

# SDS 384 11: Theoretical Statistics

## Lecture 4: Sub-gaussian and sub-exponential random variables

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# Sub-Gaussian random variables

## Theorem

For  $X_1, \dots, X_n$  independent sub-gaussian random variables with sub-gaussian parameters  $\sigma_i^2$  and  $E[X_i] = \mu_i$ , for  $\forall t > 0$ ,

$$P\left(\sum_i (X_i - \mu_i) \geq t\right) \leq e^{-\frac{t^2}{2\sum_i \sigma_i^2}}$$

- If  $X_i \in [a, b]$ ,  $E[X_i] = 0$ , using Hoeffding's lemma we have:  
 $\sigma_i^2 \leq (b - a)^2 / 4$ .
- So, the above theorem immediately gives the original Hoeffding inequality back.

$$P\left(\sum_i X_i \geq t\right) \leq e^{-\frac{2t^2}{n(b-a)^2}}$$

# Sub-exponential random variables

## Definition

$X$  is sub-exponential with parameters  $(\nu, b)$  if,  $\forall |\lambda| < 1/b$ ,

$$\log M_{X-\mu}(\lambda) \leq \frac{\lambda^2 \nu^2}{2}$$

Examples:

- Sub-Gaussian  $X$  with parameter  $\sigma^2$  is sub-exponential with parameters  $(\sigma^2, b) \forall b > 0$ .
- How about the converse?

# Sub-exponential but not sub-gaussian

## Example

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The MGF is only defined for  $\lambda < 1/2$ . So this is a sub-exponential random variable with parameter  $(2, 4)$ , but not a sub-gaussian random variable.

## Theorem

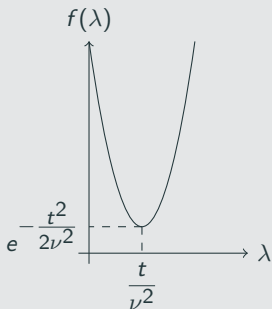
*Let  $X$  be a sub-exponential random variable with parameters  $(\nu, b)$ . Then,*

$$P(X \geq \mu + t) \leq \begin{cases} e^{-\frac{t^2}{2\nu^2}} & \text{if } 0 \leq t \leq \frac{\nu^2}{b} \\ e^{-\frac{t}{2b}} & \text{if } t \geq \frac{\nu^2}{b} \end{cases}$$

- For small  $t$  this is sub-gaussian in nature, whereas for large  $t$  the exponent decays linearly with  $t$ .

## Proof.

$$\begin{aligned}
 P(X \geq t) &\leq \inf_{\lambda \geq 0} e^{-\lambda t} E[e^{\lambda X}] \\
 &\leq \inf_{\lambda \geq 0} \underbrace{e^{-\lambda t + \lambda^2 \nu^2 / 2}}_{f(\lambda)} \quad \text{When } 0 \leq \lambda < 1/b
 \end{aligned}$$



- If  $\frac{t}{\nu^2} \leq \frac{1}{b}$ ,  

$$\inf_{\lambda \geq 0} f(\lambda) = f(t/\nu^2) = e^{-\frac{t^2}{2\nu^2}}$$
- If  $\frac{t}{\nu^2} > \frac{1}{b}$ , then  $f(\lambda)$  is minimized at the boundary  $\lambda' = 1/b$ .  

$$f(\lambda') = e^{-t/b + \nu^2/2b^2} \leq e^{-\frac{t}{2b}}$$

□



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- **Definition**

A random variable with mean  $\mu$  and variance  $\sigma^2$  satisfies the Bernstein condition with parameter  $b > 0$ , if  $E[|X - \mu|^k] \leq \frac{1}{2} k! \sigma^2 b^{k-2}$  for  $k \geq 2$ .

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- A bounded random variable with  $|X - \mu| \leq b$  satisfies the above.

# Bernstein's condition and the sub-exponential property

## Theorem

If  $X$  ( $E[X] = \mu$ ,  $\text{var}(X) = \sigma^2$ ) satisfies the Bernstein condition with parameter  $b > 0$ , then  $X$  is sub-exponential with  $(\sqrt{2}\sigma, 2b)$ .

## Proof.

$$\begin{aligned} E[e^{\lambda(X-\mu)}] &= \sum_{k=0}^{\infty} \frac{\lambda^k E[(X-\mu)^k]}{k!} \\ &= 1 + \frac{\lambda^2 \sigma^2}{2} + \sum_{k=3}^{\infty} \frac{|\lambda|^k \sigma^2 b^{k-2}}{2} \\ &= 1 + \frac{\lambda^2 \sigma^2}{2} \left( 1 + \sum_{k=1}^{\infty} (|\lambda|b)^k \right) \\ &= 1 + \frac{\lambda^2 \sigma^2}{2(1 - |\lambda|b)} \quad \text{For } |\lambda| < 1/b \\ &\leq e^{\frac{\lambda^2 \sigma^2}{2(1 - |\lambda|b)}} \leq e^{\lambda^2 \sigma^2} = e^{\frac{\lambda^2 (\sqrt{2}\sigma)^2}{2}} \quad \text{For } |\lambda| < 1/2b \end{aligned}$$

# Bernstein's inequality

## Theorem

*If  $X$  with mean  $\mu$  and variance  $\sigma^2$  satisfies the Bernstein condition with parameter  $b > 0$ , then*

$$P(|X - \mu| \geq t) \leq 2e^{-\frac{t^2}{2(\sigma^2 + bt)}} \quad (1)$$

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- Why not use Hoeffding?
- For small  $t$ , Bernstein gives us a subgaussian tail with parameter  $\sigma$
- In contrast, Hoeffding always gives us a subgaussian tail with parameter  $b \geq \sigma$ .



# Bernstein's inequality

**Proof.**

$$\begin{aligned} P(X - \mu \geq t) &\leq \inf_{\lambda \in [0, 1/b)} e^{-\lambda t + M_{X-\mu}(\lambda)} \\ &= \inf_{\lambda \in [0, 1/b)} e^{-\lambda t + \frac{\lambda^2 \sigma^2 / 2}{1 - b\lambda}} \\ &= e^{-\frac{t^2}{2(bt + \sigma^2)}} \quad \text{Setting } \lambda = \frac{t}{bt + \sigma^2} \in [0, 1/b) \end{aligned}$$



## sub-exponential property

- The sub-exponential property is preserved under summation of independent random variables.
- Consider  $X_k, k = 1, \dots, n$  independent sub-exponential  $(\nu_k, b_k)$  random variables with  $E[X_k] = \mu_k$ .
- 

$$\begin{aligned} E \left[ e^{\lambda \sum_k (X_k - \mu_k)} \right] &= \prod_{i=1}^n E \left[ e^{\lambda (X_i - \mu_i)} \right] \\ &\leq \prod_{i=1}^n e^{\frac{\lambda^2 \nu_i^2}{2}} \quad \text{For } |\lambda| \leq 1/\max_i b_i \end{aligned}$$

- So  $\sum_k (X_k - \mu_k)$  is sub-exponential with parameters  $(\sqrt{n}\nu_*, b_*)$ .

$$b_* = \max_k b_k, \text{ and } \nu_*^2 = \sum_i \nu_i^2 / n \quad (2)$$

## Concentration of sub-exponential mean

- Plugging into our previous tail bound we have:

$$P(\bar{X}_n - \mu \geq t) \leq \begin{cases} e^{-\frac{nt^2}{2\nu_*^2}} & \text{for } 0 \leq t \leq \frac{\nu_*^2}{b_*} \\ e^{-\frac{nt}{2b_*}} & \text{for } t > \frac{\nu_*^2}{b_*} \end{cases}$$

## Application: the wonders of Johnson-Lindenstrauss embedding

- Given  $m$  data points  $u_i, i = 1 : m$  in  $\mathbb{R}^d$ , one wants to compute low dimensional projections  $F(u_i), F : \mathbb{R}^d \rightarrow \mathbb{R}^n$  with  $n \ll d$ .
- The goal is to preserve distances, so that distance-based algorithms can work “almost as well” on the low dimensional space.

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- We define “almost as well” by:

$$\|u_i - u_j\|^2(1 - \epsilon) \leq \|F(u_i) - F(u_j)\|^2 \leq \|u_i - u_j\|^2(1 + \epsilon) \quad (3)$$

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- Construct a random matrix  $X \in \mathbb{R}^{n \times d}$  with  $X_{ij} \sim N(0, 1)$ .
- Define  $F(u)$  as  $Xu/\sqrt{n}$

## • Theorem

*As long as  $m > 2$ , and  $u_i \neq u_j, \forall i \neq j$  and  $n = \Omega(\log(m/\delta)/\epsilon^2)$ , Equation (3) is satisfied with probability at least  $1 - \delta$ .*

# We can do this easily with our tools

## Proof.

- $u' = u/\|u\|$ . We will assume that  $u \neq 0$ .
- Let  $Y := \frac{\|F(u)\|^2}{\|u\|^2} = \sum_i (Xu')_i^2$ .
- But  $Y_i := (Xu')_i = \sum_j X_{ij}u'_j \sim N(0, 1)$
- Note that  $Y_i^2$  is sub-exponential with parameters  $(2, 4)$ . So by the summation property,  $Y$  is sub-exponential  $(2\sqrt{n}, 4)$ .
- So  $P\left(\left|\frac{Y}{n} - 1\right| \geq t\right) \leq 2e^{-\frac{nt^2}{8}}$  for  $t \in (0, 1)$ .
- $P\left(\frac{\|F(u_i - u_j)\|^2}{\|u_i - u_j\|^2} \geq \epsilon \text{ For some } u_i \neq u_j\right) \leq 2\binom{m}{2}e^{-\frac{n\epsilon^2}{8}}$
- If  $m \geq 2$  and  $n > \frac{16}{\epsilon^2} \log(m/\delta)$ , the above probability can be made as small as  $\delta$ .