

# **SDS 384 11: Theoretical Statistics**

## **Lecture 14: Uniform Law of Large Numbers- Covering number**

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# Definitions

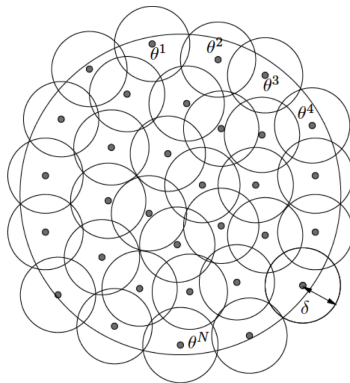
- Recall that a metric space  $(\mathcal{T}, \rho)$  consists of a nonempty set  $\mathcal{T}$  and a mapping  $\rho : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}$  that satisfies:
  - Non-negative:  $\rho(\theta, \theta') \geq 0$  for all  $(\theta, \theta')$  with equality iff  $\theta = \theta'$ .
  - Symmetric:  $\rho(\theta, \theta') = \rho(\theta', \theta)$  for all pairs  $(\theta', \theta)$ , and
  - Triangle ineq holds:  $\rho(\theta, \theta') + \rho(\theta', \theta'') \geq \rho(\theta, \theta'')$
- Examples:
  - $\mathcal{T} = \mathbb{R}^d$ ,  $\rho(\theta, \theta') = \|\theta - \theta'\|_2$
  - $\mathcal{T} = \{0, 1\}^d$  with  $\rho(\theta, \theta') = \frac{1}{d} \sum_i 1(\theta_i \neq \theta'_i)$

# Covering numbers

## Definition

A  $\delta$  cover of a set  $\mathcal{T}$  w.r.t to a metric  $\rho$  is a set  $\{\theta^1, \dots, \theta^N\}$  such that for every  $\theta \in \mathcal{T}$ ,  $\exists i \in [N]$ , s.t.  $\rho(\theta, \theta^i) \leq \delta$ . The  $\delta$  covering number  $N(\delta; \mathcal{T}, \rho)$  is the cardinality of the smallest  $\delta$  cover.

- We will consider metric spaces which are totally bounded, i.e.  $N(\delta; \mathcal{T}, \rho) < \infty$  for all  $\delta > 0$ .
- The covering number is non-increasing in  $\delta$ , i.e.  $N(\delta) \geq N(\delta')$  for all  $\delta < \delta'$
- We are interested in something called Metric entropy, which is the logarithm of the covering number.



**Figure 1:** [courtesy: Martin Wainwright's book]

- A  $\delta$  covering can be thought of as a union of balls with radius  $\delta$ .

# Covering number of a unit cube

## Example

Consider the interval  $[-1, 1]$  with  $\rho(\theta, \theta') = |\theta - \theta'|$ . We have  
$$N(\delta; [-1, 1], |\cdot|) \leq \frac{1}{\delta} + 1$$

- Divide the interval into  $L$  sub-intervals centered at  $\theta^i := -1 + (2i - 1)\delta$  for  $i \in [L]$  and each of length at most  $2\delta$ .
- By construction this is a  $\delta$  covering.
- So  $L \leq 1 + 1/\delta$

# Covering the binary hypercube

## Example

Consider a  $d$  dimensional binary hypercube  $\mathcal{T} = \{0,1\}^d$  with the Hamming metric defined before.

$$\frac{\log N(\delta; \mathcal{T}, \rho)}{\log 2} \leq \lceil d(1 - \delta) \rceil$$

- Let  $S = \{1, 2, \dots, \lceil \delta d \rceil\}$
- Consider the set of binary vectors  $\mathcal{S}(\delta) := \{\theta \in \mathcal{T} : \theta_j = 0\}$ .
- By construction, for every binary vector  $\theta' \in \mathcal{T}$ , we can find a vector  $\theta \in \mathcal{S}(\delta)$  such that  $\rho(\theta, \theta') \leq \delta$
- $N(\delta; \mathcal{T}, \rho) \leq |\mathcal{S}(\delta)| = 2^{\lceil d(1-\delta) \rceil}$

## Lower bound on Covering number of the binary hypercube

- Let  $\delta \in (0, 1/2)$
- If  $\{\theta^1, \dots, \theta^N\}$  is a  $\delta$  covering, then the (unrescaled) Hamming balls of radius  $s = \delta d$  around each  $\theta^\ell$  must contain all  $2^d$  vectors.
- Let  $s = \lfloor \delta d \rfloor$
- For each  $\theta^i$  there are exactly  $\sum_{j=0}^d \binom{d}{j}$  vectors within  $\delta d$  distance.
- So  $N \sum_{j=0}^d \binom{d}{j} \geq 2^d$

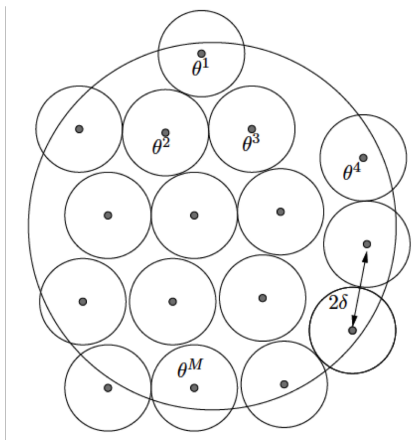
## Lower bound on Covering number of the binary hypercube

- Let  $\delta \in (0, 1/2)$
- So  $N \sum_{j=0}^s \binom{d}{j} \geq 2^d$
- Now take a Binomial  $(d, 1/2)$  random variable  $X$ .
- $P(X \leq \delta d) = \sum_{j=0}^s \binom{d}{j} / 2^d$
- So  $N \geq \frac{1}{P(X \leq \delta d)}$
- Using the Hoeffding bound gives:  $N \geq \exp(\frac{d}{2}(1/2 - \delta)^2)$
- Using the refined version in your homework gives:  
 $N \geq \exp(dKL(\delta||1/2))$



## Definition

An  $\delta$ -packing of  $\mathcal{T}$  w.r.t a metric  $\rho$  is a set  $\{\theta^1, \dots, \theta^M\}$  such that  $\rho(\theta^i, \theta^j) > \delta$  for every distinct pair  $i, j \in [M]$ . The  $\delta$  packing number  $M(\delta; \mathcal{T}, \rho)$  is the cardinality of the largest  $\delta$  packing.



**Figure 2:** [courtesy: Martin Wainwright's book]

- A  $\delta$  covering can be thought of as a union of balls with radius  $\delta$  such that no two balls touch.

# Relationship between packing and covering numbers

## Theorem

For all  $\delta > 0$ ,

$$M(2\delta; \mathcal{T}, \rho) \leq N(\delta; \mathcal{T}, \rho) \leq M(\delta; \mathcal{T}, \rho)$$

- This is saying that packing and covering numbers exhibit the same scaling behavior as  $\delta \rightarrow 0$ .

- For each element of  $N(\delta; \mathcal{T}; \rho)$ , there can be only one or less element of  $M(2\delta; \mathcal{T}; \rho)$ . Because otherwise, two elements of the  $2\delta$  packing will be within  $2\delta$  of each other via triangle inequality.
- Consider a  $\delta$  packing of  $\mathcal{T}$ . Since it is maximal, there are no more points in  $\mathcal{T}$  which can be added without falling within  $\delta$  distance of one of the elements. Hence this is also an epsilon cover. Hence the last inequality.

# Covering and Packing numbers-example

## Theorem

Let  $\rho$  be the Euclidean norm on  $\mathbb{R}^d$ . Let  $B_1(0)$  be the unit ball centered at the origin (WLOG).

$$\frac{1}{\epsilon^d} \leq N(\epsilon, B_1, \rho) \leq (1 + 2/\epsilon)^d$$

- Consider an  $\epsilon$  cover  $\{\theta^1, \dots, \theta^N\}$ . Now,

$$B_1 \subseteq \bigcup_{i=1}^N B_\epsilon(\theta^i)$$

$$\text{vol}(B_1) \leq N \text{vol}(B_\epsilon(\theta^i)) = N \epsilon^d \text{vol}(B_1)$$

$$N \geq 1/\epsilon^d$$

## Proof-upper bound

- Consider a  $\epsilon$  packing  $\{\theta^1, \dots, \theta^M\}$
- This is an union of disjoint balls of radius  $\epsilon/2$

$$\bigcup_i B_{\epsilon/2}(\theta^i) \subseteq B_{1+\epsilon/2}$$

$$M \text{vol}(B_{\epsilon/2}(\theta^i)) \leq (1 + \epsilon/2) \text{vol}(B_{1+\epsilon/2})$$

$$M(\epsilon/2)^d \text{vol}(B_1) \leq (1 + \epsilon/2)^d \text{vol}(B_1)$$

$$M \leq (1 + 2/\epsilon)^d$$

## Example-smoothly parametrized problems

- Consider the following function class parametrized by  $\theta \in \Theta$ .

$$\mathcal{F} := \{f_\theta(\cdot) : \theta \in \Theta\}$$

- Let  $\|\cdot\|_\Theta$  be the norm for  $\theta$  and  $\|\cdot\|_{\mathcal{F}}$  be the norm for  $\mathcal{F}$ .
- Say  $\|f_\theta(\cdot) - f_{\theta'}(\cdot)\|_{\mathcal{F}} \leq L\|\theta - \theta'\|_\Theta$
- Then  $N(\epsilon; \mathcal{F}, \|\cdot\|_{\mathcal{F}}) \leq N(\epsilon/L; \Theta, \|\cdot\|_\Theta)$

## Example-smoothly parametrized problems

- A Lipschitz parametrization allows us to go from cover of the  $\Theta$  space to cover of the  $f_\theta$  space with a loss of  $L$ .
- If  $\mathcal{F}$  is parametrized by a compact set of  $d$  parameters then  $N(\epsilon, \mathcal{F}) = O(1/\epsilon^d)$



## Example-Lipschitz functions on the unit interval

### Example

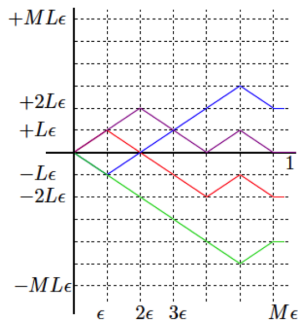
$$\mathcal{F}_L = \{g : [0, 1] \rightarrow \mathbb{R} \mid g(0) = 0, |g(x) - g(y)| \leq L|x - y|, \forall x, y \in [0, 1]\}$$

Metric entropy scales as  $\log N(\delta; \mathcal{F}_L) \asymp L/\delta$  for small enough  $\delta > 0$ .

- Its sufficient to consider a sufficiently large packing of  $\mathcal{F}_L$
- For a given  $\epsilon$  define  $M = \lfloor \frac{1}{\epsilon} \rfloor$
- Let  $x_i = (i - 1)\epsilon$  for  $i = 1, \dots, M + 1$
- 

$$\phi(x) := \begin{cases} 0 & x < 0 \\ x & x \in [0, 1] \\ 1 & x > 1 \end{cases} \quad (1)$$

- Define  $f(\beta) = \sum_{i=1} \beta_i L_{\epsilon} \phi\left(\frac{y - x_i}{\epsilon}\right)$  for  $\beta \in \{-1, 1\}^M$



**Figure 5-2.** The function class  $\{f_\beta, \beta \in \{-1, +1\}^M\}$  used to construct a packing of the Lipschitz class  $\mathcal{F}_L$ . Each function is piecewise linear over the intervals  $[0, \epsilon]$ ,  $[\epsilon, 2\epsilon]$ ,  $\dots$ ,  $[(M-1)\epsilon, M\epsilon]$  with slope either  $+L$  or  $-L$ . There are  $2^M$  functions in total, where  $M = \lceil 1/\epsilon \rceil$ .

- For any pair  $\beta \neq \beta' \in \{-1, 1\}^M$  there is at least one interval where they have the same starting point.
- So  $\|f(\beta) - f(\beta')\|_\infty \geq 2L\epsilon$
- $f_\beta \in \mathcal{F}_L$  for all  $\beta \in \{-1, 1\}^M$
- So  $f_\beta$  forms a  $2L\lfloor 1/\epsilon \rfloor$  packing.
- Making  $\epsilon L = \delta$  we see

$$N(\delta; \mathcal{F}_L, \|\cdot\|_\infty) \geq M(2L\epsilon; \mathcal{F}_L, \|\cdot\|_\infty) = 2^{\lfloor \frac{1}{\epsilon} \rfloor} = 2^{\lfloor \frac{L}{\delta} \rfloor}$$

- Also the set  $f_\beta$  also form a suitable covering of the original functions, and this gives the upper bound.

# Acknowledgment

This lecture was very much based on Martin Wainwright's unpublished book and Peter Bartlett's notes.

