

# Homework Assignment 2

Due Friday March 1st midnight

SDS 384-11 Theoretical Statistics

1. Remember Hoeffding's Lemma? We proved it with a weaker constant in class using a symmetrization type argument. Now we will prove the original version. Let  $X$  be a bounded r.v. in  $[a, b]$  such that  $E[X] = \mu$ . Let  $f(\lambda) = \log E[e^{\lambda(X-\mu)}]$ . Show that  $f''(\lambda) \leq (b-a)^2/4$ . Now use the fundamental theorem of calculus to write  $f(\lambda)$  in terms of  $f''(\lambda)$  and finish the argument.
2. Bernstein's inequality for bounded i.i.d sequences of random variables  $\{X_i\}$  with  $|X_i| \leq M$  gives:  $P(|\sum_i (X_i - E[X_i])| \geq t) \leq 2 \exp\left(\frac{-t^2/2}{\sum_i \text{var}(X_i) + Mt/3}\right)$ . There is another better inequality called Bennett's inequality, which we will prove here.

(a) Consider zero mean r.v.s  $X_i$  such that  $|X_i| \leq b$  and  $\text{var}(X_i) = \sigma_i^2$ . Prove that

$$\log E[\exp(\lambda X_i)] \leq \sigma_i^2 \lambda^2 \left( \frac{e^{\lambda b} - 1 - \lambda b}{(\lambda b)^2} \right) \quad \forall \lambda \in \mathbb{R}.$$

(b) Given independent r.v.s  $X_i, i = 1, \dots, n$  satisfying the above condition prove

$$\text{(Bennett's inequality)} \quad P\left(\sum_i X_i \geq n\delta\right) \leq \exp\left(-\frac{n\sigma^2}{b^2} h(b\delta/\sigma^2)\right),$$

where  $\sigma^2 = \frac{\sum_i \sigma_i^2}{n}$  and  $h(t) := (1+t) \log(1+t) - t$  for  $t \geq 0$ .

(c) Show that Bennett's inequality is at least as good as Bernstein's inequality.

3. Given a scalar random variable  $X$ , suppose that there are positive constants  $c_1, c_2$  such that,

$$P(|X - E[X]| \geq t) \leq c_1 \exp(-c_2 t^2) \quad \forall t \geq 0.$$

- (a) Prove that  $\text{var}(X) \leq \frac{c_1}{c_2}$
- (b) A median  $m_X$  is any number such that  $P(X \geq m_X) \geq 1/2$  and  $P(X \leq m_X) \geq 1/2$ . Show by example that the median does not need to be unique.
- (c) Show that if the mean concentration bound holds, then for any median  $m_X$ ,  $\exists$  positive constants  $c_3, c_4$  such that

$$P(|X - m_X| \geq t) \leq c_3 \exp(-c_4 t^2),$$

- (d) Conversely, show that whenever the above median concentration holds, then mean concentration holds with positive constants  $c_1 = 2c_3$  and  $c_2 = c_4/4$ .

4. Given a positive semidefinite matrix  $Q \in \mathbb{R}^{n \times n}$ , consider  $Z = \sum_{i,j} Q_{ij} X_i X_j$ . When  $X_i \sim N(0, 1)$ , prove the Hanson-Wright inequality.

$$P(Z \geq \text{trace}(Q) + t) \leq \exp\left(-\min\left\{c_1 t / \|Q\|_{op}, c_2 t^2 / \|Q\|_F^2\right\}\right),$$

where  $\|Q\|_{op}$  and  $\|Q\|_F$  denote the operator and frobenius norms respectively. *Hint: The rotation-invariance of the Gaussian distribution and sub-exponential nature of  $\chi^2$ -variables could be useful.*