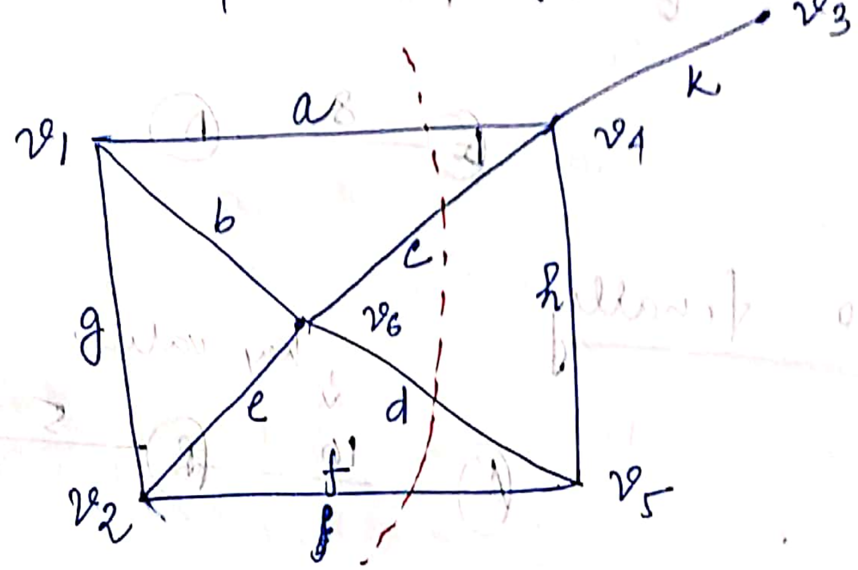
# CUT-SET



In a connected graph G, a cut-set is a set of edges, whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.

G:




example: {a, b, g} is cut set but {a, b, g, e} is not.

other cut sets.....

{a, c, d, f}

{a, c, h}

{k}

{f, d, h}

{g, b, c, h}

{g, e, d, h}

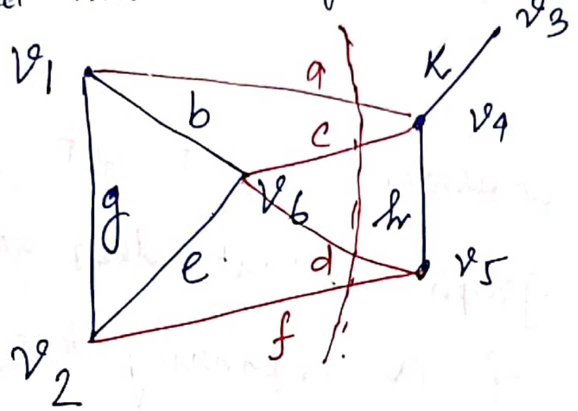
{g, e, f}

{b, c, d, e}

etc.

② A ~~gra~~ cut set disconnects a connected graph into two. So the rank $(n-k)$ decreased by 1.

For ex: w.r.t cut set $\{a, c, d, f\}$ G is divided into two graphs.



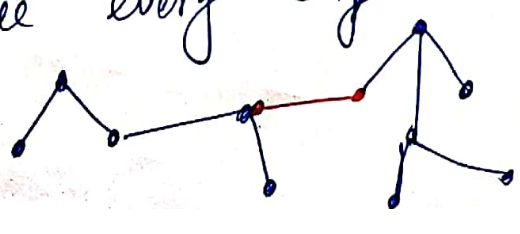

A cut set S partitions the vertex set V of G into two disjoint subsets $V_1$ and $V_2$ such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$ and any edge of S has one end vertex in $V_1$ and other end vertex in $V_2$. Each edge of a cut set is called a cross edge.

In above fig

cut set $= \{a, c, d, f\}$

$V_1 = \{v_1, v_2, v_6\}$  $V_2 = \{v_3, v_4, v_5\}$

In a tree every edge itself is a cutset



## Theorem-I

Every cut set in a connected graph must contain at least one branch of every spanning tree of G.

**Proof**

Let us consider a cut set S in a connected graph G, that does not contain any branch of a spanning tree T of G.

If we remove all cross edges of S from G then all branches of T will be there in G. As the remaining graph $G' = (G - S)$ contains T so $G'$ is not a disconnected graph.

But if S is a valid cut set then this is not possible. Hence such a cut set S does not exists.

So every cutset must contain at least one ~~sp~~ branch from every spanning tree of G.

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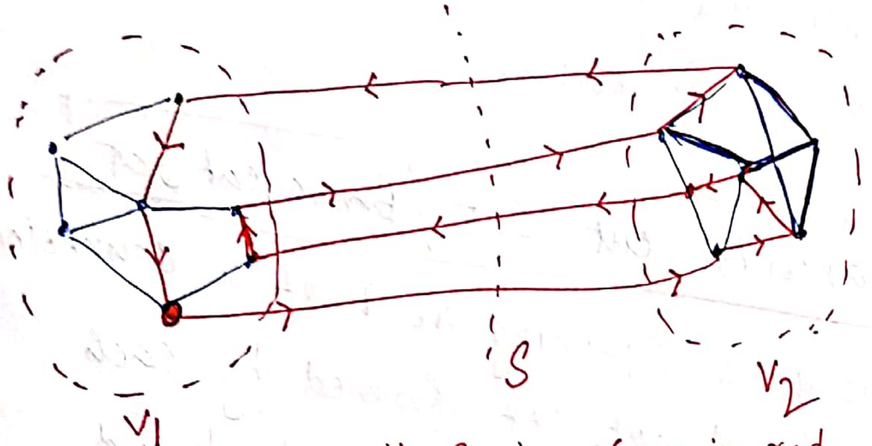
## Theorem - II

Every circuit has an even number of edges in common with any cut set.

⇒ Let us consider a connected graph G and a cut set S in G. Now S partitions vertex set V of G into two disjoint subsets $V_1$ and $V_2$ such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.




Circuit C is shown in red.

Now let us consider a circuit (closed path) C in G.

Now if vertices of C either belongs to $V_1$ or $V_2$, the intersection of ~~vertices~~ edges of C with ~~either~~ S in $\emptyset$. Hence ~~edges~~ number of edges common is 0 (i.e even).

Now if C spans both the partitions $V_1$ and $V_2$ then if we start tracing C from say $V_1$, ~~and~~ we have to take one edge from S as we go from $V_1$ to $V_2$. Again if we comes back

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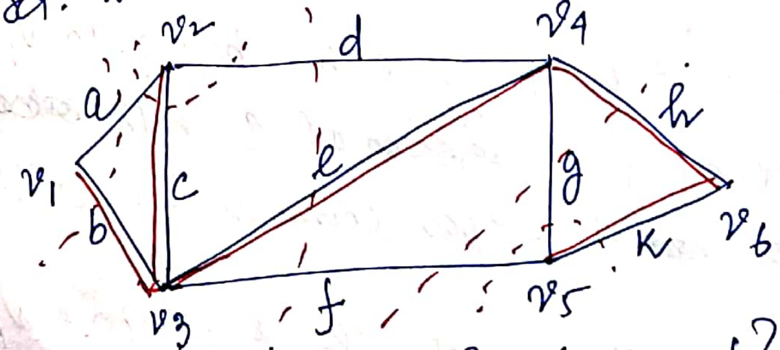
from $v_2$ to $v_1$, we have to take another edge from S. So one back and forth movement between $v_1$ and $v_2$ selects two edges from S. Now as C is closed in nature so if we start from $v_1$ we must end at $v_1$ at last. Meanwhile we can move a number of times to $v_2$. Each such movement selects 2 edges from S. Hence total number of common edges between S and C is even.

## Fundamental cut set / Basic cut set

Consider a spanning tree T of a connected graph G. A cut set formed by each branch b of T, which consists of b and chords of T, is called as fundamental cut set w.r.t T.




ex $\{a, \underline{b}\}$, $\{a, \underline{c}, d\}$, $\{d, \underline{e}, f\}$,

$\{f, g, \underline{h}\}$, $\{f, \underline{K}\}$

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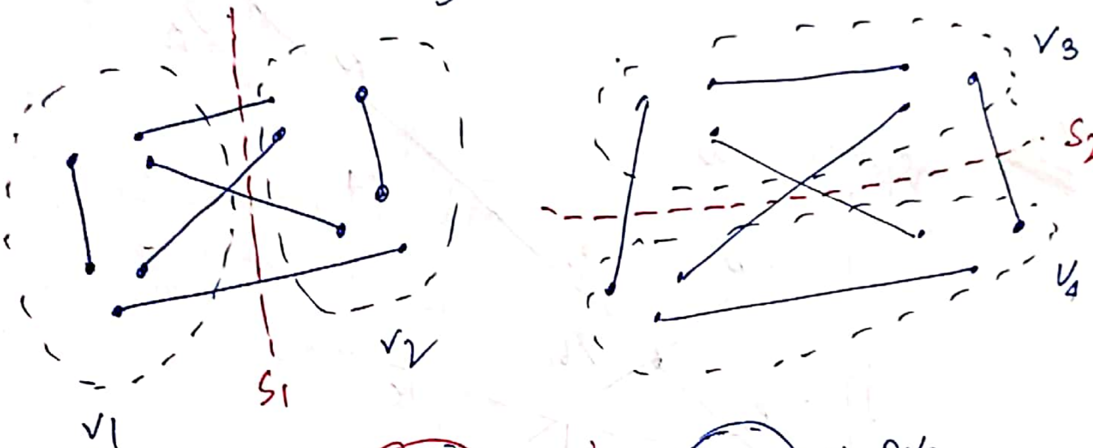
**Theorem - III** The ring sum of any two cut sets in a graph is either a third cut set or an edge disjoint union of cut sets.

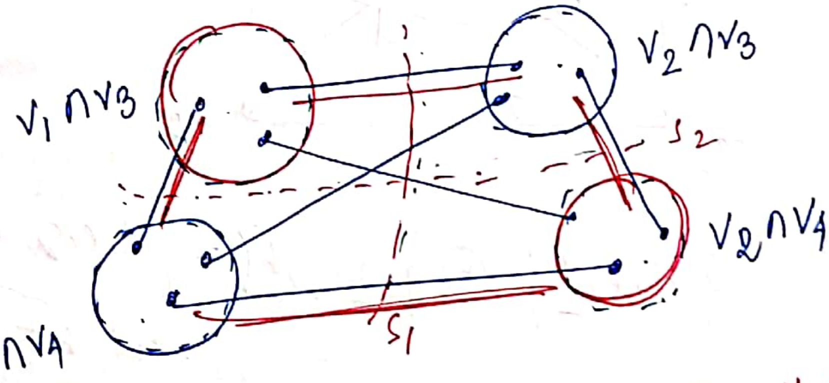
⇒ Let $S_1$ and $S_2$ be two cutsets of connected graph $G$. Let $S_1$ partitions $G$ into vertex set $V_1$ and $V_2$ and $S_2$ partitions $G$ into vertex set $V_3$ and $V_4$. so.

$$V_1 \cup V_2 = V \qquad V_1 \cap V_2 = \phi$$

$$V_3 \cup V_4 = V \qquad V_3 \cap V_4 = \phi$$




Ring sum
$A \oplus B$
$= (A \cup B) - (A \cap B)$




Let $V_5 = (V_1 \cap V_4) \cup (V_2 \cap V_3) = V_1 \oplus V_3$

and $V_6 = (V_1 \cap V_3) \cup (V_2 \cap V_4) = V_2 \oplus V_3$

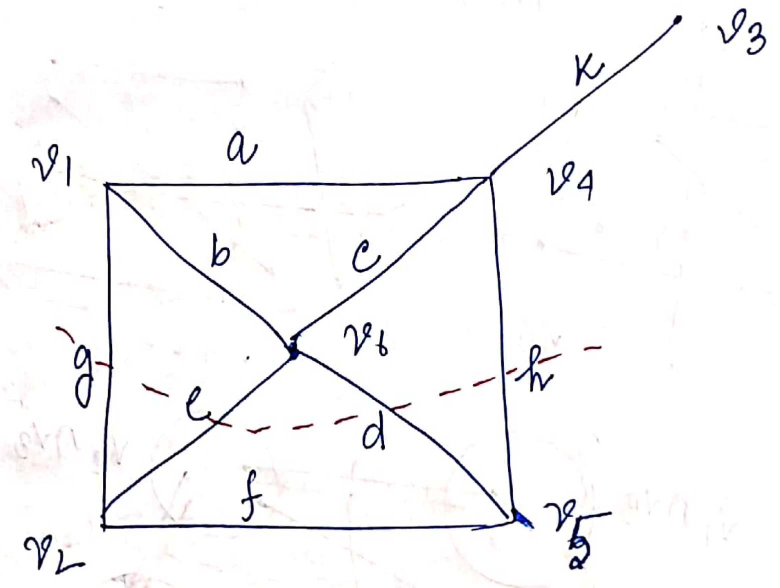
Now $S_1 \oplus S_2$ only consists of those edges that joins vertices of $V_5$ to $V_6$, and no edges outside $S_1 \oplus S_2$ joins $V_5$ to $V_6$.

So edges of $S_1 \oplus S_2$ partitions $V$ into $V_5$ and $V_6$. such that

$$V_5 \cup V_6 = V \quad \text{and} \quad V_5 \cap V_6 = \phi$$

Hence $S_1 \oplus S_2$ is a cut set if subgraphs containing $V_5$ and $V_6$ each remains connected after $S_1 \oplus S_2$ is removed from G. Otherwise $S_1 \oplus S_2$ is edge disjoint union of cut sets.

Ex




cut sets $\{d, e, \underline{f}\} \oplus \{\underline{f}, g, h\} = \{d, e, g, h\}$

= cut set

$\{a, \underline{b}\} \oplus \{\underline{b}, c, e, f\} = \{a, c, e, f\}$ cut set

$\{d, e, g, h\} \oplus \{f, g, k\} = \{d, e, f, h, k\}$

$= \{d, e, f\} \cup \{h, k\}$

edge disjoint union of cut sets.

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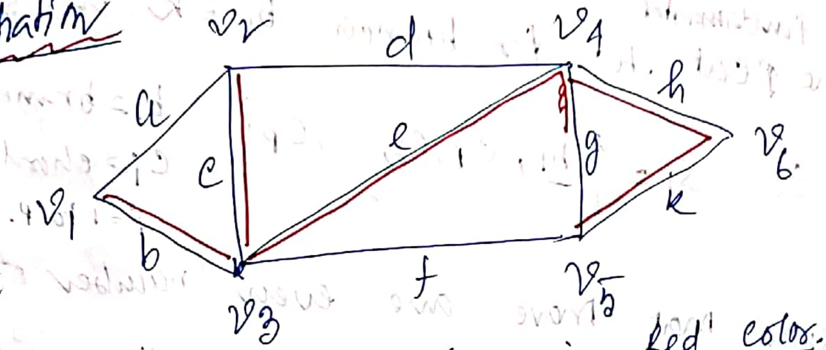
**Theorem - IV**: With respect to a given spanning tree T, a chord c, that determines a fundamental circuit K occurs in every fundamental cut set associated with the branches in K and in no other.

**explanation**




Spanning is shown in Red color.

~~Let~~ a fundamental circuit for chord 'a' is

$K = \{b, c, a\}$

branch chord

Fundamental cut set for ~~chord~~ $b = \{a, b\}$ branch of K

bran c of K $= \{a, c, d\}$

Now chord a occurs in every fundamental cut set defined by branches of K.

Now if we take any fundamental cut set that are not formed by b or c i.e. for example for $e = \{d, e, f\}$, a is not present here.

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**Proof** Let us consider a connected graph $G$, a spanning tree $T$ of $G$. Let $c$ be a chord of $T$. Now let us consider the fundamental circuit $K$ defined by chord $c$.

$$K = \{c, b_1, b_2, \ldots b_m\}$$ $c$ = chord, $b_i$ = branch $i = 1$ to $m$

Let the fundamental cut set for branch $b_1 \in K$ is $S_1$.

So $S_1 = \{b_1, c_1, c_2, \ldots, c_p\}$ $b_1$ = branch, $c_i$ = chords, $i = 1$ to $p$.

We know that there are even number of edge common between $S_1$ and $K$. Hence chord $c$ belongs to $S_1$.

Similarly if we design fundamental cut sets $S_2, S_3, \ldots, S_m$ for branches $b_1, b_2, \ldots b_m$ of $K$ respectively then $c$ occurs in every one of them.

Now we consider a fundamental cut set of $G$ that does not contain any branch of $K$. Let such a cut set be

$$S_p = \{b_s, c_1, c_2, \ldots, c_t\}$$ where

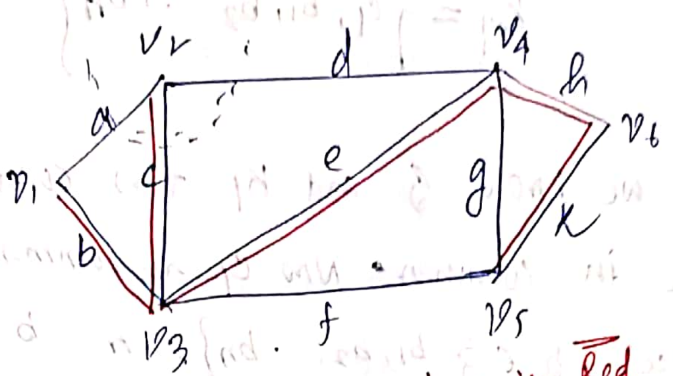
$$b \notin \{b_1, \ldots b_m\}$$

Now $K$ and $S_p$ has even number of edges common and $b_s \notin K$. Hence $c$ does not belong to $S_p$.

(Proved)

**Theorem – V** : With respect to a given spanning tree T, a branch 'b' that determine a fundamental cut set S is contained in every fundamental circuit associated with the chords in S, and in no others.

**explanation** :




spanning tree in Red

Let fundamental cut set for branch 'c', $\{a, c, d\} = S$

Now fundamental circuit for chord $a = \{a, b, c\} = C_1$

chord $d = \{c, d, e\} = C_2$

Now branch 'c' is present both in $C_1$ & $C_2$.

Now if we consider any chord other than $\{a, d\}$ say f the fundamental circuit is, $C_3 = \{e, h, k, f\}$

'c' does not belong to $C_3$.

**Proof** : Let us consider a connected graph G, a spanning tree T of G, and a branch 'b' in T. Let the fundamental cut set formed by 'b' is S.

So $S = \{b, c_1, c_2, \cdots c_m\}$ $b$ = branch, $c_i$ = chord, $i = 1, 2, \cdots m$.

Let fundamental circuit for chord $c_1 \in S$ is $K_1$.

$$K_1 = \{c_1, b_1, b_2, \dots, b_n\}$$ $c_1$ = chord, $b_i$ = branch, $i = 1, 2, \dots, n$

Now we know $S$ and $K_1$ has even number of edge in common. Now $c_1$ is common in $S$ & $K_1$.

Hence $b \in \{b_1, b_2, \cdots b_n\}$, or $b \in K_1$

Similarly if we see fundamental ~~cut sets~~ circuits $K_2, K_3, \dots, K_m$ corresponding to chords $c_2, c_3, \cdots, c_m$ of $S$ respectively we see that $b$ is present in each one of them.

Now consider a chord $c_K \notin \{c_1, c_2, \dots, c_m\}$ i.e $c_K \notin S$. Let such a fundamental circuit be $K$. Now

$$K = \{c_K, b_1', b_2', b_3', \dots, b_s'\}$$

$c_K$ = chord, $b_i'$ = branch, $i = 1, 2, \dots s$.

Now $c_K \notin S$.

So we know that $S$ and $K$ should have even number of edges in common. So $b \notin \{b_1', b_2', \dots, b_s'\}$. Hence $b \notin K$.

(Proved).

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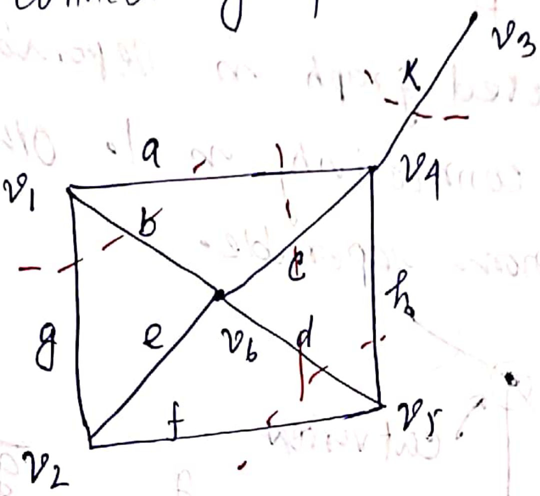
## Edge Connectivity

Each cut set of a connected graph G consists of some number of edges. The number of edges in the smallest cut-set is defined as edge connectivity of G.

ex




Cut sets.
{a, b, g}
{a, b, e, f}
{a, c, d, f}
{g, e, f}
{K}

Edge connectivity of a tree is alway 1.

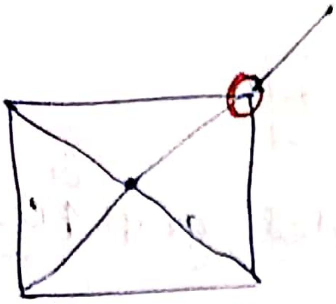
Smallest cut set = {K}

Edge connectivity = |{K}| = 1

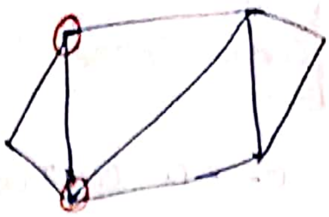
## Vertex Connectivity

Vertex connectivity of a connected graph G is defined as the minimum number of vertex whose removal from G leaves the remaining ~~grap~~ graph disconnected.

Vertex connectivity of a tree is always 1.



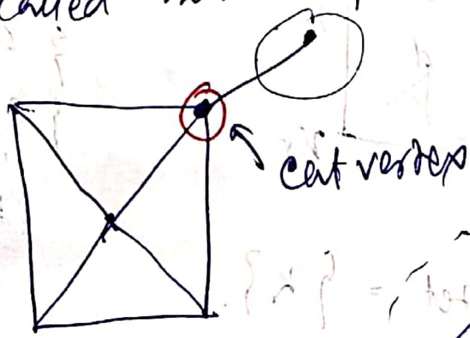
vertex connectivity = 1

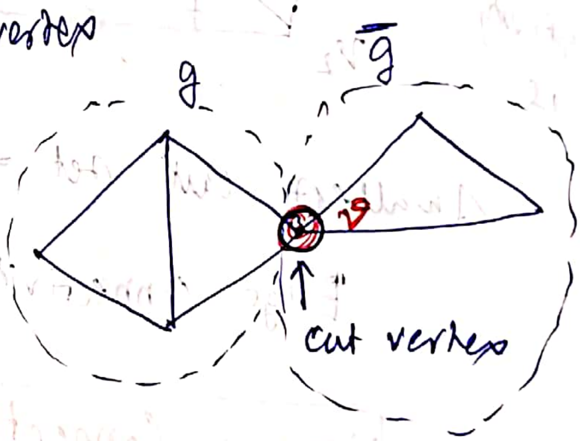


vertex connectivity = 2

## Seperable graph

A connected graph is seperable if its vertex connectivity is 1. Other graphs are called non-seperable.




In a seperable graph the vertex whose removal disconnects the graph is called <u>cut-vertex</u> or <u>articulation point</u>.

In a seperable graph G there exist a subgraph g of G such that complement of g in G i.e $\bar{g}$ and g have one vertex common (v).

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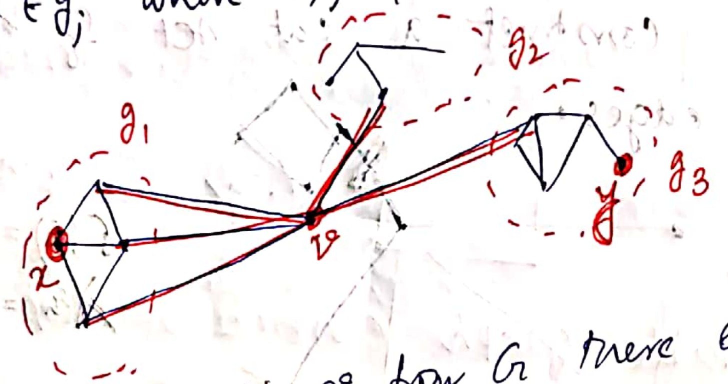
**Theorem-6** A vertex $v$ in a connected graph $G$ is a cut-vertex if and only if there exist two vertices $x$ and $y$ in $G$ such that every path between $x$ and $y$ passes through $v$.

→ Let us consider a connected graph $G$ and a cut-vertex $v$ of $G$. Now deletion of $G$ leaves graph $G$ disconnected with say $g_1, g_2 \dots g_n$ components. Now let us consider any two vertices ~~the con~~ $x \in g_i$ and $y \in g_j$ where $i, j = \{1, 2, \dots n\}$.




Now after deleting $v$ from $G$ there exists no path between $x$ & $y$ as $g_i$ and $g_j$ are different components. But as $G$ was connected there was at least one path in between $x$ & $y$. So every path between $x$ & $y$ must pass through $v$.

Conversely let us consider there exists two vertices $x$ and $y$ in $G$ where all paths between $x$ & $y$ passes through $v$. Now if we delete $v$ from $G$, there there will be no path between $x$ & $y$ in $G$. Hence $G$ will become disconnected. Hence $v$ is a cut vertex.

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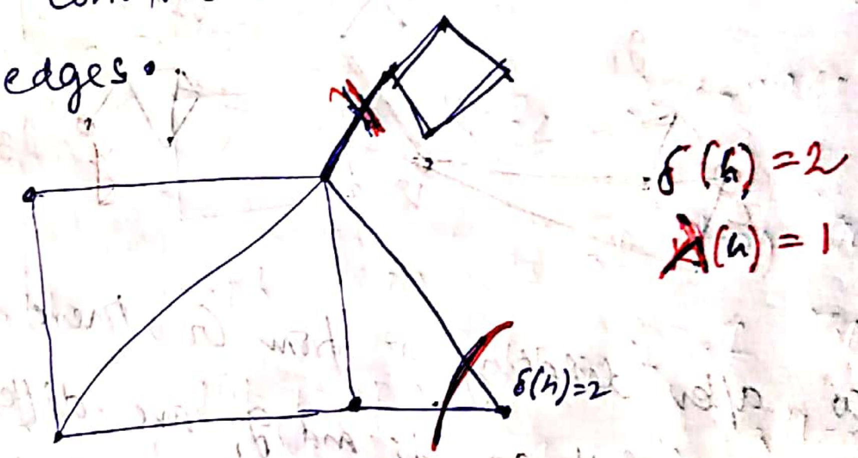
**Theorem - 7** The edge connectivity of a graph can not exceed the degree of the vertex with smallest degree in G. i.e $\lambda(G) \le \delta(G)$

⇒ Edge connectivity of a graph G (connected) defines the number of edges in the smallest cut set of G, denoted by $\lambda(G)$.

~~Smallest~~ Minimum degree of a graph is denoted by $\delta(G)$. If a vertex $v$ has degree $\delta(G)$ that means from $v$ there are $\delta(G)$ edges. Now we construct a cut set $S_\delta$ by taking these edges.




Hence $\lambda(G)$ can not be more than $\delta(G)$.

Hence $\boxed{\lambda(G) \le \delta(G)}$

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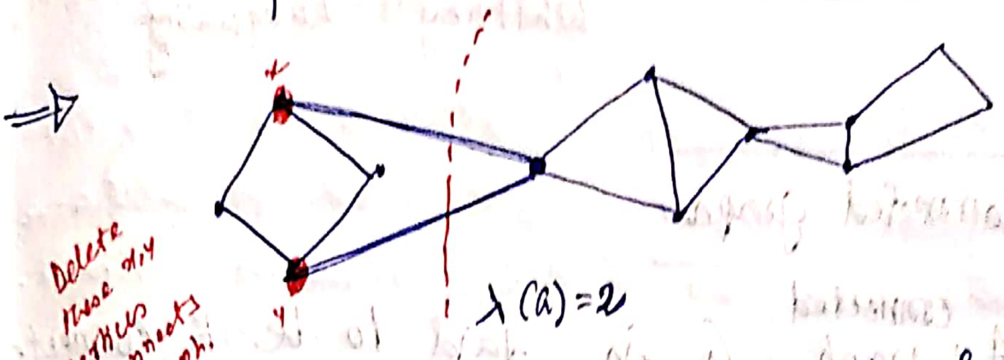
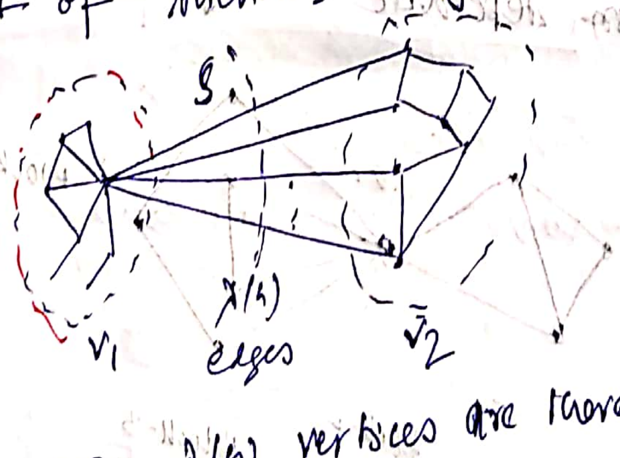
## Theorem-8

The vertex connectivity of any graph $G$ can never exceed the edge connectivity of $G$. $\boxed{K(G) \leq \lambda(G)}$

$\Rightarrow$




Let $\lambda(G)$ denote edge connectivity of a connected graph $G$. So there exist a cut set $S$ in $G$ with $\lambda(G)$ edges. Let $S$ partitions $G$ into two vertex sets $V_1$ and $V_2$. Now by removing $\lambda(G)$ ~~edges~~ edges of the cut set $S$, $G$ may be disconnected. Now each edge of $\lambda(G)$ has two end vertices one in $V_1$ & other in $V_2$. One extreme arrangement of such $S$ may be.




So at most $\lambda(G)$ vertices are there associated with cut set $S$ either in $V_1$ or $V_2$. So removing these vertices will certainly disconnects the graph. Hence vertex connectivity cannot exceed the edge connectivity. Hence

$\boxed{K(G) \leq \lambda(G)}$

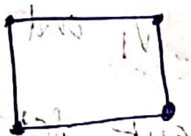
Hence: $\boxed{K(G) \le \lambda(G) \le \delta(G)}$

Whitney's Inequality

## K-connected graph

A connected graph $G$ is said to be $K$ connected if the vertex connectivity of $G$ is $K$.

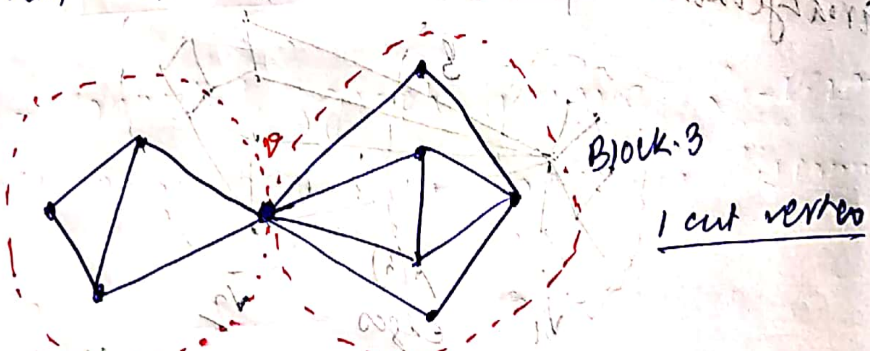
So 1-connected graph = Seperable graph

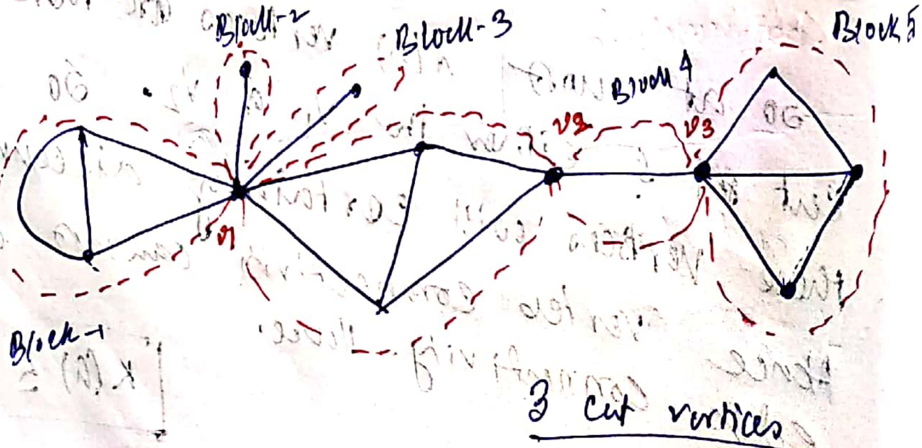


$K=2$, 2-connected

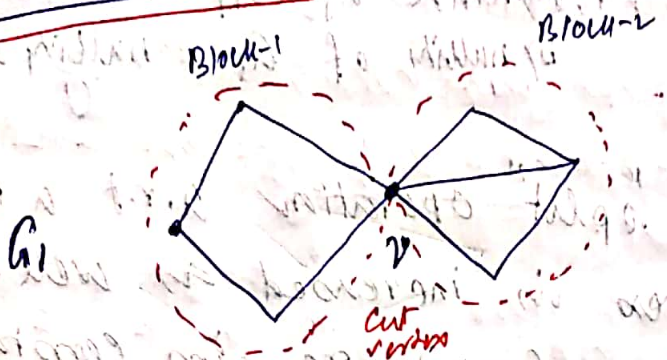
## Blocks

A seperable graph consists of two or more non-seperable subgraphs. Each of the largest non-seperable subgraphs is called a block.

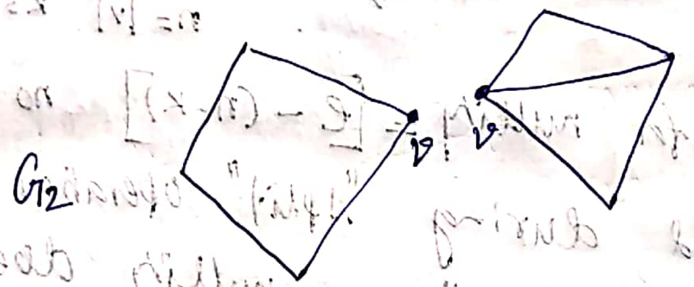


## 1- isomorphism




$G_1$ & $G_2$ are 1-isomorphic




Two graphs $G_1$ & $G_2$ are 1-isomorphic if they become isomorphic to each other under repeated application of the folling operation.

**Operation-1** Split a cut vertex into two vertices to produce two disjoint subgraphs.

If for every block of $G_1$ there exist a ~~component~~ isomorphic block ~~component~~ in $G_2$ then $G_1$ & $G_2$ are 1-isomorphic.

- For non-seperable graphs there is just one block. Hence for non-seperable graphs 1-isomorphic is equal to isomorphism.

**Theoram-9** If $G_1$ and $G_2$ are two 1-isomorphic graphs, then rank of $G_1$ = rank of $G_2$ & nullity of $G_1$ = nullity of $G_2$

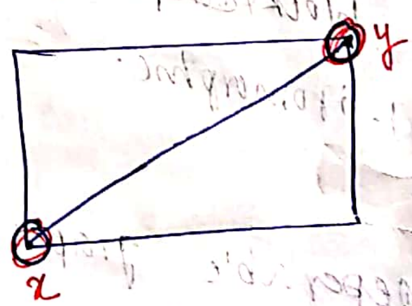
⇒ Now under "split" operation w.r.t a cut vertex, a vertex is increased as well as a component. So $(n-K)$ remains constant. $n=|V|$ $K=|\text{components}|$

Similarly for nullity = $[e-(n-K)]$ no edge is increased during "split" operation and $(n-K)$ remains invariant. Hence nullity does not change. Hence rank of $G_1$ = rank of $G_2$
nullity of $G_1$ = nullity of $G_2$.

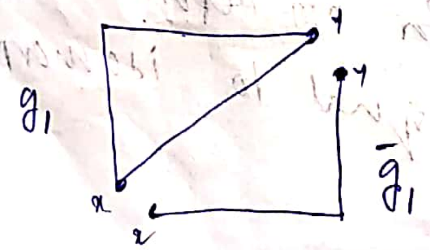
## 2-isomorphism

A 2-connected graph G have 2-vertices whose removal from G, leaves G disconnected

eg




2-connected.



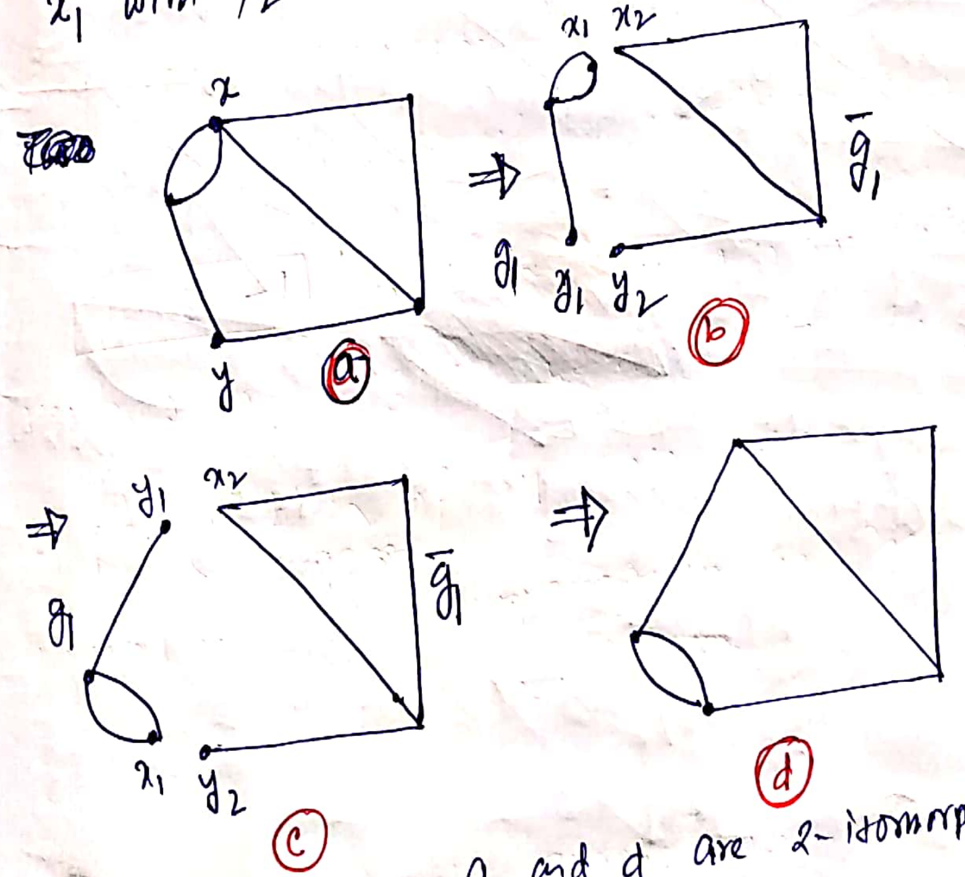

Let such two vertices are x & y.

So G consists of two subgraphs $g_1$ and $\bar{g}_1$ such that $g_1$ and $\bar{g}_1$ have exactly two vertices in common, $x$ & $y$.

Now if we perform the following operation:

Operation-2 split vertex $x$ into $x_1$ and $x_2$; $y$ into $y_1$ and $y_2$, such that a splits into $g_1$ & $\bar{g}_1$. Let vertices $x_1$ & $y_1$ go with $g_1$ and $x_2$ & $y_2$ goes with $\bar{g}_1$. Now rejoin $x_1$ with $y_2$ and $x_2$ with $y_1$.

~~Two~~




a and d are 2-isomorphic

Two graphs are 2 isomorphic if they become isomorphic after undergoing operation 1 or operation-2 or both any number of times.

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So isomorphic graphs are always 1-isomorphic. 1 isomorphic graphs are always 2-isomorphic. But 2-isomorphic graph are not necessarily 1-isomorphic and 1-isomorphic graphs are not necessarily isomorphic.

Again, two 2-isomorphic graph $G_1$ and $G_2$ have same rank & nullity.