

Homework Assignment 3

Due March 19th by noon.

SDS 384-11 Theoretical Statistics

1. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sequence of Bernoulli variables with parameter $\alpha \in (0, 1/2]$, and consider the binomial random variable $Z_n = \sum_i X_i$. We want to prove for any $\delta \in (0, \alpha)$,

$$P(Z_n \leq \delta n) \leq \exp(-nKL(\delta||\alpha)) \quad KL(\delta||\alpha) := \delta \log \frac{\delta}{\alpha} + (1 - \delta) \log \frac{1 - \delta}{1 - \alpha}$$

where $KL(p, q)$ is the Kullback-Leibler divergence between two bernoullis with parameters p, q respectively. Show that the above is strictly better than Hoeffding's inequality.

2. Now we will prove a lower bound on the binomial tail to show that indeed what you derived in the last question is sharp upto polynomial factors. Define $m = \lfloor n\delta \rfloor$ and $\delta' = \frac{m}{n}$.

- (a) Prove $\frac{1}{n} \log P(Z_n \leq \delta n) \geq \frac{1}{n} \log \binom{n}{m} + \delta' \log \alpha + (1 - \delta') \log(1 - \alpha)$.
- (b) Show that

$$\frac{1}{n} \log \binom{n}{m} \geq -\delta' \log \delta' - (1 - \delta') \log(1 - \delta') - \frac{\log(n+1)}{n}$$

Hint: Use the fact that for $Y \sim \text{Bin}(n, m/n)$ $P(Y = k)$ is maximized at $k = m$.

- (c) Now show that

$$P(Z_n \leq \delta n) \geq \frac{1}{n+1} \exp(-nKL(\delta'||\alpha))$$

3. We will use the Efron Stein inequality to obtain bounds of variances for separately convex functions whose partial derivatives exist. A separately convex function $f(x_1, \dots, x_n)$ is a convex function of its i^{th} variable, when all else are held fixed.

- (a) Let X_1, \dots, X_n be independent random variables taking values in the interval $[0, 1]$ and let $f : [0, 1]^n \rightarrow \mathbb{R}$ be a separately convex function whose partial derivatives exist. Then $f(X) := f(X_1, \dots, X_n)$ satisfies

$$\text{var}(f(X)) \leq E[\|\nabla f(X)\|^2]$$

Hint: Recall that $\text{var}(Z) \leq \sum_i (Z - E_i Z)^2 \leq \sum_i (Z - Z_i)^2$. Define $Z_i = \inf_x f(X_{1:i-1}, x, X_{i+1:n})$ and then use convexity of f .

- (b) Let A be a $m \times n$ random matrix with independent entries $A_{ij} \in [0, 1]$. Let

$$Z = \sqrt{\lambda_1(A^T A)} = \sqrt{\sup_{u \in \mathbb{R}^n: \|u\|=1} u^T A^T A u} = \sup_{u \in \mathbb{R}^n: \|u\|=1} \|Au\|$$

Show that $\text{var}(Z) \leq 1$.

4. In this question we will look at the Gaussian Lipschitz theorem. Consider $X_1, \dots, X_n \stackrel{iid}{\sim} N(0, 1)$

(a) Prove that the order statistics are 1-Lipschitz. of n iid $N(0, 1)$ random variables.

(b) Now show that

$$c\sqrt{\log n} \leq E[\max_i X_i] \leq \sqrt{2 \log n}$$

where c is some universal constant.