Question

1. Represent the given system of linear equations as

$$
\begin{aligned}
-y + z &= 2 \\
-x + y - z &= -3 \\
x - y &= -2
\end{aligned}
$$

   a. Linear Combination of Column Vector.

   b. $Ax = b$ form.

Answer:

$$x\begin{bmatrix}0\\-1\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\\-1\end{bmatrix}+z\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}2\\-3\\-2\end{bmatrix}$$

$$\begin{bmatrix}0 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\-3\\-2\end{bmatrix}$$

Question

2. For the given system of linear equations:

$$
\begin{aligned}
3x + 0y + 0z &= 6 \\
0x + 2y + 0z &= 4 \\
0x + 0y + 4z &= 16
\end{aligned}
$$

   a. Show the rough Geometric Interpretation as column vectors. (optional)

   b. Show the rough Geometric Interpretation as equations. (optional)

   c. Convert the equations to $Ax = b$ form.

   d. For which values of b does the matrix A has a solution? Why?

Answer:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 16 \end{bmatrix}$$

The set of column vectors of A are linearly independent. Hence for all b (in $\mathbb{R}^3$), A has a solution.

Question

3. Which of the following statement(s) is(are) true:
   a. Given a coefficient matrix $A_{m \times n}$, $m < n$, the number of solutions for $Ax = \vec{0}$ could either be zero or one.
   b. There exists a coefficient matrix $A_{m \times n}$, $m > n$, which doesn't give solution for any b
      .
   c. There exists a coefficient matrix $A_{m \times n}$, $m = n$, which doesn't give solution for some b.
   d. If a coefficient matrix $A_{m \times n}$ gives solution for every b, then $m = n$.

Answer:

C

A is false, since the number of solutions can not be zero for the system, $Ax = \vec{0}$.

B is false. There always exists the zero column vector for which the given A would have solution.

C is true, since $m = n$ doesn't mean that there are n linearly independent vectors in $\mathbb{R}^n$.

D is false. If A gives solution for every b, then $m \leq n$. (We'll come back to this point while discussing Rank again.)

Question

4. Given the coefficient matrix $A_{m \times n}$, what can be said about the number of solutions if (zero/unique/infinite):
   a. $m = n$
   b. $m < n$
   c. $m > n$

Answer:

There isn't much we can say about the number of solutions just by looking at the dimensions of A. There can be zero or unique or infinite in all three above cases.

Question

5. [Introduction to LA, Gilbert Strang, 5th ed., 2.1.7]

   For the given matrix A:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix}$$

   a. Find the linear combination of first and second column which obtains the third column vector.
   b. Check if there is a solution for the following b:
      i. $b = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}^T$
      ii. $b = \begin{bmatrix} 4 & 6 & 11 \end{bmatrix}^T$

Answer:

a.

$$1\begin{bmatrix}1\\1\\2\end{bmatrix}+2\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}3\\5\\8\end{bmatrix}$$

b.

i. Yes. $X = \begin{bmatrix}1 & 1 & 0\end{bmatrix}$

ii. No.

Question

6. The given matrix has solution for:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix}$$

a. All vectors b in $\mathbb{R}^3$.

b. No vector b in $\mathbb{R}^3$.

c. Some vectors b in $\mathbb{R}^3$.

d. Some vectors b in $\mathbb{R}^2$.

Answer:

C. Only for some vectors in $\mathbb{R}^3$.

Question

7. Check the number of solution (zero/unique/infinite) for the following system of linear equations

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 3 & 8 & 9 \\ 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer:

There can be more than one solutions for the above system. Solving the above system is equivalent to checking if the columns of A are linearly independent/dependent. The columns are linearly dependent. The third column is a multiple of the first column.

Question

8. The following matrix A, doesn't give solution for any b, since its columns are linearly dependent. True/False?

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & 8 & 0 \\ 3 & 9 & 0 \end{bmatrix}$$

Answer:

False. For example, $b = \begin{bmatrix} 1 & 4 & 3 \end{bmatrix}^T$ and get the solution as $X = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Question

9. Find a b such that the system $Ax = b$ has no solution.

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & 8 & 0 \\ 3 & 9 & 0 \end{bmatrix}$$

Answer:

$$b = \begin{bmatrix} 4 & 12 & 13 \end{bmatrix}.$$

Question

10. [https://www.uvm.edu/~mrombach/124_rombach_midterm1_sols.pdf Q5]

$$\begin{aligned} 2x + 2z &= 4 \\ y &= 3 \\ 2x + y + 3z &= 8 \end{aligned}$$

a. Represent the system of Linear Equations in $Ax = b$ form.
b. Check if the columns of A are linearly independent.
c. Try to find the solution to the system.

[https://www.uvm.edu/~mrombach/124_rombach_midterm1_sols.pdf Q5]

Question

11. [https://math.berkeley.edu/~nikhil/courses/54.f18/midterm1sol.pdf Q2]

Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

a. A linear system with 3 equations in 2 variables which has solution.

b. A linear system with 2 equations in 3 variables which has no solution.

c. A linear system with 2 equations in 3 variables which has solution for no b ($b \neq \vec{0}$).

Answer is here -
[https://math.berkeley.edu/~nikhil/courses/54.f18/midterm1sol.pdf Q2]