Midterm

SDS321

Spring 2016

You may use a two (2 sided) pages of notes, and you may use a calculator.

This exam consists of five questions, containing multiple sub-questions. The assigned points are noted next to each question; the total number of points is 25. You have 75 minutes to answer the questions.

Please answer all problems in the space provided on the exam. Use extra pages if needed. Of course, please put your name on extra pages.

Read each question carefully, show your work and clearly present your answers. Note, the exam is printed two-sided - please don't forget the problems on the even pages!

Good Luck!

Name: ____

UTeid: ____

1. (5 pts) 60% of the students at a certain school take neither calculus nor statistics. 20% percent takes calculus and 30% takes statistics. If one of the students is chosen randomly, what is the probability that this student takes

   (a) (2 pts) calculus and statistics?

   Solution: $P(C) = 0.2$ *and* $P(S) = .3$.*So* $P(C \cup S) = 1 - .6 = .4$. $P(C \cap S) = P(C) + P(S) - P(C \cup S) = .1$

   Grading: 1 pt for correct neither nor. 1 pt correct formula of union. Take 1/2 pt of for silly mistakes.

   (b) (2 pts) Are the events {A randomly picked student takes calculus} and {A randomly picked student takes statistics} independent?

   Solution: $P(C \cap S) = .1 \neq P(C)P(S)$. *So No.*

   Grading: 1 pt for understanding that they have to use the formula of independence. 1 pt for correct answer. Of course if they do it using conditional prob thats fine too.

   (c) (1 pts) either calculus or statistics (but not both)?

   Solution: $P(C \cup S) - P(C \cap S) = .4 - .1 = .3$

   Grading: 1/2 pt for the right formula. 1/2 pt for right answer.

2. (5 pts) How many solutions are there to the equation $x_1 + x_2 + x_3 = 8$ where:

   (a) (1 pt) x_1, x_2 and x_3 are non-negative integers, i.e. $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$?

   Solution: $\binom{8+3-1}{3-1} = \binom{10}{2} = 45$

   Grading: They should know this. 1 pt for correct answer.

   (b) (3 pts) x_1, x_2 and x_3 are non-negative integers such that $x_1 > 2$?

   Solution: *Same as number of solutions to $y_1 + x_2 + x_3 = 5$, where each is a non negative integer.* $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$.

   Grading: 1 pt for for understanding this is $x_1 \geq 3$. 1pt for understanding this is the same as counting non-negative integral solutions by subtracting 3 off from the right hand side and 1 pt for calculating the last bit. 1/2 pt off for silly mistakes. If they try out a ton of different examples manually, give them 1 pt. If they answer it correctly but take $x_1 \geq 2$ then take 1 pt pt off. If they calculate the case with $x_1 \leq 2$ first and then subtract from the total, full score.

   (c) (1 pt) x_1, x_2 and x_3 are non-negative integers such that $x_1 \leq 2$? *Hint: no further calculation necessary.*

   Solution: *1 pt for writing this is a-b. Full score if they calculate by taking $x_1 = 0$, $x_1 = 1$ and $x_1 = 2$, i.e.* $9 + 8 + 7 = 24$

3. (5 pts) The joint probability distribution $P_{X,Y}(x, y)$ of two discrete r.v's X and Y is as follows, with $P(X = 1, Y = 1) = a$, $P(X = 1, Y = 2) = 0.3$, etc.

y \ x	1	2
1	a	0.2
2	0.3	b

(a) (1 pt) Find $P(X = x)$ for each $x \in \{1, 2\}$ in terms of a and b.

Solution: $P(X = 1) = a + .3$, $P(X = 2) = b + .2$

Grading: If they sum up the wrong column take 1 pt off. But don't carry the mistake over.

(b) (2 pts) If $E[X] = 1.3$, what are the values a and b?

Solution: $a + b = 0.5$ *since all the values should sum upto one.* $(a + .3) + 2(b + .2) = 1.3$ *and so* $a + 2b = .6$. $b = .1$, $a = .4$.

Grading: 1/2 pt for setting up $E[X]$ ***correctly. 1 pt for normalization. 1/2 pt for correct a and b.***

(c) (2 pts) Find $P(XY \geq 2)$. This should be a numeric answer, i.e. not in terms of a and b.

Solution: $P(XY \geq 2) = P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 2, Y = 2) = .2 + .3 + b = .5 + b = .6$

Grading: 1/2 pt each for 3 different possibilities. 1/2 for correct answer.

4. (5 pts) The probability of a royal flush in a poker hand is $p = 1/649740$. Below, let X denote the number of royal flushes in n hands.

   (a) (2 pts) Find the probability of no royal flush in $n = 1000000$ hands. You can leave your answer in terms of n and p if you want.

      Solution: $X \sim Poisson(np)$. $P(X = 0) = \exp(-np)$.

      Grading: 1/2 pt for understanding its a Poisson. 1/2 pt for correct parameter. 1 pt for correct $P(X = 0)$

   (b) (2 pts) How large must n be to render the probability of having no royal flush in n hands smaller than $1/e$? That is, how large must n be for $P(X = 0) < 1/e$?

      Solution: $\exp(-np) < \exp(-1) \to np > 1 \to n > 1/p = 649740$.

      Grading: p(0) = 1/e; using the (exact) binomial prob is okay, $(1 - p)^n < 1/e \to n > -1/\log(1 - p) = 649739.5$.

   (c) (1 pt) How large must n be to have at least two royal flushes on average. That is, how large must n be for $E[X] \geq 2$?

      Solution: $E[X] = np \geq 2 \to n \geq 2/p$.

      Grading: 1/2 pt for correct expectation.

5. (5 pts) Each day, Alice's boss gives her 10 umbrellas to sell that day, and takes back any she doesn't sell. She gets a bonus on a day if and only if she sells at least one umbrella that day. On a rainy day Alice sells each umbrella independently with probability 1. On a sunny day Alice sells each umbrella independently with probability 0.2. The probability that it would rain on a particular day is 0.1.

   (a) (1 pt) If it's a rainy day, find the probability that Alice gets a bonus.

   Solution: *Let* $R = \{rain\}$*,* X *denote the number of umbrellas.* $P(B|R) = P(X \geq 1|R) = 1 - P(X = 0|R) = 1$

   Grading: Intuitive explanation of why its 1 is ok.

   (b) (1 pt) If it's a sunny day, find the probability that Alice gets a bonus.

   Solution: $P(B|R^c) = 1 - P(X = 0|R^c) = 1 - (1 - .2)^{10} = 1 - .8^{10} = .8926$.

   Grading: 1/2 pt for writing at least as one minus probability of none. 1/2 pt for correct binomial representation. Don't take anything if they don't calculate it and leave it as $1 - .8^{10}$***.***

   (c) (2 pts) Given that Alice got a bonus, what is the probability that it rained that day?

   Solution: $P(R|B) = \frac{P(B|R)P(R)}{P(B)} = .1/(1 \times .1 + .8926 \times .9) = 0.11$

   Grading: 1 pt for correct Bayes rule. 1 pt for calculating $P(B)$ ***with total probability rule. 1/2 pt off for silly mistakes.***

   (d) (1 pt) What is the expected number of bonuses Alice gets in a week?

   Solution: *If* Y *is the number of bonuses Alice gets in a week. Then* $Y = \sum_i Z_i$ *where* Z_i *is a bernoulli with parameter* $P(B)$*. So the expectation of this is* $7 \times .9 = 6.3$

   Grading: 1/2 point for understanding this is the expectation of a binomial. 1/2 pt for correct answer. Even if they do $10 \times P(B)$ ***with the wrong*** $P(B)$ ***give full score. Any other explanation, like total expectation theorem is ok.***