Homework Assignment 2

Due by 4pm at GDC 7.504, February 9

SDS 321 Intro to Probability and Statistics

1. (2+2 pts) With probability .4, the present was hidden by mom; with probability .35, it was hidden by dad and with probability .25 by grandma. When mom hides the present, she hides it upstairs 70 percent of the time and downstairs 30 percent of the time. Dad is equally likely to hide it upstairs or downstairs. Grandma always hides it downstairs.

   (a) What is the probability that the present is upstairs?
   Solution: Let U = the event that the present is upstairs. Let $X = 1$ indicate it being hidden by mom, $X = 2$ by dad, and $X = 3$ by grandma. By the law of total probability,

   $$P(U) = P(U|X = 1)P(X = 1) + P(U|X = 2)P(X = 2) + P(U|X = 3)P(X = 3)$$

   or

   $$P(U) = 0.7 \cdot 0.4 + 0.5 \cdot 0.35 + 0 \cdot 0.25 = 0.455$$

   Rubric: 1 pt for writing total probability rule correctly. 1 pt for putting everything together correctly.

   (b) Given that it is downstairs, what is the probability it was hidden by grandma?
   Solution: Let D = the event that the present is downstairs. By Bayes' rule,

   $$P(X = 3|D) = \frac{P(D|X = 3)P(X = 3)}{P(D)} = \frac{1 \cdot 0.25}{1 - P(U)} = 0.459...$$

2. A gambler has a fair die and a die with all sides equal to 6 in his pocket. He selects one of the die at random; when he rolls it, it shows a 6.

   (a) (2 pt) What is the probability that it is the fair die?
   Solution: Let F = the die is fair, U = the die is unfair, X = the result of the first roll. The first step is to apply Bayes' rule, and the next one is to apply the law of total probability to determine $P(X = 6)$.

   $$\begin{aligned} P(F|X = 6) &= \frac{P(X = 6|F)P(F)}{P(X = 6)} = \frac{\frac{1}{6} \cdot \frac{1}{2}}{P(X = 6|F)P(F) + P(X = 6|U)P(U)} \\ &= \frac{\frac{1}{12}}{\frac{1}{12} + 1 \cdot \frac{1}{2}} = \frac{\frac{1}{12}}{\frac{7}{12}} = \frac{1}{7} \end{aligned}$$

   (b) (3 pts) Suppose that he rolls the same die a second time and, again, it shows a 6. Now what is the probability that it is the fair die? *Hint: use conditional independence.*

Solution: Let Y = the result of the second roll. This will be similar to the last problem, but with a slight trick: $P(F|X=6,Y=6) = \frac{P(X=6,Y=6|F)P(F)}{P(X=6,Y=6)}$.
Now, we need to calculate $P(X=6,Y=6|F)$ and $P(X=6,Y=6)$. By the definition of conditional probability,

$$P(X=6,Y=6|F) = P(X=6|Y=6,F)P(Y=6|F)$$

Now we invoke conditional independence: $P(X=6|Y=6,F) = P(X=6|F)$ (given that the die is fair, the probability that we roll a 6 is independent of the result of any other roll). So:

$$P(X=6,Y=6|F) = P(X=6|F)P(Y=6|F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Similarly,

$$\begin{aligned} P(X=6,Y=6) &= P(X=6,Y=6|F)P(F) + P(X=6,Y=6|U)P(U) \\ &= P(X=6|F)P(Y=6|F)P(F) + P(X=6|U)P(Y=6|U)P(U) \\ &= P(X=6|F)P(Y=6|F)P(F) + P(X=6|U)P(Y=6|U)P(U) \\ &= \frac{1}{72} + 1 \cdot 1 \cdot \frac{1}{2} = \frac{37}{72} \end{aligned}$$

Thus, $P(F|X=6,Y=6) = \frac{1/72}{37/72} = \frac{1}{37}$.

(c) (2 pts) Suppose that he rolls the same die a third time and it shows a 1. Now what is the probability that it is the fair die?
Solution: Let Z = the result of the third roll. This problem is nearly identical to part (b). The following is an expression for $P(F|X=6,Y=6,Z=1)$:

$$\frac{P(Z=1|F)P(Y=6|F)P(X=6|F)P(F)}{P(Z=1|F)P(Y=6|F)P(X=6|F)P(F) + P(Z=1|U)P(Y=6|U)P(X=6|U)P(U)}$$

However, $P(Z=1|U) = 0$ so the second term in the denominator is simply 0, so the whole expression simplifies to $P(F|X=6,Y=6,Z=1) = 1$.

3. A true/false question is to be posed to a father-and-son team on a quiz show. Both the father and the son will independently give the correct answer with probability q. They think of the following two strategies to come up with an answer.

(a) (1 pt) Choose one of them uniformly at random and let that person answer the question. What is the probability that the team answers correctly using this strategy?
Solution: Let SC and FC be events that corresponding to the son and father being correct, each with probability q of being equal to 1 and probability $1-q$ of being equal to 0. By the strategy given, $P(\text{correct}) = \frac{1}{2} \cdot P(S=1) + \frac{1}{2} \cdot P(F=1) = q$

(b) Have them both consider the question, and then either give the common answer if they agree or, if they disagree, flip a fair coin to determine which answer to give.

i. (3 pts) What is the probability that the two answers don't match up?
Solution: We can compute the joint probabilities (since S and F are independent):

$$\begin{aligned}
P(SC, FC) &= q^2 \\
P(SC, FC^c) &= q(1-q) \\
P(SC^c, FC) &= (1-q)q \\
P(SC^c, FC^c) &= (1-q)^2
\end{aligned}$$

The probability that the answers don't match up is then:
$P(\text{mismatch}) = P\left(SC, FC^c \cup SC^c, FC)\right) = 2q(1-q)$.

ii. (1 pt) What is the probability that the two answers match up and the common answer is correct?
Solution: As computed in the previous part, $P(SC, FC) = q^2$.

iii. (3 pts) What is the probability that the team answers correctly using this strategy?
Solution: This is
$P(\text{correct}) = P(\text{match and correct}) + P(\text{mismatch and correct is chosen})$.
$P(\text{mismatch and correct is chosen}) = P(\text{correct}|\text{mismatch})P(\text{mismatch})$.
When mismatched, exactly one of the father and son is correct, and we will select that person with a fair coin. As such, $P(\text{correct}|\text{mismatch}) = \frac{1}{2}$.
Therefore,

$$\begin{aligned}
P(\text{correct}) &= P(SC, FC) + P(\text{correct}|\text{mismatch})P(\text{mismatch}) \\
&= q^2 + \frac{1}{2} \cdot 2q(1-q) = q^2 + q(1-q) = q^2 + q - q^2 = q
\end{aligned}$$

(c) (1 pt) Which strategy should they take?
Solution: Because the probability of answering correctly is the same in either strategy, they could use either.