Probability Theory - Part IV

Course on Engineering Mathematics for GATE - CSE

Rahul • Lesson 10 • June 23, 2024

$P(E) = \frac{\text{Number of favourable outcomes}}{\text{possible outcomes}}$

$P(E) = \frac{n(E)}{n(S)}$

$\Downarrow$

Single Event

= Two events OR more than Two events

E_1, E_2 E_1, E_2, E_3

n Events $E_1, E_2 - - - E_n$

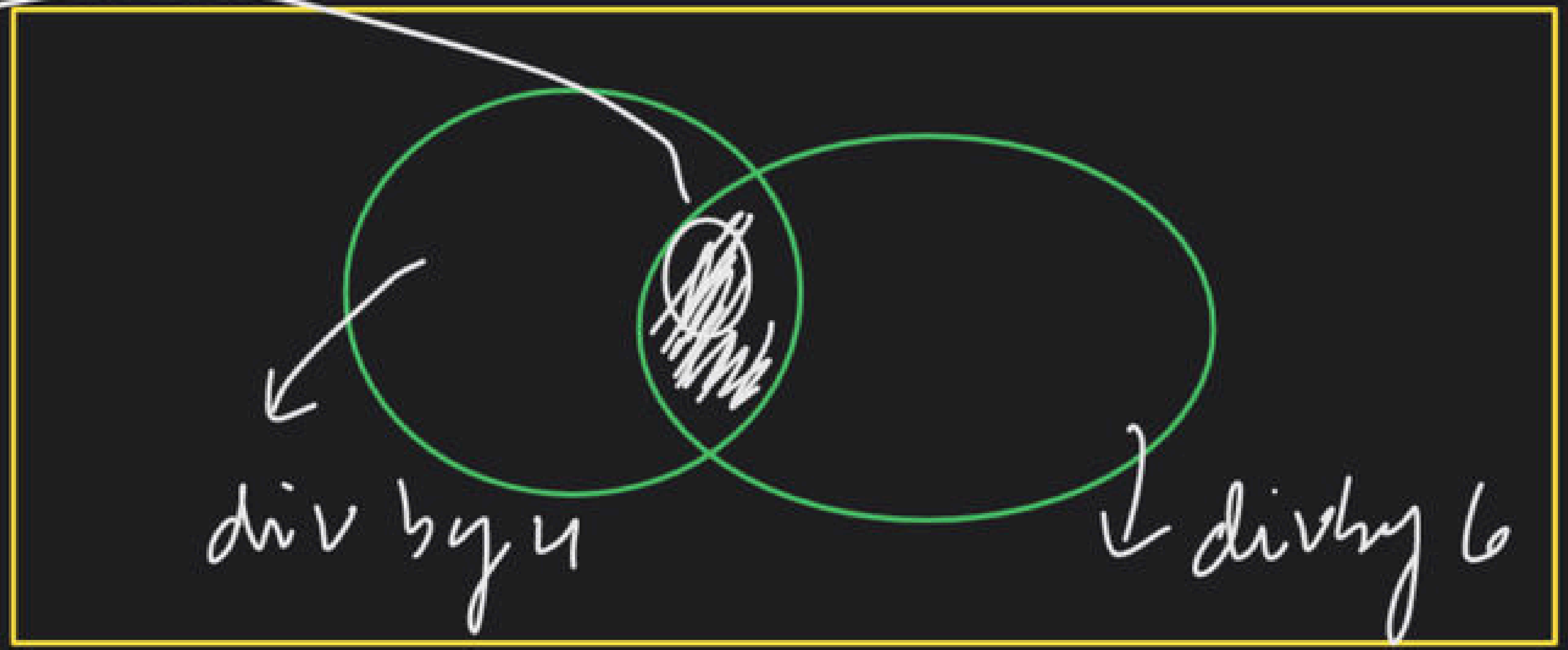
Two events OR more Than Two events

→ "compound events"

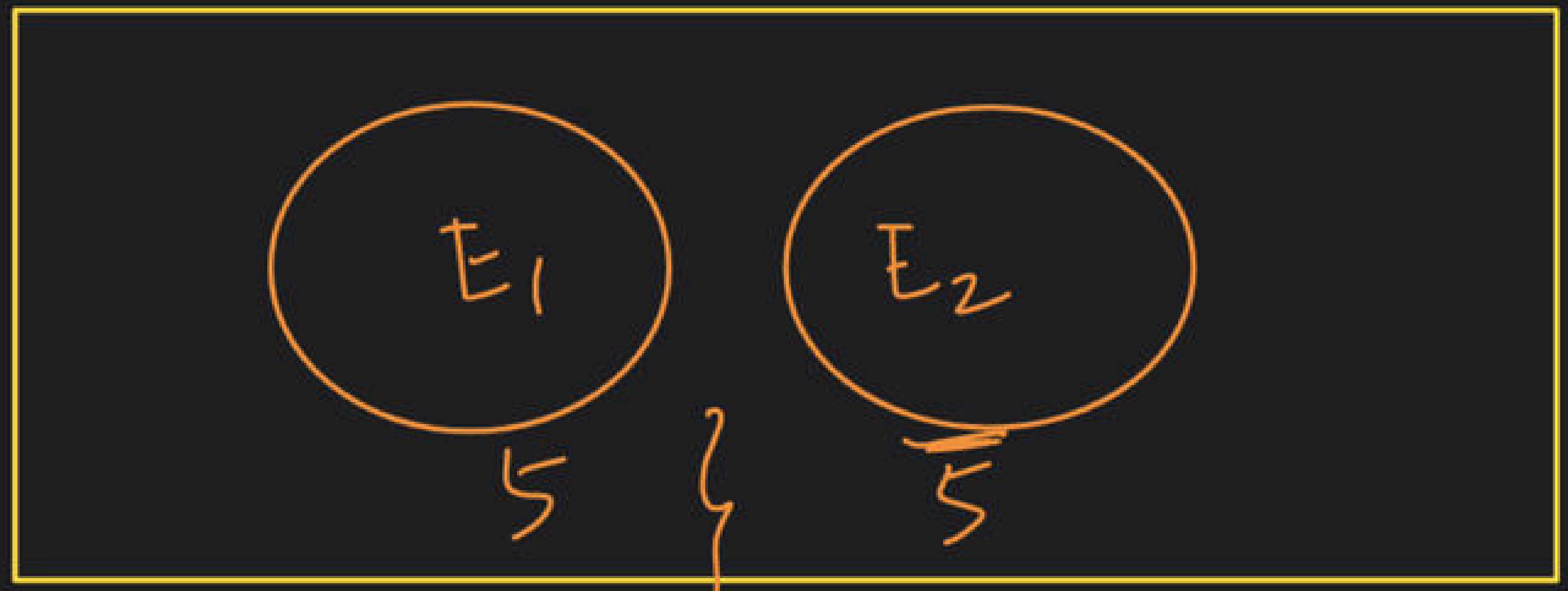
#

common element

(something is common)



"Nothing is common"

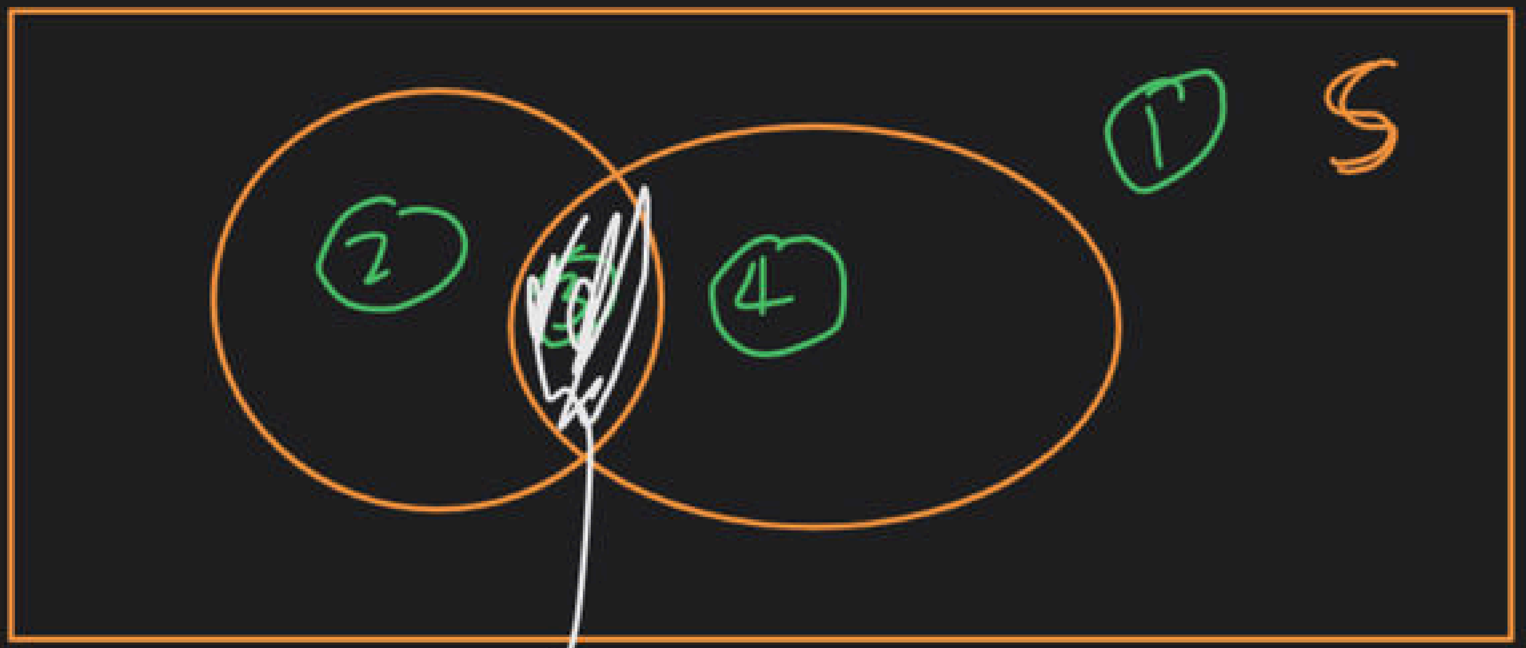


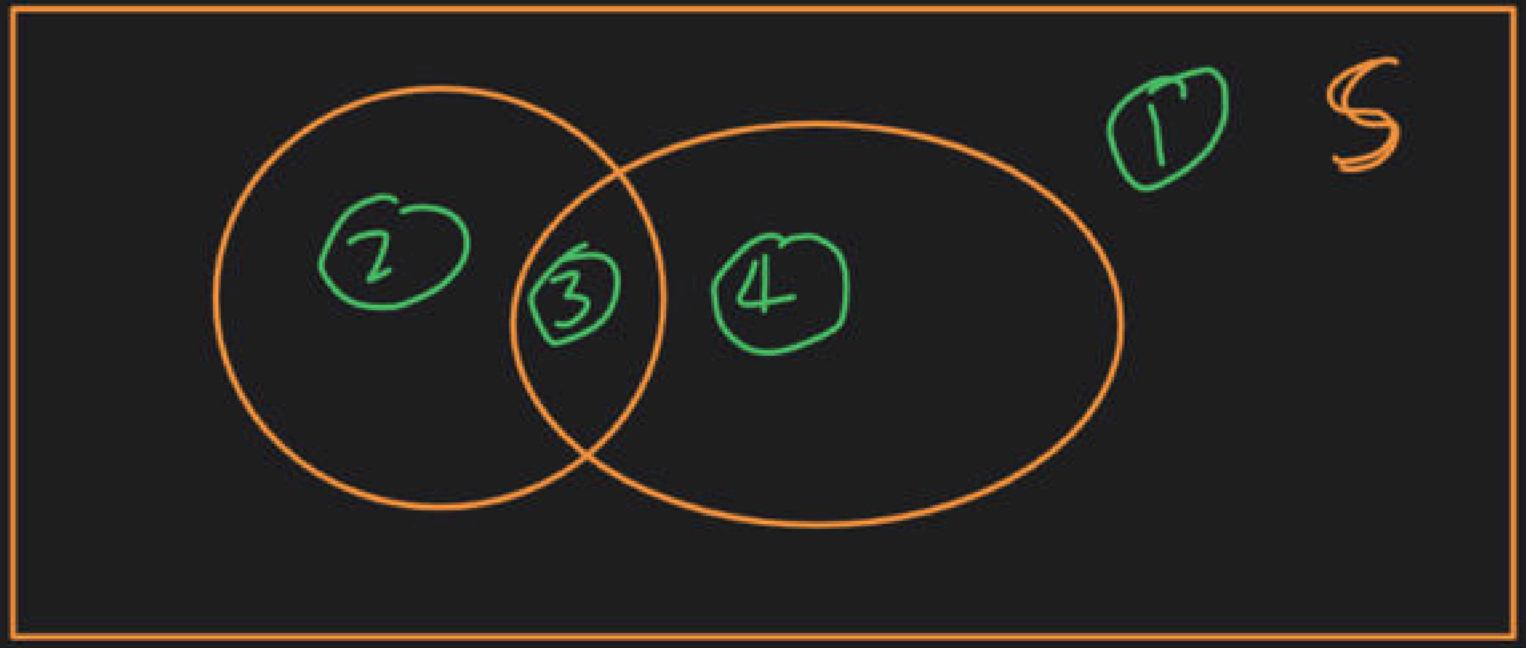
11 Time

T late

CASE (1) something is common

(A)





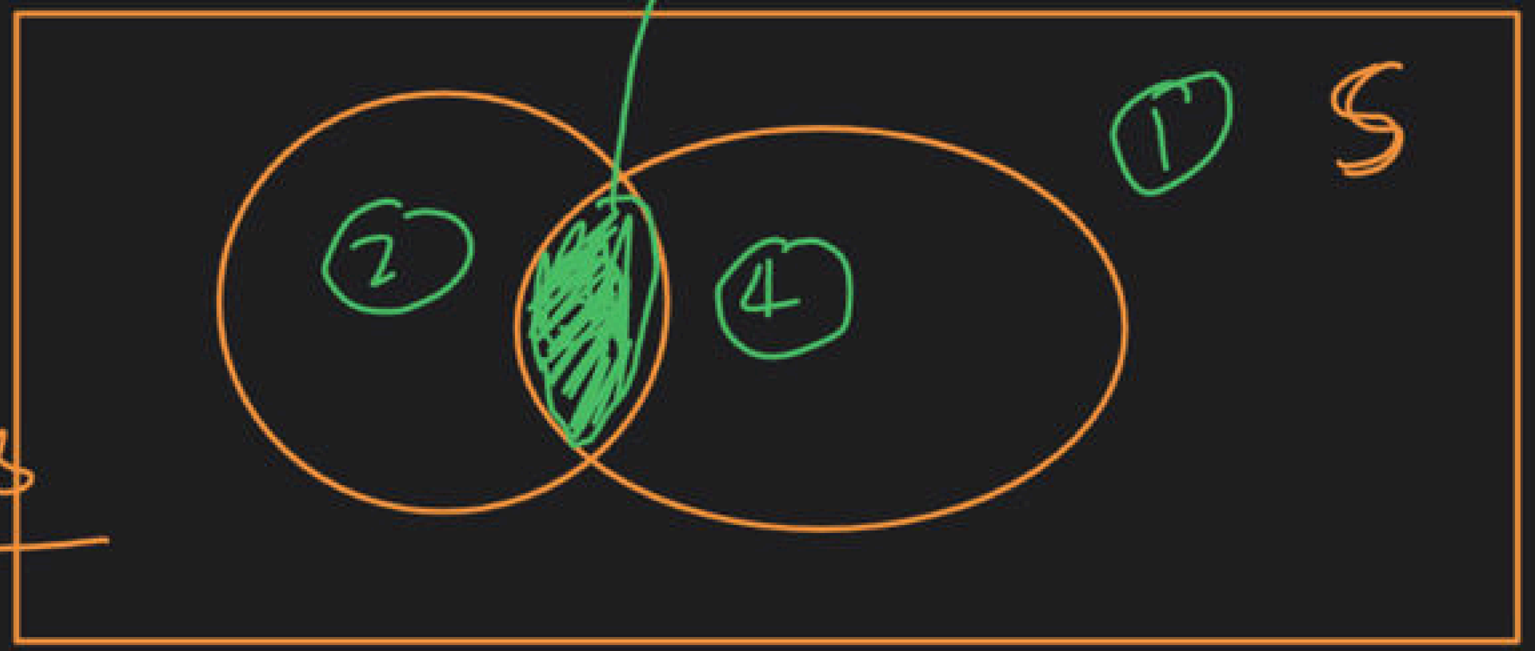
$\cap$ = common Event

Simultaneously occur E_1 and E_2

$(E_1 \cap E_2)$

region 3 = common

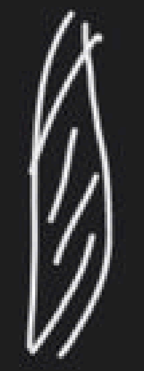
$(E_1$ and $E_2)$ common element



$$P(E_1 \cap E_2) = \frac{\text{Number of fav items}}{\text{Total outcomes}}$$

$P(E_1$ and $E_2) = P($both occurs$) = P(E_1 \cap E_2) =$

$P(E_1 \cap E_2) = \dfrac{\text{Number of element } E_1 \cap E_2}{\text{Total } n(S)}$



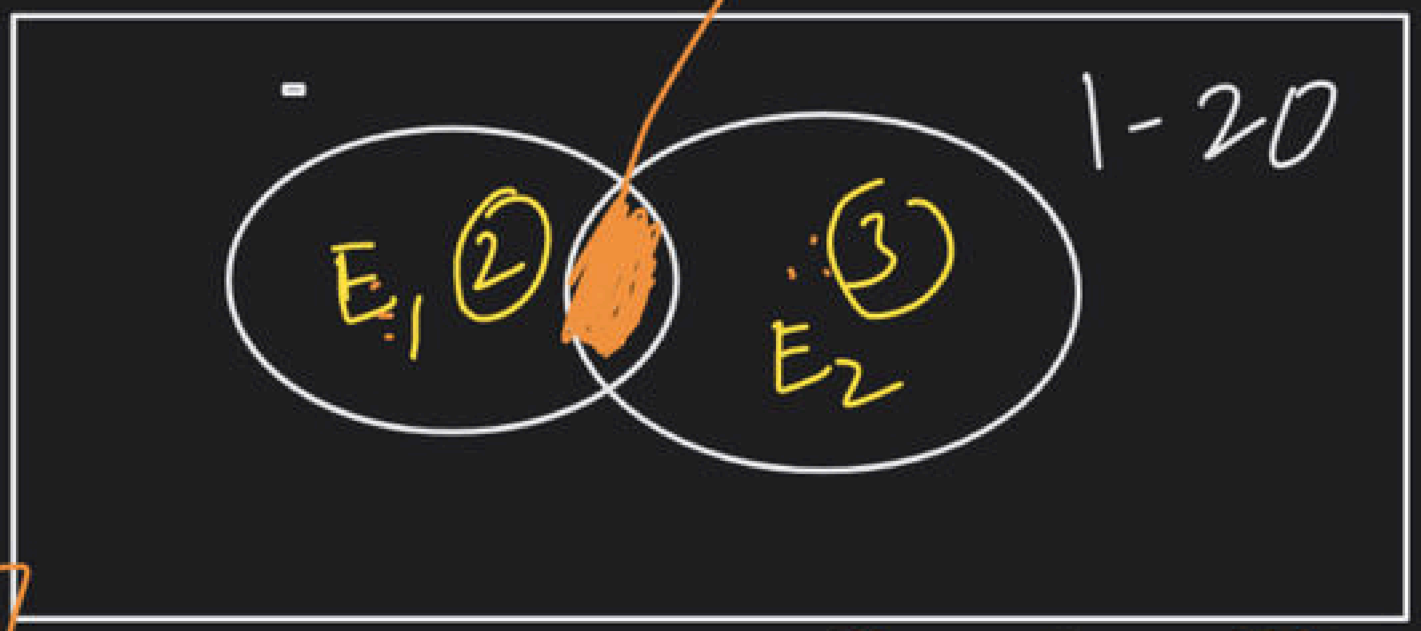
$n(E_1 \cap E_2)$

$$\boxed{P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)}}$$

$E_1 =$ div by 2 $\quad E_2 =$ div by 3

what is the prob $P(E_1 \cap E_2)$

$P(\text{div by } 2 \cap \text{div by } 3)$



3 elements common

$= \dfrac{3}{20}$

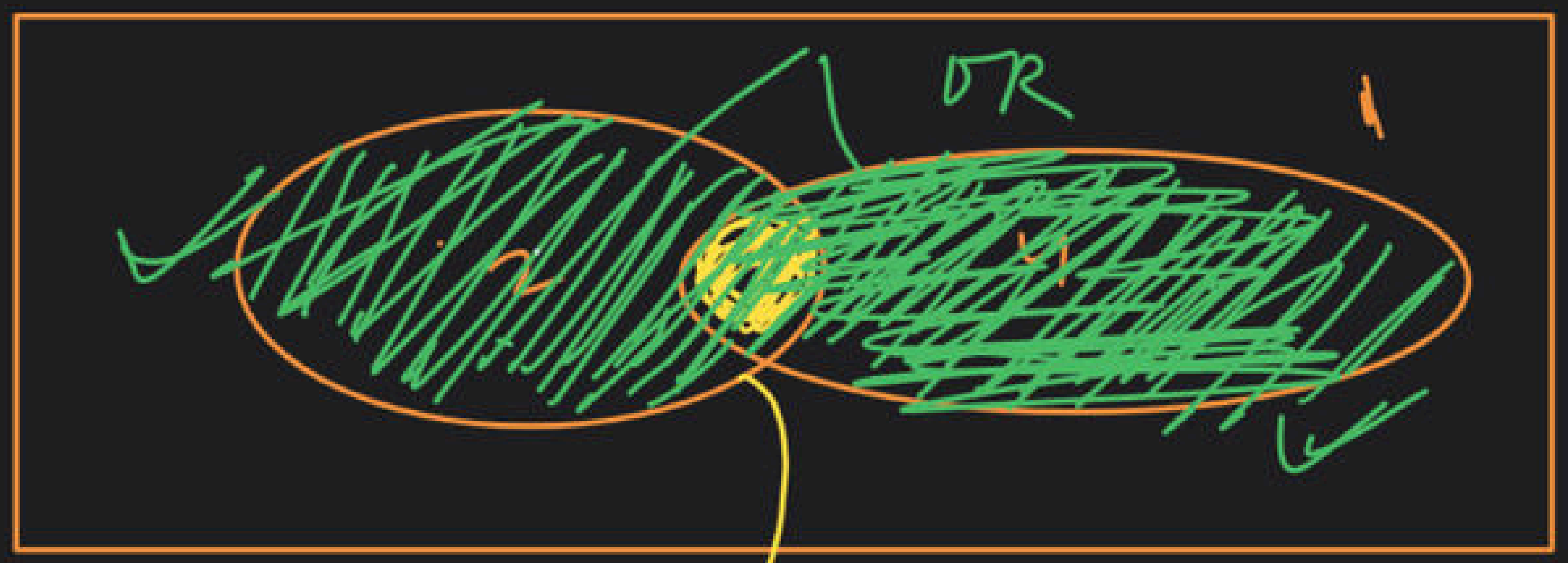
$$P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{3}{20}$$

$2 \rightarrow 2, 4, 6, 8, 10, 12, 14, 16, 18, 20$

$3 \rightarrow 3, 6, 9, 12, 15, 18$

$P(E_1 \text{ or } E_2) = P(\text{atleast ONE occur})$

$P(E_1 \vee E_2) = \dfrac{\text{No. of favourable}}{\text{Total outcomes}}$



2 OR 4
$\underline{(2 \vee 4)}$

both occur
E_1 and E_2
$(E_1 \cap E_2)$

$P(E_1 \vee E_2) = \dfrac{n(E_1 \vee E_2)}{n(S)}$

$= \dfrac{\text{green region}}{\text{Total region}} = \dfrac{n(A) + n(B) - n(A \cap B)}{n(S)}$

$= \dfrac{n(A)}{n(S)} + \dfrac{n(B)}{n(S)} - \dfrac{n(A \cap B)}{n(S)}$

$$\boxed{P(E_1 \vee E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)}$$

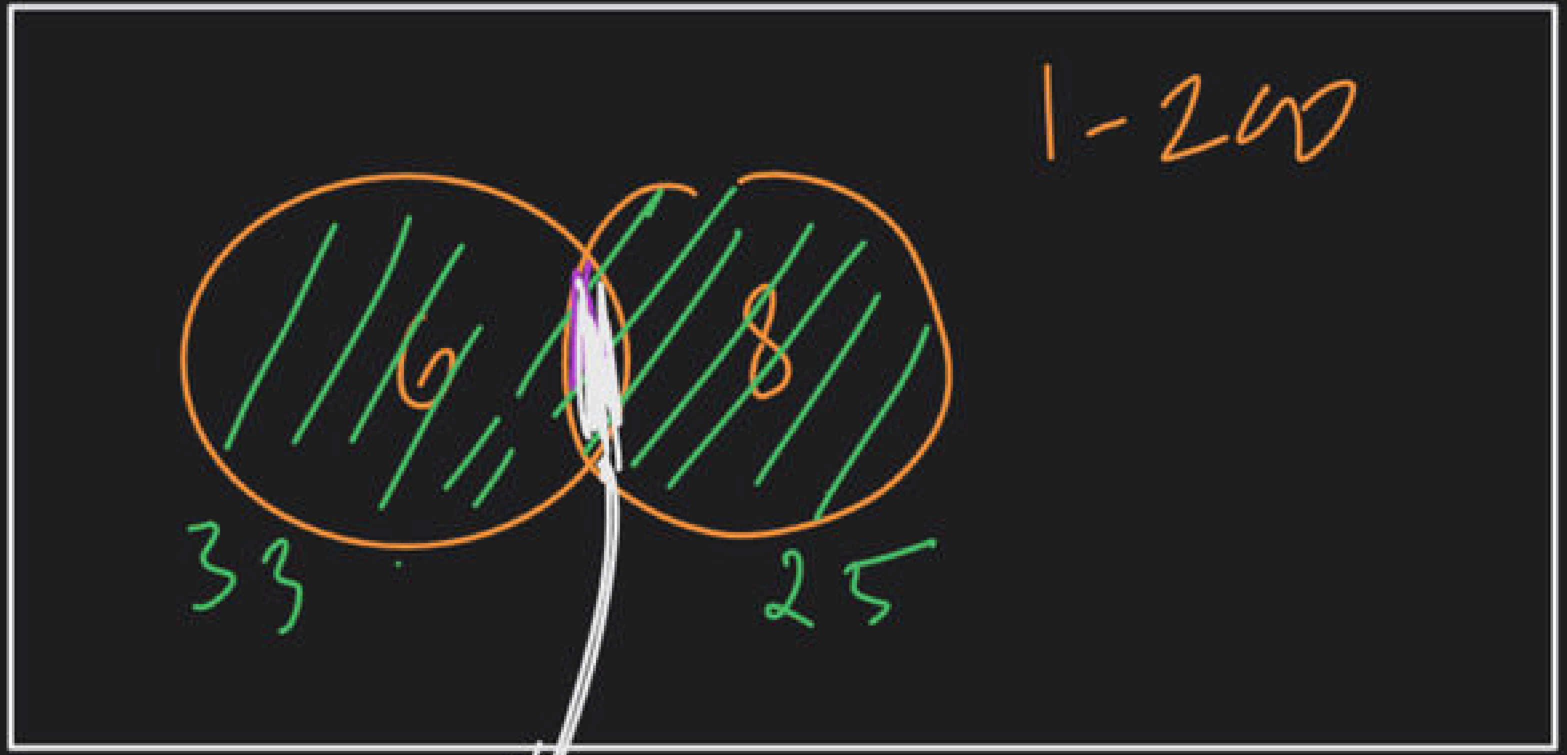
something is common

$E_1 =$ div by 6

$E_2 =$ div by 8

What is prob. P(div by 6 or 8)

$P(6 \vee 8)$



$$= P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{33 + 25 - 8}{200}$$

8

$$= \frac{50}{200} = \frac{1}{4}$$

$= P(E_1 \vee E_2)$

$= P(E_1$ or $E_2)$

$\frac{200}{6} = 33$

$\frac{200}{6} = 25$

$\frac{200}{6 \times 8} = \frac{200}{24} = 8$

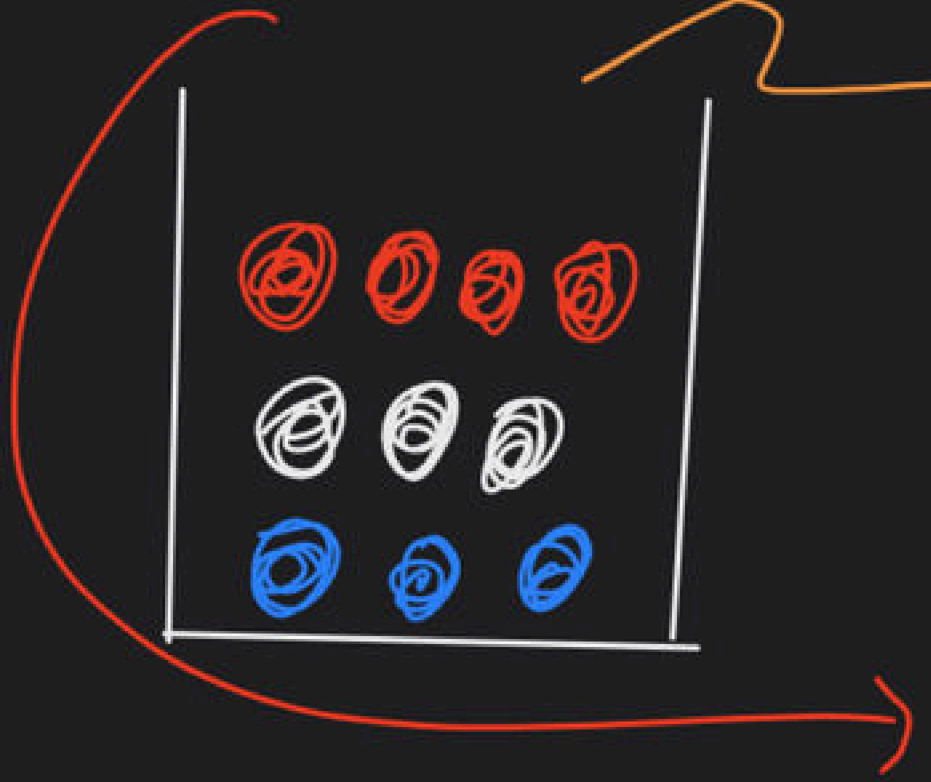
6×8

24

dependent events vs Independent events

✓ **without Replacement**

DECREASE — Dependent
Constant — Independent

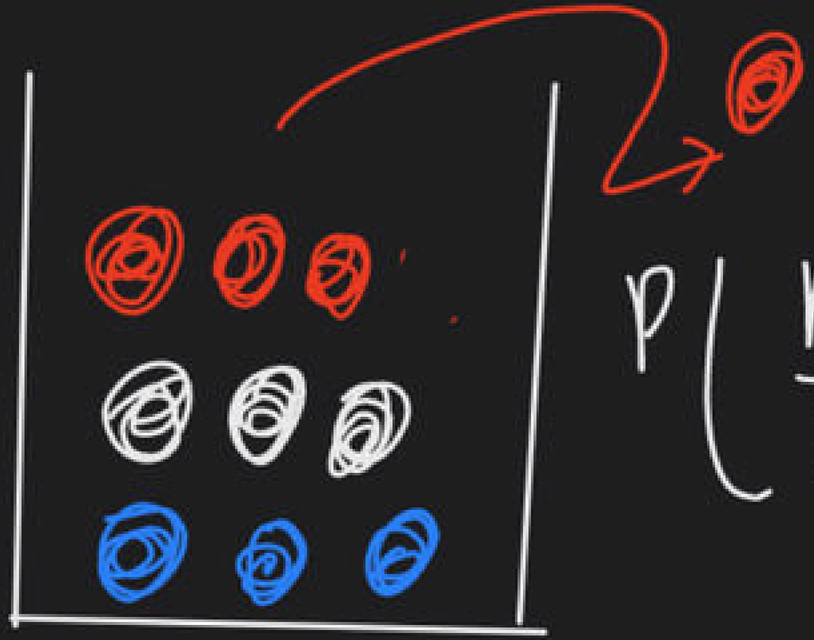


What is the Prob.

P(3 Red balls are drawn at random ONE at time)

counting

Ist red $P(R_1) = \frac{4}{10}$

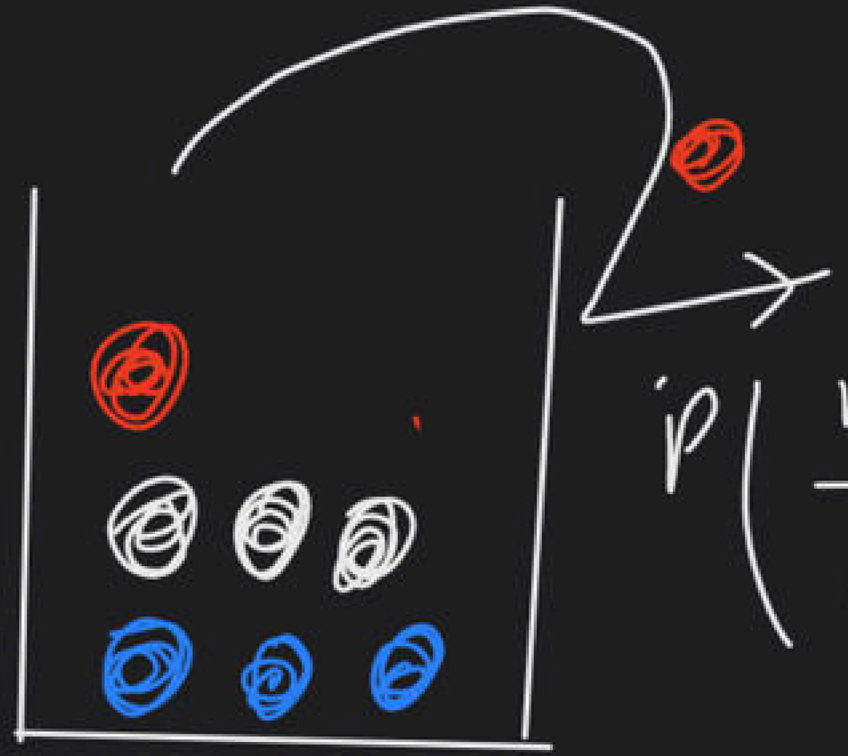


$P\left(\frac{R_2}{R_1}\right) = \frac{3}{9}$

R_1 = Happened

R_2 = Happening

$P\left(\frac{R_3}{R_1, R_2}\right) = \frac{2}{8}$



$P\left(\frac{R_3}{R_1, R_2, R_3}\right) = \frac{1}{7}$

dependent

without Replacement

E_1

→ What is the prob.
P(4 Red balls are drawn at random one at time)

{ DECREASE — Dependent
{ Constant — Independent

counting

Ist red $P(R_1) = \frac{4}{10} =$

$R_1 \; R_2 \; R_3$

simultaneously occur
and

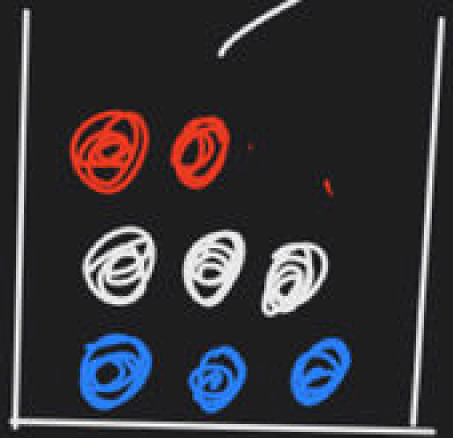
E_1 and E_2 and E_3 and E_4

$\cap$

E_2

$P\left(\frac{R_2}{R_1}\right) = \frac{3}{9}$

R_1 = Happened
R_2 = Happening



E_3

$P\left(\frac{R_3}{R_1, R_2}\right) = \frac{2}{8}$

E_4

$P\left(\frac{R_4}{R_1, R_2, R_3}\right) = \frac{1}{7}$

$$P(R_1 \wedge R_2 \wedge R_3 \wedge R_4) \Rightarrow P(R_1)\, P\left(\frac{R_2}{R_1}\right) P\left(\frac{R_3}{R_1, R_2}\right) P\left(\frac{R_4}{R_1, R_2, R_3}\right)$$

$$P(R_1 \wedge R_2 \wedge R_3 \wedge R_4) \Rightarrow P(R_1)\, P\left(\frac{R_2}{R_1}\right) P\left(\frac{R_3}{R_1, R_2}\right) P\left(\frac{R_4}{R_1, R_2, R_3}\right)$$

$$\boxed{P(R_1 \cap R_2 \cap R_3 \wedge R_4) = P(R_1)\, P\left(\frac{R_2}{R_1}\right) P\left(\frac{R_3}{R_1 \cap R_2}\right) P\left(\frac{R_4}{R_1 \cap R_2 \wedge R_3}\right)}$$

$R_1 \to E_1$ $\quad R_2 \to E_2$

$R_3 \to E_3$ $\quad R_4 \to E_4$

$E_1 \quad E_2 \quad E_3 \quad \bar{E}_4$ $\qquad E_1 \cap E_2 \cap E_3 \cap E_4$

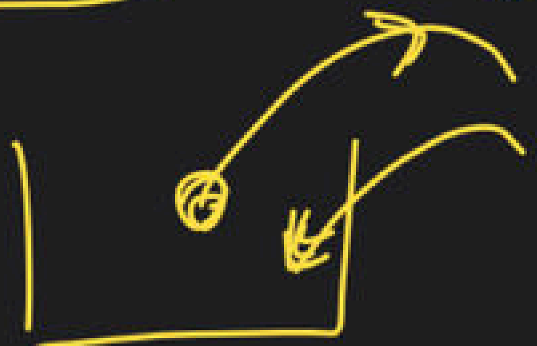
$$\boxed{P(E_1 \cap E_2 \wedge E_3 \wedge E_4) = P(E_1)\, P\left(\frac{E_2}{E_1}\right) P\left(\frac{E_3}{E_1 \cap E_2}\right) P\left(\frac{E_4}{E_1 \cap E_2 \cap E_3}\right)}$$

" Dependent events "

If Two Events are Dependent

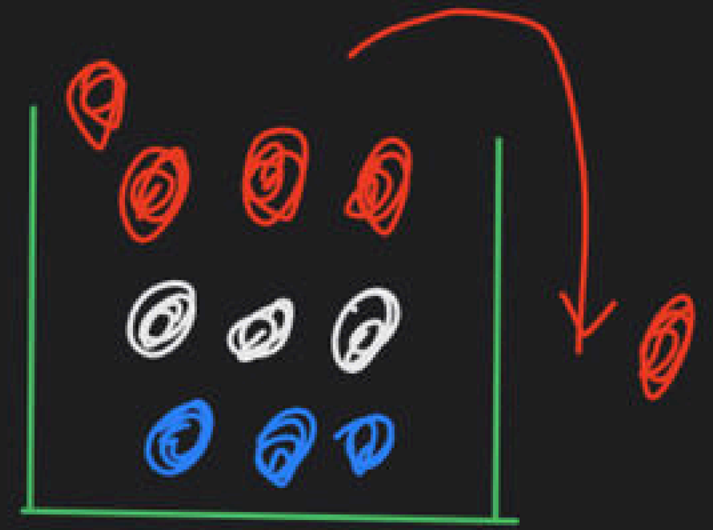
$$\boxed{P(E_1 \cap E_2) = P(E_1)\,P\left(\frac{E_2}{E_1}\right)}$$

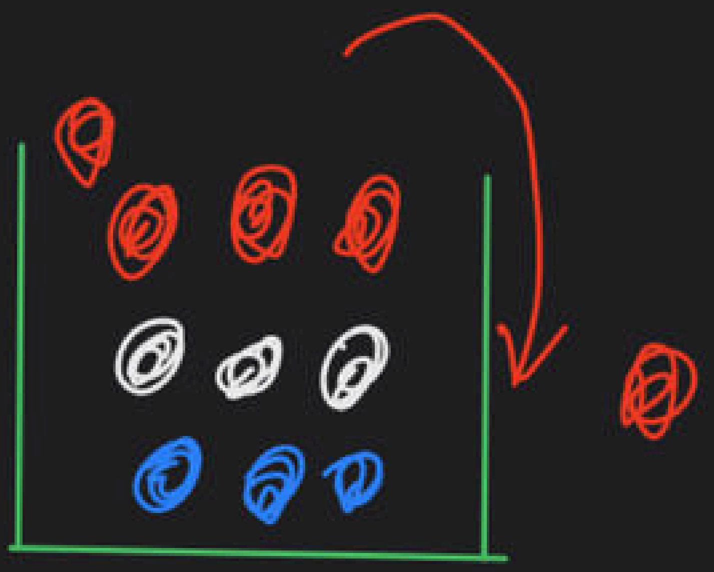
dependent events

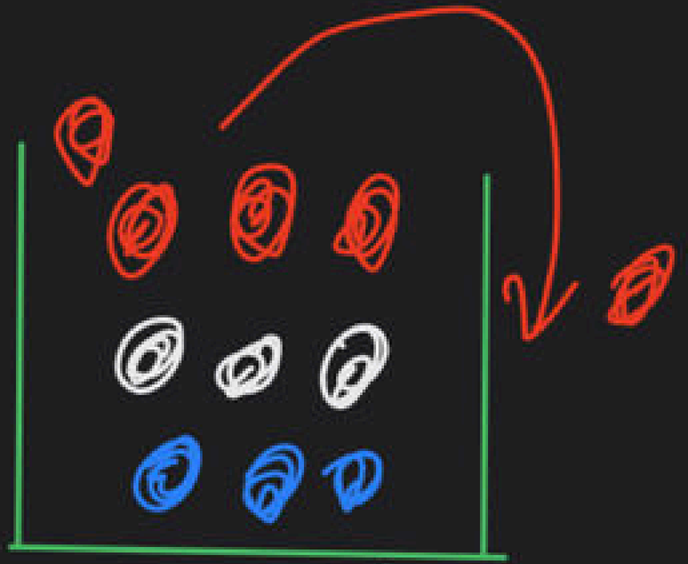


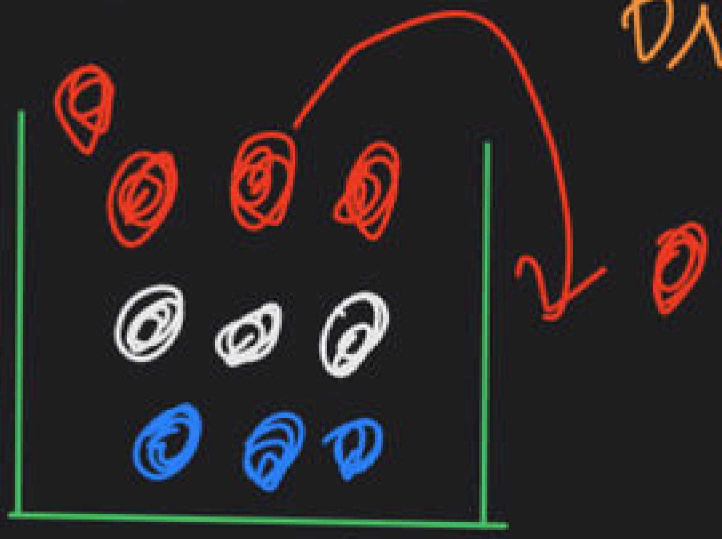
CASE 02 (with Replacement)

What is the probl 4 red ball are Drawn)









$$P(R_1) = \left(\frac{4}{10}\right)$$

$$P\left(\frac{R_2}{R_1}\right) = \frac{4}{10}$$

$$P\left(\frac{R_3}{R_1 \cap R_2}\right) = \frac{4}{10}$$

$$P\left(\frac{R_4}{R_1 \cap R_2 \cap R_3}\right) = \frac{4}{10}$$

unacademy

$$P(R_1 \cap R_2 \cap R_3 \cap R_4) = P(R_1)\, P\left(\frac{R_2}{R_1}\right) P\left(\frac{R_3}{R_1 \cap R_2}\right) P\left(\frac{R_4}{R_1 \cap R_2 \cap R_3}\right)$$

$$P\left(\frac{R_2}{R_1}\right) = P(R_2)$$

$$P\left(\frac{R_3}{R_1 \cap R_2}\right) = P(R_3)$$

$$P\left(\frac{R_4}{R_1 \cap R_2 \cap R_3}\right) = P(R_4)$$

$$P(R_1 \cap R_2 \cap R_3 \cap R_4) = P(R_1)\, P(R_2)\, P(R_3)\, P(R_4)$$

$R_1 \to E_1 \quad R_2 \to E_2$

$R_3 \to E_3 \quad R_4 \to E_4$

Independent events

✓ $$\boxed{P(E_1 \cap E_2 \cap E_3 \cap E_4) = P(E_1)\, P(E_2)\, P(E_3)\, P(E_4)}$$

✓ DEfault → $\boxed{E_1 \text{ and } E_2 \text{ ARE Independent}}$ ✓

Tossing A Two coins Re1, Re2 (simult)

both coins (Events) are Dependent / independent

A Re $<$ H T   Re2 $<$ H T

B

$S = \{HH, HT, TH, TT\}$

① ②

A B

P1 P1

E_1 H H E_2 $(E_1 \cap E_2)$

H T $H \cap T$

T H $T \cap H$

T T $T \cap T$

$P(H_A \cap H_B) = \frac{1}{4}$ $P(T_A \cap H_B) = \frac{1}{4}$

$P(H_A \cap T_B) = \frac{1}{4}$ $P(T_A \cap T_B) = \frac{1}{4}$

simultaneously occur

Die A   Die B

$5 \cap 6$

$5 \vee 6$

$P(H_A \cap H_B) = \frac{1}{4}$ $P(T_A \cap H_B) = \frac{1}{4}$

$P(H_A \cap T_B) = \frac{1}{4}$ $P(T_A \cap T_B) = \frac{1}{4}$

$P(E_1 \cap E_2)$

$= P(E_1)\,P(E_2)$

$P(H_A \cap H_B) = P(H_A)\,P(H_B)$

$\frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\underbrace{Re\,1,\ Re\,2,\ Re\,3}_{\text{Independent}}$

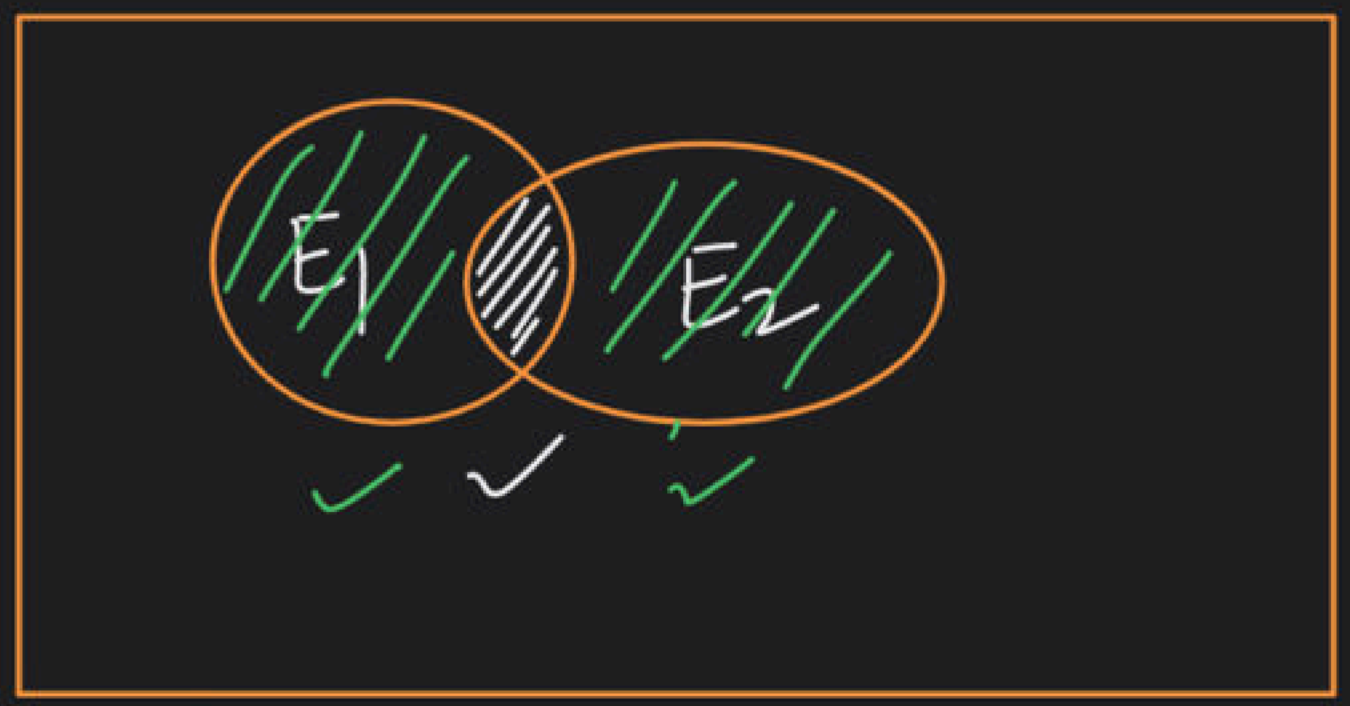
$P(T_A \cap H_B) = P(T_A) \times P(H_B)$

$\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$ ✓ $\underbrace{\text{Independent}}$

$P(H_A \cap T_B) = P(H_A)\,P(T_B)$ ✓

(Two Die A, B Independent)

#



only E_1 (orange region)

only E_2 (white region)

"E_1, E_2 independent events"

"something is common"

$$P(\text{only } E_1) = P(E_1) - P(E_1 \cap E_2)$$

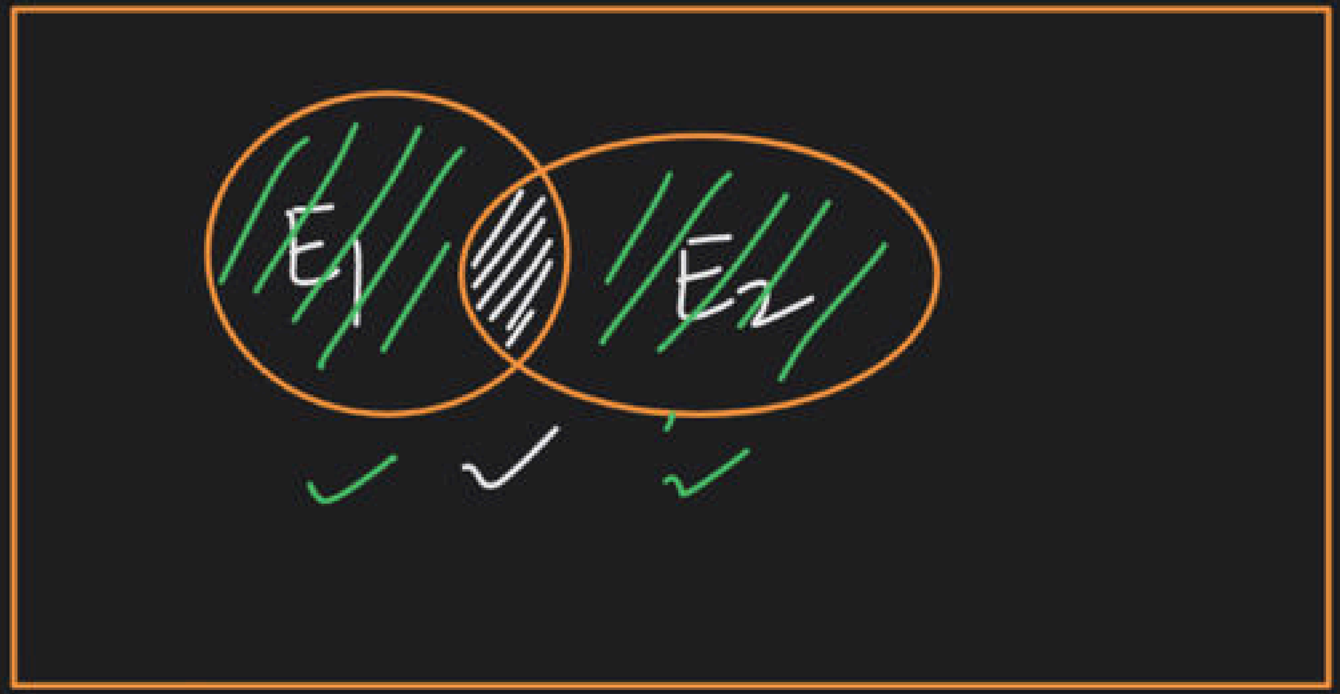
$$= \text{orange} - \text{yellow}$$

E_1 and E_2 ARE independent

$$\boxed{P(\text{only } E_1) = P(E_1) - P(E_1)P(E_2)}$$

only E_1 (orange region)

only E_2 (white region)



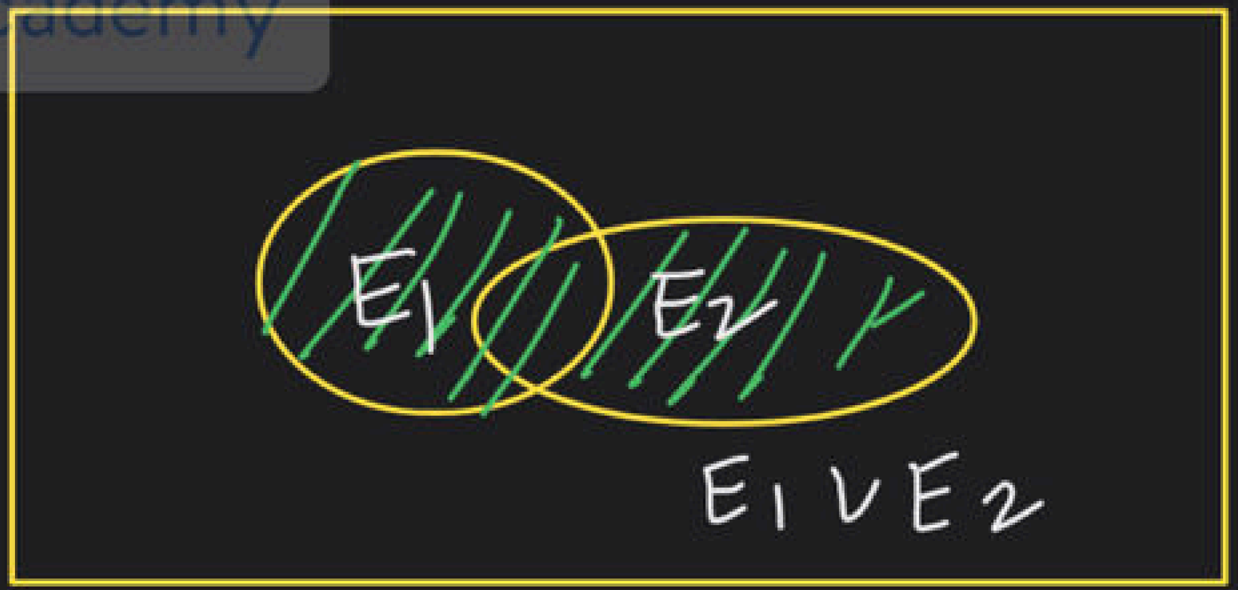
"E_1, E_2 independent events"

"something is common"

$$P(\text{only } E_2) = \text{white region} - \text{yellow region}$$
$$= P(E_2) - P(E_1 \cap E_2)$$

E_1, E_2 indep.

$$\boxed{P(\text{only } E_2) = P(E_2) - P(E_1)P(E_2)}$$

#


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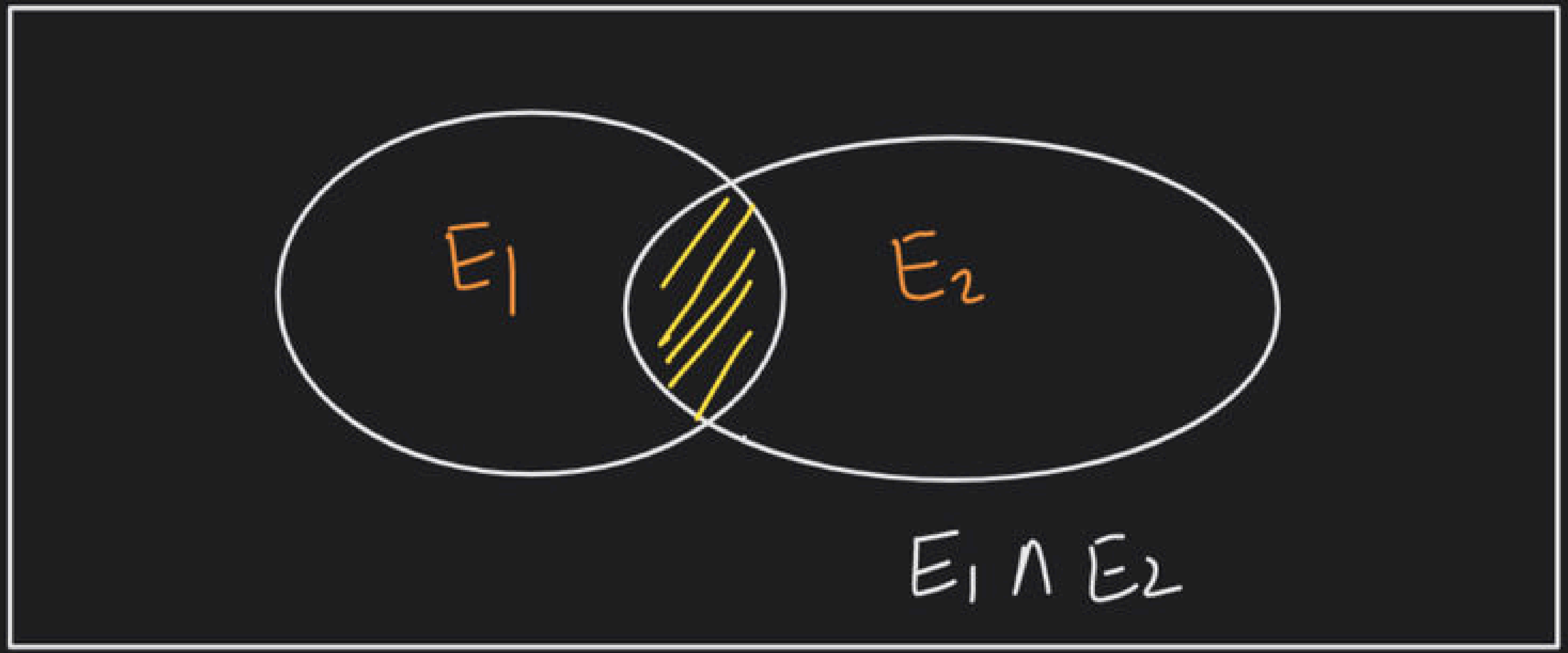
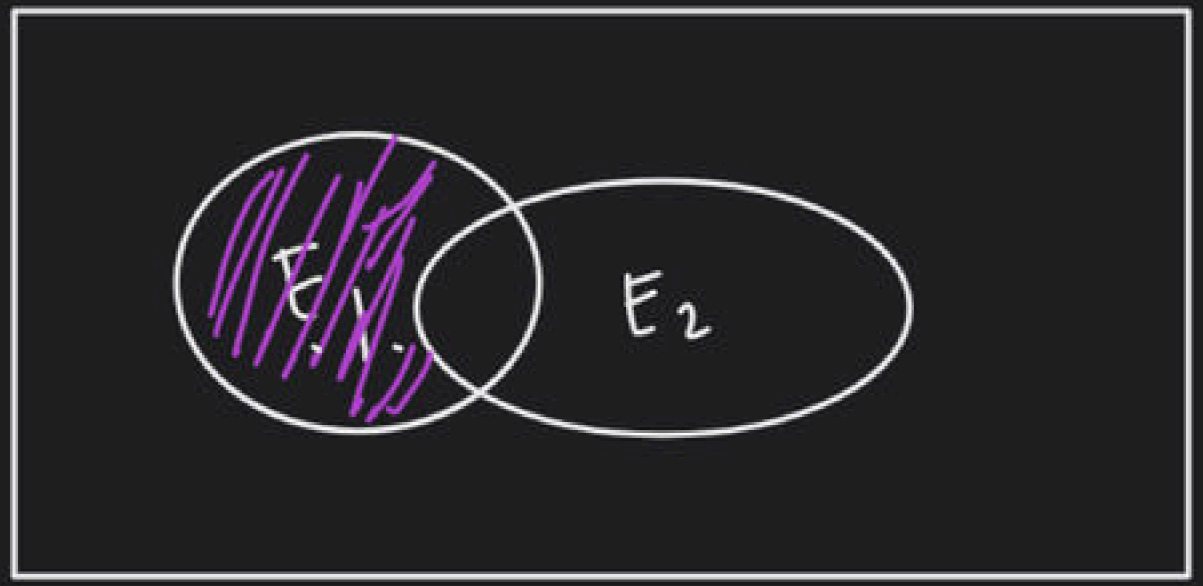
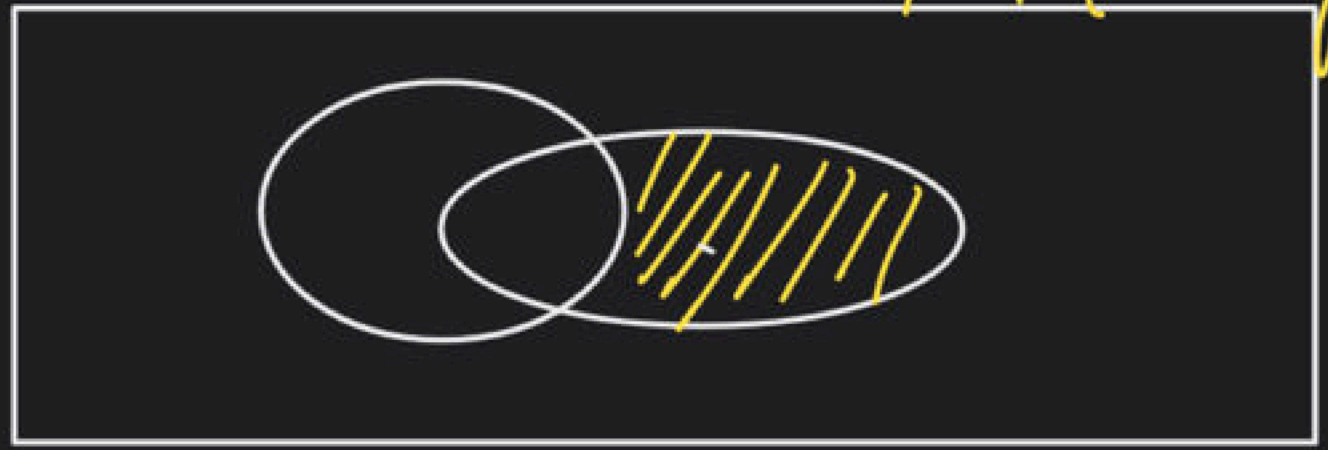
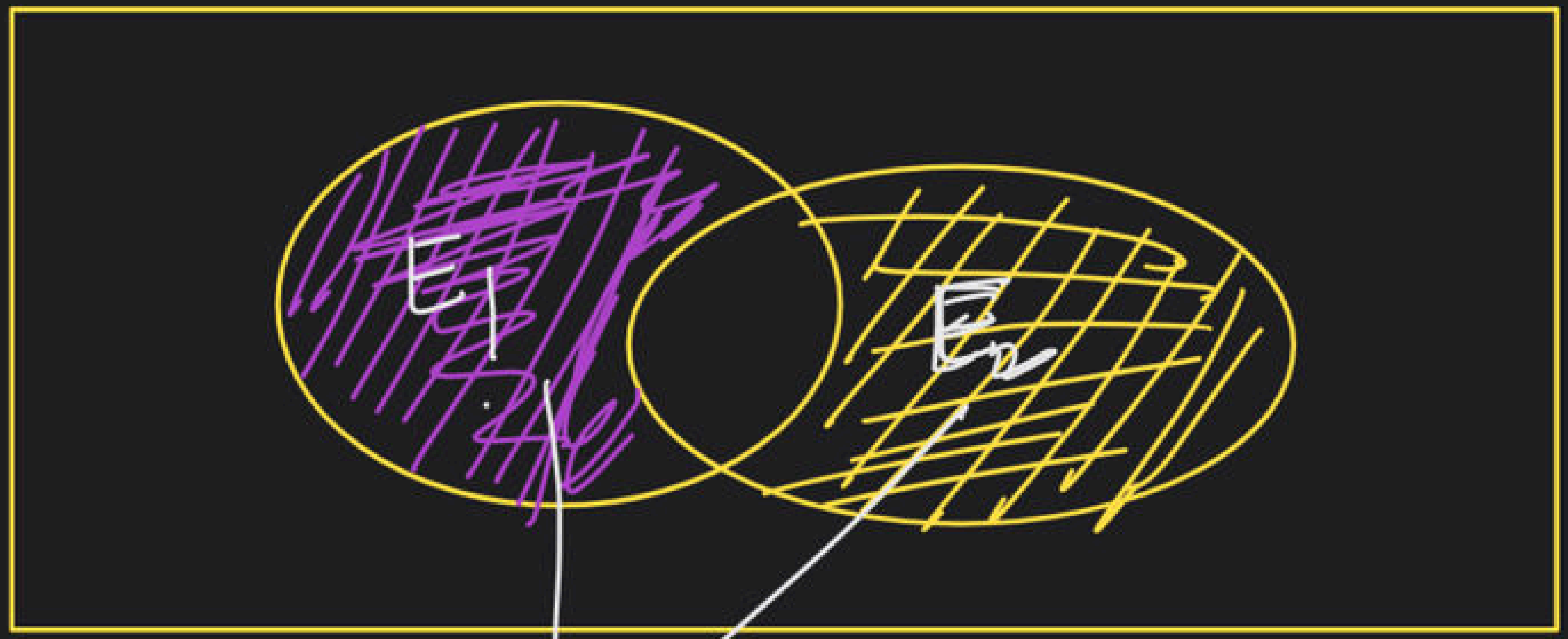




P(only E_1)

P(only E_2)





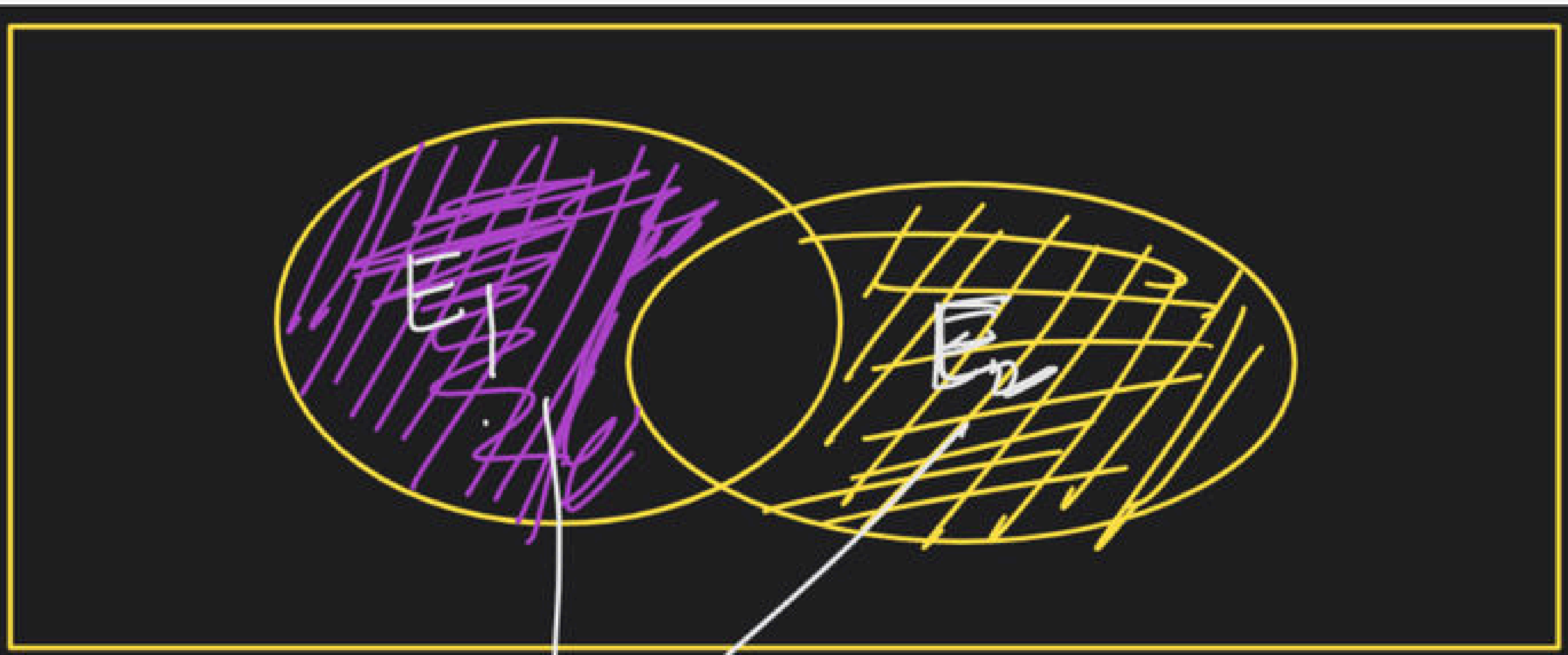
Exactly ONE

P(only E_1) + P(only E_2)

P(only E_1) OR P(only E_2)

P(exactly ONE)

$\Rightarrow P[\text{only } E_1) + P(\text{only } E_2)$



only E_1 $\left\langle \begin{matrix} E_1 \\ \overline{E_2} \end{matrix} \right]$ $(E_1 \cap \overline{E_2})$

only one job

Exactly ONE

$P(\text{only } E_1) + P(\text{only } E_2)$

$P(\text{only } E_1)$ OR $P(\text{only } E_2)$

$\Rightarrow P(E_1 \cap \overline{E_2}) + P(E_2 \cap \overline{E_1})$ XOR - GATE

only E_1 + only E_2

$\Rightarrow P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$ = add = exactly ONE

$P(\text{exactly ONE})$

$= P(\text{only } E_1) + P(\text{only } E_2)$

$= P(E_1 \cap \overline{E_2}) + P(E_2 \cap \overline{E_1})$

$= P(E_1) - P(E_1 \cap E_2) + P(E_2) - P(E_1 \cap E_2)$

$= P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$

"Something is common"

$P(\text{Aditi}) = E_1$

$P(\text{sumit}) = E_2$

company — job — Aditi, sumit

single job

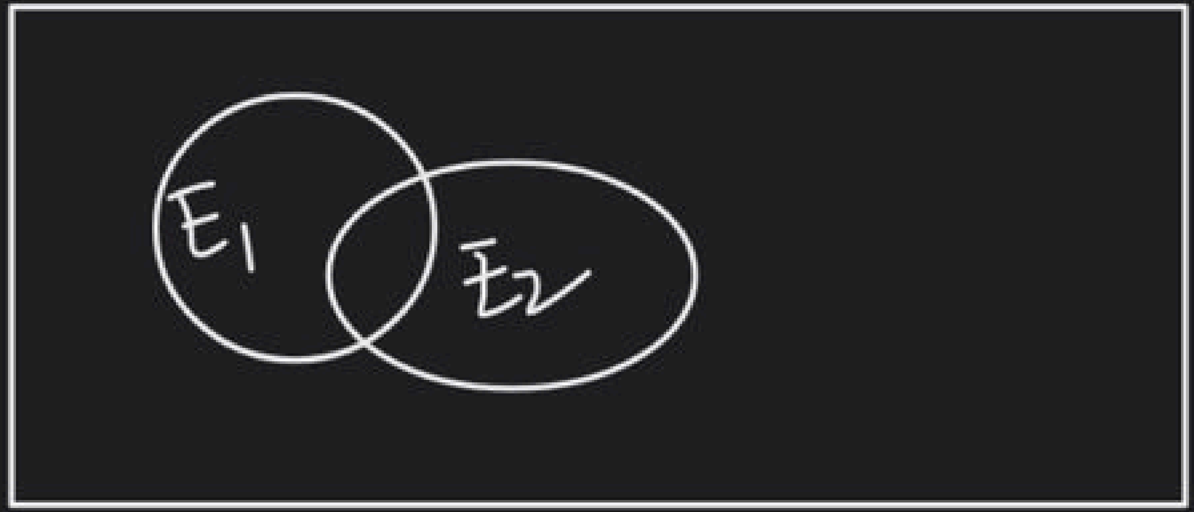
$= \text{Aditi} (\overline{\text{sumit}})$ OR $+$ $\text{sumit} (\overline{\text{Aditi}})$

$P(\text{only Aditi}) + P(\text{sumit only})$

$P(\text{only } E_1) = P(E_1 \cap \overline{E_2})$

If E_1 and E_2 ARE independent events

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$



$\left.\begin{matrix}\overline{E_1} \\ \overline{E_2}\end{matrix}\right]$ Independent events

$$P(\overline{E_1} \cap \overline{E_2}) = P(\overline{E_1})P(\overline{E_2})$$

$$P(H_A \cap H_B) = P(H_A)P(H_B)$$

$$\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$$

compl.

$$P(T_A \cap T_B) = P(T_A)P(T_B)$$

$$P(T_A) = \frac{1}{2} \quad P(T_B) = \frac{1}{2}$$

$$P(T_A \cap T_B) = \frac{1}{4}$$

E_1, E_2 Are indep. events

$\overline{E_1}, \overline{E_2}$ Are also indep.

$P(\text{only } E_1) = P(E_1 \cap \bar{E_2})$

$$= P(E_1)\,P(\bar{E_2})$$

$$= P(E_1)\,(1 - P(E_2))$$

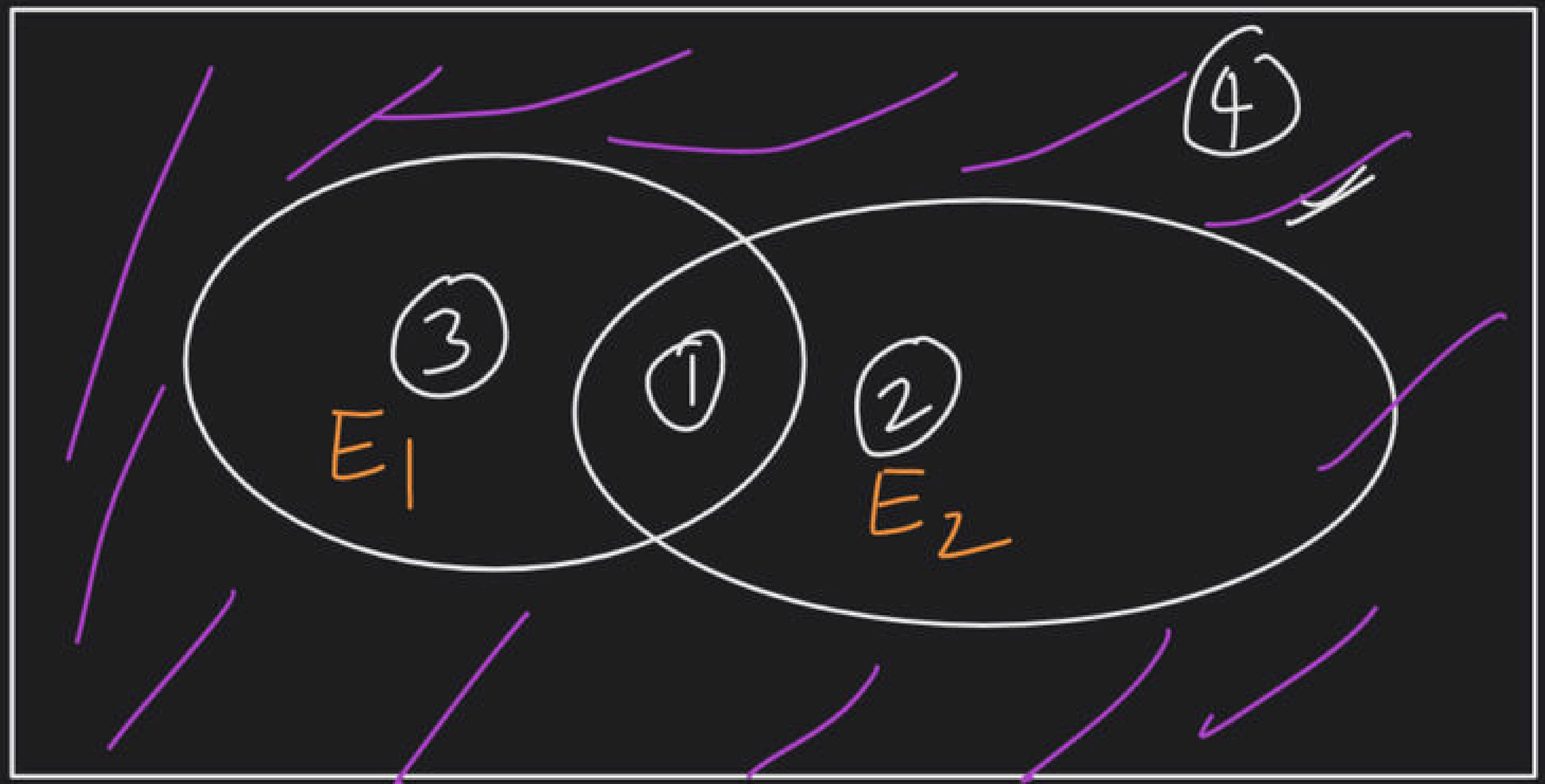
$$\boxed{P(\text{only } E_1) = P(E_1) - P(E_1)P(E_2)}$$

$P(\text{only } E_2) = P(\bar{E_1} \cap E_2) = P(\bar{E_1})\,P(E_2)$

$$\boxed{P(\bar{E_1} \cap E_2) = P(E_2) - P(E_1)P(E_2)}$$

$P(\text{exactly one}) = P(\text{only } E_1) + P(\text{only } E_2)$

$$= P(E_1) + P(E_2) - 2P(E_1)\,P(E_2)$$



$$P(\text{Neither } E_1 \text{ Nor } E_2) = P(\bar{E_1} \cap \bar{E_2})$$

$$= P(\bar{E_1})\,P(\bar{E_2})$$

$$\boxed{P(\bar{E_1} \cap \bar{E_2}) = [1 - P(E_1)]\,[1 - P(E_2)]}$$

E_1 (Happening) $\qquad$ E_2 (Happ)

$\overline{E_1}$ (does Not Happ) $\qquad$ $\overline{E_2}$ (does Not Happ)

$E_1 \to E_2 = E_1 \cap E_2$ (both occur)

$E_1 \to \overline{E_2} = E_1 \cap \overline{E_2}$ (only E_1)

$\overline{E_1} \to E_2 = \overline{E_1} \cap E_2$ (only E_2)

$\overline{E_1} \to \overline{E_2} = \overline{E_1} \cap \overline{E_2}$ (Neither E_1 Nor E_2)

	E_2	$\overline{E_2}$
E_1	$E_1 \cap E_2$ both occur	$E_1 \cap \overline{E_2}$ only E_1
$\overline{E_1}$	$\overline{E_1} \cap E_2$ only E_2	$\overline{E_1} \cap \overline{E_2}$ Neither

(exactly ONE)

FOR THREE EVENTS

E_1, E_2, E_3 ARE THREE events

What is the Prob

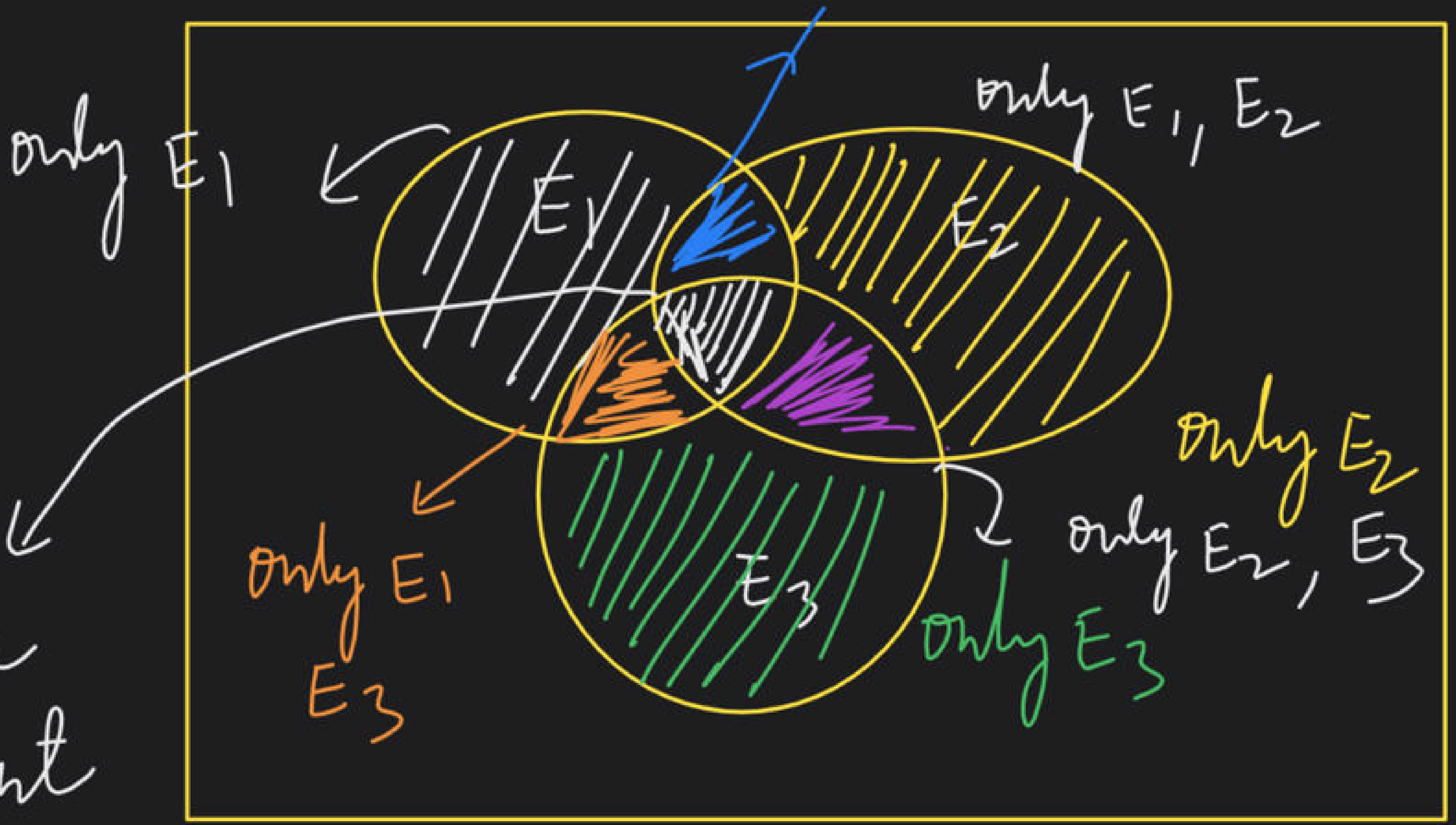
P[E_1 occur OR
E_2 occur
OR
E_3 occur]

E_1
E_2
E_3

$E_1 \cap E_2 \cap E_3$

common element

$P(E_1 \cap E_2 \cap E_3)$



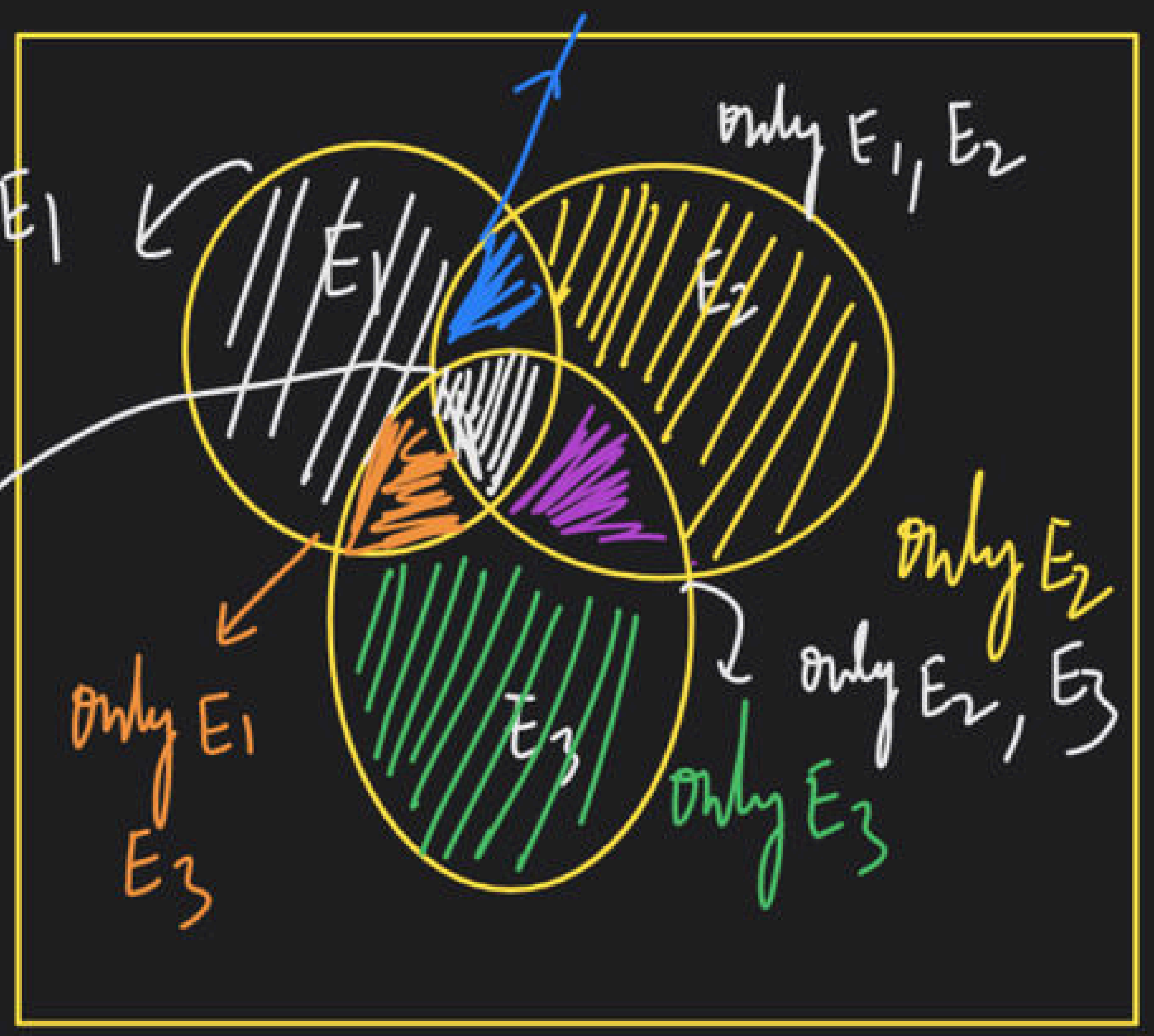
$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

What is the Prob

P[E_1 occur OR
E_2 occur
OR
E_3 occur]

1
2
3

E_1 or E_2 or E_3



common element

$P(E_1 \cap E_2 \cap E_3)$

E_1 or E_2 or E_3

E_1 $E_1 E_2$
E_2 $E_2 E_3$
E_3 $E_3 E_1$

$E_1 E_2 E_3$

only E_1

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

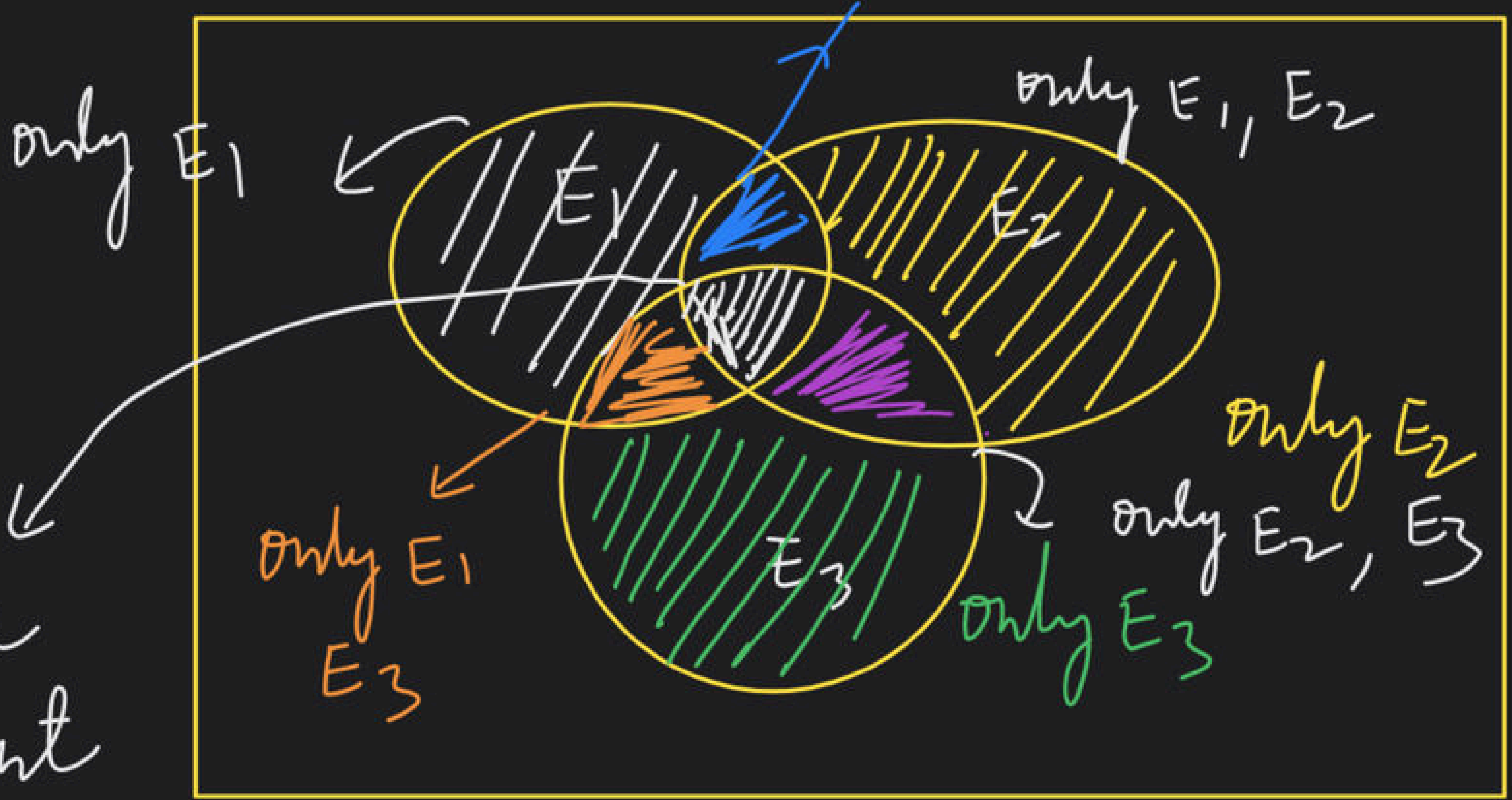
$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

What is the Prob

P[E_1 occur OR
E_2 occur
OR
E_3 occur]

1
2
3

$E_1 \cap E_2 \cap E_3$



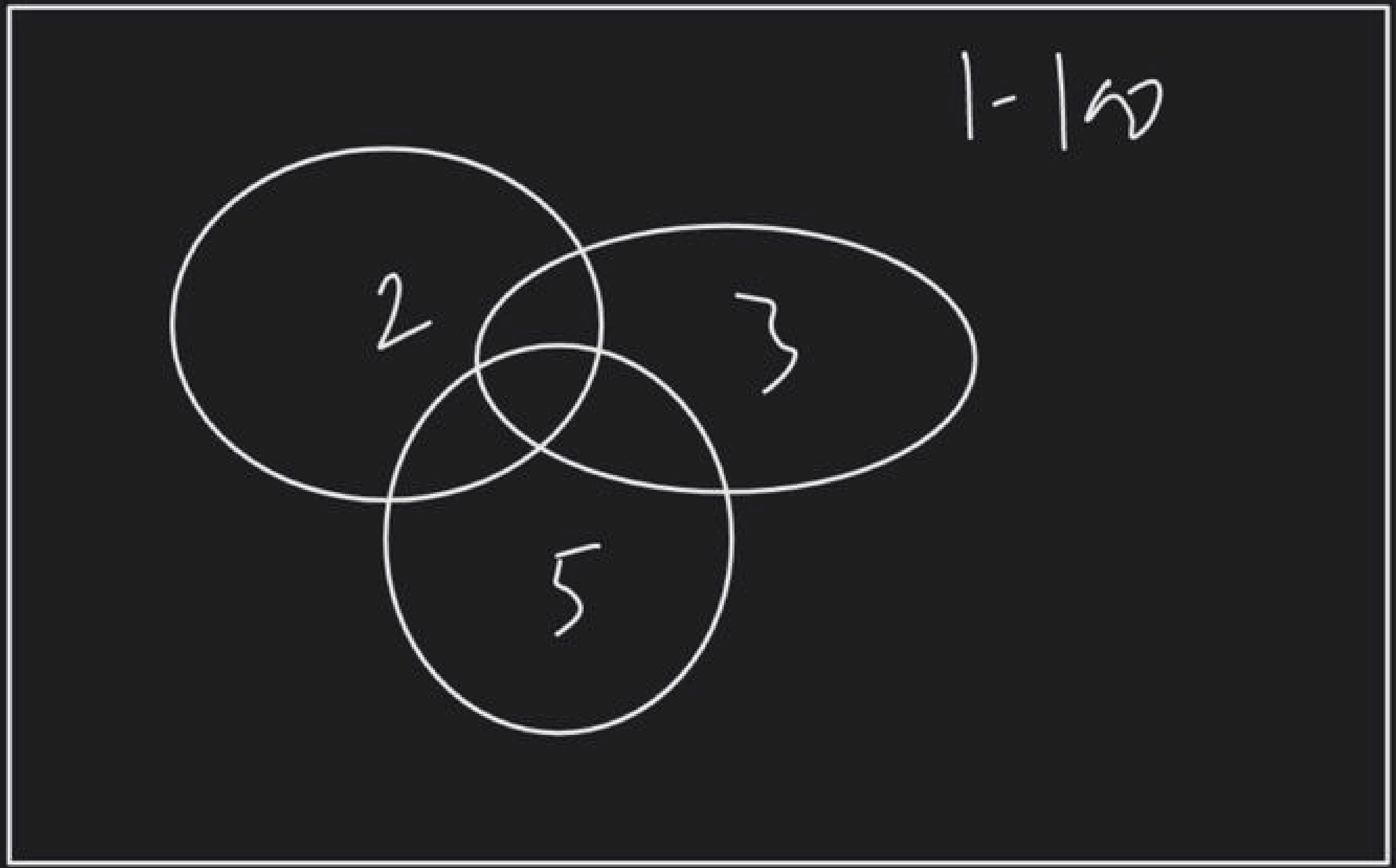
common element

$P(E_1 \cap E_2 \cap E_3)$

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

What is the Prob. div by 2 or 3 or 5

2 ✓	2, 3	2, 3, 5
3 ✓	3, 5	
5	2, 5	



$$P(2 \cup 3 \cup 5) = P(2) + P(3) + P(5) - P(2 \cap 3) - P(3,5) - P(2,5) + P(2,3,5)$$

$$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$$

$$P(2 \vee 3 \vee 5) = P(2) + P(3) + P(5) - P(2,3) - P(3,5) - P(2,5) + P(2,3,5)$$

$$= \left\{ P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3) \right\}$$

$$P(2 \vee 3 \vee 5) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100}$$

$$= \frac{74}{100} \quad \text{done}$$

$$P(\text{Not divisible}) = 1 - \frac{74}{100} = \frac{26}{100} \quad \text{done}$$

E_1, E_2, E_3 (THREE of THEM)

$E_1 \to E_2 \to E_3 = E_1 \cap E_2 \cap E_3 = E_1, E_2, E_3$ (Intersection

$E_1 \to E_2 \to \overline{E_3} = E_1 \cap E_2 \cap \overline{E_3}$ = only $\{E_1, E_2\}$

$E_1 \to \overline{E_2} \to E_3 = E_1 \cap \overline{E_2} \cap E_3$ = only $\{E_1, E_3\}$

$E_1 \to \overline{E_2} \to \overline{E_3} = E_1 \cap \overline{E_2} \cap \overline{E_3}$ = only $\{E_1\}$

$\overline{E_1} \to E_2 \to E_3 = \overline{E_1} \cap E_2 \cap E_3$ = only $\{E_2, E_3\}$

$\overline{E_1} \to E_2 \to \overline{E_3} = \overline{E_1} \cap E_2 \cap \overline{E_3}$ = only $\{E_2\}$

$\overline{E_1} \to \overline{E_2} \to E_3 = \overline{E_1} \cap \overline{E_2} \cap E_3$ = only $\{E_3\}$

$\overline{E_1} \to \overline{E_2} \to \overline{E_3} = \overline{E_1} \cap \overline{E_2} \cap \overline{E_3}$ = neither $\{E_1, E_2 E_3\}$

8 Events

E_1, E_2, E_3

(THREE OF THEM)

$E_1 \to E_2 \to E_3$: $= E_1 \cap E_2 \cap E_3 = E_1, E_2, E_3$ (Intersection) $= 7$ region

$E_1 \to E_2 \to \overline{E_3}$: $= E_1 \cap E_2 \cap \overline{E_3} =$ only $\{E_1, E_2\}$ $=$ only $1 + 2$

$E_1 \to \overline{E_2} \to E_3$: $= E_1 \cap \overline{E_2} \cap E_3 =$ only $\{E_1, E_3\}$

$E_1 \to \overline{E_2} \to \overline{E_3}$: $= E_1 \cap \overline{E_2} \cap \overline{E_3} =$ only $\{E_1\}$

$\overline{E_1} \to E_2 \to E_3$: $= \overline{E_1} \cap E_2 \cap E_3 =$ only $\{E_2, E_3\}$

$\overline{E_1} \to E_2 \to \overline{E_3}$: $= \overline{E_1} \cap E_2 \cap \overline{E_3} =$ only $\{E_2\}$

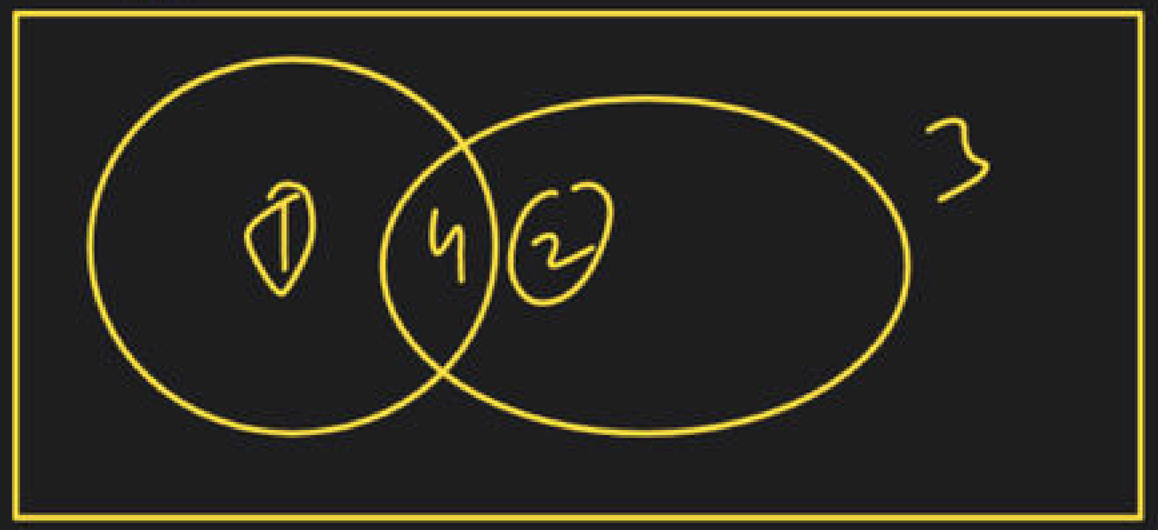
$\overline{E_1} \to \overline{E_2} \to E_3$: $= \overline{E_1} \cap \overline{E_2} \cap E_3 =$ only $\{E_3\}$

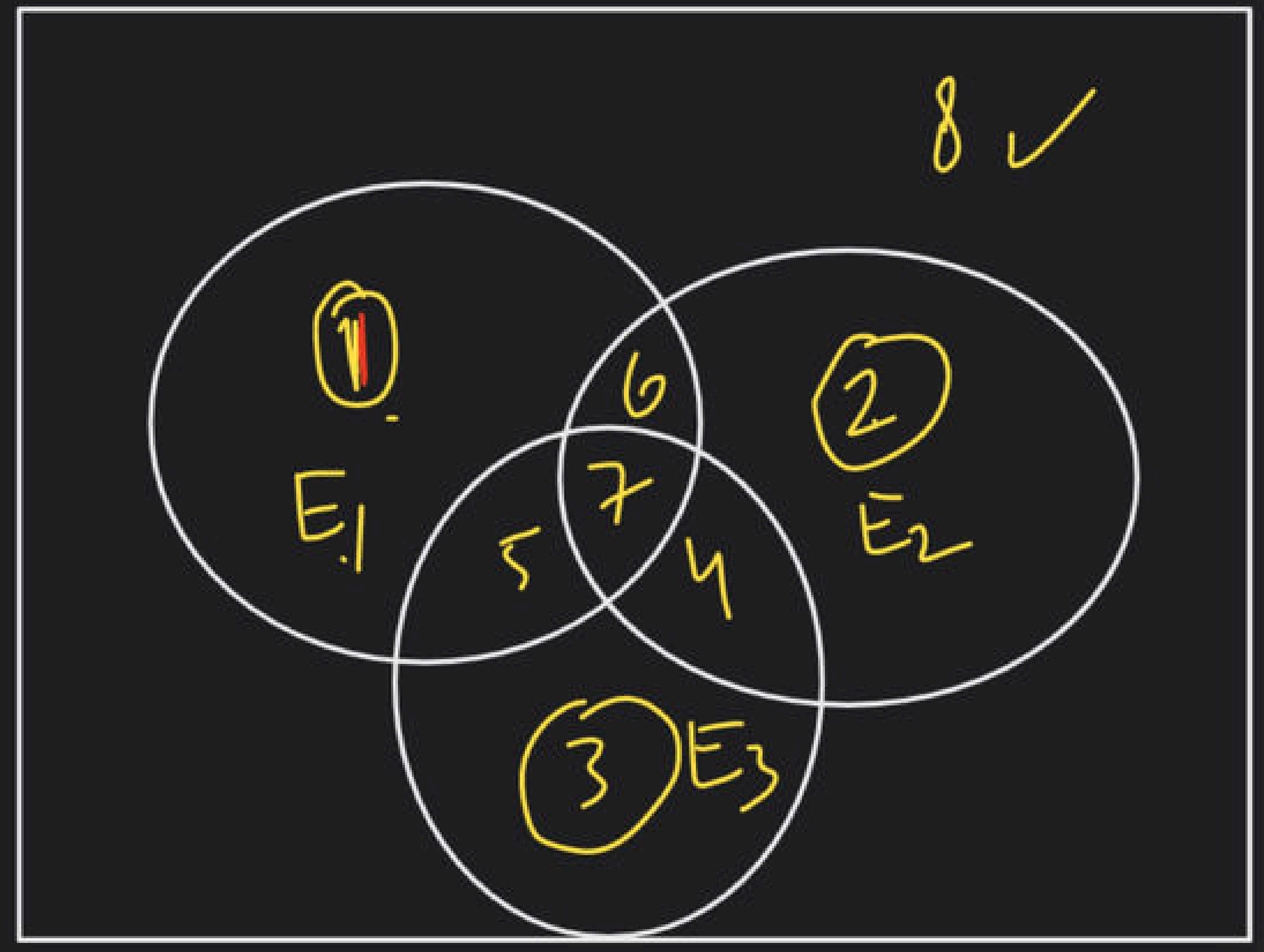
$\overline{E_1} \to \overline{E_2} \to \overline{E_3}$: $= \overline{E_1} \cap \overline{E_2} \cap \overline{E_3} =$ neither $\{E_1, E_2, E_3\}$

or

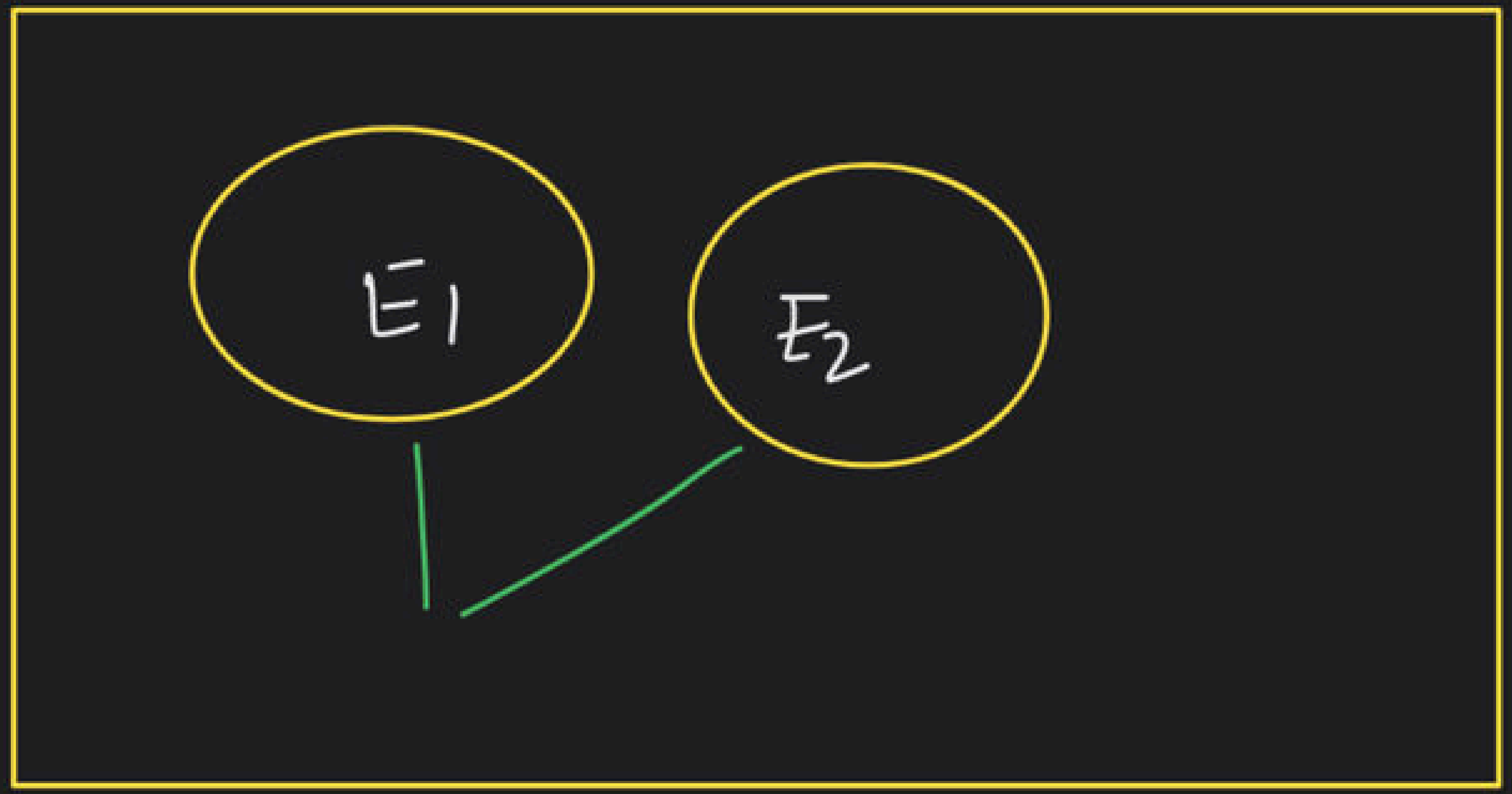
Nor

8 Events





Nothing is common = "Mutually exclusive events"

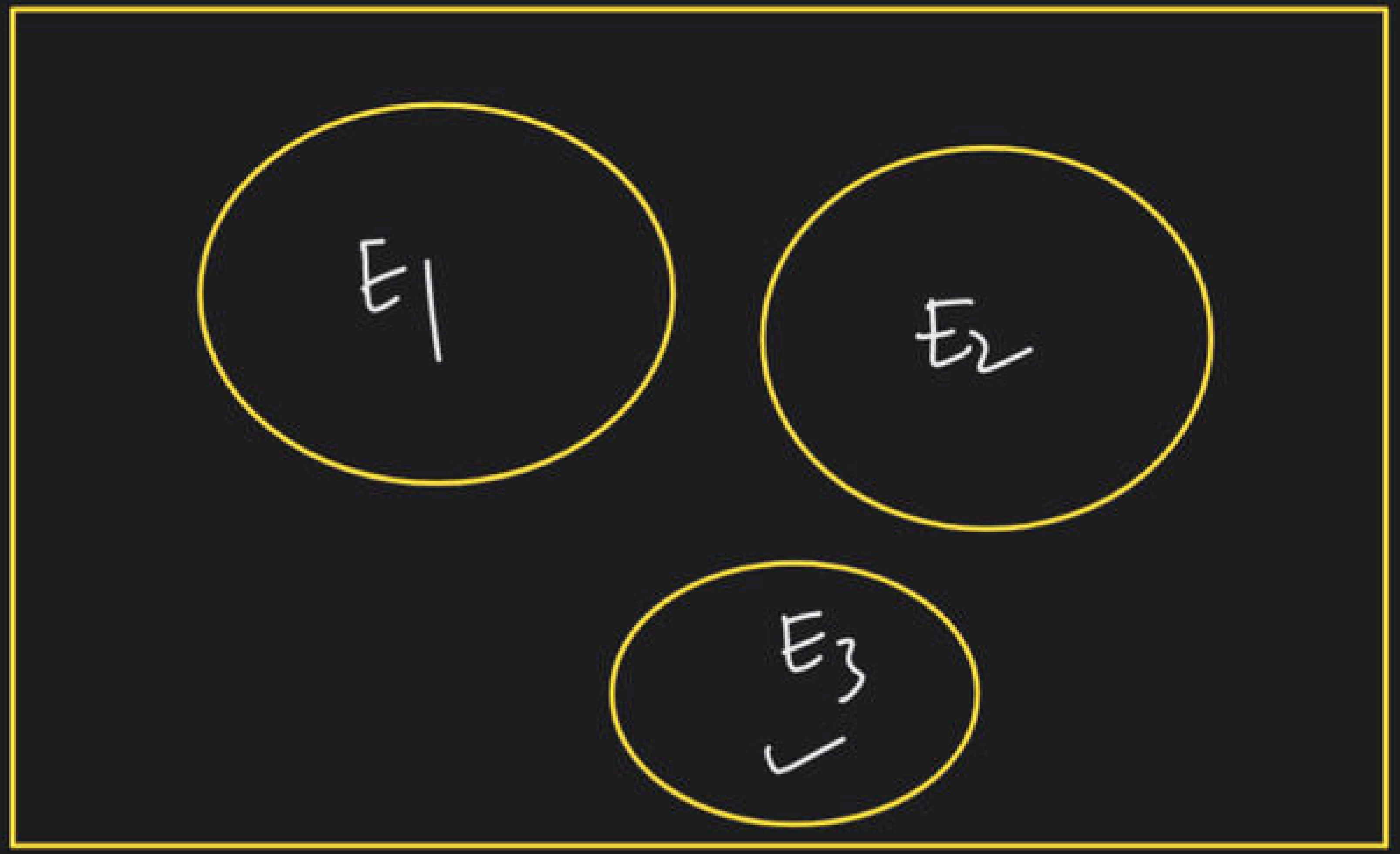


If E_1 and E_2 ARE disjoint events

$$P(E_1 \cap E_2) = 0$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$\boxed{P(E_1 \cup E_2) = P(E_1) + P(E_2)}$$

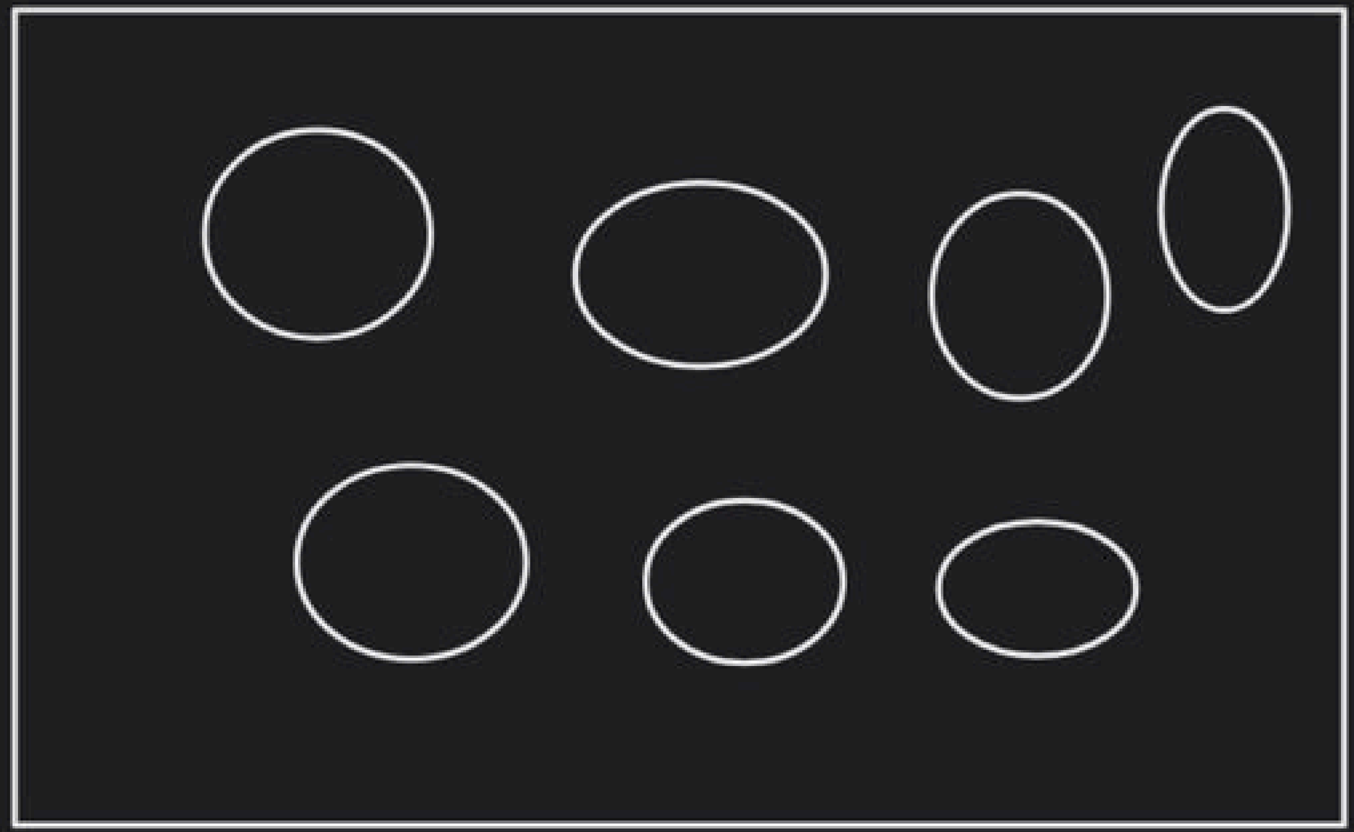


$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

Indep. $P(E_1 \cap E_2) = P(E_1)P(E_2) \Rightarrow$

$E_1, E_2, E_3, E_4 ---- E_n$ ARE Mutually exclusive events

✓ # $\boxed{P(E_1 \cup E_2) = 1}$



Indep $P(E_1 \cap E_2) = P(E_1)\,P(E_2)$
mutually $P(E_1 \cap E_2) = 0$

$\boxed{P(E_1 \cup E_2 \cup E_3 -- E_n) = P(E_1) + P(E_2) + P(E_3) + -- + P(E_n)}$

$E_1, E_2, E_3 -- E_n$ disjoint EVENTS (nothing is common)