Sabanci University, FENS
CS419 Digital Image and Video Analysis, Fall 2023
Assignment 1 Answers

Pelinsu Saraç, 28820

November 2023

Question 1

With the given definition of convolution for two continuous functions $f(x)$ and $g(x)$:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

Prove that the convolution operation is commutative:

$$(f * g) = (g * f)$$

We can define a new variable u as $u = x - y$. With this definition, y can be expressed as $y = x-u$. Taking the derivative of both sides shows that $dy = -du$. Now, u can be used to change variables in the convolution definition:

$$-\int_{\infty}^{-\infty} f(x-u)g(u)du$$

where ∞ in the lower boundary comes from $u = x + \infty = \infty$ when $y = -\infty$ and $-\infty$ in the upper boundary comes from $u = x - \infty = -\infty$ when $y = \infty$. Minus in front of the integral can be used to invert the boundaries:

$$\int_{-\infty}^{\infty} f(x-u)g(u)du$$

If we rewrite this integral, we can see that it fits into the convolution formulation:

$$\int_{-\infty}^{\infty} g(u)f(x-u)du = (g * f)(x)$$

In the end, $(f * g)(x) = (g * f)(x)$. Thus, the convolution operation is commutative.

Prove that the cross-correlation is not commutative:

$$(f \star g) = \int_{-\infty}^{\infty} f(y)g(x+y)dy$$

We can follow a similar path like the first part, in other words we can change variables. We can define u as $u = x + y$, then $y = u - x$ and $dy = du$:

$$\int_{-\infty}^{\infty} f(u-x)g(u)du$$

where $-\infty$ in the lower boundary comes from $u = x - \infty = -\infty$ when $y = -\infty$ and ∞ in the upper boundary comes from $u = x + \infty = \infty$ when $y = \infty$. This expression cannot be organized in a way such that it is equal to $(g \star f)$. We can try the same from the other way around:

$$(g \star f) = \int_{-\infty}^{\infty} g(y)f(x+y)dy$$

$$u = x + y, y = u - x, dy = du$$

$$\int_{-\infty}^{\infty} g(u-x)f(u)du$$

This expression also cannot be reorganized into $(f \star g)$. It can be observed that:

$$\int_{-\infty}^{\infty} f(u-x)g(u)du \neq \int_{-\infty}^{\infty} g(u-x)f(u)du$$

thus, $(f \star g) \neq (g \star f)$, in other words, cross-correlation operation is not commutative.

Prove that the convolution operation is associative:

$$(f * g) * h = f * (g * h)$$

We can start by writing the expression $(f*g)*h(x)$ according to the definition of convolution:

$$\int_{-\infty}^{\infty} (f * g)(y)h(x-y)dy$$

We can extend the expression with $(f * g)(x)$:

$$\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(u)g(y-u)du)h(x-y)dy$$

Then, we can rewrite the expression as double integral and change the order of differentials:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(y-u)h(x-y)dudy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(y-u)h(x-y)dydu$$

After the change of the order of differentials, we can isolate the inner integral as:

$$\int_{-\infty}^{\infty} f(u)(\int_{-\infty}^{\infty} g(y-u)h(x-y)dy)du$$

For this inner integral, we can define $z = y-u$. Then, $y = u+z$ and $dy = dz$. Now, we can replace $y-u$ with z in the inner integral:

$$\int_{-\infty}^{\infty} g(z)h(x-u-z)dz$$

where $-\infty$ in the lower boundary comes from $z = -\infty - u = -\infty$ when $y = -\infty$ and ∞ in the upper boundary comes from $z = \infty - u = \infty$ when $y = \infty$. Resulting expression fits the formulation of convolution operation:

$$\int_{-\infty}^{\infty} g(z)h((x-u)-z)dz = (g*h)(x-u)$$

Now, we can replace the isolated inner integral with $(g*h)(x-u)$. After the insertion, outer integral also becomes a convolution operation:

$$\int_{-\infty}^{\infty} f(u)(g*h)(x-u)du = f*(g*h)(x)$$

In the end, $(f*g)*h(x) = f*(g*h)(x)$. Thus, the convolution operation is associative.

Question 2

Prove that the Laplacian operator is rotation invariant, which is defined as:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Rotation matrix for (x,y) is given in the question as:

$$x' = xcos(\theta) - ysin(\theta)$$
$$y' = xsin(\theta) + ycos(\theta)$$

We can start with the first partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial y}$$

We can find the derivatives of x' and y' with respect to x and y:

$$\frac{\partial x'}{\partial x} = cos(\theta)$$

$$\frac{\partial y'}{\partial x} = sin(\theta)$$

$$\frac{\partial x'}{\partial y} = -sin(\theta)$$

$$\frac{\partial y'}{\partial y} = cos(\theta)$$

Now, we can insert these derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \cdot cos(\theta) + \frac{\partial f}{\partial y'} \cdot sin(\theta)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y'} \cdot cos(\theta) - \frac{\partial f}{\partial x'} \cdot sin(\theta)$$

We can continue with the second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \cdot (\frac{\partial f}{\partial x'} \cdot cos(\theta) + \frac{\partial f}{\partial y'} \cdot sin(\theta))$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \cdot (\frac{\partial f}{\partial y'} \cdot cos(\theta) - \frac{\partial f}{\partial x'} \cdot sin(\theta))$$

We can extend these expressions further:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x'} \cdot cos(\theta) + \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y'} \cdot sin(\theta)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y'} \cdot cos(\theta) - \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial x'} \cdot sin(\theta)$$

They can be rewritten as:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x'} \cdot \frac{\partial f}{\partial x} \cdot cos(\theta) + \frac{\partial}{\partial y'} \cdot \frac{\partial f}{\partial x} \cdot sin(\theta)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y'} \cdot \frac{\partial f}{\partial y} \cdot cos(\theta) - \frac{\partial}{\partial x'} \cdot \frac{\partial f}{\partial y} \cdot sin(\theta)$$

Corresponding expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ were found previously, thus in the next step the expressions become:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x'} \cdot \frac{\partial f}{\partial x'} \cdot cos^2(\theta) + \frac{\partial}{\partial x'} \cdot \frac{\partial f}{\partial y'} \cdot cos(\theta) \cdot sin(\theta) + \frac{\partial}{\partial y'} \cdot \frac{\partial f}{\partial x'} \cdot cos(\theta) \cdot sin(\theta) + \frac{\partial}{\partial y'} \cdot \frac{\partial f}{\partial y'} \cdot sin^2(\theta)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y'} \cdot \frac{\partial f}{\partial y'} \cdot cos^2(\theta) - \frac{\partial}{\partial y'} \cdot \frac{\partial f}{\partial x'} \cdot cos(\theta) \cdot sin(\theta) - \frac{\partial}{\partial x'} \cdot \frac{\partial f}{\partial y'} \cdot cos(\theta) \cdot sin(\theta) + \frac{\partial}{\partial x'} \cdot \frac{\partial f}{\partial x'} \cdot sin^2(\theta)$$

We can sum up these two expressions and simplify:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x'} \cdot \frac{\partial f}{\partial x'} (sin^2(\theta) + cos^2(\theta)) + \frac{\partial}{\partial y'} \cdot \frac{\partial f}{\partial y'} (cos^2(\theta) + sin^2(\theta))$$

By trigonometry, it is known that $sin^2(\theta) + cos^2(\theta) = 1$:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} \cdot 1 + \frac{\partial^2 f}{\partial y'^2} \cdot 1 = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

Thus, Laplacian operator is rotation invariant.

Question 3

$h(x, y) = 3f(x, y) + 2f(x - 1, y) + 2f(x + 1, y) - 17f(x, y - 1) + 99f(x, y + 1)$

Is $h(x, y)$ a linear filter? For h to be a linear filter, additivity and homogeneity must be satisfied. In other words, the equality below must hold:

$$h(\alpha f(x, y) + \beta g(x, y)) = \alpha h(f(x, y)) + \beta h(g(x, y)$$

We can start with $h(\alpha f(x, y) + \beta g(x, y))$:

$$h(\alpha f(x, y) + \beta g(x, y)) = 3[\alpha f(x, y) + \beta g(x, y)] + 2[\alpha f(x - 1, y) + \beta g(x - 1, y)]$$

$$+2[\alpha f(x+1, y)+\beta g(x+1, y)]-17[\alpha f(x, y-1)+\beta g(x, y-1)]+99[\alpha f(x, y+1)+\beta g(x, y+1)]$$

The sum can be further organized as:

$$\begin{gathered} h(\alpha f(x, y) + \beta g(x, y)) = 3\alpha f(x, y) + 3\beta g(x, y) \\ +2\alpha f(x - 1, y) + 2\beta g(x - 1, y) \\ +2\alpha f(x + 1, y) + 2\beta g(x + 1, y) \\ -17\alpha f(x, y - 1) - 17\beta g(x, y - 1) \\ +99\alpha f(x, y + 1) + 99\beta g(x, y + 1) \end{gathered}$$

If the sum reorganized according to α and β coefficients, expected result for linearity is achieved:

$$= \alpha[3f(x,y) + 2f(x-1,y) + 2f(x+1,y) - 17f(x,y-1) + 99f(x,y+1)]$$

$$+\beta[3g(x,y) + 2g(x-1,y) + 2g(x+1,y) - 17g(x,y-1) + 99g(x,y+1)]$$

$$= \alpha h(f(x,y)) + \beta h(g(x,y)) = h(\alpha f(x,y) + \beta g(x,y))$$

Thus, the filter h is linear. Corresponding convolution mask can be constructed by looking at the elements of h, as each of them corresponds to either central or a neighboring pixel:

$$\begin{bmatrix} 0 & -17 & 0 \\ 2 & 3 & 2 \\ 0 & 99 & 0 \end{bmatrix}$$