Continuous r.v practice problems

SDS 321 Intro to Probability and Statistics

1. (2+2+1+1 = 6 pts) The annual rainfall (in inches) in a certain region is normally distributed with mean 40 and standard deviation 4.

   (a) What is the probability that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches?
   Let R denote the amount of rainfall. $P(R \geq 50) = P(\frac{R-40}{4} \geq 2.5) = 1 - \Phi(2.5) = .006$. Let X be the number of years before we see over 50 inches of rainfall. $P(X \geq 10) = P(\text{None of the first 10 years have more than 50 inches of rain}) = (1 - .006)^{10} = .94$.

   (b) What is the probability that at least 4 out of the next 50 years will have a rainfall of over 50 inches?
   This is a binomial probability with $n = 50, p = .006$ and $np = .3$. May be better to use $X \sim Poisson(3)$ where X denotes number of years with rainfall more than 50 inches.
   $$P(X \geq 4) = 1 - (\sum_{i=0}^{3} P(X = i)) = 1 - .9997 = .0003$$

   (c) What is the expected number of years with over 50 inches of rainfall in the next 50 years?
   Expectation of a Poisson: $50 \times .006 = 0.3$.

   (d) What assumptions are you making?
   The rainfall on each day is independent of each other.

2. (2+1+3+1+3 = 10pts) The joint pdf of two random variables X and Y are given by:
   $$f_{X,Y}(x, y) = \begin{cases} 24xy & x, y \in [0, 1], 0 \leq x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $f_{X,Y}(x, y)$ is a valid joint probability density function.
   $$\int_0^1 \int_0^{1-x} 24xydxdy = 12 \int_0^1 x(1-x)^2dx = 12 \int_0^1 (x - 2x^2 + x^3) = 1$$

(b) Find $f_X(x)$.

$$f_X(x) = \int_y f_{X,Y}(x,y)dxdy = \int_0^{1-x} 24xydy$$
$$= 12x(1-x)^2 \qquad \text{When } x \geq 0 \text{ and zero otherwise.}$$

(c) Find $E[X]$ and $\text{var}(X)$.

$$E[X] = \int_0^1 x f_X(x)dx = 12\int_0^1 x^2(1-x)^2 dx$$
$$= 12\int_0^1 (x^2 - 2x^3 + x^4)dx = 12(1/3 - 1/2 + 1/5) = 2/5$$
$$\text{var}(X) = E[X^2] - E[X]^2 = \int_0^1 x^2 f_X(x)dx - (2/5)^2$$
$$= \int_0^1 (x^3 - 2x^4 + x^5)dx - 4/25 = .66$$

(d) Find $f_Y(y)$.
By symmetry, this is $f_Y(y) = 12y(1-y)^2$ when $y \geq 0$ and zero otherwise.

(e) Find $E(Y)$ and $\text{var}(Y)$.
Since the pdf's are the same, the expectation and variances are too.

3. (2+1+1+2+2=8pts) The random variables X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} cxy(1-x) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c.

$$c\int_0^1\int_0^1 xy(1-x)dxdy = c\int_0^1 x(1-x)dx \int_0^1 ydy = \frac{c}{12} = 1$$
$$c = 12$$

(b) Find $E[X]$.
$f_X(x) = \int_0^1 f_{X,Y}(x,y)dy = 6x(1-x)$. So $E[X] = \int_0^1 6x^2(1-x)dx = .5$

(c) Find $E[Y]$.
$f_Y(y) = \int_0^1 f_{X,Y}(x,y)dx = 2y$. So $E[Y] = \int_0^1 2y^2 dy = 2/3$.

(d) Find $Var(X)$. $\text{var}(X) = E[X^2] - .25 = \int_0^1 6x^3(1-x)dx - .25 = 3/10 - 1/4 = 1/20$

(e) Find $Var(Y)$. $\text{var}(Y) = E[Y^2] - 4/9 = \int_0^1 2y^3 dy - 4/9 = 1/2 - 4/9 = 1/18$

4. The random variables X and Y have a joint density function given by:

$$f(x,y) = \begin{cases} 2e^{-2x}/x & 0 \le x < \infty,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

   (a) What are $f_X(x)$ and $f_Y(y)$?

   (b) What are $E[X]$ and $E[Y]$? We have: $f_X(x) = \int_{y=0}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 2e^{-2x}/x dy = 2e^{-2x}$. So

$$E[X] = \int_0^\infty x2e^{-2x}dx = 1/2$$

   And

$$f_Y(y) = \int_{x=0}^{\infty} f_{X,Y}(x,y)dx = \int_y^\infty 2e^{-2x}/x dx$$

   So,

$$\begin{aligned} E[Y] &= \int_0^\infty y f_Y(y)dy = \int_0^\infty y\Big(\int_y^\infty \frac{2e^{-2x}}{x}dx\Big)dy \\ &= \int_0^\infty \frac{2e^{-2x}}{x}\Big(\int_0^x y dy\Big)dx = \int_0^\infty xe^{-2x}dx = 1/4\int_0^\infty ve^{-v}dv = 1/4 \end{aligned}$$

   In the last step, we changed the order of the integrals. Originally the outer integral was over $0 \le y < \infty$ and inner was over $y \le x < \infty$. But for ease of integration, the outer integral is now over $0 \le x < \infty$ and the inner is over $0 \le y \le x$.

5.

$$f_X(x) = \begin{cases} C(2x - x^3) & 0 < x < 5/2 \\ 0 & x \le 0 \end{cases}$$

   Could f be a probability density function? If so find C. We want $1 = \int_0^{5/2} C(2x-x^3)dx = C(2.5^2 - (2.5)^4/4) = C2.5^2(1 - 25/16) = -3.51C$. Now say we use $C = -1/3.51 = -.28$. If $f(x) > 0$ for all x and integrates to one, then we call it a valid pdf. If you can find a x where it is negative then this is not. $f(1) = C(x^2 - x^4/4) = 3C/4$. If $C < 0$ then this is negative and thats not okay. So no, this cannot be a probability density function.

6. Consider the density function

$$f_X(x) = \begin{cases} C(2 - x) & 0 < x < 2 \\ 0 & x \le 0 \end{cases}$$

   Could f be a probability density function? If so find C. We know that $\int_0^2 C(2-x)dx = 1$ and so $C(4 - (2)^2/2) = 2C = 1$ and so $C = 1/2$. Also $(2 - x)/2 \ge 0$ for $0 < x < 2$. So this is a valid pdf.

   (a) We know that $\int_0^2 C(2 - x)dx = 1$ and so $C(4 - (2)^2/2) = 2C = 1$ and so $C = 1/2$.

   (b) Also $(2 - x)/2 \ge 0$ for $0 < x < 2$.

   (c) So this is a valid pdf.

7. Let U be an uniform $[0,1]$ r.v and let $a < b$ be constants. Show that:

   (a) If $b > 0$ then $bU \sim Uniform([0,b])$.

   (b) $a + U \sim Uniform([a, a+1])$

   (c) What function of U is distributed as $Uniform([a,b])$

   (d) Show that $\min(U, 1-U) \sim Uniform(0, 1/2)$

   (a) Let $X = bU$. Fist note the values $X = bU$ can take. $X \in [0,b]$ When $F_X(x) = P(X \le x) = P(bU \le x)$

$$F_X(x) = P(U \le x/b) = \begin{cases} 0 & \text{For } x < 0 \\ \int_0^{x/b} du = u/b & \text{For } 0 \le x/b \le 1 \\ 1 & \text{For } x/b > 1 \end{cases}$$

   So $f_X(x) = 1/b$ when $x \in [0,b]$ and 0 otherwise. But this is the pdf of a $Uniform([0,b])$.

   (b) First note that $a + U \in [a, a+1]$.
   Now,

$$F_{a+U}(x) = P(a + U \le x) = P(U \le x - a) = (x - a),$$

   when $x \in [a, a+1]$, 0 when $x < a$ and 1 when $x > a$.
   So $f_{a+U}(x) = 1$ when $x \in [a, a+1]$ and 0 otherwise. This is Uniform$[a, a+1]$.

   (c) From the last two exercises we see that adding a constant shifts the Uniform distribution, and multiplying by a constant stretches it. To convert $Uniform([0,1])$ to $Uniform([a,b])$ we need both shifting and stretching. Let $X = \alpha U + \beta \sim Uniform([a,b])$. $X \in [\beta, \alpha + \beta]$. $\beta = a$ and $\alpha + \beta = b$ and so $\alpha = b - a$. So $(b-a)U + a \sim Uniform([a,b])$

   (d) Let $X = \min(U, 1-U)$. First note that X has to lie in $[0, 1/2]$.

$$\begin{aligned} F_X(x) &= P(\min(U, 1-U) \le x) = 1 - P(U \ge x, 1 - U \ge x) \\ &= 1 - P(x \le U \le 1 - x) = 1 - (1 - 2x) = 2x \quad \text{If } x \le 1/2 \text{ and 0 otherwise} \end{aligned}$$

   And $f_X(x) = 2$ if $x \in [0, 1/2]$. This is $Uniform([0, 1/2])$.

8. The joint density of X and Y are given by:

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are X and Y independent?

   (b) What are $f_X(x)$ and $f_Y(y)$?

   (c) What is $P(X + Y \le 2)$?

   (a) Check if the joint factorizes for all x, y? Yes. So independent. Always remember, check if the bounds on x involve y, that can lead to dependence. Here it does not.

   (b) $f_X(x) = \int_{y=0}^{\infty} xe^{-(x+y)} dy = xe^{-x}$

   (c) $f_Y(y) = \int_{x=0}^{\infty} xe^{-(x+y)} dy = e^{-y} \int_0^{\infty} xe^{-x} dx = e^{-y}$.

(d) This is convolution, because the random variables are independent.

$$\begin{aligned}
P(X+Y \leq 2) &= \int_{x=0}^{\infty} \int_{y=0}^{2-x} f_X(x) f_Y(y) dx dy \\
&= \int_{x=0}^{2} x e^{-x} \int_{y=0}^{2-x} e^{-y} dy dx \\
&= \int_{x=0}^{2} x e^{-x} (1 - e^{-(2-x)}) dx = \int_{x=0}^{2} \\
&= x e^{-x} dx - e^{-2} \int_0^2 x dx = (1 - 3/e^2) - 2/e^2 = 1 - 5/e^2
\end{aligned}$$

Note that after you plug in $f_Y(y)$, you basically say that $2 - x \geq 0$ so $x \leq 2$ and that changes the limits on x.

9. The joint density function of X and Y is:

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?

(b) Find $f_X(x)$

(c) Find $P(X+Y < 1)$

(a) No, the joint density does not factorize into a product of functions of x and y.

(b) $f_X(x) = \int_{y=0}^{1} (x+y) dy = x + 1/2$ and $f_Y(y) = y + 1/2$

(c)

$$\begin{aligned}
P(X+Y \leq 1) &= \int_{(x,y): x+y \leq 1} f_{X,Y}(x,Y) dx dy \\
&= \int_{x=0}^{1} \int_{y=0}^{1-x} (x+y) dx dy \\
&= \int_0^1 (x(1-x) + (1-x)^2/2) dx \\
&= \int_0^1 (x - x^2 + 1/2 - x + x^2/2) \\
&= \int_0^1 (1/2 - x^2/2) = 1/2 - 1/6 = 1/3
\end{aligned}$$

10. The running time in seconds of an algorithm on a medium sized data set is approximately normally distributed with mean 30 and variance 25.

(a) What is the probability that the running time of a run selected at random will exceed 25 seconds?

(b) What is the probability that the running time of at least one of four randomly selected runs will exceed 25 seconds?

(c) What is the probability that the running time of all runs will exceed 25 seconds given at least one of four randomly selected runs will exceed 25 seconds?

(a) 0.8413

(b) $1 - (1 - 0.8413)^4$

(c) $.8413^4/(1 - (1 - .8413)^4)$

11. (2+2+1+1 + 1+1= 8pts) The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c & 0 < x < y, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is c?

(b) What is $f_Y(y)$?

(c) What is $E[Y]$?

(d) What is the conditional pdf $f_{X|Y}(x|y)$?

(e) Are X and Y independent?

(f) What is $E[Y|X = x]$?

(g) Calculate $E[Y]$ using the total expectation theorem. Does this match your answer from part (c)?

(a) The pdf has to normalize to one.

$$\int_{xy} f_{X,Y}(x,y)dxdy = \int_{x=0}^{1}\int_{y=x}^{1} cdxdy = c\int_{x=0}^{1}(1-x)dx = c/2 = 1$$

$$c = 2$$

(b)

$$f_Y(y) = \begin{cases} \int_x f_{X,Y}(x,y)dx = \int_{x=0}^{y} 2dx = 2y & \text{When } 0 < y < 1 \\ 0 & \text{Otherwise} \end{cases}$$

(c)

$$E[Y] = \int_0^1 y f_Y(y)dy = \int_0^1 2y^2 dy = 2/3$$

(d)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y} \qquad \text{When } 0 < x < y, 0 < y < 1 \text{ and } 0 \text{ otherwise.}$$

(e)

$$f_X(x) = \begin{cases} \int_y f_{X,Y}(x,y)dy = \int_{y=x}^{1} 2dy = 2(1-x) & \text{When } 0 < x < 1. \\ 0 & \text{otherwise} \end{cases}$$

No, since $0 < x < y < 1$, there are values such as $x = .5$, $y = .1$, where the pdf is zero but $f_X(x)$ and $f_Y(y)$ is nonzero.

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990