THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics
Lecture 9: Discrete random variables

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

www.cs.cmu.edu/∼psarkar/teaching

Summing up

- Last time, we looked at the probability of a random variable taking on a given value:
$$p_X(x) = P(X = x)$$
- We also looked at plots of various PMFs of Uniform, Bernoulli, Binomial, Poisson and Geometric.
- Often, we want to make predictions for the value of a random variable
  - How many heads do I expect to get if I toss a fair coin 10 times?
  - How many lottery tickets should Alice expect buy until she wins the jackpot?
- We may also be interested in how far, on average, we expect our random variable to be from these predictions.

- Today we will talk about means and variances of these random variables.

Mean

You want to calculate average grade points from hw1. You know that 20 students got 30/30, 30 students got 25/30, and 50 students got 20/30. Whats the average?

- The average grade point is

$$\frac{30 \times 20 + 25 \times 30 + 20 \times 50}{100} = 30 \times 0.2 + 25 \times 0.3 + 20 \times 0.5$$

- Let X be a random variable which represents grade points of hw1.
- How will you calculate $P(X = 30)$?
  - See how many out of 100 students got 30 out of 30 points.
  - $P(X = 30) \approx 0.2$
  - $P(X = 25) \approx 0.3$
  - $P(X = 20) \approx 0.5$
- So roughly speaking,
  average grade $\approx 30 \times P(X = 30) + 25 \times P(X = 25) + 20 \times P(X = 20)$

Expectation

We define the expected value (or expectation or mean) of a discrete random variable X by

$$E[X] = \sum_x xP(X = x).$$

- X is a Bernoulli random variable with the following PMF:

$$P(X = x) = \begin{cases} p & X = 1 \\ 1 - p & X = 0 \end{cases}$$

So $E[X] =$.

Expectation

We define the expected value (or expectation or mean) of a discrete random variable X by

$$E[X] = \sum_x xP(X = x).$$

- X is a Bernoulli random variable with the following PMF:

$$P(X = x) = \begin{cases} p & X = 1 \\ 1 - p & X = 0 \end{cases}$$

  So $E[X] = 1 \times p + 0 \times (1 - p) = p$.
- Expectation of a Bernoulli random variable is just the probability that it is one.
- You will also see notation like μ_X.

Expectation: example

You are tossing 4 fair coins independently. Let X denote the number of heads. What is $E[X]$?

- Any guesses? Well, on an average we should see about 2 coin tosses. No?
- Lets write down the PMF first.

Expectation: example

You are tossing 4 fair coins independently. Let X denote the number of heads. What is $E[X]$?

- Any guesses? Well, on an average we should see about 2 coin tosses. No?
- Lets write down the PMF first.
- $P(X = x) = \begin{cases} 1/2^4 & X = 0 \\ 4/2^4 & X = 1 \\ 6/2^4 & X = 2 \\ 4/2^4 & X = 3 \\ 1/2^4 & X = 4 \end{cases}$

Expectation: example

You are tossing 4 fair coins independently. Let X denote the number of heads. What is $E[X]$?

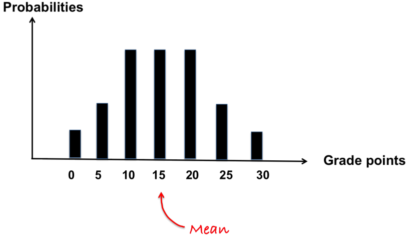
- Any guesses? Well, on an average we should see about 2 coin tosses. No?
- Lets write down the PMF first.
- $P(X = x) = \begin{cases} 1/2^4 & X = 0 \\ 4/2^4 & X = 1 \\ 6/2^4 & X = 2 \\ 4/2^4 & X = 3 \\ 1/2^4 & X = 4 \end{cases}$
- So $E[X] = \frac{4}{2^4} + 2\frac{6}{2^4} + 3\frac{4}{2^4} + 4\frac{1}{2^4} = \frac{32}{16} = 2$.

Expectation of a function of a random variable

Lets say you want to compute $E[g(X)]$. Example, I know average temperature in Fahrenheit, but I now want it in Celsius.

- $E[g(X)] = \sum_x g(x) P(X = x)$.
- Follows from the definition of PMF of functions of random variables.
- Look at page 15 of Bersekas-Tsitsiklis and derive it at home!
- So $E[X^2] = \sum_x x^2 P(X = x)$. **Second moment of X**
- So $E[X^3] = \sum_x x^3 P(X = x)$. **Third moment of X**
- So $E[X^k] = \sum_x x^k P(X = x)$. k^{th} **moment of X**
- We are assuming "under the rugs" that all these expectations are well defined.

Expectation



- Think of expectation as center of gravity of the PMF or a representative value of X.
- How about the spread of the distribution? Is there a number for it?

Variance

Often, you may want to know the spread or variation of the grade points for homework1.

- If everyone got the same grade point, then variation is?
- If there is high variation, then we know that many students got grade points very different from the average grade point in class.
- Formally we measure this using variance of a random variable X.
- $\mathrm{var}(X) = E[(X - E[X])^2]$
- The standard deviation of X is given by $\sigma_X = \sqrt{\mathrm{var} X}$.
- Its easier to think about σ_X, since its on the same scale.
- The grade points have average 20 out of 30 with a standard deviation of 5 grade points. Roughly this means, most of the students have grade points within $[20 - 5, 20 + 5]$.

Variance

Often, you may want to know the spread or variation of the grade points for homework1.

- If everyone got the same grade point, then variation is? 0
- If there is high variation, then we know that many students got grade points very different from the average grade point in class.
- Formally we measure this using variance of a random variable X.
- $\mathrm{var}(X) = E[(X - E[X])^2]$
- The standard deviation of X is given by $\sigma_X = \sqrt{\mathrm{var} X}$.
- Its easier to think about σ_X, since its on the same scale.
- The grade points have average 20 out of 30 with a standard deviation of 5 grade points. Roughly this means, most of the students have grade points within $[20 - 5, 20 + 5]$.

Computing the variance

- $\mathrm{var}(X) = E[(X - E[X])^2] = \sum_x (x - E[X])^2 \, P(X = x)$
- Always remember! $E[X]$ or $E[g(X)]$ **do not depend on any particular value of** x. You can treat it as a constant. It only depends on the PMF of X.

- This can actually be made simpler.
- $\mathrm{var}(X) = E[X]^2 - (E[X])^2$.

- So you can calculate $E[X^2]$ (second moment) and then subtract the square of $E[X]$ to get the variance!

A tiny bit of algebra

$$\mathrm{var}(X) =$$

A tiny bit of algebra

$\mathrm{var}(X) =$

$$\sum_x (x - E[X])^2 \, P(X = x) = \sum_x \left(x^2 + (E[X])^2 - 2xE[X]\right) P(X = x)$$

A tiny bit of algebra

$$\mathrm{var}(X) =$$

$$\sum_x (x - E[X])^2 \, P(X = x) = \sum_x \left(x^2 + (E[X])^2 - 2xE[X]\right) P(X = x)$$

$$= \sum_x x^2 P(X = x) + \sum_x (E[X])^2 P(X = x) - 2 \sum_x xE[X]P(X = x)$$

A tiny bit of algebra

$$\mathrm{var}(X) =$$

$$\sum_x (x - E[X])^2 \, P(X = x) = \sum_x \left(x^2 + (E[X])^2 - 2xE[X]\right) P(X = x)$$

$$= \sum_x x^2 P(X = x) + \sum_x (E[X])^2 P(X = x) - 2\sum_x xE[X]P(X = x)$$

$$= \sum_x x^2 P(X = x) + \underbrace{(E[X])^2 \sum_x P(X = x)}_{(E[X])^2 \text{ does not depend on } x} - 2E[X] \underbrace{\sum_x xP(X = x)}_{\text{This is } E[X]}$$

A tiny bit of algebra

$$\mathrm{var}(X) =$$

$$\sum_x (x - E[X])^2 \, P(X = x) = \sum_x \left(x^2 + (E[X])^2 - 2xE[X]\right) P(X = x)$$

$$= \sum_x x^2 P(X = x) + \sum_x (E[X])^2 P(X = x) - 2 \sum_x x E[X] P(X = x)$$

$$= \sum_x x^2 P(X = x) + \underbrace{(E[X])^2 \sum_x P(X = x)}_{(E[X])^2 \text{ does not depend on } x} \underbrace{-2E[X] \sum_x x P(X = x)}_{\text{This is } E[X]}$$

$$= \sum_x x^2 P(X = x) + (E[X])^2 - 2(E[X])^2 = E[X^2] - (E[X])^2$$

Some simple rules– Expectation

Say you are looking at a linear function (or transformation) of your random variable X.

- $Y = aX + b$. Remember celsius to fahrenheit conversions? They are linear too!
- $E[Y] = E[aX + b] = aE[X] + b$, as simple as that! why?
  - $$\begin{aligned} E[aX + b] &= \sum_x (ax + b)P(X = x) \\ &= a\sum_x xP(X = x) + b\sum_x P(X = x) \end{aligned}$$

Some simple rules– Expectation

Say you are looking at a linear function (or transformation) of your random variable X.

- $Y = aX + b$. Remember celsius to fahrenheit conversions? They are linear too!
- $E[Y] = E[aX + b] = aE[X] + b$, as simple as that! why?
  - $$E[aX+b] = \sum_x (ax+b)P(X=x) \\ = a\sum_x xP(X=x) + b\sum_x P(X=x) \\ = aE[X] + b$$

Some simple rules– Expectation

How about $E[Y]$ for $Y = aX^2 + bX + c$?

- $E[Y] = E[aX^2 + bX + c] = aE[X^2] + bE[X] + c$, as simple as that! why?
  - $$E[aX^2 + bX + c] = \sum_x (ax^2 + bx + c)P(X = x)$$
    $$= a\sum_x x^2 P(X = x) + b\sum_x xP(X = x) + c\sum_x P(X = x)$$

Some simple rules– Expectation

How about $E[Y]$ for $Y = aX^2 + bX + c$?

- $E[Y] = E[aX^2 + bX + c] = aE[X^2] + bE[X] + c$, as simple as that! why?
  - $$E[aX^2 + bX + c] = \sum_x (ax^2 + bx + c)P(X = x)$$
    $$= a\sum_x x^2 P(X = x) + b\sum_x xP(X = x) + c\sum_x P(X = x)$$
    $$= aE[X^2] + bE[X] + c$$

Some simple rules– Expectation

How about $E[Y]$ for $Y = aX^2 + bX + c$?

- $E[Y] = E[aX^2 + bX + c] = aE[X^2] + bE[X] + c$, as simple as that! why?
  - $E[aX^2 + bX + c] = \sum_x (ax^2 + bx + c)P(X = x)$
    $= a\sum_x x^2 P(X = x) + b\sum_x xP(X = x) + c\sum_x P(X = x)$
    $= aE[X^2] + bE[X] + c$

- $Y = aX^3 + bX^2 + cX + d$. Can you guess what $E[Y]$ is?

Some simple rules– Expectation

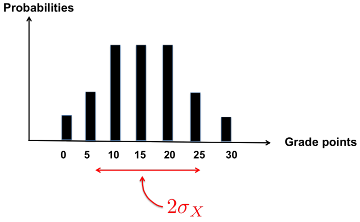
How about $E[Y]$ for $Y = aX^2 + bX + c$?

- $E[Y] = E[aX^2 + bX + c] = aE[X^2] + bE[X] + c$, as simple as that! why?
  - $$E[aX^2 + bX + c] = \sum_x (ax^2 + bx + c)P(X = x)$$
    $$= a\sum_x x^2 P(X = x) + b\sum_x xP(X = x) + c\sum_x P(X = x)$$
    $$= aE[X^2] + bE[X] + c$$
- $Y = aX^3 + bX^2 + cX + d$. Can you guess what $E[Y]$ is?

  $E[Y] = aE[X^3] + bE[X^2] + cE[X] + d$.

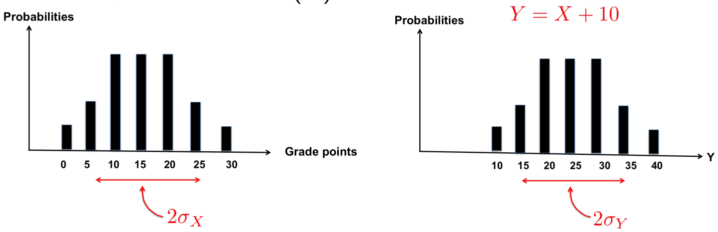
Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?



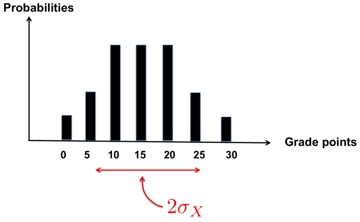
Some simple rules– Variance

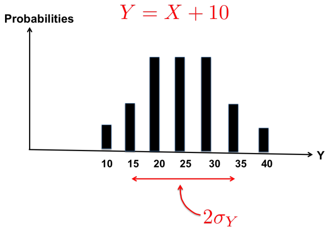
Let $Y = X + b$. What is $\mathrm{var}(Y)$?



Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?





- Intuitively? Well you are just shifting everything by the same number.
- So? the spread of the numbers should stay the same!
- Prove it at home.

Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?

Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?

Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?

- Proof:
- $\mathrm{var}(X + b) = E\left[(X + b)^2\right] - (E[X + b])^2$

Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?

- Proof:
- $$\begin{aligned} \mathrm{var}(X+b) &= E\left[(X+b)^2\right] - (E[X+b])^2 \\ &= E\left[X^2 + 2bX + b^2\right] - (E[X]+b)^2 \end{aligned}$$

Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?

- Proof:
- $$\begin{aligned} \mathrm{var}(X+b) &= E\left[(X+b)^2\right] - (E[X+b])^2 \\ &= E\left[X^2 + 2bX + b^2\right] - (E[X]+b)^2 \\ &= E[X^2] + 2bE[X] + b^2 - ((E[X])^2 + 2bE[X] + b^2) \end{aligned}$$

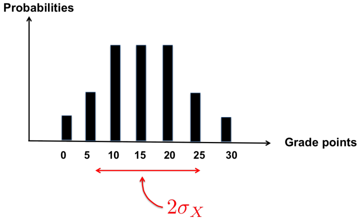
Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?

- Proof:
- $$\begin{aligned}\mathrm{var}(X+b) &= E\left[(X+b)^2\right] - (E[X+b])^2 \\ &= E\left[X^2 + 2bX + b^2\right] - (E[X]+b)^2 \\ &= E[X^2] + 2bE[X] + b^2 - ((E[X])^2 + 2bE[X] + b^2) \\ &= E[X^2] - (E[X])^2 = \mathrm{var}(X)\end{aligned}$$

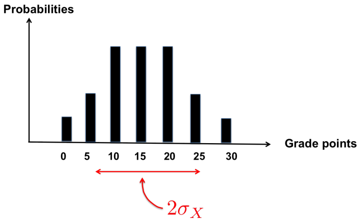
Some simple rules– Variance

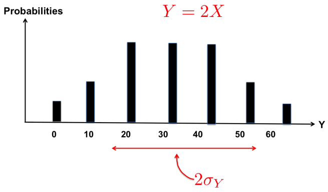
Let $Y = aX$. What is $\mathrm{var}(Y)$?



Some simple rules– Variance

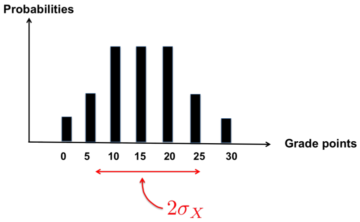
Let $Y = aX$. What is $\mathrm{var}(Y)$?

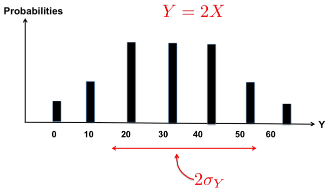




Some simple rules– Variance

Let $Y = aX$. What is $\mathrm{var}(Y)$?





- Intuitively? Well you are just scaling everything by the same number.
- So? the spread should increase if $a > 1$!

Some simple rules– Variance

Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2 \mathrm{var}(X)$.

Some simple rules– Variance

Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2\mathrm{var}(X)$.

- Proof:
-

Some simple rules– Variance

Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2\mathrm{var}(X)$.

- Proof:
- $\mathrm{var}(aX) = E\left[(aX)^2\right] - (E[aX])^2$

Some simple rules– Variance

Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2\mathrm{var}(X)$.

- Proof:
- $$\begin{aligned}\mathrm{var}(aX) &= E\left[(aX)^2\right] - (E[aX])^2\\ &= E\left[a^2X^2\right] - (aE[X])^2\end{aligned}$$

Some simple rules– Variance

Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2\mathrm{var}(X)$.

- Proof:
- $$\begin{aligned}\mathrm{var}(aX) &= E\left[(aX)^2\right] - (E[aX])^2 \\ &= E\left[a^2X^2\right] - (aE[X])^2 \\ &= a^2E\left[X^2\right] - a^2(E[X])^2\end{aligned}$$

Some simple rules– Variance

Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2\mathrm{var}(X)$.

- Proof:
- $$\begin{aligned}\mathrm{var}(aX) &= E\left[(aX)^2\right] - (E[aX])^2 \\ &= E\left[a^2X^2\right] - (aE[X])^2 \\ &= a^2E\left[X^2\right] - a^2(E[X])^2 \\ &= a^2(E[X^2] - (E[X])^2) = a^2\mathrm{var}(X)\end{aligned}$$

Some simple rules– Variance

Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2 \mathrm{var}(X)$.

- Proof:
- $$\begin{aligned} \mathrm{var}(aX) &= E\left[(aX)^2\right] - (E[aX])^2 \\ &= E\left[a^2 X^2\right] - (aE[X])^2 \\ &= a^2 E\left[X^2\right] - a^2 (E[X])^2 \\ &= a^2 (E[X^2] - (E[X])^2) = a^2 \mathrm{var}(X) \end{aligned}$$
- In general we can show that $\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$.

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

- First lets get $E[X^2]$. This is

  $$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

  We see that $E[X^2] = E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values:

.

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

- First lets get $E[X^2]$. This is

$$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

  We see that $E[X^2] = E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values: 0 and 1
  - $P(X^2 = 1)$.

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

- First lets get $E[X^2]$. This is

$$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

  We see that $E[X^2] = E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values: 0 and 1
  - $P(X^2 = 1) = P(X = 1) = p$.

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

- First lets get $E[X^2]$. This is

$$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

  We see that $E[X^2] = E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values: 0 and 1
  - $P(X^2 = 1) = P(X = 1) = p$. $P(X^2 = 0) = P(X = 0) = 1 - p$.

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

- First lets get $E[X^2]$. This is

$$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

  We see that $E[X^2] = E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values: 0 and 1
  - $P(X^2 = 1) = P(X = 1) = p$. $P(X^2 = 0) = P(X = 0) = 1 - p$.
  - X and X^2 have identical PMF's! **They are identically distributed.**

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

- First lets get $E[X^2]$. This is

  $$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

  We see that $E[X^2] = E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values: 0 and 1
  - $P(X^2 = 1) = P(X = 1) = p$. $P(X^2 = 0) = P(X = 0) = 1 - p$.
  - X and X^2 have identical PMF's! **They are identically distributed.**

- $\mathrm{var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1 - p)$.

Mean and Variance of a Binomial

Let $X \sim Bin(n, p)$.

- $E[X] = np$ and $\mathrm{var}(X) = np(1-p)$.
- We will derive these in the next class.

Mean and Variance of a Poisson

X has a Poisson(λ) distribution. What is its mean and variance?

- One can use algebra to show that $E[X] = \lambda$ and also $\mathrm{var}(X) = \lambda$.
- How do you remember this?
- Hint: mean and variance of the Binomial approach that of a Poisson when n is large and p is small, such that $np \approx \lambda$? Anything yet?

Mean and variance of a geometric

- The PMF of a geometric distribution is $P(X = k) = (1 - p)^{k-1} p$.
  - $E[X] = 1/p$
  - $\mathrm{var}(X) = (1 - p)/p^2$
  - We will also prove this later.