

SDS 321: Practice questions

1 Discrete

1. My partner and I are one of 10 married couples at a dinner party. The 20 people are given random seats around a round table.
 - (a) What is the probability that I am seated next to my spouse?
 - (b) What is the expected number of couples that are seated next to each other?
2. How many unique combinations can you get by rearranging the letters MISSISSIPPI?
3. Combinatorics question:
 - (a) How many different solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2 and x_3 are positive integers? (count “ $x_1 = 1, x_2 = 2, x_3 = 7$ ” and “ $x_1 = 2, x_2 = 1, x_3 = 7$ ” as two separate solutions).
 - (b) How many different solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where $x_1 < x_2$?
 - (c) How many different solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where $x_1 < x_2 < x_3$?

4. I have three envelopes, each containing two objects. In one, there is a silver square and a gold disk. In another, there are a gold square and a gold disk. In the third, there are a silver square and a silver disk.
 - (a) I pick an envelope (at random), and take out an object (at random). It is gold. What is the probability that the second object is silver?
 - (b) I put the objects back in their envelope and shuffle the envelopes. I again pick an envelope (at random), and take out an object (at random). It is a gold disk. What is the probability that the second object is silver?
5. On the first day of a non-leap-year, I put \$1 in a box. On the second day, I put \$2 in the box. On the third day, I put \$3 in. And so on. At the end of the year (365 days), how much money is in the box?
6. Alice and Bob are playing rock-paper-scissors. If both Alice and Bob play the same hand, they play again. What is the expected number of turns before someone wins?

2 Continuous

1. Let X be a normal random variable with mean 3 and variance 1, and let Y be a normal random variable with mean 4 and variance 2.
 - (a) What is the distribution of $Z = X + Y$?
2. I am waiting for a bus, that I know will arrive at some time between 1pm and 2pm, with all times being equally likely. It gets to 1:30, and the bus has still not arrived. What is the probability that it arrives before 1:40?
3. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 0.125x + 0.125 & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

What is the PDF of $Z = |X|$?

1. Let X be a continuous random variable, with PDF:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \leq x < 1 \\ ce^{-x} & x \geq 1 \end{cases}$$

- (a) What is c ?
- (b) What is the conditional expectation of X , given $X < 1$?
- (c) What is the conditional expectation of X , given $X \geq 1$?

(d) What is the expectation of X ?

2. Let X and Y be random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{ay}{x^2} & x \geq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is a ?

(b) What is the conditional PDF $f_{Y|X}(y|x)$ of Y given $X = x$?

(c) What is the conditional expectation of Y given X ?

(d) What is the expected value of Y ?

3. Let X and Y be uniform random variables between 0 and 1. What is the probability that:

(a) $X < Y$

(b) $X < 2Y$

(c) $X + Y < 0.5$

(d) $\max\{X, Y\} \leq 0.7$

4. Let X be the number of ice-cream cones a vendor sells on a day. If the average temperature of a summer day is a random variable Y (in Fahrenheit), where $Y \sim \text{Uniform}([95, 105])$. We also have $X \sim \text{Poisson}(Y^2/10 + Y/5 + 5)$.

- (a) What is $E[X|Y]$?
 - (b) What is $E[X]$?
5. If there are no distractions, it takes me 30 minutes to walk to the store. However, if I pass someone with a dog, I stop and pet the dog and chat to the owner. The number Y of dogs I pass is a Poisson random variable with mean 2. Each time I stop, the number of minutes I spend petting the dog and chatting is an exponential random variable with PDF:

$$f_X(x) = 0.5e^{-0.5x}$$

- (a) If I see a single dog, what is the expectation and variance of the time spent petting the dog and chatting to its owner?
- (b) What is the conditional expectation of the total time spent petting dogs and chatting to their owners, as a function of Y ?
- (c) Using the law of iterated expectation calculate $E[X]$.

3 Statistics

1. Consider n iid Poisson random variables $X_i \sim \text{Poisson}(\lambda)$.
 - (a) What is the MLE $\hat{\lambda}$ of λ ?
 - (b) Is it unbiased or biased?
2. Consider n iid Exponential random variables $X_i \sim \text{Exp}(\lambda)$.
 - (a) What is the MLE $\hat{\lambda}$ of λ ?
 - (b) Is $1/\hat{\lambda}$ an unbiased or biased estimator of $1/\lambda$?
 - (c) What is the variance of $1/\hat{\lambda}$?
 - (d) Using CLT approximate the distribution of $1/\hat{\lambda}$.
 - (e) Using CLT, compute how big n had to be in order to have $P(|1/\hat{\lambda} - 1/\lambda| \leq 0.1/\lambda) = .95$.

- (f) Using Chebyshev's inequality, use the n you found in the last question to upper bound $P(|1/\hat{\lambda} - 1/\lambda| \geq 0.1/\lambda)$.
3. On a given day, a $Poisson(100)$ number of insects fly through my yard. Using an appropriate approximation, what is the probability that, over the month of May (31 days), the average number of insects is between 98 and 102? You may use the fact that a $Poisson(\lambda)$ random variable has mean and variance λ .
4. I am interested in seeing whether a sequence of 16 observations has zero mean. My null hypothesis is $H_0 : \mu = 0$, and my alternative is $H_1 : \mu \neq 0$. I know the variance is 1. What is an appropriate rejection region for the null hypothesis at 0.05 significance?
5. I set up a motion sensor to look for intruders. The signal when there is no intruder is a normal random variable with mean 0 and variance 1. The signal where there is an intruder is a normal random variable with mean 1 and variance 2.
- (a) If I raise the alarm if the signal is greater than 0.7, what is my Type I error?
- (b) What is the corresponding Type II error?
6. I have a coin with unknown probability of heads θ . I throw the coin n times, and count the number k of heads. What is the MLE estimator of θ ? Is it biased or unbiased? *hint: we can write the likelihood of a single Bernoulli random variable as $f_X(x_i; \theta) = \theta^{x_i}(1 - \theta)^{1-x_i}$*

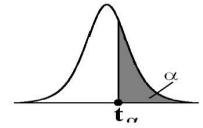
7. I have 10 samples x_1, \dots, x_{10} from a normal distribution with unknown mean μ and variance 4. The mean $\bar{x} = \frac{\sum_{i=1}^{10} x_i}{10}$ of the samples is 5.33.
- What is the mean and variance of \bar{x} ?
 - Give a 95% confidence interval for the mean.
 - If I see 10 more observations, will the confidence interval be narrower or wider?
8. I want to estimate the mean water level of a lake. I assume that, on any given day, the water level will be a normal random variable with unknown mean corresponding to the mean water level, and unknown variance σ^2 .
- Show that $\hat{s}_n^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$ is an unbiased estimator for the variance.
 - You record the water height over three days as 115, 121, 118. Calculate an unbiased estimator for the mean and variance of the water level.
 - Using the attached t -distribution tables, give a 0.95 confidence region for the true mean.
9. I have 500 samples from a Poisson distribution with pdf $p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$. My null hypothesis is that $\lambda = 1$; my alternative hypothesis is that $\lambda > 1$.
- Is this a one-sided or two-sided test?
 - Under the null hypothesis, use a normal approximation to estimate the probability that $\bar{X} < 0.9$. You may use the fact that the mean and variance of a Poisson random variable are both λ .

- (c) We want to choose a critical value ξ such that we reject the null hypothesis if $\bar{X} > \xi$. Find a value of ξ that gives a significance of 0.01.
10. I have two independent sequences, X and Y (which are also mutually independent), with known variance σ^2 and unknown means μ_X and μ_Y . I want to test the null hypothesis that $\mu_X = 2\mu_Y$ using a two-sided significance test.
- (a) Suggest an appropriate statistic – specifically, a function of \bar{X} and \bar{Y} – that has zero mean under the null hypothesis.
- (b) What is the variance of this statistic?
- (c) Construct an appropriate significance test for the null hypothesis, at significance level $\alpha = 0.05$

Standard normal table

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Percentage Points of the t Distribution; $t_{\nu, \alpha}$
 $P(T > t_{\nu, \alpha}) = \alpha$



ν	α													
	0.40	0.30	0.20	0.15	0.10	0.05	0.025	0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706	15.895	21.205	31.821	42.434	63.657	127.322	636.590
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303	4.849	5.643	6.965	8.073	9.925	14.089	31.598
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052	2.158	2.291	2.473	2.598	2.771	3.057	3.690
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045	2.150	2.282	2.462	2.586	2.756	3.038	3.659
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960	2.054	2.170	2.326	2.432	2.576	2.807	3.291