THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
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# SDS 384 11: Theoretical Statistics

## Lecture 12: Uniform Law of Large Numbers-Rademacher Complexity

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# Proof of the GC theorem

- We will work on a proof that can handle general function classes $\mathcal{F}$ with bounded functions. WLOG let $|f(X_i)| \leq 1$ for $f \in \mathcal{F}$.
- Recall that we want to bound $\|\hat{P}_n - P\|_{\mathcal{F}}$ $(:= \sup_{f \in \mathcal{F}} |\frac{1}{n} \sum_i f(X_i) - E[f]|)$
- The proof has three components:
  - Concentration inequality to bound $\|\hat{P}_n - P\|_{\mathcal{F}} - E[\|\hat{P}_n - P\|_{\mathcal{F}}]$

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  - Symmetrization to relate $E[\|\hat{P}_n - P\|_{\mathcal{F}}]$ to Rademacher complexity
  - Bound this complexity using the effective "size" of the function class.

# Concentration

- First note that we cannot apply Hoeffding/Chernoff here.
- Let $X := \{X_1, \ldots, X_n\}$
- Let $g(X) = \|\hat{P}_n - P\|_{\mathcal{F}}$. Let $Y$ be another sample $\{Y_1, \ldots, Y_n\}$, where $Y_i = X_i, \forall i \neq 1$.

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- Let $f_1$ maximize $g(X)$, and $f_2$ maximize $g(Y)$
- $$\begin{aligned} g(X) - g(Y) &= \left| \frac{\sum_i f_1(X_i)}{n} - Ef_1[X_1] \right| - \left| \frac{\sum_i f_2(Y_i)}{n} - Ef_2[X_1] \right| \\ &\leq \left| \frac{\sum_i f_1(X_i)}{n} - Ef_1[X_1] \right| - \left| \frac{\sum_i f_1(Y_i)}{n} - Ef_1[X_1] \right| \\ &\leq \frac{2}{n} \end{aligned}$$

# Concentration

- Using McDiarmid's inequality, we get:

$$P\left(g(X) - E[g(X)] \geq \epsilon\right) \leq \exp(-\epsilon^2 n/2)$$

- So, with probability $1 - \exp(-\epsilon^2 n/2)$,

$$\|\hat{P}_n - P\|_{\mathcal{F}} \leq E[\|\hat{P}_n - P\|_{\mathcal{F}}] + \epsilon.$$

- So, we need to bound $E[\|\hat{P}_n - P\|_{\mathcal{F}}]$.

# Symmetrization

- Consider an iid copy of $X'$ of $X$

$$
\begin{aligned}
E\|\hat{P}_n - P\|_{\mathcal{F}} &= E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - E[f(X_i)]) \right| \\
&= E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - E[f(X_i')]) \right| \\
&= E_X \sup_{f \in \mathcal{F}} \left| E_{X'} \left[ \frac{1}{n} \sum_i (f(X_i) - f(X_i')) \right] \right| \\
&\leq E_{X,X'} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - f(X_i')) \right| \\
&= E_{X,X'} \|\hat{P}_n - \hat{P}_n'\|_{\mathcal{F}}
\end{aligned}
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&= E_X \sup_{f\in\mathcal{F}} \left| E_{X'} \left[ \frac{1}{n} \sum_i (f(X_i) - f(X_i')) \right] \right| \\
&\le E_{X,X'} \sup_{f\in\mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - f(X_i')) \right| \\
&= E_{X,X'} \|\hat{P}_n - \hat{P}_n'\|_{\mathcal{F}}
\end{aligned}$$

# Symmetrize again

- Let $\epsilon_i \in \{1, -1\}$.
- Note that $f(X_i) - f(X_i')$ is symmetric
- For a symmetric random variable $R$, and a random variable $\epsilon \in \{-1, 1\}$ (independent of $R$)

$$\begin{aligned} P(\epsilon R \leq t) &= P(R \leq t)P(\epsilon = 1) + P(R \geq -t)P(\epsilon = -1) \\ &= P(R \leq t) \end{aligned}$$

- Hence $\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - f(X_i')) \right|$ and $\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i (f(X_i) - f(X_i')) \right|$ have the same distribution, and expectation
- We will choose $\epsilon_i$'s uniformly, i.e. we will consider Rademacher random variables.

# Rademacher complexity

$$
\begin{aligned}
E\|\hat{P}_n - P\|_{\mathcal{F}} &\leq E_{X,X'} \sup_{f\in\mathcal{F}} \left| \frac{1}{n}\sum_i (f(X_i) - f(X_i')) \right| \\
&= E_{X,X',\epsilon} \sup_{f\in\mathcal{F}} \left| \frac{1}{n}\sum_i \epsilon_i (f(X_i) - f(X_i')) \right| \\
&\leq E_{X,\epsilon} \sup_{f\in\mathcal{F}} \left| \frac{1}{n}\sum_i \epsilon_i f(X_i) \right| + E_{X',\epsilon} \sup_{f\in\mathcal{F}} \left| \frac{1}{n}\sum_i \epsilon_i f(X_i')) \right| \\
&= 2E \sup_{f\in\mathcal{F}} \left| \frac{1}{n}\sum_i \epsilon_i f(X_i) \right| =: 2\mathcal{R}_{\mathcal{F}}
\end{aligned}
$$

- $\mathcal{R}_{\mathcal{F}}$ is also called the Rademacher complexity of the function class.

# Why the Rademacher complexity?

- We have now shown that $\|\hat{P}_n - P\|_{\mathcal{F}} \leq 2\mathcal{R}_{\mathcal{F}} + \epsilon$ with prob. $1 - e^{-n\epsilon^2/2}$.
- $\mathcal{R}_{\mathcal{F}}$ measures the maximum possible correlation (over all $f \in \mathcal{F}$) between the vector $(f(X_1), \ldots, f(X_n))$ and the "noise vector" $(\epsilon_1, \ldots, \epsilon_n)$.
- If a function class has some function which has a high correlation with a random noise vector, then we should not expect concentration.
- If $\mathcal{R}_{\mathcal{F}}$ is $o(1)$ then the Borel Cantelli lemma gives $\|\hat{P}_n - P\|_{\mathcal{F}} \stackrel{a.s.}{\to} 0$.

## Size of a function class $\mathcal{F}$

- Let $\mathcal{F}(X) = \{(f(X_1), \ldots, f(X_n)) : f \in \mathcal{F}\}$
- $\mathcal{R}_{\mathcal{F}} = E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i f(X_i) \right| = E \left[ E \sup_{f \in \mathcal{F}} |\frac{1}{n} \sum_i \epsilon_i f(X_i)| \middle| X_1, \ldots, X_n \right]$
- In the next slide we will bound this using the cardinality of $\mathcal{F}(X)$

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**Theorem**

*Let* $A \subseteq \mathbb{R}^n$, $R = \max_{a \in A} \|a\|$,

$$E \sup_{a \in A} \langle \epsilon, a \rangle \leq \sqrt{2R^2 \log |A|}.$$

*And,*

$$E \sup_{a \in A} |\langle \epsilon, a \rangle| \leq \sqrt{2R^2 \log |2A|}.$$

# Proof

**Proof.**

$$
\begin{aligned}
\exp\left(\lambda E \sup_{a\in A}\langle\epsilon, a\rangle\right) &\le E\exp\left(\lambda \sup_{a\in A}\langle\epsilon, a\rangle\right) \\
&= E \sup_{a\in A} \exp\left(\lambda\langle\epsilon, a\rangle\right) \\
&\le \sum_{a\in A} E\exp\left(\lambda\langle\epsilon, a\rangle\right) \\
(\langle\epsilon, a\rangle \sim \text{Subgaussian}(\|a\|_2^2)) \quad &\le \sum_{a\in A}\exp\left(\frac{\lambda^2\|a\|_2^2}{2}\right) \\
&\le |A|\exp\left(\frac{\lambda^2 R^2}{2}\right)
\end{aligned}
$$

Take $\lambda = 2\log|A|/R^2$. □

# Size of a function class $\mathcal{F}$

- Note that in this case $\mathcal{A}$ contains of vectors $(f(X_1)/n, \ldots, f(X_n)/n)$, where $f$ is a indicator function, i.e. $f(X_i) = 1(X_i \leq t)$.
- So $R^2 = 1/n$.

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- So $R^2 = 1/n$.
- The question is for a given dataset $X_1, \ldots X_n$, how many distinct points are there in $\mathcal{A}$?

$$\begin{aligned}
|\mathcal{A}| = |\mathcal{F}(X)| &= |\{(f(X_{(1)}), \ldots, f(X_{(n)})) : f \in \mathcal{F}\}| \\
&= |\{(1(X_{(1)} \le t), \ldots, 1(X_{(n)} \le t)) : t \in \mathbb{R}\}| \\
&\le n + 1 \qquad \text{(HUH!!)}
\end{aligned}$$

# Glivenko Cantelli

**Proof.**

If $\mathcal{F}$ is the set of one sided indicator functions, then

$$\begin{aligned} \|\hat{P}_n - P\|_{\mathcal{F}} &\leq 2\mathcal{R}_{\mathcal{F}} + \epsilon = 2E[E[\sup_{f\in\mathcal{F}} \sum_i \epsilon_i f(X_i)/n]|X] + \epsilon \\ &\leq \sqrt{8R^2 \log(n+1)} + \epsilon \\ &\leq \sqrt{\frac{8\log(n+1)}{n}} + \epsilon \end{aligned}$$

By Borel Cantelli, $\|\hat{P}_n - P\|_{\mathcal{F}} \overset{a.s.}{\to} 0$ $\square$