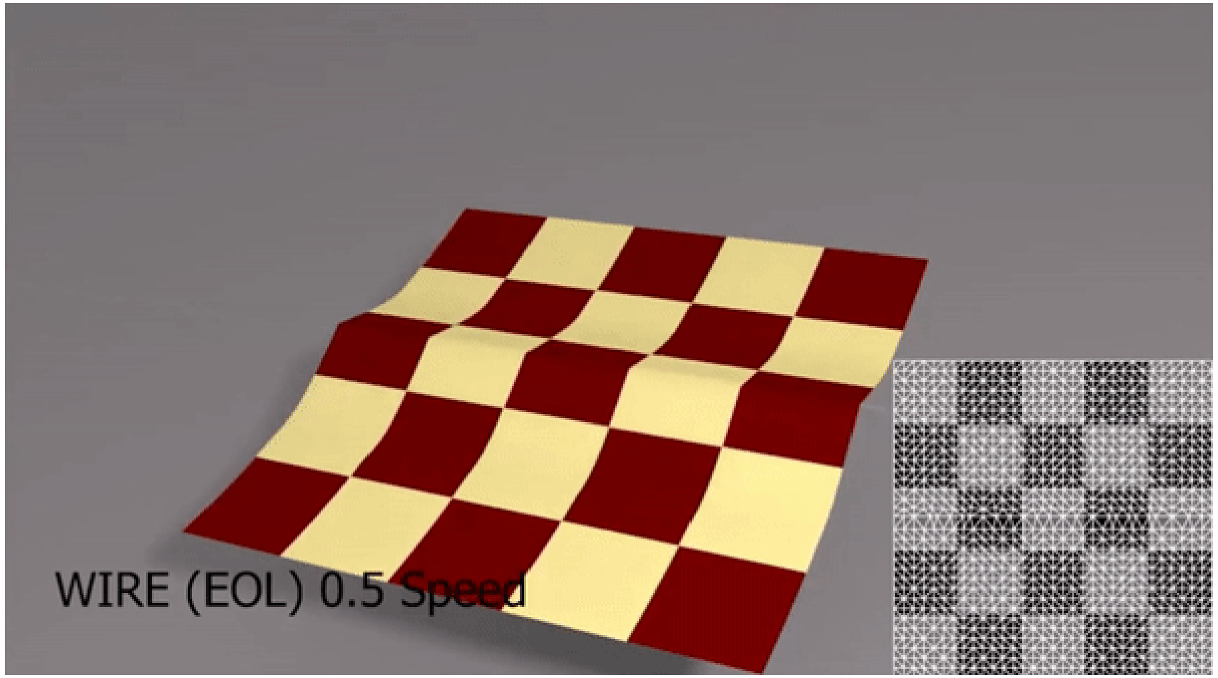
Physics Based Animation: Mass-Spring Systems



Some Slides/Images adapted from Marschner and Shirley and David Levin

Announcements

Assignment 7 due Sunday 2 August, ask questions in tutorial on Friday

Office hours today after lecture (half hour)

No class next Monday 3 August

Final exam Saturday 22 August

Bonus Assignment

Goal: make the coolest image or video using the tools we learned in the course

Will be scored 0-5. Add this number to your final mark. e.g. you have a 78% in the course and get 3 points on the bonus assignment. Your mark is now 81%.

I'll be marking it! Email me your idea if you are unsure how many points it might be worth.

Any Questions?

Physics-Based Animation

Monday:

Newton's Laws of Motion
The Mass-Spring System
Time Integration via Optimization

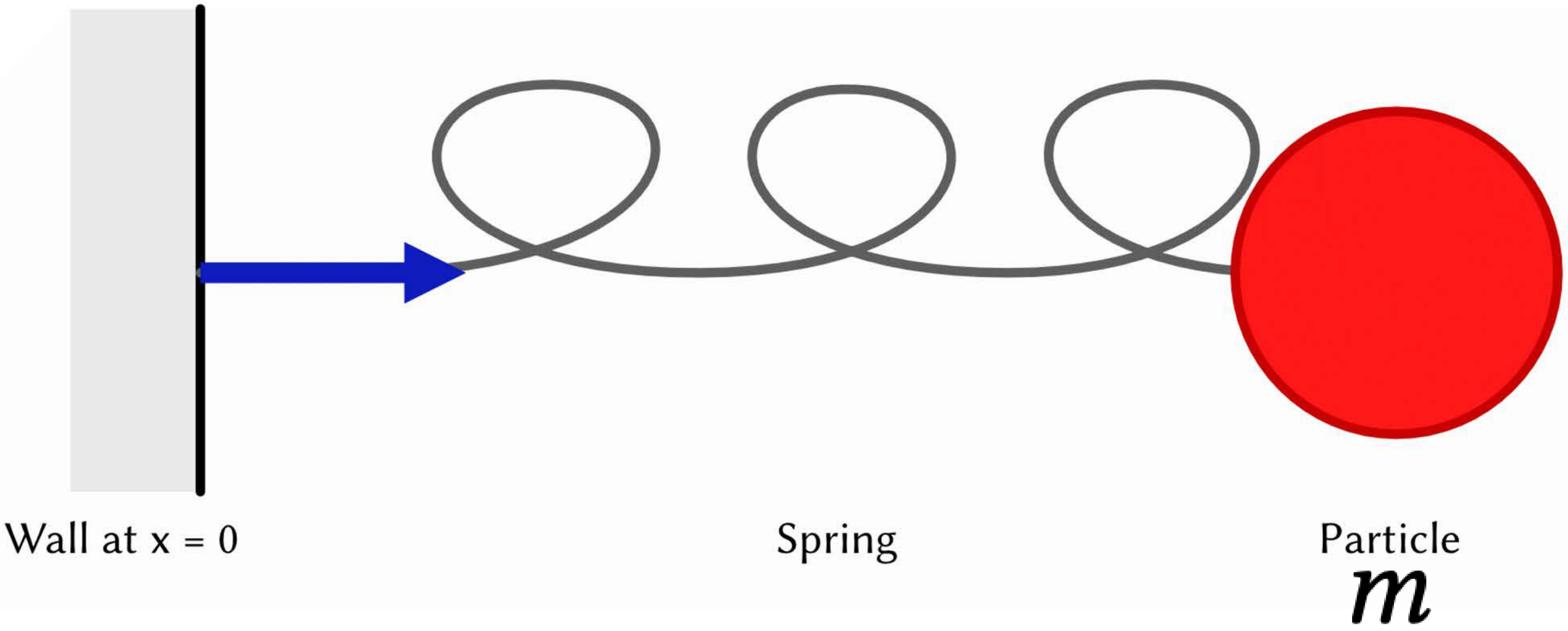
Today:

Review Mass-Spring Systems
Implicit Time Integration via Optimization (Energy formulation)
A Local-Global Solver for Fast-Mass Springs
Fixed Points
Dense and Sparse Matrices

Good ole Newton's Second Law

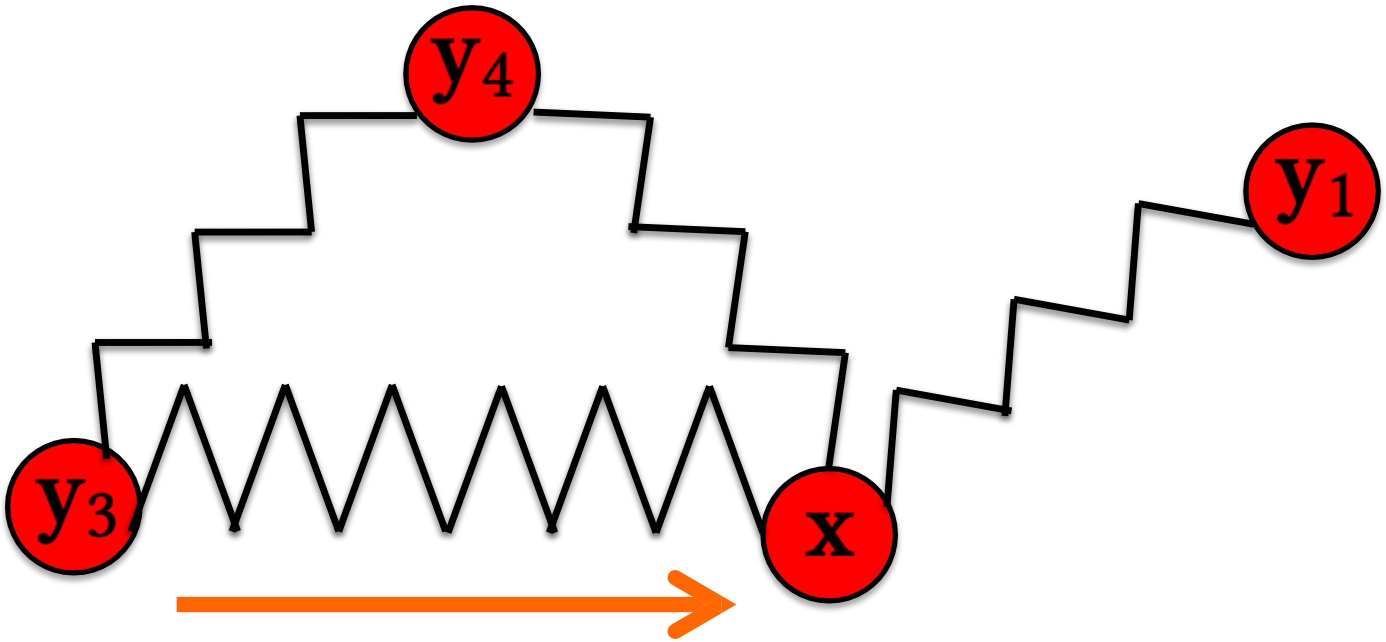
force $\mathbf{f} = m\mathbf{a}$ Acceleration

Mass



$$f = -kx$$

The Mass-Spring System: Force



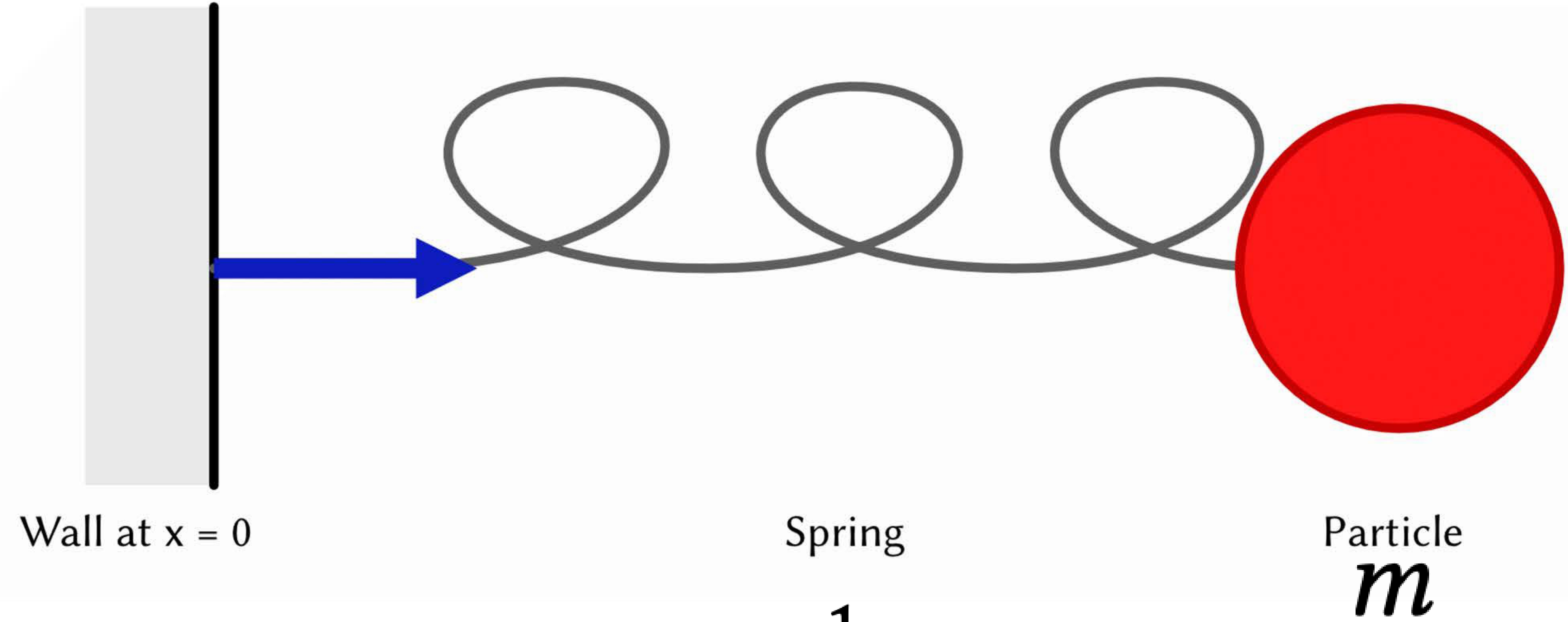
$$\mathbf{x} - \mathbf{y}_i$$

Force on the spring

Rest Length

Direction of Spring

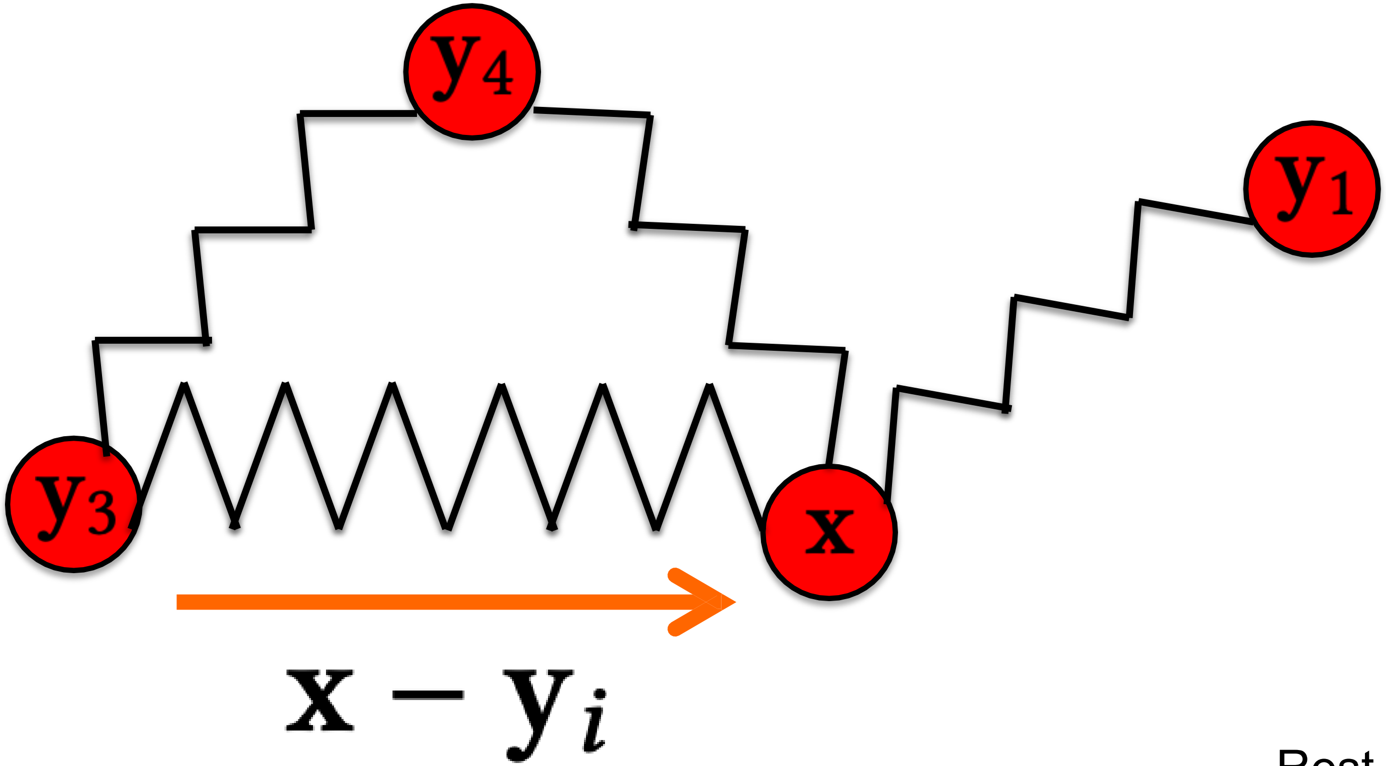
$$\boxed{\mathbf{f}_x(\mathbf{y}_i)} = -k\underbrace{(\|\mathbf{x} - \mathbf{y}_i\| - r_{ix})}_{\text{Change in Length}} \frac{\mathbf{x} - \mathbf{y}_i}{\|\mathbf{x} - \mathbf{y}_i\|}$$



$$V = \frac{1}{2}kx^2$$

https://www.physicsclassroom.com/class/waves/Lesson-0/Motion-of-a-Mass-on-a-Spring

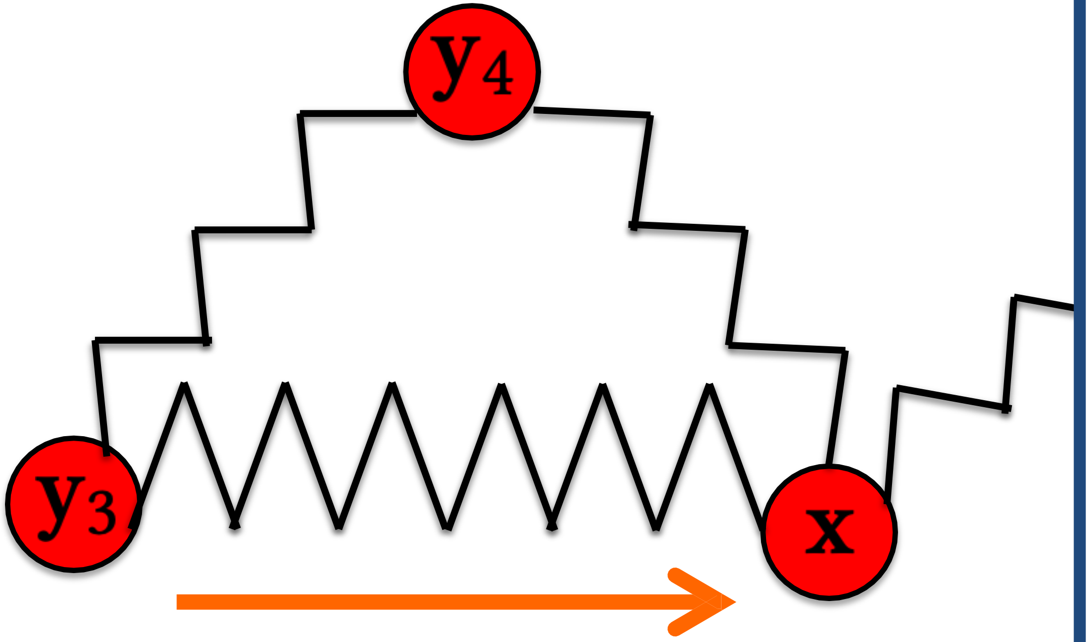
The Mass-Spring System: Potential Energy



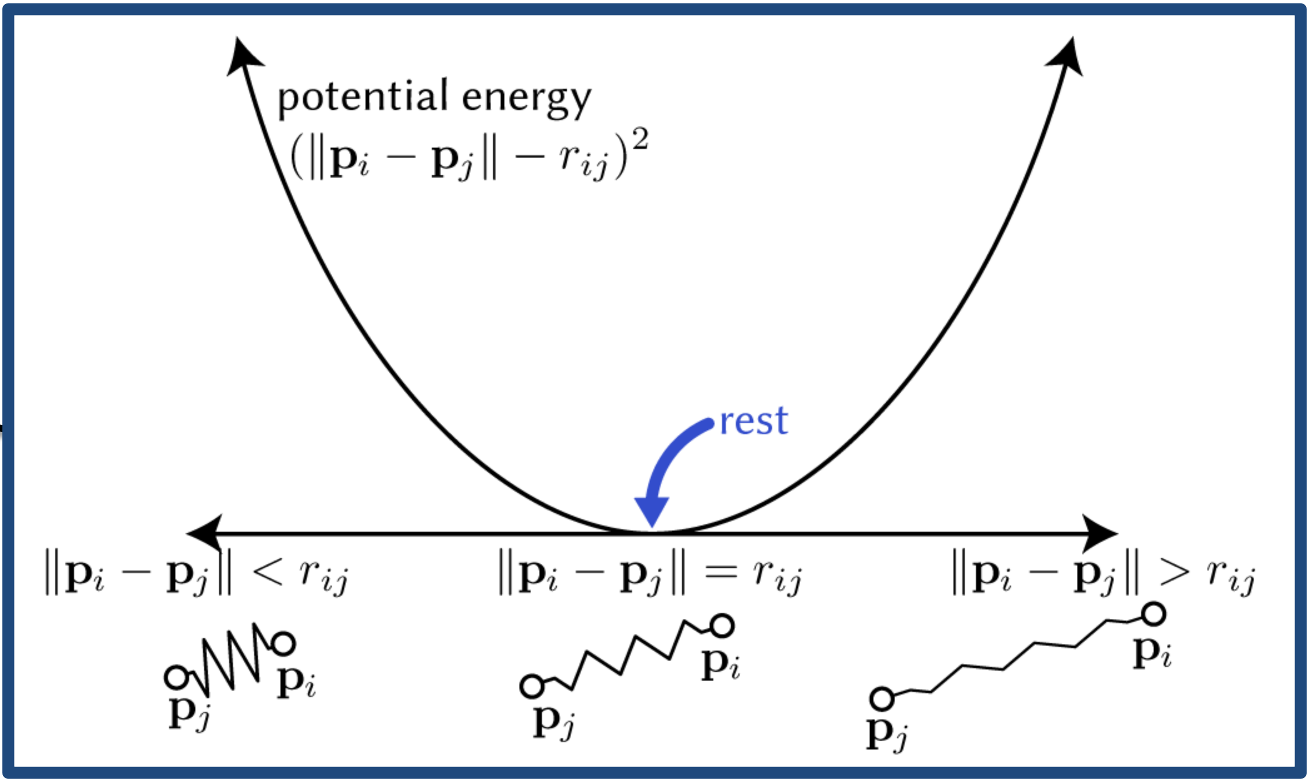
Potential energy of the spring

$$\boxed{V(\mathbf{y}_i, \mathbf{x})} = \frac{1}{2}k\underbrace{(\|\mathbf{y}_i - \mathbf{x}\| - \overset{\text{Rest Length}}{\uparrow\, r_{ix}})^2}_{\text{Change in Length}}$$

The Mass-Spring System



$$\mathbf{x} - \mathbf{y}_i$$

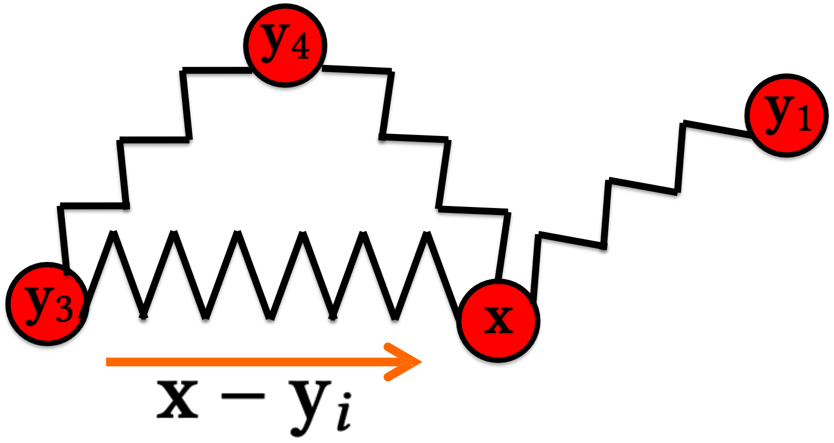


Rest Length

Potential energy of the spring

$$\boxed{V(\mathbf{y}_i, \mathbf{x})} = \frac{1}{2}k\underbrace{(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2}_{\text{Change in Length}}$$

The Mass-Spring System



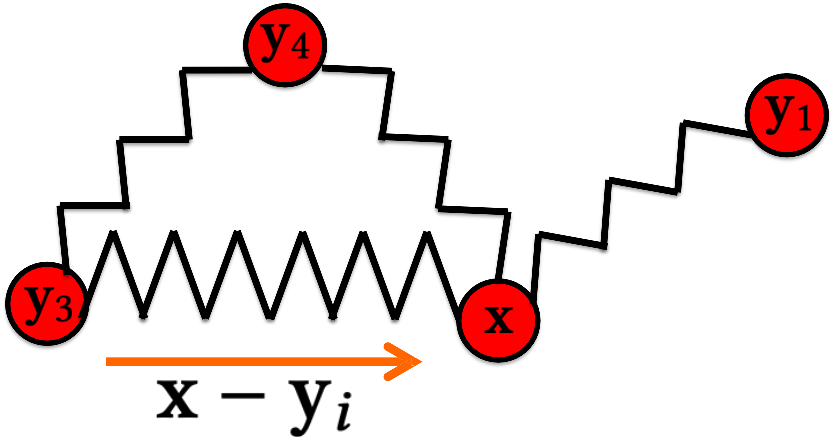
Force on the spring

$$\boxed{\mathbf{f}_x(\mathbf{y}_i)} = -k\underbrace{(\|\mathbf{x}-\mathbf{y}_i\| - r_{ix})}\frac{\mathbf{x}-\mathbf{y}_i}{\|\mathbf{x}-\mathbf{y}_i\|}$$

Potential energy of the spring

$$\boxed{V(\mathbf{y}_i, \mathbf{x})} = \frac{1}{2}k\underbrace{(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})}^2$$

The Mass-Spring System



$$\boxed{\mathbf{f}_{ix}(\mathbf{y}_i) = -\frac{\partial V}{\partial \mathbf{y}_i} \in \mathbb{R}^3}$$

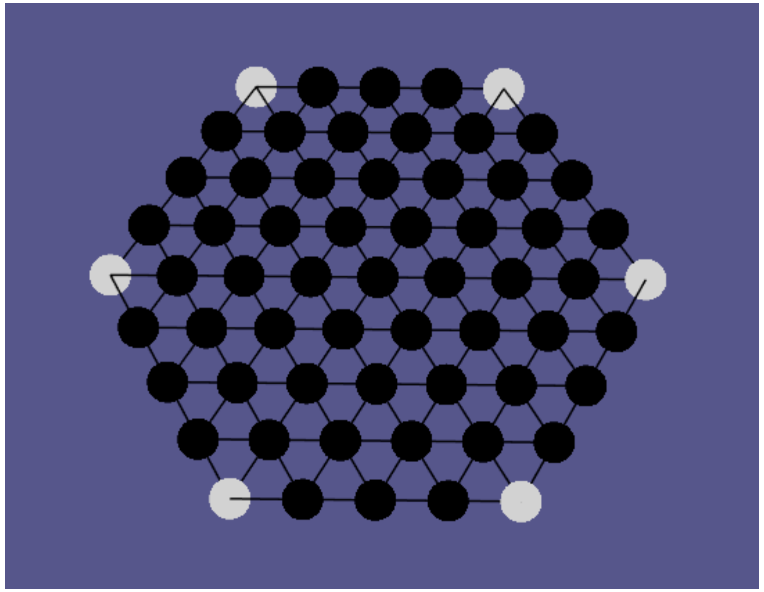
Force on the spring

$$\boxed{\mathbf{f}_x(\mathbf{y}_i)} = -k(\|\mathbf{x}-\mathbf{y}_i\| - r_{ix})\frac{\mathbf{x}-\mathbf{y}_i}{\|\mathbf{x}-\mathbf{y}_i\|}$$

Potential energy of the spring

$$\boxed{V(\mathbf{y}_i, \mathbf{x})} = \frac{1}{2}k(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2$$

The Mass-Spring System: Initial Conditions



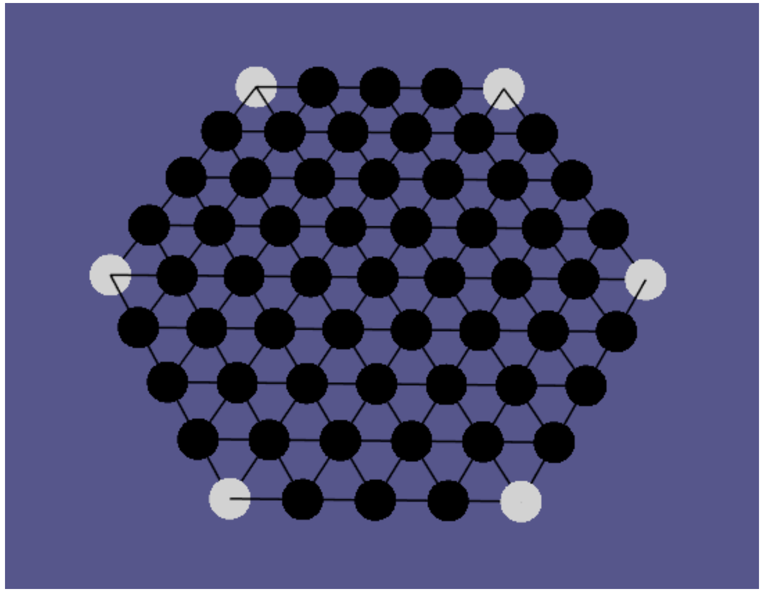
Initial conditions: the initial state of the system

$$\mathbf{y}_i^t \in \mathbb{R}^3$$

$$\dot{\mathbf{y}}_i^t = \frac{\partial \mathbf{y}_i(t)}{\partial t} \in \mathbb{R}^3$$

when $t = 0$

The Mass-Spring System: Timesteps



Initial conditions: the initial state of the system

$$\mathbf{y}_i^t \in \mathbb{R}^3$$

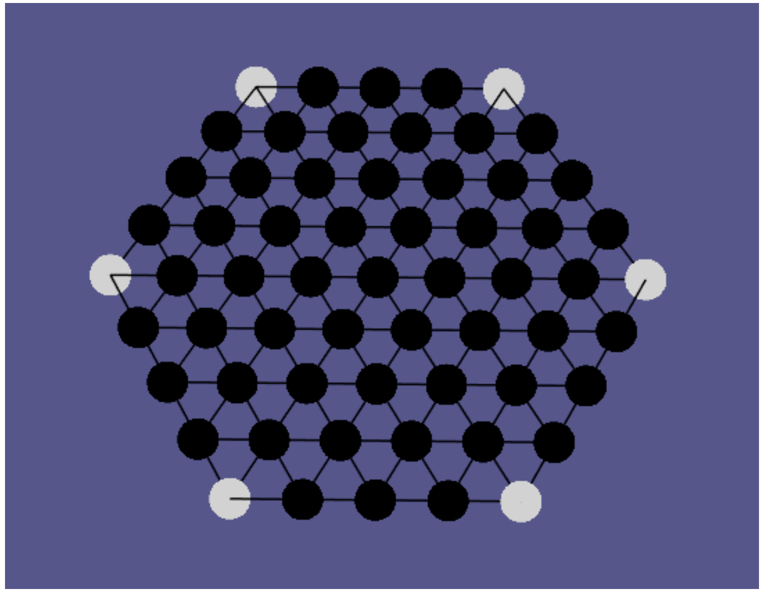
$$\dot{\mathbf{y}}_i^t = \frac{\partial \mathbf{y}_i(t)}{\partial t} \in \mathbb{R}^3$$

when $t = 0$

Want to solve for future system states
while following the laws of physics

$$\mathbf{f} = m\mathbf{a}$$

The Mass-Spring System: Time Integration



Initial conditions: the initial state of the system

$$\mathbf{y}_i^t \in \mathbb{R}^3$$

$$\dot{\mathbf{y}}_i^t = \frac{\partial \mathbf{y}_i(t)}{\partial t} \in \mathbb{R}^3$$

when $t = 0$

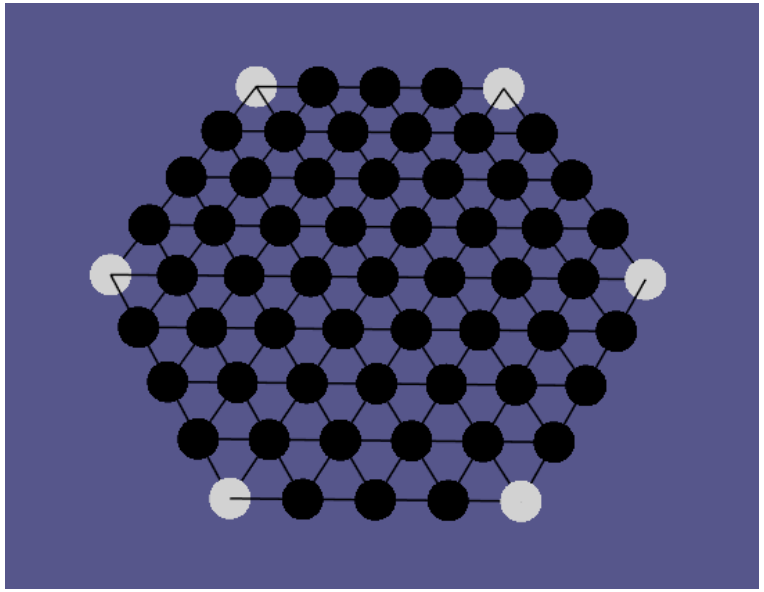
Want to solve for future system states
while following the laws of physics

$$\mathbf{f} = m\mathbf{a}$$

$$\mathbf{a}_i^t = \ddot{\mathbf{y}}_i^t = \frac{d^2\mathbf{y}_i(t)}{dt^2} \approx \frac{\mathbf{y}_i^{t+1} - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}$$

https://en.wikipedia.org/wiki/Finite_difference#Higher-order_differences

The Mass-Spring System: Time Integration



Initial conditions: the initial state of the system

$$\mathbf{y}_i^t \in \mathbb{R}^3$$

$$\dot{\mathbf{y}}_i^t = \frac{\partial \mathbf{y}_i(t)}{\partial t} \in \mathbb{R}^3$$

$$\text{when } t = 0$$

Want to solve for future system states
while following the laws of physics

$$\mathbf{f} = m\mathbf{a}$$

$$\mathbf{a}_i^t = \ddot{\mathbf{y}}_i^t = \frac{d^2\mathbf{y}_i(t)}{dt^2} \approx \frac{\boxed{\mathbf{y}_i^{t+1}} - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}$$

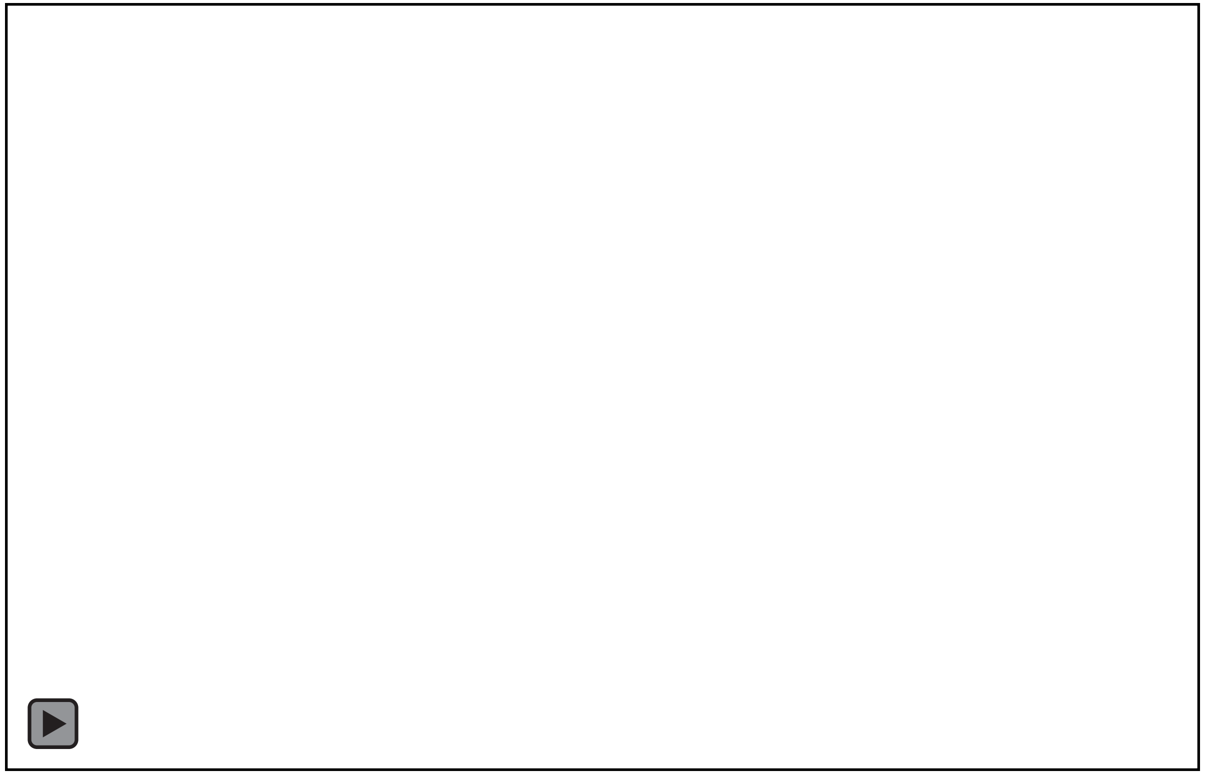
https://en.wikipedia.org/wiki/Explicit_and_implicit_methods

Aside: Time Integration Schemes

Forward Euler Integration

$$v^{t+1} = v^t - \Delta t \frac{k}{m} x^{\boxed{t}}$$

$$x^{t+1} = x^t + \Delta t v^{\boxed{t}}$$

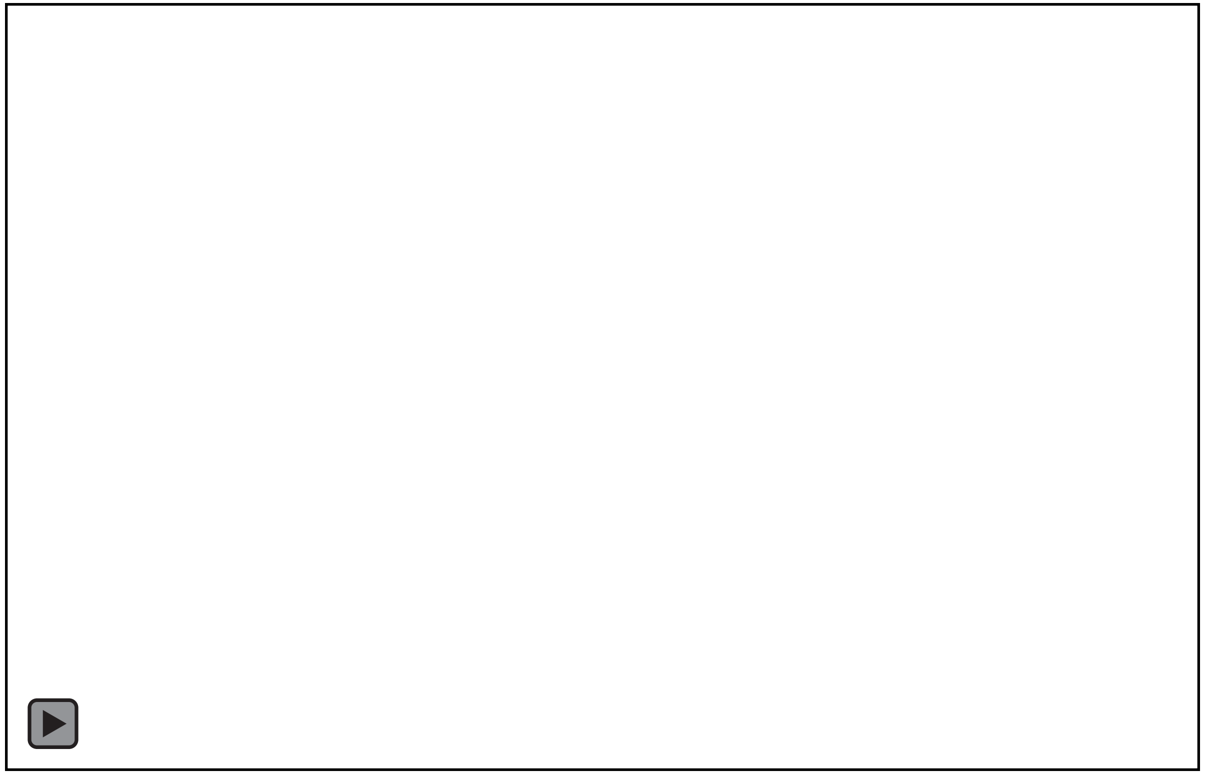


Aside: Time Integration Schemes

Backward (Implicit) Euler Integration

$$v^{t+1} = v^t - \Delta t \frac{k}{m} x^{\boxed{t+1}}$$

$$x^{t+1} = x^t + \Delta t v^{\boxed{t+1}}$$

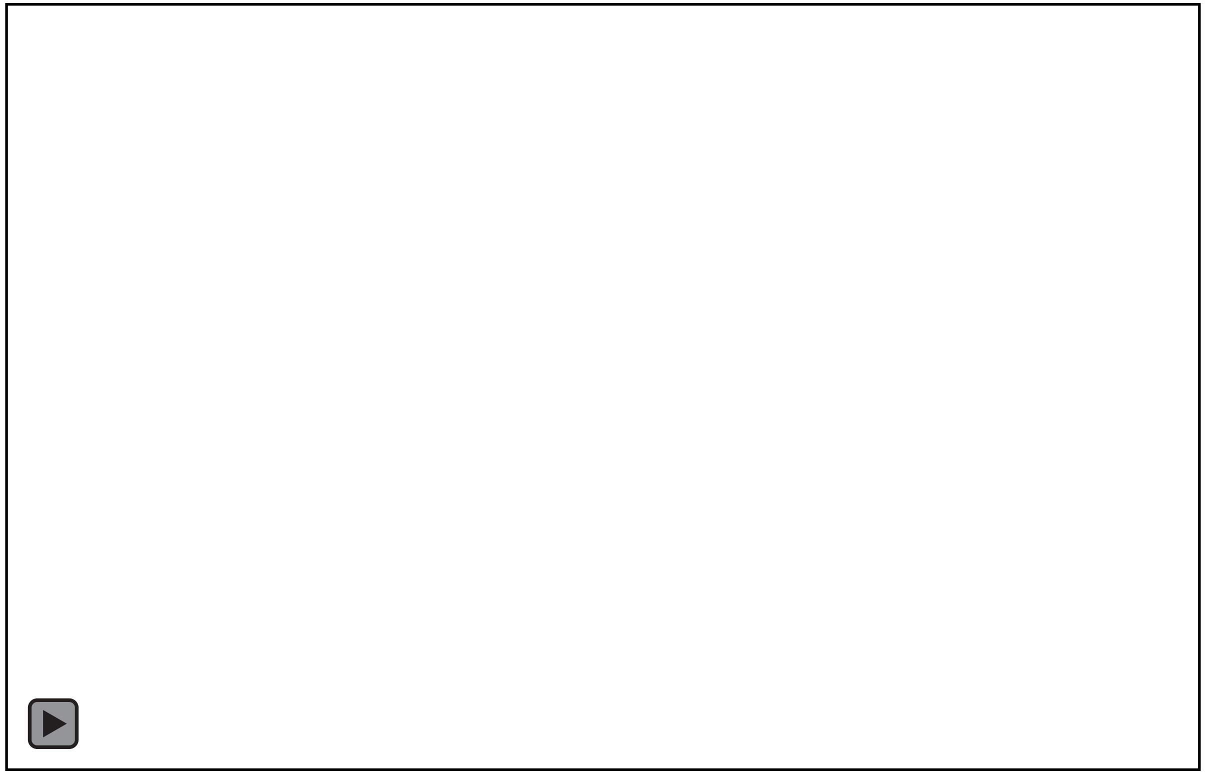


Aside: Time Integration Schemes

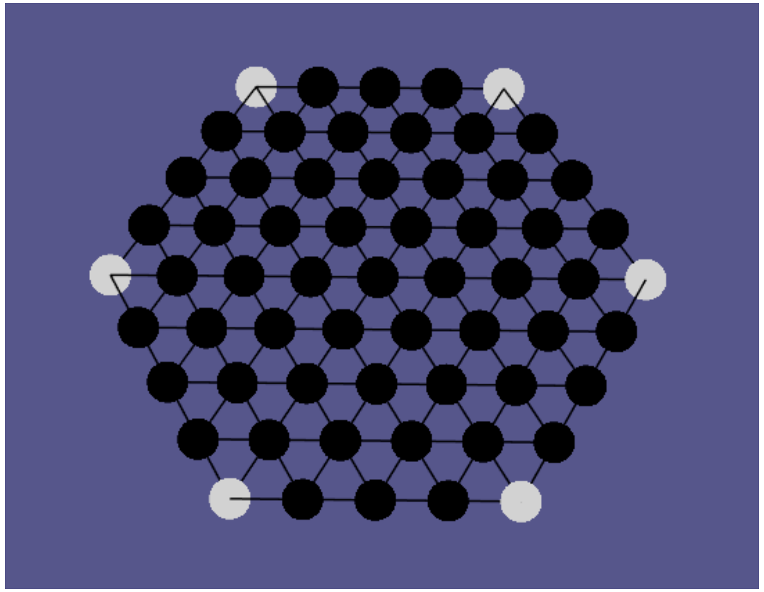
Symplectic Euler Integration

$$v^{t+1} = v^t - \Delta t \frac{k}{m} x^{\boxed{t}}$$

$$x^{t+1} = x^t + \Delta t v^{\boxed{t+1}}$$



The Mass-Spring System: Time Integration

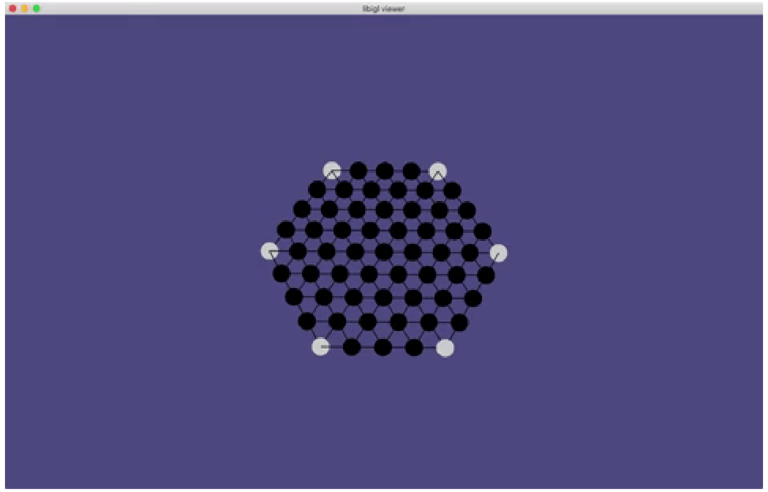


Initial conditions: the initial state of the system

$$\mathbf{y}_i^t \in \mathbb{R}^3$$

$$\dot{\mathbf{y}}_i^t = \frac{\partial \mathbf{y}_i(t)}{\partial t} \in \mathbb{R}^3$$

when $t = 0$



Want to solve for future system states while following the laws of physics

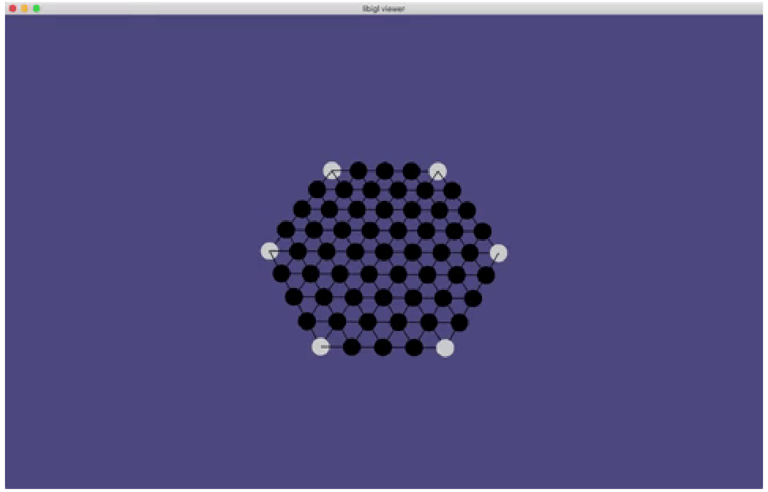
$$\mathbf{f} = m\mathbf{a}$$

$$\mathbf{a}_i^t = \ddot{\mathbf{y}}_i^t = \frac{d^2\mathbf{y}_i(t)}{dt^2} \approx \frac{\mathbf{y}_i^{t+1} - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}$$

Building an Energy

Construct some function $E(x)$ such that the minimizer of it will satisfy $\frac{\partial E}{\partial \mathbf{y}} = 0$

$$\mathbf{f} = m\mathbf{a} \longrightarrow \frac{\partial E}{\partial \mathbf{y}} = \mathbf{f} - m\mathbf{a}$$



Building an Energy

Construct some function $E(x)$ such that the minimizer of it will satisfy $\dfrac{\partial E}{\partial \mathbf{y}} = 0$

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2\right) - \Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{E(\mathbf{y})}$$

Building an Energy

Construct some function $E(x)$ such that the minimizer of it will satisfy $\frac{\partial E}{\partial \mathbf{y}} = 0$

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\boxed{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2\right)} - \Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{E(\mathbf{y})}$$

Potential energy force

$$V(\mathbf{y}_i, \mathbf{x}) = \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2$$

Building an Energy

Construct some function $E(x)$ such that the minimizer of it will satisfy $\frac{\partial E}{\partial \mathbf{y}} = 0$

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2\right) - \Delta t^2 \boxed{\left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right)} - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{E(\mathbf{y})}$$

Potential energy force

$$V(\mathbf{y}_i, \mathbf{x}) = \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2$$

$$\mathbf{a}_i^t = \ddot{\mathbf{y}}_i^t = \frac{d^2 \mathbf{y}_i(t)}{dt^2} \approx \frac{\mathbf{y}_i^{t+1} - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}$$

ma^2

force keeping masses in the same direction

Building an Energy

Construct some function $E(x)$ such that the minimizer of it will satisfy $\frac{\partial E}{\partial \mathbf{y}} = 0$

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2 \right) - \Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2} \right)^2 \right) - \boxed{\left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}} \right)}}_{E(\mathbf{y})}$$

External forces

Potential energy force

$$V(\mathbf{y}_i, \mathbf{x}) = \frac{1}{2} k(\|\mathbf{y}_i - \mathbf{x}\| - r_{ix})^2$$

$$\mathbf{a}_i^t = \ddot{\mathbf{y}}_i^t = \frac{d^2 \mathbf{y}_i(t)}{dt^2} \approx \frac{\mathbf{y}_i^{t+1} - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}$$

ma^2

force keeping masses in the same direction

Building an Energy

Construct some function $E(x)$ such that the minimizer of it will satisfy $\frac{\partial E}{\partial \mathbf{y}} = 0$

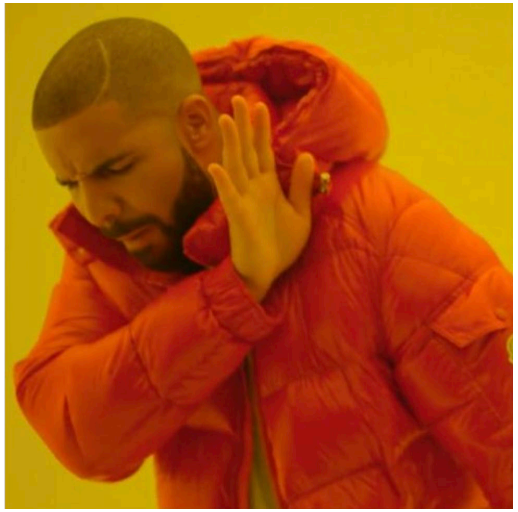
$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k \boxed{(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})}^2 \right) - \Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2} \right)^2 \right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}} \right)}_{E(\mathbf{y})}$$

Non linear :(

Building an Energy

Construct some function $E(x)$ such that the minimizer of it will satisfy $\frac{\partial E}{\partial \mathbf{y}} = 0$

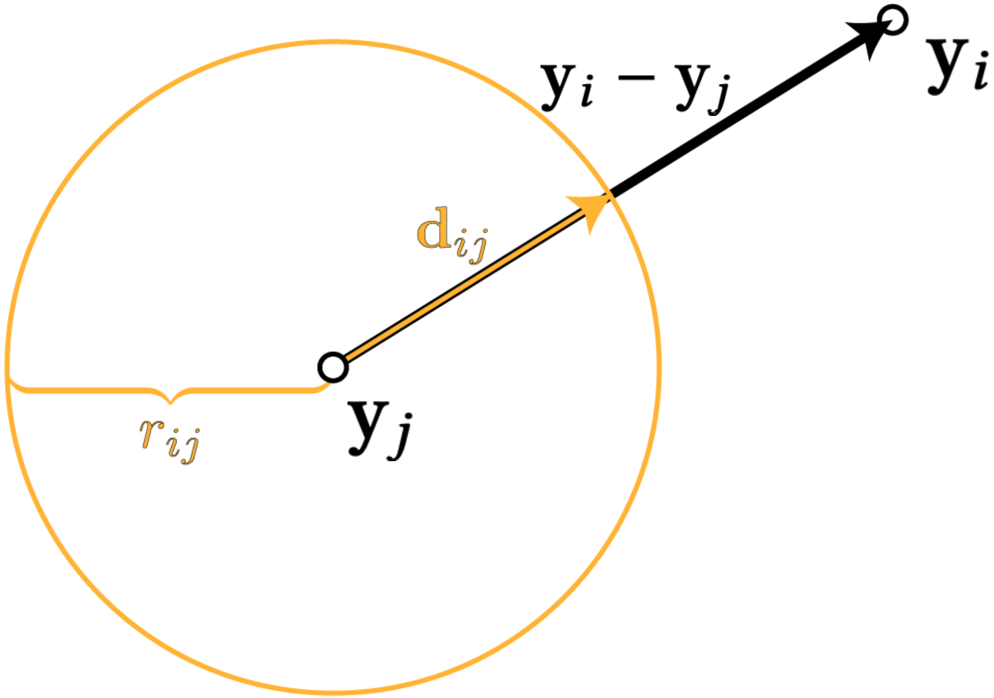
$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k \boxed{\left(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij}\right)}^2\right) - \Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{E(\mathbf{y})}$$



Non linear :(

Observation!

$$(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2 = \min_{\mathbf{d}_{ij} \in \mathbb{R}^3,\, \|\mathbf{d}\| = r_{ij}} \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2$$



Observation!

$$(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2 = \min_{\mathbf{d}_{ij} \in \mathbb{R}^3, \|\mathbf{d}\| = r_{ij}} \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2$$

$$\mathbf{y}^{t+1} = \operatorname*{argmin}_{\mathbf{y}} \underbrace{\left(\sum_{ij} \frac{1}{2} k \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2\right) - \Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{\tilde{E}(\mathbf{y})}$$

Quadratic!

Observation!

$$(\|\mathbf{y}_i - \mathbf{y}_j\| - r_{ij})^2 = \min_{\mathbf{d}_{ij} \in \mathbb{R}^3, \|\mathbf{d}\| = r_{ij}} \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2$$

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \underbrace{\left(\sum_{ij} \frac{1}{2} k \|(\mathbf{y}_i - \mathbf{y}_j) - \mathbf{d}_{ij}\|^2\right) - \Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2}\right)^2\right) - \left(\sum_i \mathbf{y}_i^\top \mathbf{f}_i^{\text{ext}}\right)}_{\tilde{E}(\mathbf{y})}$$

Quadratic!

Local-Global Solvers for Mass-Spring Systems

WHILE Not done

For Each Spring

Local Optimization Given current values of **y** determine $\mathbf{d}_{ij}$ for each spring

Global Optimization Given all $\mathbf{d}_{ij}$ vectors, find positions that minimize **E**

END

$$\frac{d\tilde{E}}{d\mathbf{y}} = 0$$

Local-Global Solvers for Mass-Spring Systems

WHILE Not done — A fixed number of iterations (50)

For Each Spring

Local Optimization — Given current values of **y** determine $\mathbf{d}_{ij}$ for each spring

Global Optimization — Given all $\mathbf{d}_{ij}$ vectors, find positions that minimize **E**

END

$$\frac{d\tilde{E}}{d\mathbf{y}} = 0$$

Local Step

Given current values of **y** determine $\mathbf{d}_{ij}$ for each spring

$$\arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}|=r_{ij}} \sum_{ij} \underbrace{\frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j}\|^2 - (\mathbf{y}_i - \mathbf{y_j})^T \mathbf{d}_{ij} + \mathbf{d}_{ij}^T \mathbf{d}_{ij}}_{\text{Each } \mathbf{d} \text{ acts on a spring independently!}}$$

Can be minimized by visiting each spring and finding **d** such that

$$E_{ij} = \arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}|=r_{ij}} \underbrace{\frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j} - \mathbf{d}_{ij}\|^2}_{\text{No sum anymore!}}$$

Global Step

Given all $\mathbf{d}_{ij}$ vectors, find positions that minimize **E**

$$\Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2} \right)^2 \right)$$

$$\frac{1}{\Delta t^2} \operatorname{tr} \left(\mathbf{y} - 2\mathbf{y}^t + \mathbf{y}^{t-1} \right)^\top \mathbf{M} \left(\mathbf{y} - 2\mathbf{y}^t + \mathbf{y}^{t-1} \right)$$

Global Step

Given all $\mathbf{d}_{ij}$ vectors, find positions that minimize **E**

$$\Delta t^2 \left(\sum_i m_i \left(\frac{\mathbf{y}_i - 2\mathbf{y}_i^t + \mathbf{y}_i^{t-1}}{\Delta t^2} \right)^2 \right)$$

$$\frac{1}{\Delta t^2} \operatorname{tr} \left(\mathbf{y} - 2\mathbf{y}^t + \mathbf{y}^{t-1} \right)^\top \mathbf{M} \left(\mathbf{y} - 2\mathbf{y}^t + \mathbf{y}^{t-1} \right)$$

Stacked matrix

Mass matrix

$$\mathbf{y}, \mathbf{y}^t, \mathbf{y}^{t-1} \in \mathbb{R}^{n \times 3}$$

Matrices

Please see README for details. There are a lot of smaller matrices when you expand this expression.

$$\mathbf{y}^{t+1} = \operatorname*{argmin}_{\mathbf{y}} \frac{1}{2} \operatorname{tr}\left(\mathbf{y}^\top \mathbf{Q} \mathbf{y}\right) - \operatorname{tr}\left(\mathbf{y}^\top \mathbf{b}\right) y$$

Precomputation

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \frac{1}{2} \operatorname{tr}\left(\mathbf{y}^\top \mathbf{Q} \mathbf{y}\right) - \operatorname{tr}\left(\mathbf{y}^\top \mathbf{b}\right) y$$

$$\mathbf{Q}\mathbf{y} = \mathbf{b}$$

$$O(n^3)$$

Would have to invert Q ☹ (possible but slow ***and*** unstable)
Insight: can precompute Q since it is the same every iteration ☺

Precomputation

$$\mathbf{y}^{t+1} = \underset{\mathbf{y}}{\operatorname{argmin}} \frac{1}{2} \operatorname{tr}\left(\mathbf{y}^\top \mathbf{Q} \mathbf{y}\right) - \operatorname{tr}\left(\mathbf{y}^\top \mathbf{b}\right) y$$

$$\mathbf{Q}\mathbf{y} = \mathbf{b}$$

Would have to invert Q ☹ (possible but slow)
Insight: can precompute Q since it is the same every iteration ☺

$$\boxed{\mathbf{Q} = \mathbf{L}\mathbf{L}^\top}$$

Local-Global Solvers for Mass-Spring Systems

$\mathbf{Q} = \mathbf{L}\mathbf{L}^\top$ Compute Q and its factorization

WHILE Not done

For Each Spring

Local Optimization Given current values of **y** determine $\mathbf{d}_{ij}$ for each spring

Global Optimization Given all $\mathbf{d}_{ij}$ vectors, find positions that minimize **E**

END

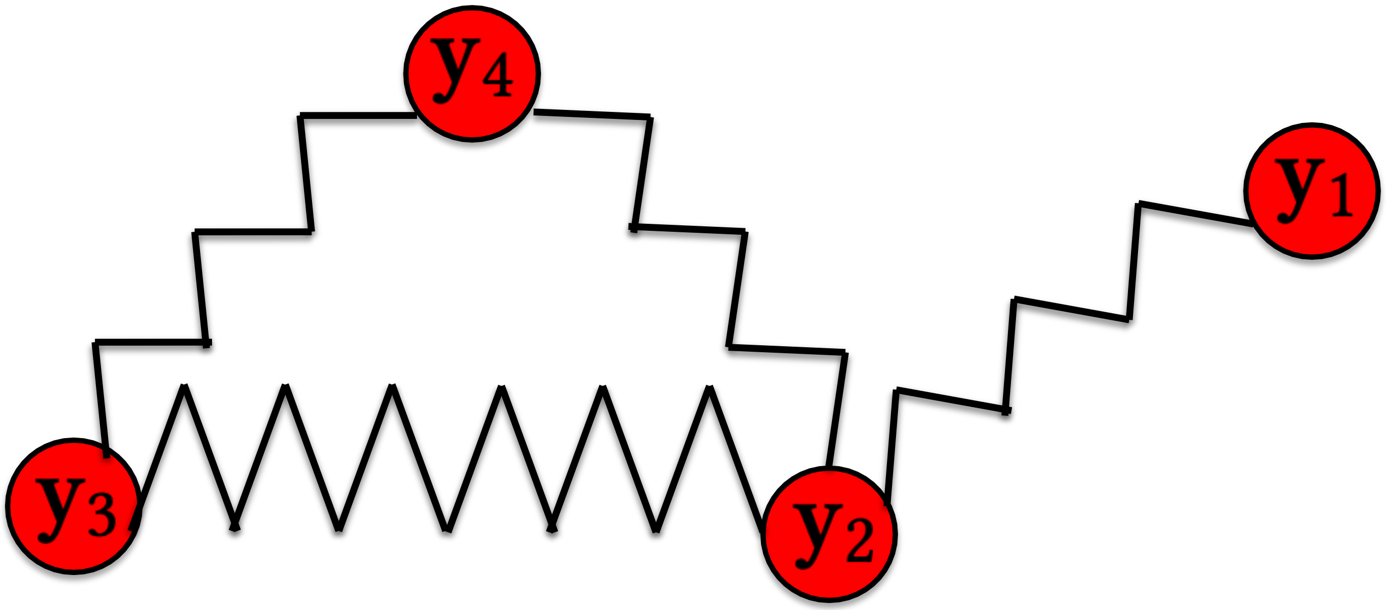
What would happen to our mass spring system now?

What would happen to our mass spring system now?

Our spring system would just move
downward and off the screen!

What can we do?

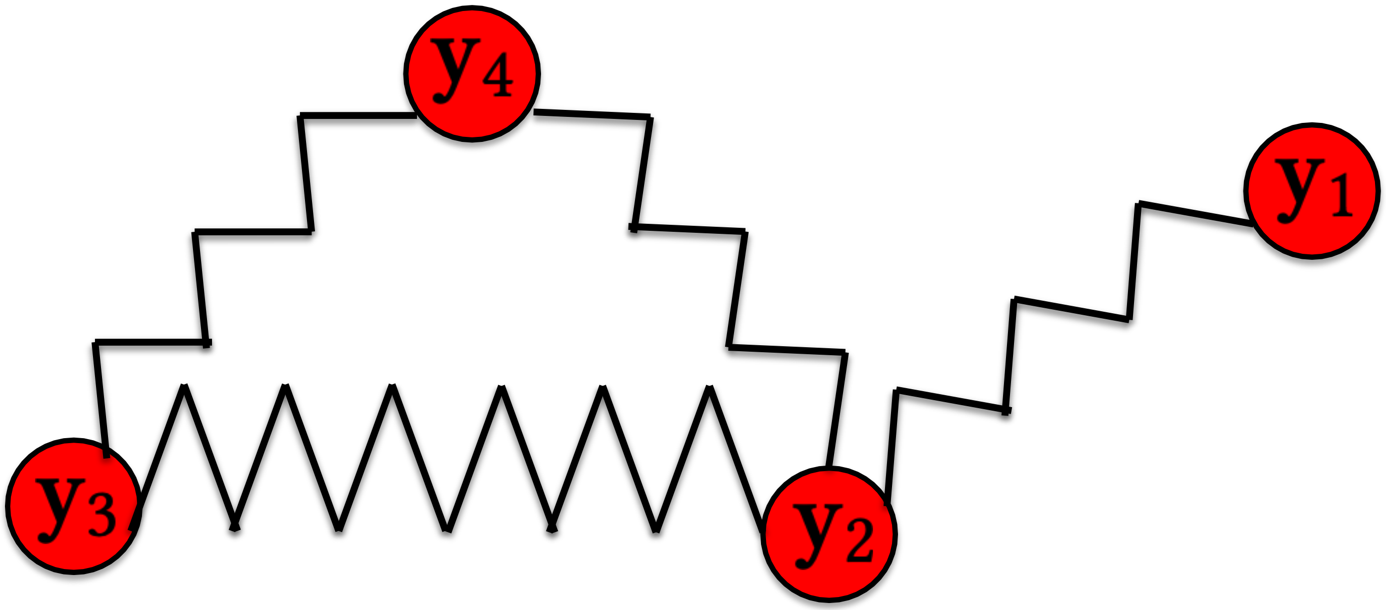
Fixed Points



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

Let’s say we never want $\mathbf{y}_3$ to move

Fixed Points

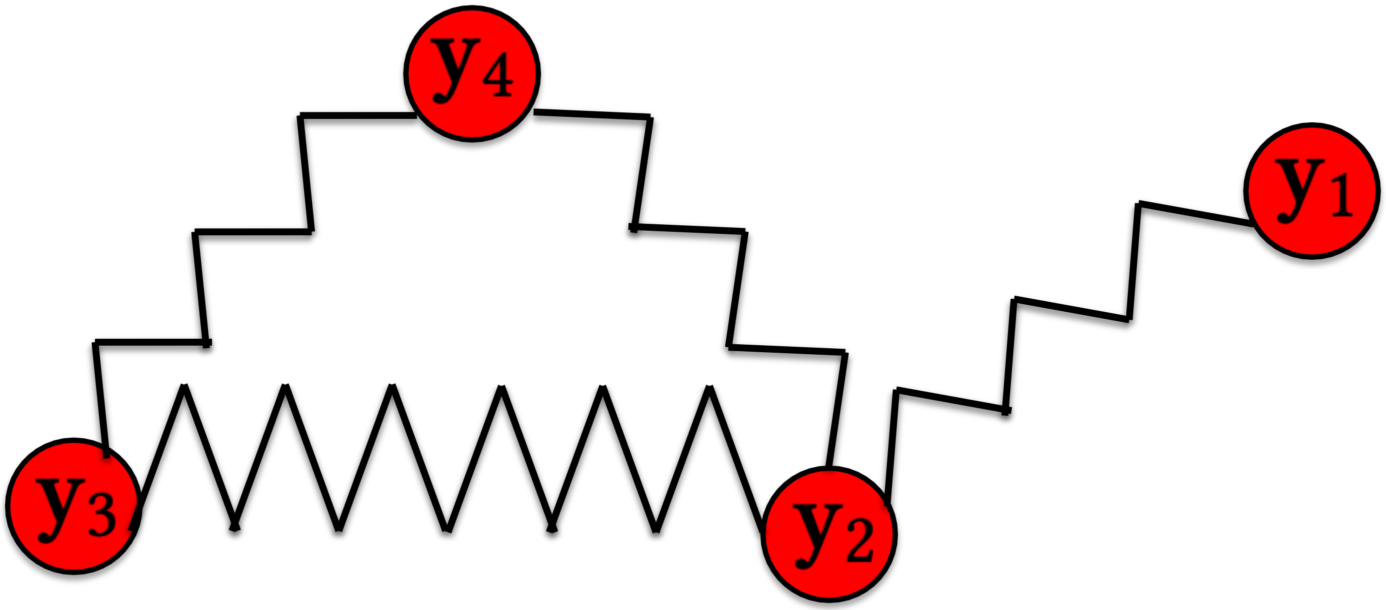


$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

Let's say we never want $\mathbf{y_3}$ to move

i.e. $\mathbf{y_3 = y_3^{rest}}$ forever and always

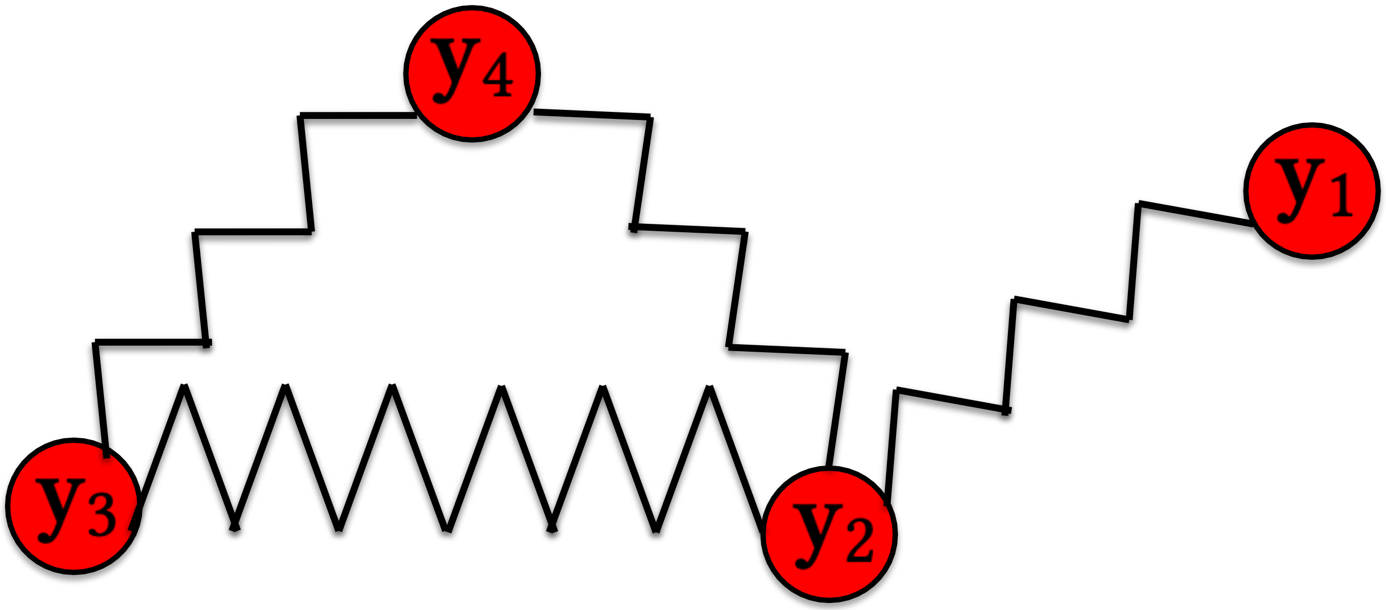
Fixed Points



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

add an additional quadratic energy term which is minimized when our pinning constraints are satisfied

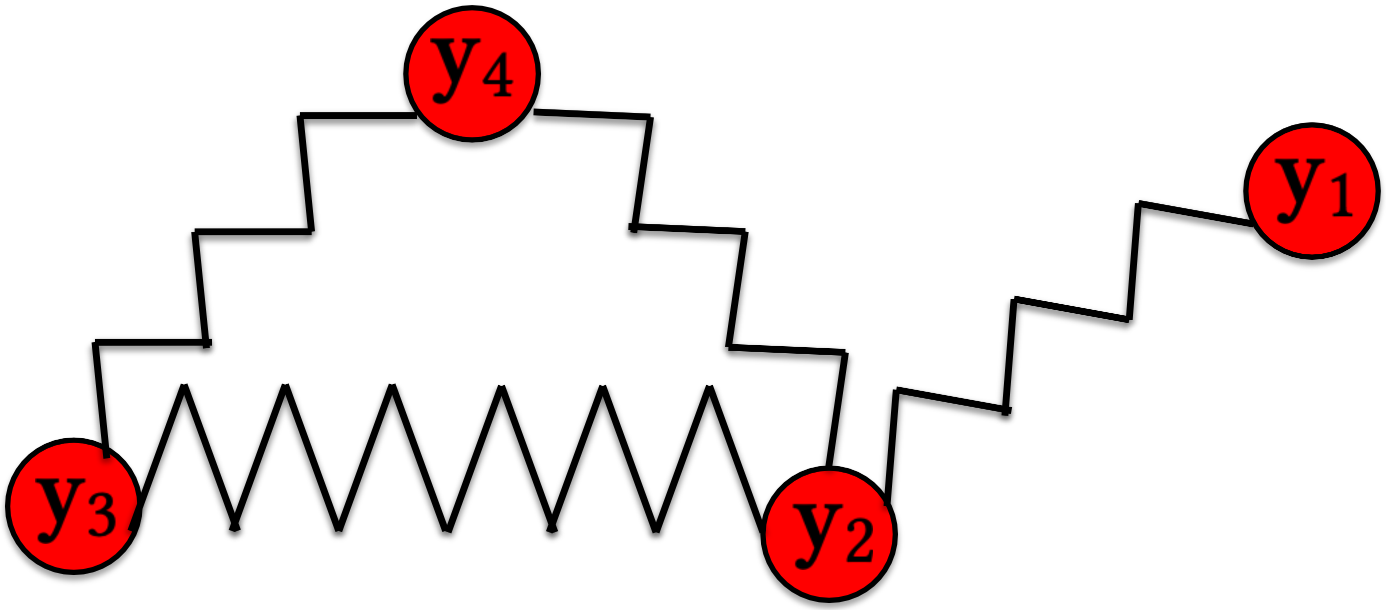
Fixed Points



add an additional quadratic energy term which is minimized when our pinning constraints are satisfied

$$\frac{w}{2} \sum_{i \text{ in pinned vertices}} \|\mathbf{y}_i - \mathbf{y}_i^{\text{rest}}\|^2$$

Fixed Points



add an additional quadratic energy term which is minimized when our pinning constraints are satisfied

$$\frac{w}{2} \sum_{i \text{ in pinned vertices}} \|\mathbf{y}_i - \mathbf{y}_i^{\text{rest}}\|^2$$

$$\frac{w}{2} \|\mathbf{y}_3 - \mathbf{y}_i^{\text{rest}}\|^2$$

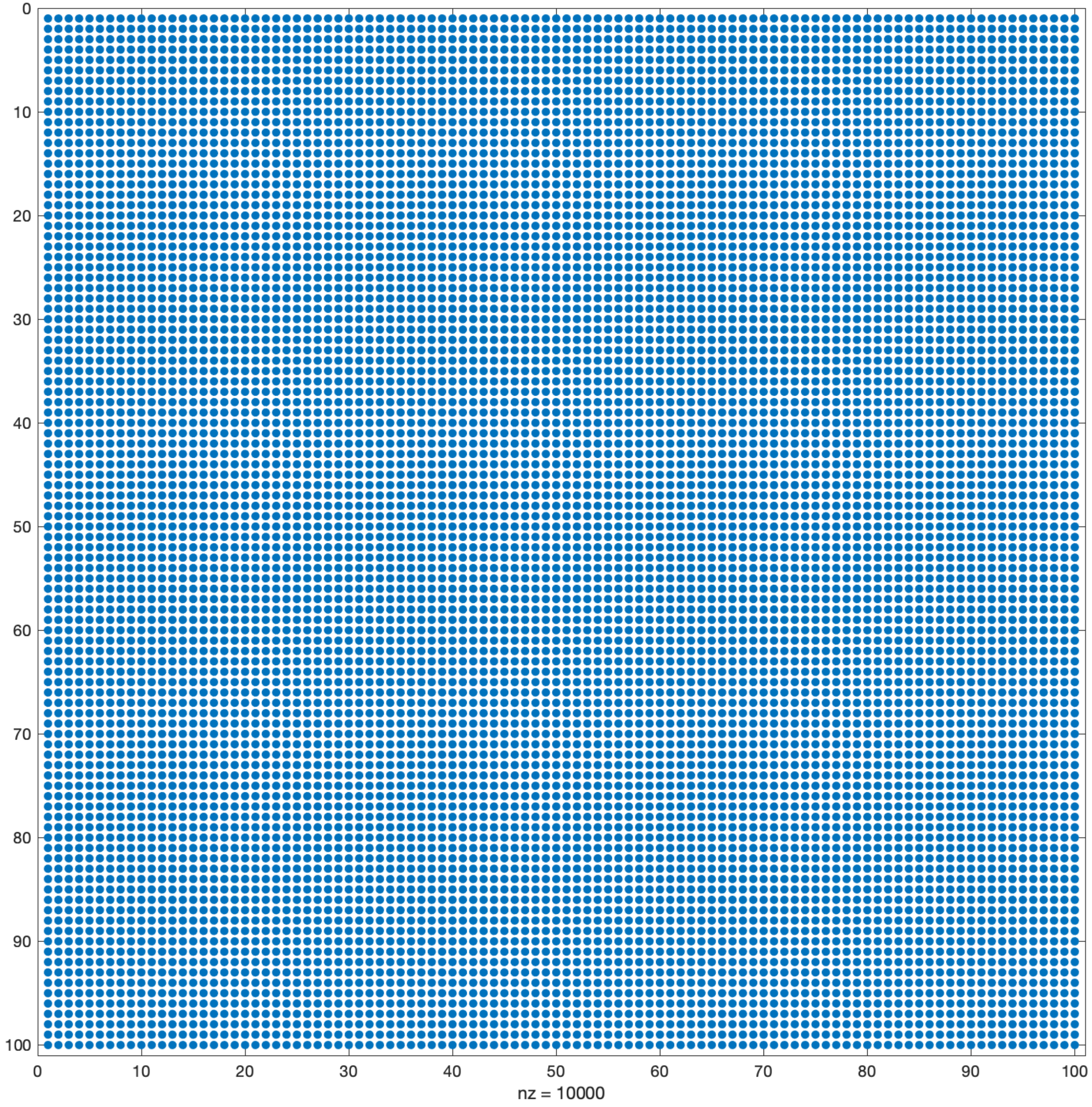
Sparse Matrices!

$\mathbf{Q} = \mathbf{L}\mathbf{L}^\top$ takes $O(n^3)$

$\mathbf{L}\mathbf{L}^\top\mathbf{y} = \mathbf{b}$ takes $O(n^2)$

https://en.wikipedia.org/wiki/Triangular_matrix#Forward_and_back_substitution

Sparse Matrices!



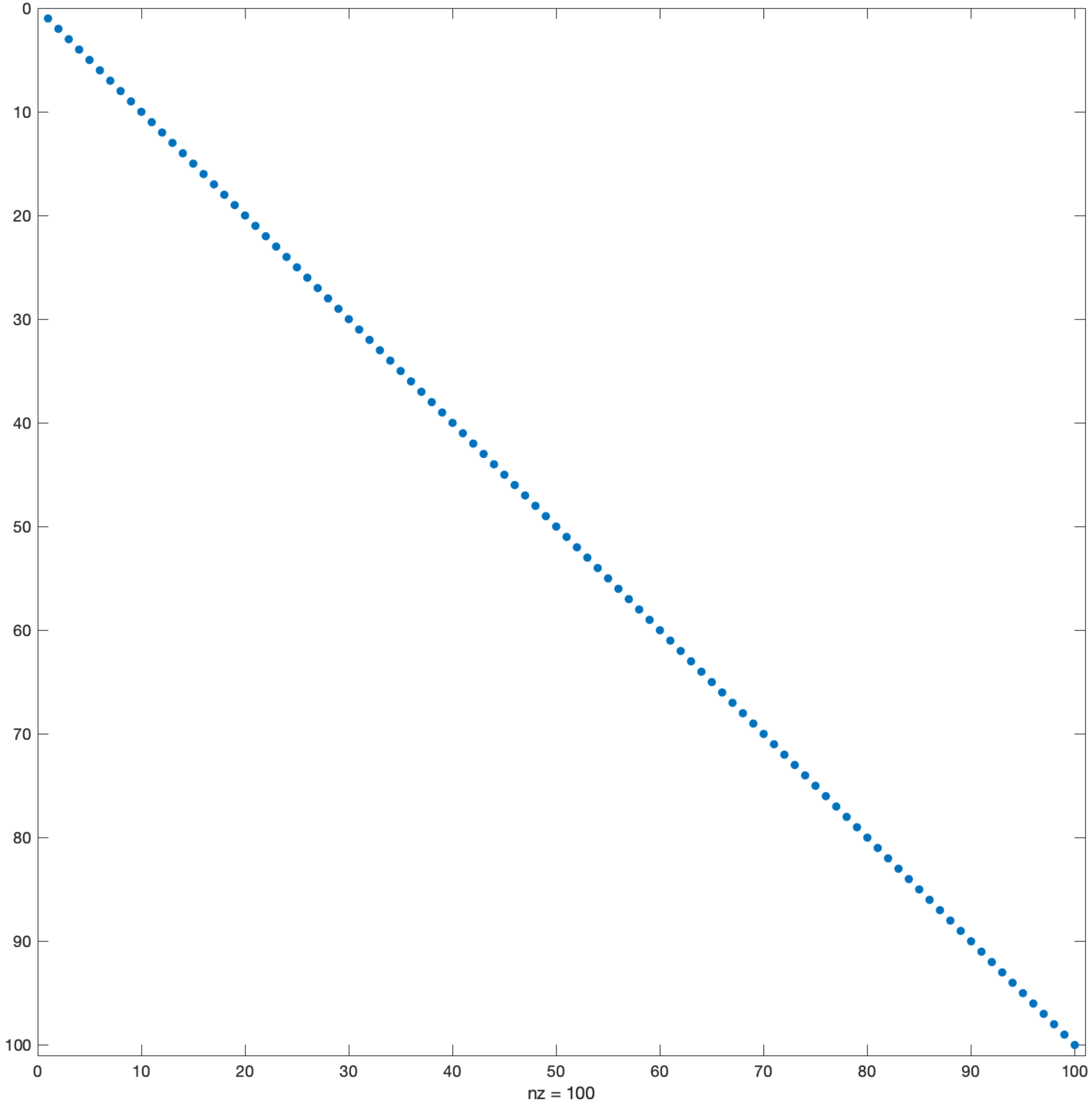
Dense: all numbers (even 0’s) are stored in memory

For example, our mass matrix:

$$\mathbf{M}_{ij} = \begin{cases} m_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Storing $n^2 - n$ zeros

Sparse Matrices!

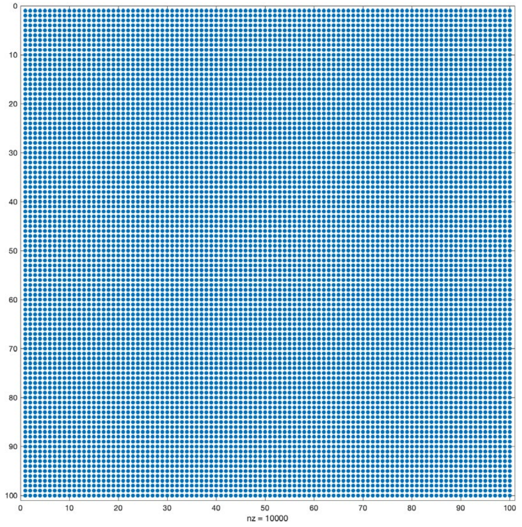


Sparse: only non-zero values are stored

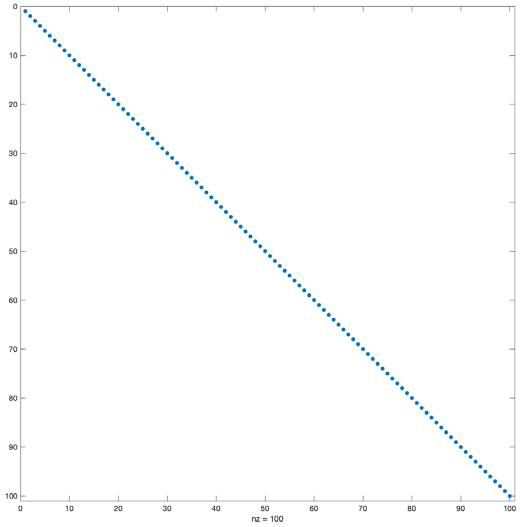
For example, our mass matrix is diagonal!!

$$\mathbf{M}_{ij} = \begin{cases} m_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Sparse Matrices!




```
Afull = zero(m,n)
for each pair i j
    Afull(i,j) += v end
```



Stored as list of triplets

$$(i, j, \text{value})$$

```
triplet_list = []
for each pair i j
    triplet_list.append(i,j,v)
end
Asparse = construct_from_triplets(triplet_list)
```

Sparse Matrices!

$\mathbf{Q} = \mathbf{L}\mathbf{L}^\top$ takes $O(n^3)$

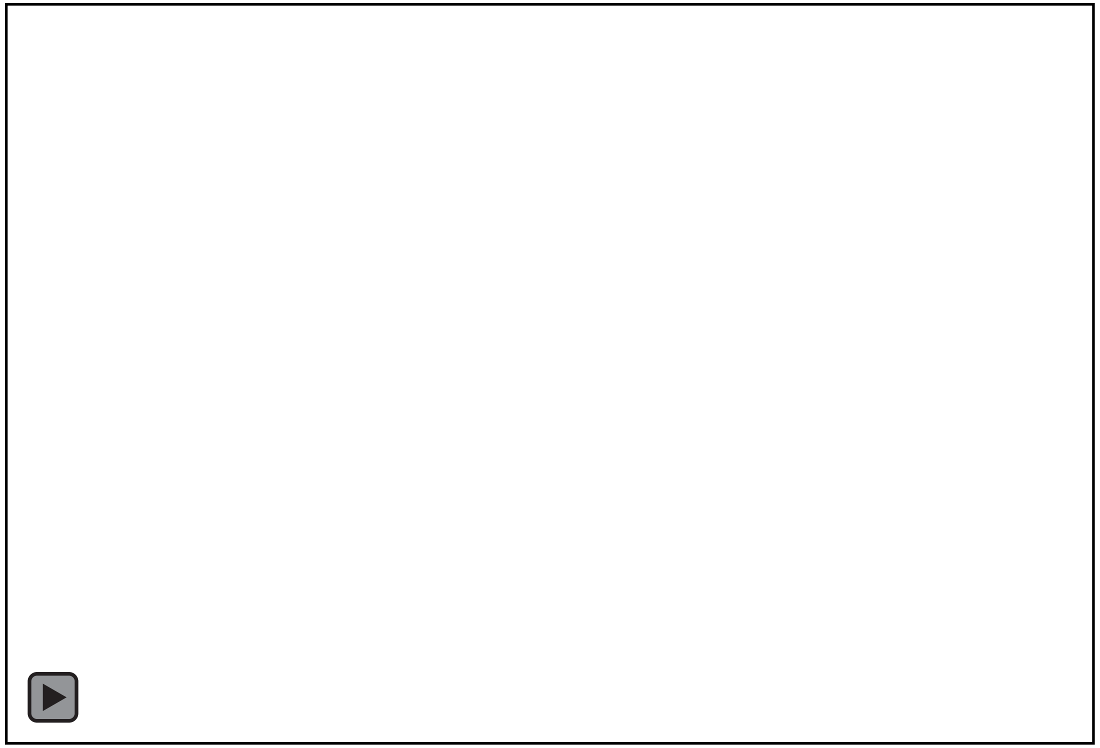
$\mathbf{L}\mathbf{L}^\top \mathbf{y} = \mathbf{b}$ takes $O(n^2)$

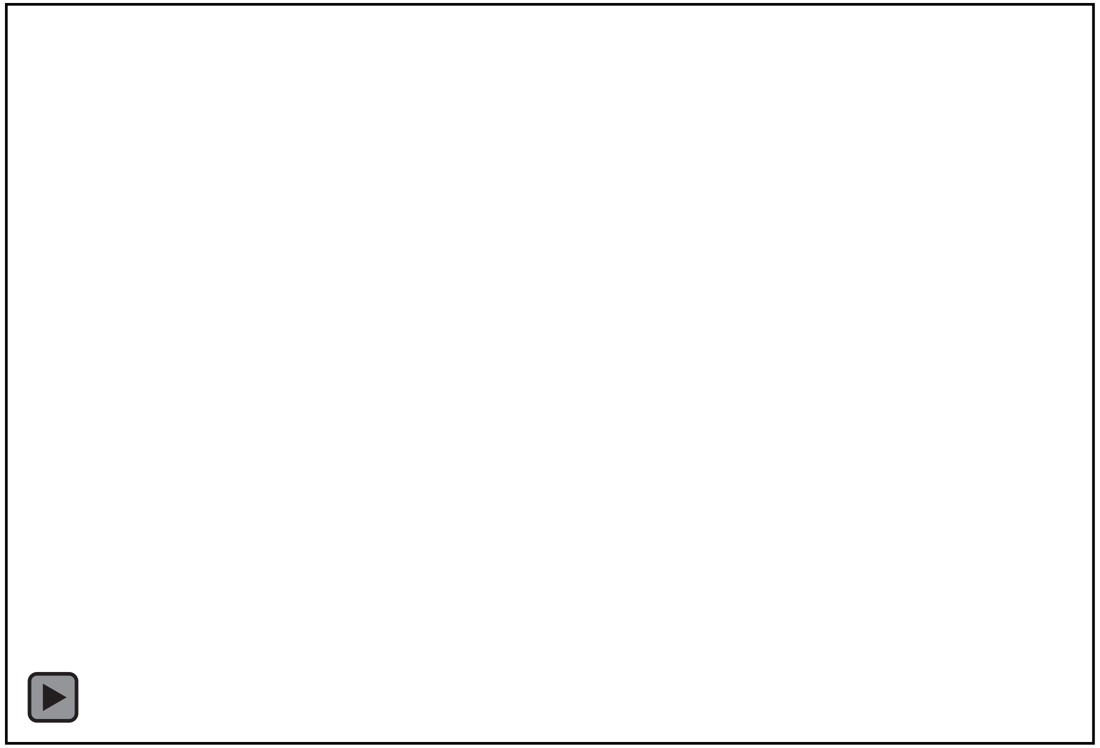
$\mathbf{Q} = \mathbf{L}\mathbf{L}^\top$ takes $O(n^{\approx 1.5})$

$\mathbf{L}\mathbf{L}^\top \mathbf{y} = \mathbf{b}$ takes $O(n)$

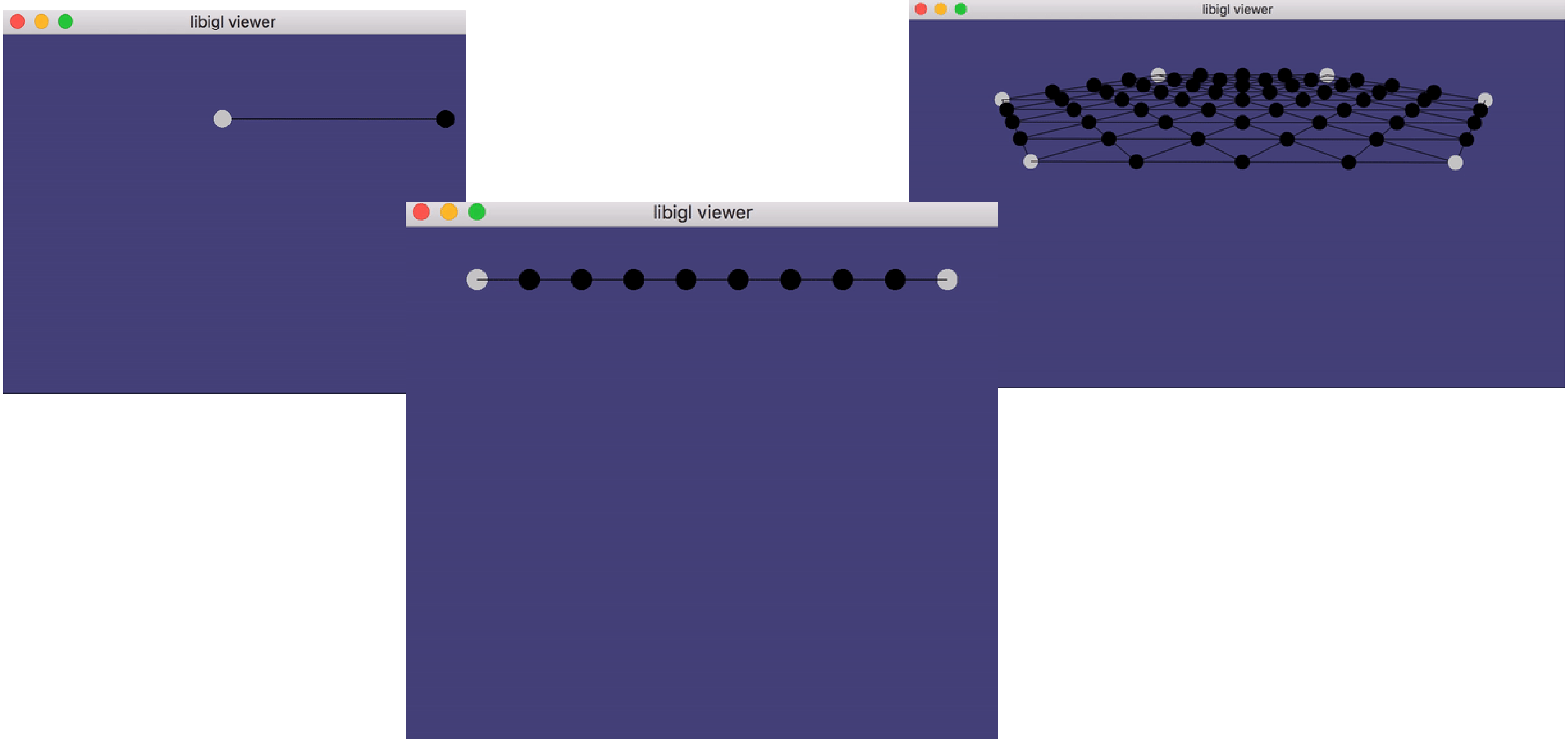
https://en.wikipedia.org/wiki/Triangular_matrix#Forward_and_back_substitution

Lots more on the Assignment Page





Lots more on the Assignment Page



Simpler to debug the dense code first!

Lots more on the Assignment Page

So please read it carefully when doing the assignment

Done for Today

Office hours: Right now