Solution to Assignment 1

Due via canvas Feb 17th

Thanks to Brandon Carter

1. (4 pts) Consider a sequence of iid random variables $\{X_n\}$ such that $X_i \sim Beta(\theta, 1)$, where $\theta > 0$. Let $\bar{X}_n$ denote the sample mean. The method of moments estimator of θ is $\hat{\theta}_n = \bar{X}_n/(1 - \bar{X}_n)$. Derive the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$.

   Solution.

$$
\begin{aligned}
\sqrt{n}(\hat{\theta} - \theta) &= \sqrt{n}\left(\frac{\bar{X}_n}{1 - \bar{X}_n} - \theta\right) \\
&= \sqrt{n}\left(\frac{\bar{X}_n - (1 - \bar{X}_n)\theta}{1 - \bar{X}_n}\right) \\
&= \frac{1}{1 - \bar{X}_n}\sqrt{n}\left(\bar{X}_n(\theta + 1) - \theta\right) \\
&= \frac{\theta + 1}{1 - \bar{X}_n}\sqrt{n}\left(\bar{X}_n - \frac{\theta}{\theta + 1}\right)
\end{aligned}
$$

   The mean of a $Beta(\theta, 1)$ is $\frac{\theta}{\theta+1}$, from the weak law of large numbers (WLLN) we know that $\bar{X}_n \xrightarrow{p} \frac{\theta}{\theta+1}$.

   By the CLT we have $\sqrt{n}\left(\bar{X}_n - \frac{\theta}{\theta+1}\right) \xrightarrow{d} N(0, \sigma^2)$, where $\sigma^2 = \frac{\theta}{(\theta+1)^2(\theta+2)}$.

   Using the continuous mapping theorem we can show $\frac{\theta+1}{1-\bar{X}_n} \xrightarrow{p} \frac{\theta+1}{1-\frac{\theta}{\theta+1}} = (\theta + 1)^2$. This is a constant, which allows us to use Slutsky's theorem and conclude:

$$
\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \sigma^2(\theta + 1)^4) = N\left(0, \frac{\theta(\theta + 1)^2}{\theta + 2}\right)
$$

2. We will do some examples of convergence in distribution and convergence in probability here.

   (a) (1 pt) Let $X_n \sim N(0, 1/n)$. Does $X_n \xrightarrow{d} 0$?

Solution. Yes, first we show that X_n converges in probability to 0.

$$\begin{aligned}\lim_{n\to\infty} \Pr(|X_n - 0| > \epsilon) &= \lim_{n\to\infty} \Pr(|X_n|^2 > \epsilon^2)\\ &\le \lim_{n\to\infty} \frac{\text{var}(X_n)}{\epsilon^2}\\ &= \lim_{n\to\infty} \frac{1}{n\epsilon^2}\\ &= 0\end{aligned}$$

Thus $X_n \xrightarrow{p} 0$ which implies that $X_n \xrightarrow{d} 0$.

(b) (3pts) Let $\{X_n\}$ be independent r.v's such that $P(X_n = n^\alpha) = 1/n$ and $P(X_n = 0) = 1 - 1/n$ for $n \ge 1$, where $\alpha \in (-\infty, infty)$ is a constant. For what values of α, will you have $X_n \xrightarrow{q.m} 0$? For what values will you have $X_n \xrightarrow{p} 0$?

Solution.

Convergence in quadratic mean:

$$\begin{aligned}\text{E}[|X_n|^2] &= (n^\alpha)^2 \frac{1}{n}\\ &= \frac{n^{2\alpha}}{n}\end{aligned}$$

The above will converge to zero if $2\alpha < 1$, or $\alpha < \frac{1}{2}$.

Convergence in Probability:

For $\epsilon \ge n^\alpha$ we have $\Pr(|X_n| > \epsilon) = 0$. For $\epsilon < n^\alpha$ we have $\Pr(|X_n| > \epsilon) = \frac{1}{n}$. This probability converges to zero for all values of α.

3. (1+1+1+1) If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

(a) $g(x) = 1(x \in (0, 10))$

Solution. It is not necessarily true that $E[g(X_n)] \to E[g(X)]$ because there is a discontinuity in g at 0 and 10. Take the following counter example:

Let $X_n = X + \frac{1}{n}$. It is simple to show that $X_n \xrightarrow{p} X$, thus $X_n \xrightarrow{d} X$.

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = \lim_{n\to\infty} \Pr(|X + \frac{1}{n} - X| > \epsilon) = \lim_{n\to\infty} \Pr(\frac{1}{n} > \epsilon) = 0$$

Now we need to show that $E[g(X_n)] \not\to E[g(X)]$. We pick a convenient λ to show that the result doesn't hold. For simplicity let $\lambda \to 0$. Thus $E[g(X)] = \Pr(X \in (0, 10)) = 0$ as $\lambda \to 0$, however

$$E[g(X_n)] = \Pr(X_n \in (0, 10)) = 1$$

as $\lambda \to 0$, thus $E[g(X_n)] \not\to E[g(X)]$.

(b) $g(x) = e^{-x^2}$

Solution. Yes, from the Portmanteau Theorem it is true that $E[g(X_n)] \to E[g(X)]$ because $g(x)$ is bounded by 0 and 1 and continuous on the real line.

(c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

Solution. Yes it is true that $E[g(X_n)] \to E[g(X)]$. First the function $g(x)$ is bounded as it only takes on the values $(-1, 0, 1)$. Second the discontinuities only occur when $\cos(x) = 0$ which can be defined by the set $A = \{\pi(\frac{1}{2} + n)\}_{n=0}^{\infty}$. We define $C(g) = \mathbb{R} - A$ to be the set of values on which $g(x)$ is continuous. Since X only takes on integer values $\Pr(X \in) C(g) = 1$, thus by the Portmanteau theorem $E[g(X_n)] \to E[g(X)]$.

(d) $g(x) = x$

Solution. Since $g(x)$ is not bounded, it is not necessarily true that $E[g(X_n)] \to E[g(X)]$. Take the following counter example: Let

$$= \begin{cases} X & \text{w.pr. } \frac{n-1}{n} \\ n & \text{w.pr. } \frac{1}{n} \end{cases} \tag{1}$$

Clearly $X_n \xrightarrow{d} X$, however

$$\begin{aligned} \lim_{n\to\infty} E[g(X_n)] &= \lim_{n\to\infty} E[X_n] \\ &= \lim_{n\to\infty} (\frac{1}{n}n + E[X]\frac{n-1}{n}) \\ &= 1 + E[X] \\ &\neq E[X] \end{aligned}$$

Therefore $E[g(X_n)] \not\to E[g(X)]$.

4. (4 pts) Let $X_1, \ldots, X_n$ be independent r.v's with mean zero and variance $\sigma_i^2 := E[X_i^2]$ and $s_n^2 = \sum_i \sigma_i^2$. If $\exists \delta > 0$ s.t. as $n \to \infty$,

$$\frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0,$$

then $\sum_i X_i/s_n$ converges weakly to the standard normal.

Proof. We want to show that

$$\frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}}$$

is an upper bound for the Lindeberg condition. If the above condition converges to 0 as $n \to \infty$ then the Lindeberg must also be satisfied and thus $\sum_i X_i/s_n \xrightarrow{d} N(0,1)$.

$$\begin{aligned} \frac{1}{s_n^2}\sum_{i=1}^{n} E[|X_i|^2 1(|X_i| \geq \epsilon s_n)] &= \frac{1}{s_n^2}\sum_{i=1}^{n} E\left[|X_i|^2 1(|X_i|^\delta \geq \epsilon^\delta s_n^\delta)\right] \\ &\leq \frac{1}{s_n^2}\sum_{i=1}^{n} E\left[|X_i|^2 \frac{|X_i|^\delta}{\epsilon^\delta s_n^\delta}\right] \\ &= \frac{1}{\epsilon^\delta}\frac{\sum_{i=1}^{n} E|X_i|^{2+\delta}}{s_n^{2+\delta}} \end{aligned}$$

We used the fact that for a positive number X,

$$1(X \geq \epsilon) \leq X/\epsilon.$$

The constant $\frac{1}{\epsilon^\delta}$ will not affect the convergence on the right hand side. If

$$\frac{\sum_{i=1}^n E|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0$$

as $n \to \infty$, then

$$0 \leq \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^n E[|X_i|^2 1(|X_i| \geq \epsilon s_n)] \leq 0$$

Which proves that the Lindeberg condition is also met and thus $\sum_i X_i/s_n \xrightarrow{d} N(0,1)$. □

5. (2+2) Recall the converse of the Lindeberg Feller theorem. We will gather some intuition about that here. Let $X_1, \ldots, X_n$ be independent r.v's with mean zero and variance $\sigma_i^2 := E[X_i^2]$ and $s_n^2 = \sum_i \sigma_i^2$.

   (a) If $\max_i \sigma_i^2/s_n^2$ does not converge to zero as $n \to \infty$, then the Lindeberg condition does not hold.

   Proof. We shall prove the above statement by proving the contraposition, that is if the Lindeberg condition holds, then $\max_i \sigma_i^2/s_n^2 \to 0$.

$$\begin{aligned}
\frac{\sigma_i^2}{s_n^2} &= \frac{1}{s_n^2} E|X_i|^2 \\
&= \frac{1}{s_n^2} E[|X_i|^2 1(|X_i| \geq \epsilon s_n)] + \frac{1}{s_n^2} E[|X_i|^2 1(|X_i| < \epsilon s_n)] \\
&\leq \frac{1}{s_n^2} E[|X_i|^2 1(|X_i| \geq \epsilon s_n)] + \frac{1}{s_n^2} E[|s_n \epsilon|^2 1(|X_i| < \epsilon s_n)] \\
&\leq \frac{1}{s_n^2} E[|X_i|^2 1(|X_i| \geq \epsilon s_n)] + \epsilon^2
\end{aligned}$$

$\lim_{n\to\infty} \frac{1}{s_n^2} E[|X_i|^2 1(|X_i| \geq \epsilon s_n)] + \epsilon^2 = \epsilon^2,$ by the Lindeberg condition.

   So, for all $\epsilon > 0$, we have

$$\frac{1}{s_n^2} E[\sum_i |X_i|^2 1(|X_i| \geq \epsilon s_n)] \geq \max_i \frac{\sigma_i^2}{s_n^2} \geq 0$$

   So in order to have the Lindeberg condition hold, one must have $\max_i \frac{\sigma_i^2}{s_n^2} \to 0$. □

(b) Construct an example where the above is true, but still we have $\sum_i X_i/s_n$ converges weakly to $N(0,1)$. This shows that the Lindeberg condition is not necessary. You can show this by showing that the moment generating function converges to that of a standard normal.

Solution.

Let $X \sim N(0, \sigma_i^2)$ where $\sigma_i^2 = 2^{1-i}$. Let $s_n^2 = \sum_{i=1}^n \sigma_i^2$ Thus we have:

$$\max_j \frac{\sigma_j^2}{s_n} = \frac{1}{\sum_{i=1}^n 2^{1-i}} = \frac{1}{\sum_{i=1}^n 2^{1-i}} \to \frac{1}{2}$$

Therefore $\max_j \frac{\sigma_j^2}{s_n^2} \not\to 0$, however we know that

$$Z_n = \sum_i X_i \sim N(0, \sum_i \sigma_i^2) \xrightarrow{d} N(0, 1/2)$$

Thus using the continuous mapping theorem and Slutsky's theorem we get that

$$\sqrt{s_n^2} \xrightarrow{p} \frac{1}{\sqrt{2}}, \quad \frac{Z_n}{s_n} \xrightarrow{d} N(0,1)$$