Homework Assignment 2

Due Friday March 1st midnight

SDS 384-11 Theoretical Statistics

1. Remember Hoeffding's Lemma? We proved it with a weaker constant in class using a symmetrization type argument. Now we will prove the original version. Let X be a bounded r.v. in $[a, b]$ such that $E[X] = \mu$. Let $f(\lambda) = \log E[e^{\lambda(X-\mu)}]$. Show that $f''(\lambda) \leq (b-a)^2/4$. Now use the fundamental theorem of calculus to write $f(\lambda)$ in terms of $f''(\lambda)$ and finish the argument.

2. Bernstein's inequality for bounded i.i.d sequences of random variables $\{X_i\}$ with $|X_i| \leq M$ gives: $P(|\sum_i (X_i - E[X_i])| \geq t) \leq 2\exp\left(\frac{-t^2/2}{\sum_i \mathrm{var}(X_i) + Mt/3}\right)$. There is another better inequality called Bennett's inequality, which we will prove here.

   (a) Consider zero mean r.v.s X_i such that $|X_i| \leq b$ and $\mathrm{var}(X_i) = \sigma_i^2$. Prove that
   $$\log E[\exp(\lambda X_i)] \leq \sigma_i^2 \lambda^2 \left(\frac{e^{\lambda b} - 1 - \lambda b}{(\lambda b)^2}\right) \quad \forall \lambda \in \mathbb{R}.$$

   (b) Given independent r.v.s $X_i, i = 1, \ldots, n$ satisfying the above condition prove
   $$\text{(Bennett's inequality)} \qquad P\left(\sum_i X_i \geq n\delta\right) \leq \exp\left(-\frac{n\sigma^2}{b^2} h(b\delta/\sigma^2)\right),$$
   where $\sigma^2 = \sum_i \sigma_i^2$ and $h(t) := (1+t)\log(1+t) - t$ for $t \geq 0$.

   (c) Show that Bennett's inequality is at least as good as Bernstein's inequality.

3. Given a scalar random variable X, suppose that there are positive constants c_1, c_2 such that,
   $$P(X - E[X] \geq t) \leq c_1 \exp(-c_2 t^2) \qquad \forall t \geq 0.$$

   (a) Prove that $\mathrm{var}(X) \leq \frac{c_1}{c_2}$

   (b) A median m_X is any number such that $P(X \geq m_X) \geq 1/2$ and $P(X \leq m_X) \geq 1/2$. Show by example that the median does not need to be unique.

   (c) Show that if the median concentration bound holds, then for any median m_X,
   $$P(|X - m_X| \geq t) \leq c_3 \exp(-c_4 t^2),$$
   where $c_3 = 4c_1$ and $c_4 = c_2/8$.

   (d) Conversely, show that whenever the above median concentration holds, then mean concentration holds with $c_1 = 2c_3$ and $c_2 = c_4/4$.

4. Given a positive semidefinite matrix $Q \in \mathbb{R}^{n\times n}$, consider $Z = \sum_{i,j} Q_{ij} X_i X_j$. When $X_i \sim N(0,1)$, prove the Hanson-Wright inequality.

$$P\left(Z \geq \mathrm{trace}(Q) + t\right) \leq 2\exp\left(-\min\left\{c_1 t/\|Q\|_{op}, c_2 t^2/\|Q\|_F^2\right\}\right),$$

where $\|Q\|_{op}$ and $\|Q\|_F$ denote the operator and frobenius norms respectively. *Hint: The rotation-invariance of the Gaussian distribution and sub-exponential nature of χ^2-variables could be useful.*