THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 6: Conditional Independence and counting introduction

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Conditional Independence

- Recall, we said two events A and B were independent if
  - $P(A|B) = P(A)$ - knowing B tells us nothing about the probability of A.
  - This means that $P(A \cap B) = P(A)P(B)$.

- We can extend this definition to conditional probabilities. We say two events A and B are *conditionally* independent *given* some event C if
  - $P(A|B \cap C) = P(A|C)$.
  - We write this as $A \perp\!\!\!\perp B|C$.
  - Like before, this boils down to: $P(A \cap B|C) = P(A|C)P(B|C)$
  - Can you prove it?

Conditional Independence: Urn example

- Consider two urns, each containing 100 balls.
- The first urn contains all red balls.
- The second urn contains all blue balls.
- We select an urn at random. Let A be the event that the first urn is chosen.
- We select a ball from the urn, note it's color, and put it back. We then select another ball from the urn, note it's color, and put it back.
- Let A be the event that the first urn was chosen, let R_1 be the event that the first ball was red, and let R_2 be the event that the second ball was red.
- *Are R_1 and R_2 independent?*

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Think about it intuitively first.

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- Then you know for sure that the first urn is picked.
- So the second ball has to be red as well.
- Knowing about the first ball tells you a lot about the color of the second ball.
- Clearly they are not independent!

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  - Let's compare: $P(R_1)P(R_2) = 0.5 \times 0.5 = 0.25$
  - So, $P(R_1 \cap R_2) \neq P(R_1)P(R_2)$ – i.e. $R_1 \not\perp R_2$.

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- Our gut says yes... let's just double check the math.

$$P(R_1|A) = P(R_2|A) = 1$$

$$P(R_1 \cap R_2|A) = 1 \times 1 = 1 = P(R_1|A)P(R_2|A)$$

- So, $R_1 \perp\!\!\!\perp R_2|A$
- Knowing which urn was used tells us something about how likely it is that they are both red!
- **Conditional independence does not imply independence!**

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Summing up

So far we have done–

- Sets, sample spaces
- Axioms of probability
- Conditional probability and Bayes rule
- Independence and conditional independence
- Today we will start counting. Reading-Ross chapter 1.

Ordered Samples – Sampling with replacement

There are total n different elements in a *population* or set. You want to create an (ordered) arrangement of r elements.

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- Total number of arrangements is n^r.

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- The first element can be any of n elements.
- The second can be any of the remaining $n-1$ elements.
- The third can be any of the $n-2$ elements.
- So the total number of arrangements is $n(n-1)(n-2)\ldots(n-r+1)$. This is also denoted by $(n)_r$ (or $P(n,r)$) and called n permute r.

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- Total $n \times n \times \cdots \times n$ repeated r times. So n^r different ways.
- Sometimes you will hear words like distinguishable or indistinguishable balls/bins.
  - Distinguishable basically means that each have a unique identifier/color/number on them. Here often the ordering matters.
  - In this case the balls and bins are both distinguishable.

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- How many three digit numbers are there with digits from $\{1, 2, 3\}$ and no repeated digit?
- 123,132,213,231,312,321
- $3! = 3 \times 2 = 6$.

Practice problem

Find out the number of ways:

1. 3 boys and 3 girls can sit in a row?
2. 3 boys and 3 girls can sit in a row if the boys and girls are each to sit together?
3. 3 boys and 3 girls can sit in a row if only the boys must sit together?
4. 3 boys and 3 girls can sit in a row if no two people of the same sex are allowed to sit together?

Combinations

So far we have been talking only about ordered samples. For many applications we only need an unordered sample. In particular we want to figure out how many ways one can form groups of size r from a set of size n.

- Q: How many groups of size r are there of a given population of size n?

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- How many ways can I choose two digits without replacement from $\{1, 2, 3\}$?
- 12,23,13. So $\binom{3}{2}=3$.

Subpopulations and partitions

- Earlier we learned about n permute r. This is how we choose r elements without replacement, but the **order matters**.
- Let consider all ordered samples of size 2 picked without replacement from 3 numbers.
- Now we want the same. But order does not matter. So $\{1, 2\}$ and $\{2, 1\}$ are the same.

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- $\underbrace{(1, 2), (2, 1)}_{2 times}, \underbrace{(1, 3), (3, 1)}_{2 times}, \underbrace{(2, 3), (3, 2)}_{2 times}$
- Say we are picking 3 out of 4 numbers. Consider all ordered arrangements.
  $\underbrace{123, 132, 231, 213, 312, 321}_{\text{(1,2,3) appears 6 times}}, \underbrace{143, 134, 431, 413, 314, 341}_{\text{(1,3,4) appears 6 times}}, \ldots$