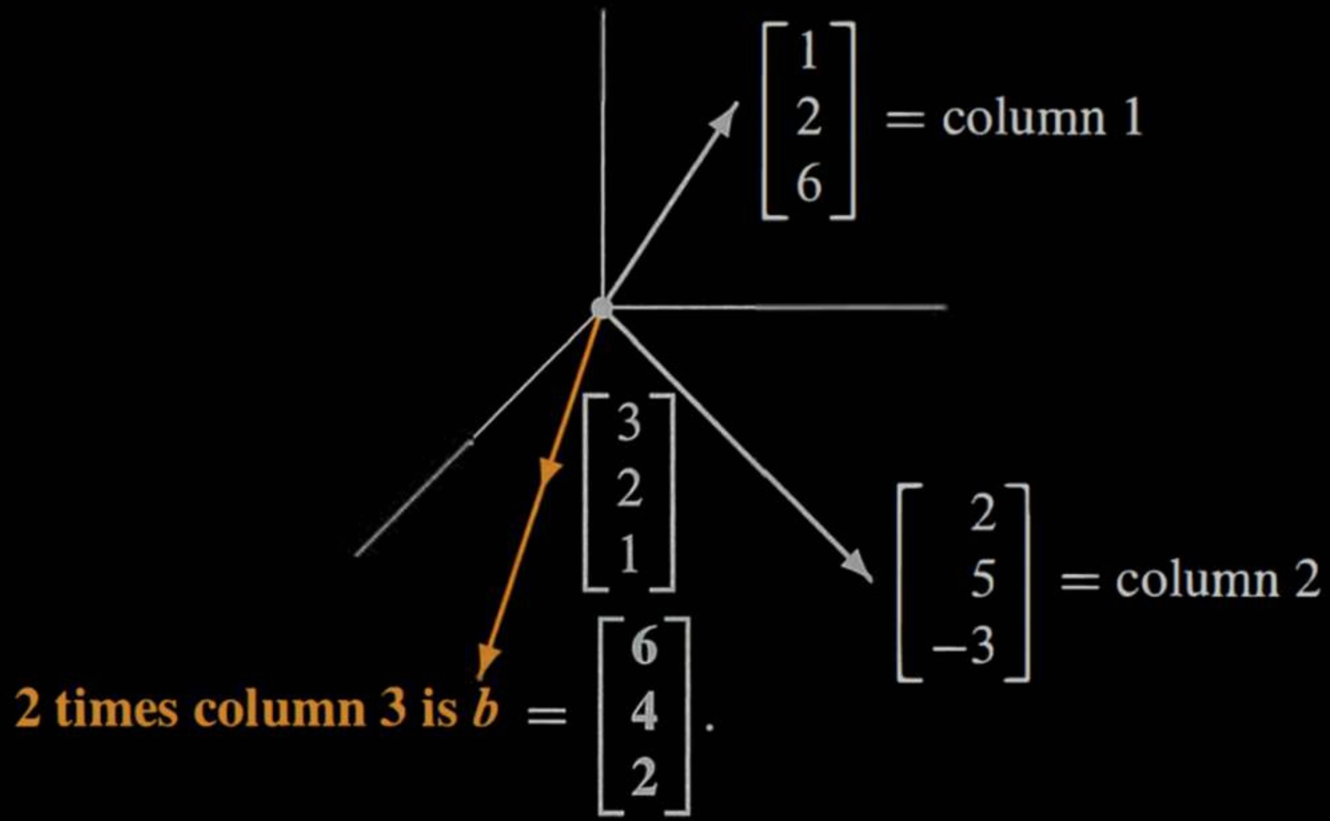
$$\begin{aligned} x &+ 2y + 3z = 6 \\ 2x &+ 5y + 2z = 4 \\ 6x &- 3y + z = 2 \end{aligned}$$

ROW *The row picture shows three planes meeting at a single point.*

COLUMN *The column picture combines three columns to produce* $b = (6, 4, 2)$.



Row picture: Two planes meet at a line L. Three planes meet at a point.



Column picture: Combine the columns with weights $(x, y, z) = (0, 0, 2)$.

interesting

Do we always have solution of $Ax = b$?

Do we always have solution?

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

Example1

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

Example2

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

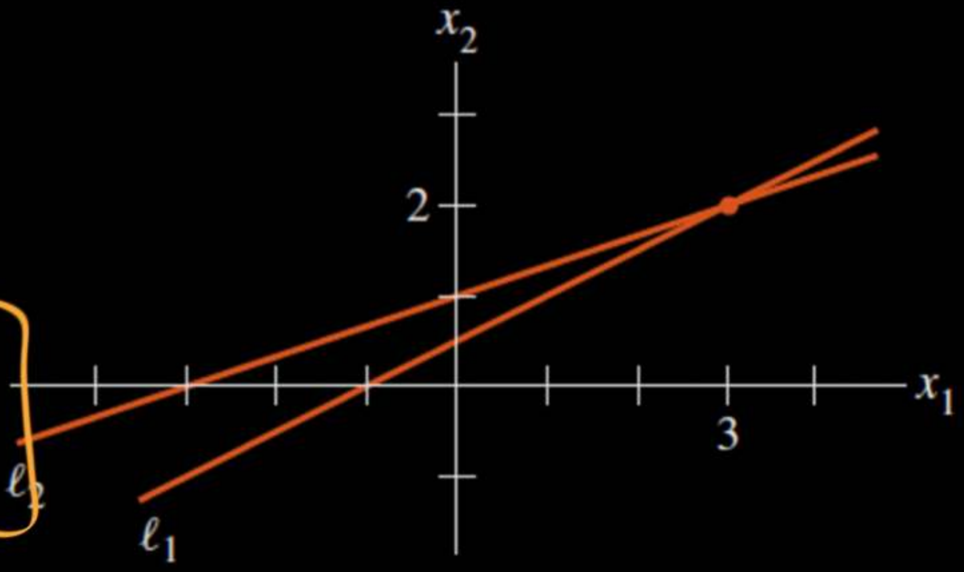
Example3

Case 1: Unique solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

Example1

$$\begin{bmatrix}1\\-1\end{bmatrix} \qquad \begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

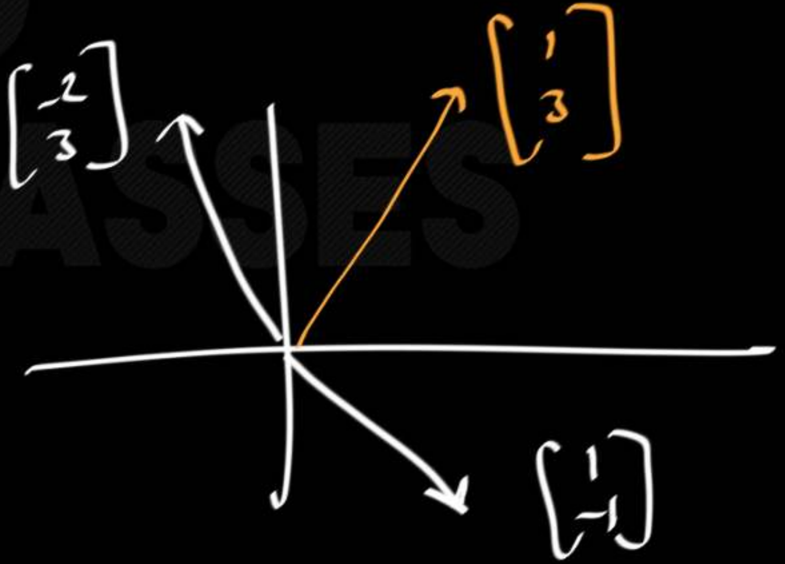


Case 1: Unique solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

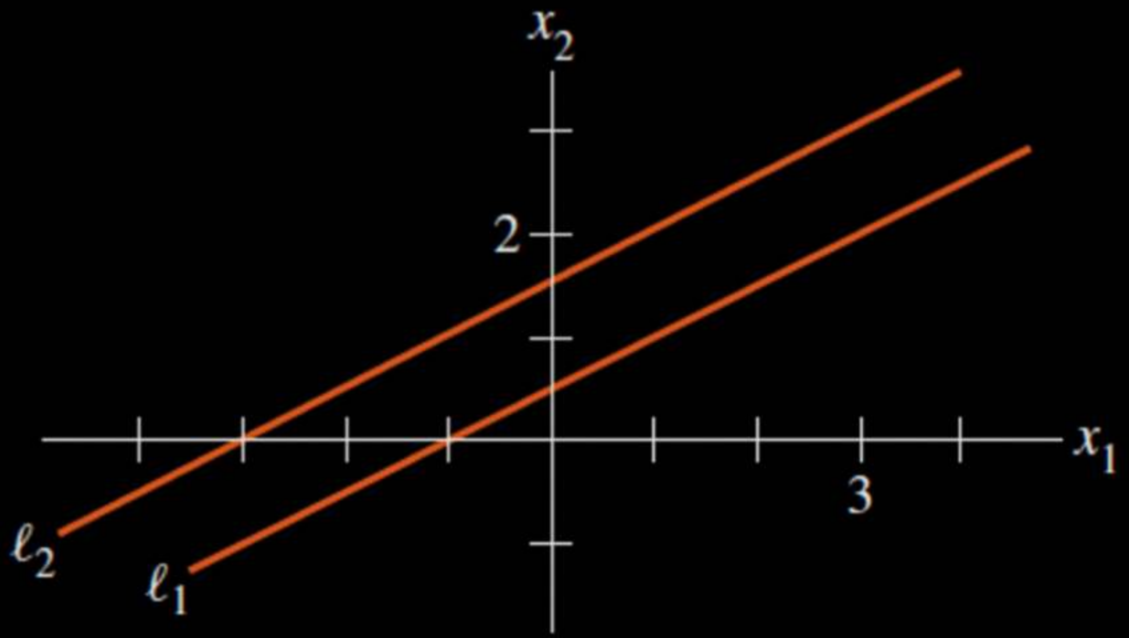
Example1

$$\begin{bmatrix}1\\-1\end{bmatrix} \quad \begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$



Case 2: No solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$



$$\left.\begin{cases} x+y=3 \\ x+y=5 \end{cases}\right\}$$ Not agreeing to each other

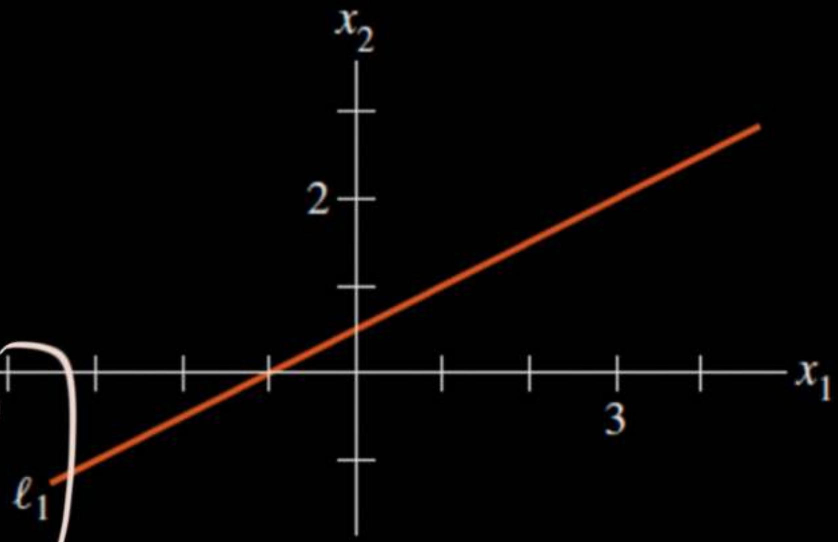
↳ No sol^n

Case 3: Infinite Solution

$$
\begin{aligned}
x_1 - 2x_2 &= -1 \\
-x_1 + 2x_2 &= \ \ 1
\end{aligned}
$$

-1

$$-1\begin{bmatrix}1\\-1\end{bmatrix} + 0\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$



$x = k$

$y = 5 - k \quad (k, 5-k)$

$$\begin{bmatrix} x + y = 5 \\ \cancel{2x + 2y = 10} \end{bmatrix}$$

$x + y = 5$

x	y
1	4
2	3
-1	6

$(1, 4)$

$(2, 3)$

$(-1, 6)$

System of linear equations

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- A step by step method to find Solution for $Ax=b$ (Gaussian Elimination)
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    - Parametric form of solution

Understanding $Ax = b$ intuitively

$Ax = b$

Matrix

vector

vector

$$\underset{A}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}} \underset{x}{\begin{bmatrix} \ \\ \ \end{bmatrix}} = \underset{b}{\begin{bmatrix} \ \\ \ \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} \uparrow \\ a_1 \\ \downarrow \end{bmatrix} x_1 + \begin{bmatrix} \uparrow \\ a_2 \\ \downarrow \end{bmatrix} x_2 + \begin{bmatrix} \uparrow \\ a_3 \\ \downarrow \end{bmatrix} x_3}$$

$$\underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} \rightarrow \text{linear combination of columns of } A.$$

$$\underbrace{\begin{bmatrix} \uparrow \\ a_1 \\ \downarrow \end{bmatrix} x_1 + \begin{bmatrix} \uparrow \\ a_2 \\ \downarrow \end{bmatrix} x_2 + \begin{bmatrix} \uparrow \\ a_3 \\ \downarrow \end{bmatrix} x_3}$$

$Ax = 0$ has some sol^n.

$$c_1 \begin{bmatrix} \uparrow \\ a_1 \\ \downarrow \end{bmatrix} + c_2 \begin{bmatrix} \uparrow \\ a_2 \\ \downarrow \end{bmatrix} + c_3 \begin{bmatrix} \uparrow \\ a_3 \\ \downarrow \end{bmatrix} = 0$$

$Ax = 0$

$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

Scalar

vector

vector

$Ax = 0$

$\underline{c_1} a_1 + \underline{c_2} a_2 + \underline{c_3} a_3 = 0$

$x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

is there any Non zero c_i ?

Yes → LD

NO → LI

If $Ax = 0$ has some solution then what can you say about linearly dependency of columns of A.

Nothing can be inferred.

True/False

If $Ax = 0$ has some nontrivial solution then columns of A are linearly dependent.

$$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$$

True/False

If $Ax = 0$ has some nontrivial solution then columns of A are linearly dependent.

$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

there is some nonzero c_i

let $c_2 \neq 0$ $\qquad a_2 = -\frac{c_1}{c_2} a_1 - \frac{c_3}{c_2} a_3$

True/False

If $Ax = 0$ has ONLY trivial solution then columns of A are linearly dependent.

$$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$$

$\exists$ a non zero $c_i = ?$

$x = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$

True/False

If $Ax = 0$ has ONLY trivial solution then columns of A are linearly dependent.

$$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$$

$x = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$

$\exists$ a non zero $c_i = ?$

<u>No</u>

linearly indep.

Question: Suppose a matrix A_{3x3} contains 3 linearly independent columns. What can you say about the solutions to $Ax = b$? (here $b \neq \mathbf{0}$)

YES, sol^n always exists

$$\overset{A}{\begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix}}_{3\times 3} \overset{x}{\begin{bmatrix} \ \end{bmatrix}} = \overset{b}{\begin{bmatrix} \ \end{bmatrix}}$$

$\in R^3$ $\in R^3$ $\in R^3$

3 LI columns in R^3

$\Rightarrow$ it will fill whole space

(we can generate any b)

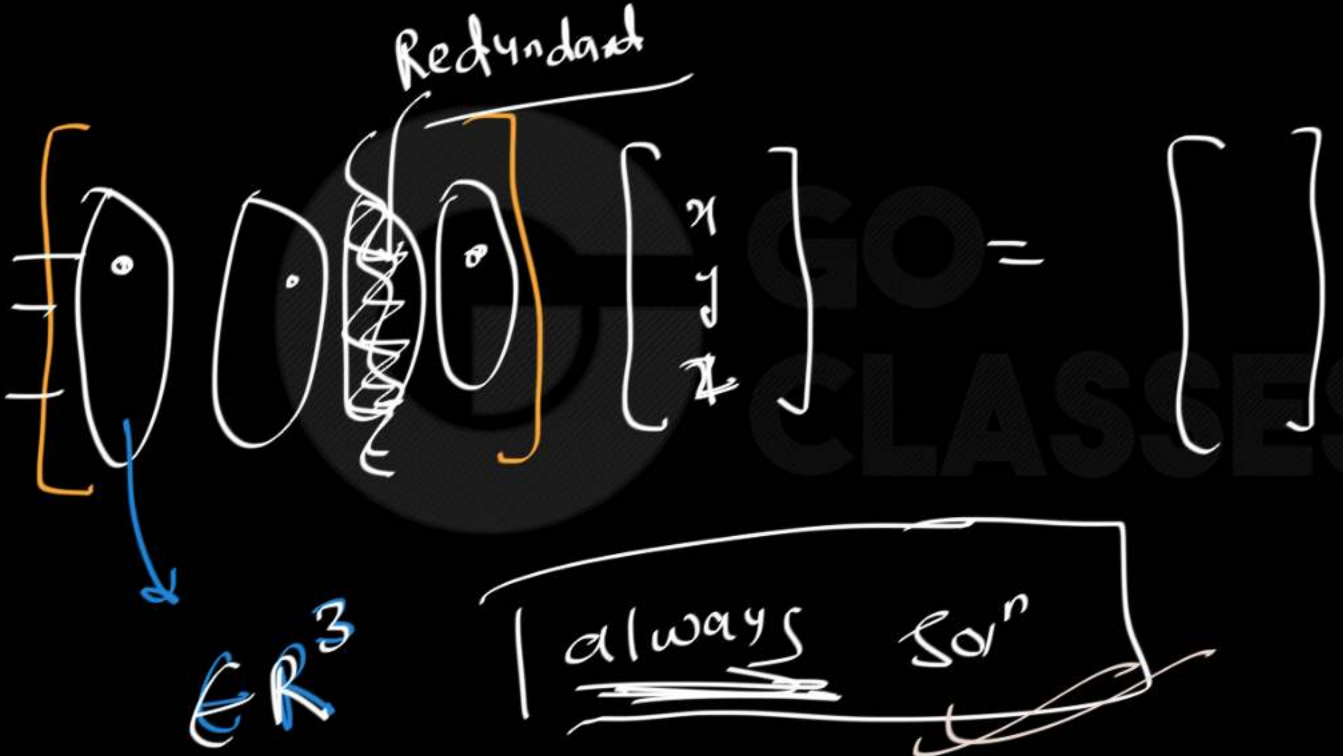
$$\underbrace{\begin{bmatrix} \ & \ & \ \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \ \end{bmatrix}}_{b}$$

$\in R^3$ $\quad \in R^3$ $\quad R^3$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right. \Rightarrow 3 \text{ LI vectors in } \mathbb{R}^3$$

$$\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \underline{5} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underline{6} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Question: Suppose a matrix A_{3x4} contains 3 linearly independent columns. What can you say about the solutions to $Ax = b$? (here $b \neq \mathbf{0}$)



$\boxed{\text{always sol}^n}$

Question: Suppose a matrix A_{5x4} contains 3 linearly independent columns. What can you say about the solutions to Ax = b ? (here $b \neq \mathbf{0}$)

Redundant

b

$$[\ \cdot\ \cdot\ \cdot\ \cdot\ ][\ ] = [\ b\ ]$$

$\in R^5$

depends on b. if b is a linear combination of col^ns of A ⇒ yes

otherwise ⇒ no

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 10 \\ 3 & 6 & 15 & 15 \\ 4 & 9 & 12 & 20 \\ 5 & 10 & 15 & 25 \end{bmatrix} \begin{bmatrix} \; \\ \; \\ \; \\ \; \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

L.I. Col^{n}s

$$\begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

Yes sol^n exists

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 10 \\ 3 & 6 & 15 & 15 \\ 4 & 9 & 12 & 20 \\ 5 & 10 & 15 & 25 \end{bmatrix} \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 25 \end{bmatrix}$$

L.I. Col^{n}s

Not a linear combination of these 3

Question: Suppose a matrix A_{5x2} contains 2 linearly independent columns. What can you say about the solutions to Ax = b ? (here $b \neq \mathbf{0}$)

$\in R^5$ $\in R^5$

=

may or may not

$$\begin{bmatrix}1 & 2\\ 2 & 4\\ 3 & 6\\ 4 & 8\\ 5 & 20\end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\\ 4\\ 5\end{bmatrix}$$

$$\begin{bmatrix}1\\ 0\end{bmatrix}$$

NO sol^n

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \\ 5 & 20 \end{bmatrix} \begin{bmatrix} \; \\ \; \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \\ \underbrace{28} \end{bmatrix}$$

$$\begin{bmatrix} \; \\ \; \end{bmatrix}$$

(assume) not a linear comb of these two col^ns

GO CLASSES

① if you can fill space

$\Downarrow$

Always sol^n

② if you can not fill space

$\Downarrow$

may or may not be

in $\mathbb{R}^n$ if we have n LI col^{n}s then only we can fill space

Question: Suppose a matrix A_{2x5} contains 3 linearly independent columns. What can you say about the solutions to Ax = b ? (here $b \neq \mathbf{0}$)

$\in R^2$

NOT even possible

bcoz in R^2, we can

not have 3 LI col^ns

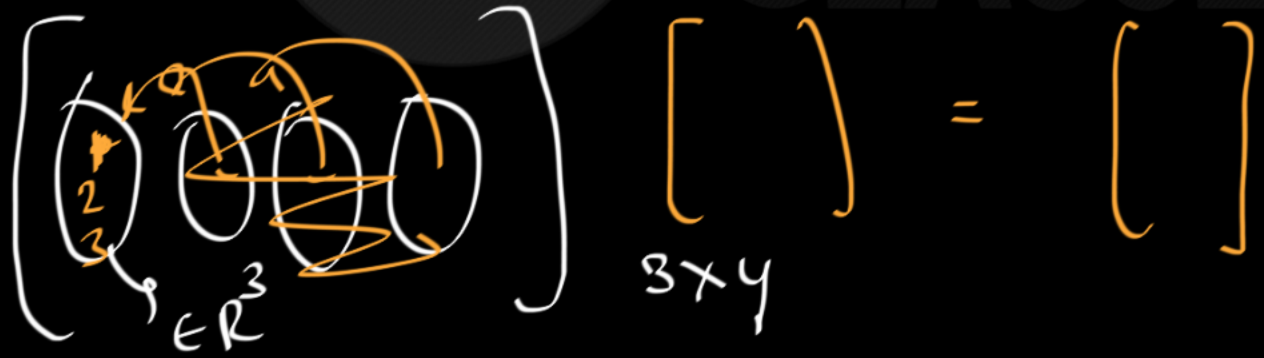
$Ax = \mathbf{0}$ always have one trivial solution which is $x = \mathbf{0}$.

Consider a matrix A with dimension $m \times n$. For system $Ax = b$, what can you say about below statement ?

(Note that b may or may not be zero)

A & b

Statement: if $\overset{3}{m} < \overset{4}{n}$ then $Ax = b$ always have solution (True/False)



Consider a matrix A with dimension $m \times n$. For system $Ax = b$, what can you say about below statement ?

(Note that b may or may not be zero)

(A, b)

Statement: if $\overset{2}{m} < \overset{3}{n}$ then there exist such system $Ax = b$ which has solution even if b is nonzero vector. (True/False)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\in \mathbb{R}^2$

Consider a matrix A with dimension $m{\times}n$. For system $Ax = b$, what can you say about below statement ?

(Note that b may or may not be zero)

A, b

Statement: if $m > n$ then none of the system $Ax = b$ has solution (True/False)

$3 \quad 2$

$$\begin{bmatrix} \ & \ \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

$\in R^3$

Consider a matrix A with dimension $m \times n$. For system $Ax = b$, what can you say about below statement ?

(Note that b may or may not be zero)

Statement: for $\overset{2}{m} < \overset{3}{n}$ there is a matrix A which gives solution of $Ax = b$ for any given b (True/False)

2 LI col to ALWAYS have sol^n

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

$\in R^2$

Consider a matrix A with dimension $m \times n$. For system $Ax = b$, what can you say about below statement ?

(Note that b may or may not be zero)

there is no matrix which can fill whole space.

Statement: for $\overset{3}{m} > \overset{2}{n}$ there is a matrix A which gives solution of $Ax = b$ for any given b (True/False)

$\in \mathbb{R}^3$

Consider a matrix A with dimension $m \times n$. For system $Ax = b$, what can you say about below statement ?

(Note that b may or may not be zero)

Statement: for $m = n$ there is a matrix A which gives solution of $Ax = b$ for any given b (True/False)

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$\in R^3$