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SDS 384 11: Theoretical Statistics

Lecture 10: U Statistics cont.

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U Statistics

- We will see many interesting examples of U statistics.
- Interesting properties
  - Unbiased (done)
  - Reduces variance (done)
  - Concentration (via McDiarmid) (done)
  - Asymptotic variance (done)
  - Asymptotic distribution (today)

Normal Convergence of U statistics-proof

- Trick: find some $\hat{U}$ such that $\hat{U}$ is asymptotically equivalent to U.
- Make sure $\hat{U}$ is easy to analyze.

Theorem

If $X_n \xrightarrow{d} X$ and $|Y_n - X_n| \xrightarrow{P} 0$, then $Y_n \xrightarrow{d} X$.

- In our case we will use $\hat{U}$ as a sum of functions of X_i
- Then use CLT on $\hat{U}$
- We will find the functions using Hajék projections.

Hajék Projections – Setup

- Let $\{X_1, \dots, X_n\}$ be independent random vectors.
- Consider a linear space $\mathcal{S}$ of random variables.
    - E.g. $\mathcal{S}$ can be the set of all random variables of the form

$$\sum_{i=1}^{n} g_i(X_i)$$

    - g_i are arbitrary measurable functions $g_i : \mathbb{R}^d \to \mathbb{R}$ with $E[g_i(X_i)^2] < \infty$, for $i \in [n]$
- $ES^2 < \infty, \forall S \in \mathcal{S}$
- Consider a random variable T with $E[T^2] < \infty$

Hájek projections

- Define by the projection $\hat{S} = \arg\inf_{S \in \mathcal{S}} E[(T - S)^2]$

Theorem

$\hat{S}$ is a projection of T onto a linear space $\mathcal{S}$ with finite second moments, iff, $\hat{S} \in \mathcal{S}$ and

$$E[(T - \hat{S})S] = 0, \qquad \textit{For every } S \in \mathcal{S}. \qquad \textit{Orthogonality}$$

Every two projections of T onto $\mathcal{S}$ are equal a.s. If $\mathcal{S}$ contains the constant variables, then $E[T] = E[\hat{S}]$ and cov$(T - \hat{S}, S) = 0$ for every $S \in \hat{S}$.

Hájek projections

Proof.

- First note that

$$E(T-S)^2 = E[(T-\hat{S})^2] + 2E[(T-\hat{S})(\hat{S}-S)] + E[(S-\hat{S})^2]$$

- If the orthogonality condition is satisfied, then the middle term is zero.
- So $E(T-S)^2 \geq E(T-\hat{S})^2$, and this inequality is strict unless $E(\hat{S}-S)^2 = 0$. This proves uniqueness.

$\square$

Hájek projections-converse

Proof.

- For any number α

$$E(T - \hat{S} - \alpha S)^2 = E[(T - \hat{S})^2] - 2\alpha E[(T - \hat{S})S] + \alpha^2 E[S^2]$$

- If $\hat{S}$ is the projection, then $\forall\alpha$ and $\forall S \in \mathcal{S}$,

$$\alpha^2 E[S^2] - 2\alpha E[(T - \hat{S})S] \geq 0$$

- So for $\alpha > 0$, $E[(T - \hat{S})S] \leq \alpha E[S^2]/2$
- for $\alpha < 0$, $E[(T - \hat{S})S] \geq -|\alpha| E[S^2]/2$
- So the orthogonality condition must hold.

$\square$

Hájek projections-proos cont.

- If constants are in $\mathcal{S}$, then the orthogonality condition with $S = 1$ gives $E[T] = E[\hat{S}]$.
- So, $\text{cov}(T - \hat{S}, S) = E[(T - \hat{S})S] - E[T - \hat{S}]E[S] = 0$
- The first term is zero using orthogonality.
- The second term is zero because $E[T] = E[\hat{S}]$.
- Hájek projections do not always exist, i.e. the $\inf\limits_{S \in \mathcal{S}}$ may not be achievable.
- However it is typically easy to establish existence directly

Projections and asymptotic equivalence

- By the orthogonality, we have $E[T^2] = E[(T - \hat{S})^2] + E[\hat{S}^2]$
- If $\mathcal{S}$ contains constants, then $E[T] = E[\hat{S}]$
- So $\text{var}(T) = \text{var}(T - \hat{S}) + \text{var}(\hat{S})$
- So if $\mathcal{S}$ has constants, and $\text{var}(T) = \text{var}(\hat{S})$, then $\hat{S} = T$ a.s.
- What if the variances are not equal, but almost (or asymptotically) equal?

Projections and asymptotic equivalence

Theorem

Consider linear spaces of random variables with finite second moment $\mathcal{S}_n$ that contains constants. Let T_n be random variables with projections $\hat{S}_n$ onto $\mathcal{S}_n$. If $var(T_n)/var(S_n) \to 1$, then,

$$\frac{T_n - E[T_n]}{sd(T_n)} - \frac{\hat{S}_n - E[\hat{S}_n]}{sd(\hat{S}_n)} \xrightarrow{P} 0,$$

where $sd(X)$ is $\sqrt{var(X)}$.

Projections and asymptotic equivalence-proof

Proof.

- We will prove convergence in second mean.
- Let $D_n = \dfrac{T_n - E[T_n]}{\text{sd}(T_n)} - \dfrac{\hat{S}_n - E[\hat{S}_n]}{\text{sd}(\hat{S}_n)}$
- $E[D_n] = 0$
- So the variance calculation gives:

$$
\begin{aligned}
\text{var}(D_n) &= 2 - 2\frac{\text{cov}(T_n, \hat{S}_n)}{\text{sd}(T_n)\text{sd}(\hat{S}_n)} \\
&= 2 - 2\frac{\text{cov}(T_n - \hat{S}_n, \hat{S}_n) + \text{var}(\hat{S}_n)}{\text{sd}(T_n)\text{sd}(\hat{S}_n)} \\
&= 2 - 2\frac{\text{var}(\hat{S}_n)}{\text{sd}(T_n)\text{sd}(\hat{S}_n)} \to 0
\end{aligned}
$$

$\square$

How to get a Hájek projection

- Let $\{X_1, \ldots, X_n\}$ be independent random vectors.
- Consider a linear space $\mathcal{S}$ of random variables.
    - E.g. $\mathcal{S}$ can be the set of all random variables of the form $\sum_{i=1}^{n} g_i(X_i)$.
    - g_i are arbitrary measurable functions $g_i : \mathbb{R}^d \to \mathbb{R}$ with $E[g_i(X_i)^2] < \infty$, for $i \in [n]$

Theorem

The Hájek projection of an arbitrary random variable $T(X_1, \ldots, X_n)$ with finite second moment onto $\mathcal{S}$ is given by

$$\hat{S} = \sum_{i=1}^{n} E[T|X_i] - (n-1)E[T].$$

How to get a Hájek projection

Proof.

- First note that $\hat{S} \in \mathcal{S}$
- All that remains is to check the orthogonality condition.
-

$$\begin{aligned} E[(T-\hat{S})S] &= E[(T-\hat{S})\sum_i g_i(X_i)] \\ &= \sum_i E[(T-\hat{S})g_i(X_i)] \\ &= \sum_i E_{X_i} E[(T-\hat{S})g_i(X_i)|X_i] \\ &= \sum_i E g_i(X_i) E[T-\hat{S}|X_i] \end{aligned}$$

- But $E[\hat{S}|X_i] = E[\sum_j E[T|X_j]|X_i] - (n-1)E[T] = E[T|X_i]$. □

What if X_i's are iid?

- If $X_1, \ldots, X_n$ are iid,
- So in this case,
  $$\begin{aligned} E[T|X_i = x] &= E[T(X_1, \ldots, X_{i-1}, x, X_i, \ldots)] \\ &= E[T(x, X_2, \ldots, X_n)] \end{aligned}$$
- Thus the Hájek projections can be computed by taking a projection on a smaller set $\mathcal{S}' \subset \mathcal{S}$
- $\mathcal{S}'$ contains random variables of the form $\sum_{i=1}^{n} g(X_i)$ where g is some arbitrary measurable function with $E[g(X_i)^2] < \infty$

Normal Convergence of U statistics-proof

- Recall $U := \frac{1}{\binom{n}{r}} \sum_{S \in \mathcal{I}_r} h(X_S)$
- Define the Hájek projection as

$$\begin{aligned}\hat{U} &:= \sum_{i=1}^{n} E[U - \theta | X_i] \\ &= \frac{1}{\binom{n}{r}} \sum_{i=1}^{n} \sum_{S \in \mathcal{I}_r} E[h(X_S) - \theta | X_i]\end{aligned}$$

- Note that

$$E[h(X_S) - \theta | X_i = x] = \begin{cases} E[h(x, X_2, \ldots, X_r)] - \theta =: g(x) & \text{When } i \in S \\ 0 & \text{o.w.} \end{cases}$$

Normal Convergence of U statistics-proof

- Define the Hájek projection as

$$
\begin{aligned}
\hat{U} &:= \sum_{i=1}^{n} E[U - \theta | X_i] \\
&= \frac{1}{\binom{n}{r}} \sum_{i=1}^{n} \sum_{S \in \mathcal{I}_r} E[h(X_S) - \theta | X_i] \\
&= \frac{1}{\binom{n}{r}} \sum_{i=1}^{n} \sum_{S \in \mathcal{I}_r : X_i \in S} E[h(X_S) - \theta | X_i] \\
&= \frac{1}{\binom{n}{r}} \sum_{i=1}^{n} \binom{n-1}{r-1} g(X_i) \\
&= \frac{r}{n} \sum_{i=1}^{n} g(X_i)
\end{aligned}
$$

Normal Convergence of U statistics-proof

- Ok. So we got a projection. Now we need to move to asymptotics
- So let us calculate the variance of $\hat{U}$

$$\begin{aligned} \mathrm{var}(\hat{U}) &= \frac{r^2}{n}\mathrm{var}(g(X_1)) \\ &= \frac{r^2}{n}\mathrm{var}(E[h(X_S)|X_1]) = \frac{r^2}{n}\xi_1 \end{aligned}$$

- Now CLT gives, $\sqrt{n}(\hat{U} - \theta) \xrightarrow{d} N(0, r^2\xi_1)$

Normal Convergence of U statistics-proof

- $\sqrt{n}\hat{U} \xrightarrow{d} N(0, r^2\xi_1)$
- We already proves $\dfrac{\text{var}(U)}{\text{var}(\hat{U})} \to 1$
- So $\sqrt{n}(\hat{U} - (U - \theta)) \xrightarrow{P} 0$
- So $\sqrt{n}(U - \theta) \xrightarrow{d} N(0, r^2\xi_1)$