Group work

SDS 321 Intro to Probability and Statistics

1. Calculate c and $E[X]$ and $E[Y]$ for the following.

   (a)

$$f_{X,Y}(x,y) = \begin{cases} ce^{-(x+y)/5} & x, y \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

   $f_X(x)$ looks like e^{-5x} so $f_X(x) = 5e^{-5x}$. Similarly $f_Y(y) = 5e^{-5y}$. $E[X] = E[Y] = 1/5$ from your knowledge of the Exponential(λ) r.v.

   (b)

$$f_{X,Y}(x,y) = \begin{cases} ce^{-(2x+3y)} & x, y \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

   Just match the patterns!! $f_X(x) = 2e^{-2x}$ and $f_Y(y) = 3e^{-3y}$. So $E[X] = 1/2$ and $E[Y] = 1/3$ and $c = 6$.

   (c)

$$f_{X,Y}(x,y) = ce^{-(x^2+y^2)/2}$$

   Just match the patterns!! The moment you see something like $e^{-(x-\mu)^2/2\sigma^2}$, you think gaussian/normal distribution with mean μ and variance σ^2 for pattern matching!. $f_X(x) = f_Y(y) = e^{-x^2/2}/\sqrt{2\pi}$. This means $X \sim N(0,1)$ and $Y \sim N(0,1)$ So $E[X] = 0$ and $E[Y] = 0$ and $c = 1/(2\pi)$.

2. Evaluate the following integrals. We will again just pattern match, and not really do integration by parts. These will come in handy for continuous random variables. Show your calculations.

   (a) $\int_0^\infty x\exp(-2x)dx$ Remember $E[X]$ for $X \sim \mathit{Exponential}(1)$? $E[X] = \int_0^\infty xe^{-x}dx = 1$. Thats what we will use. For $\int_0^\infty x\exp(-2x)dx$, do a substitution, $v = 2x$. So $dx = 1/2dv$.

$$\int_0^\infty x\exp(-2x)dx = 1/4\int_0^\infty ve^{-v}dv = 1/4E[X] = 1/4.$$

   (b) $\int_0^\infty x\exp(-x/2)dx$ Now do a substitution, $v = x/2$. So $dx = 2dv$.

$$\int_0^\infty x\exp(-x/2)dx = 4\int_0^\infty ve^{-v}dv = 4E[X] = 4.$$

(c) $\int_0^\infty (1+x)^2 \exp(-2x)dx$ For this exercise we will use $E[X^2] = \text{var}(X) + E[X]^2 = 2$ for $X \sim Exponential(1)$. Now do a substitution, $v = 2x$. So $dx = dv/2$. We will use, for $X \sim Exponential(1)$,

$$\int_0^\infty \exp(-v)dv = 1 \tag{1}$$

$$\int_0^\infty v\exp(-v)dv = E[X] = 1 \tag{2}$$

$$\int_0^\infty v^2\exp(-v)dv = E[X^2] = 2 \tag{3}$$

$$\begin{aligned}
\int_0^\infty (1+x)^2 \exp(-2x)dx &= \int_0^\infty (1+2x+x^2)\exp(-2x)dx \\
&= 1/2\int_0^\infty (1+v+v^2/4)\exp(-v)dv \\
&= 1/2(\underbrace{\int_0^\infty \exp(-v)dv}_{1 \text{ by eq(1)}} + \underbrace{\int_0^\infty v\exp(-v)dv}_{1 \text{ by eq(2)}} + 1/4\underbrace{\int_0^\infty v^2\exp(-v)dv}_{2 \text{ by eq(3)}}) \\
&= 1/2(1+1+1/2) = 5/4
\end{aligned}$$