Homework Assignment 1

Due via canvas Feb 17th

SDS 384-11 Theoretical Statistics

1. Consider a sequence of iid random variables $\{X_n\}$ such that $X_i \sim Beta(\theta, 1)$, where $\theta > 0$. Let $\bar{X}_n$ denote the sample mean. The method of moments estimator of θ is $\hat{\theta}_n = \bar{X}_n/(1-\bar{X}_n)$. Derive the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$.

2. We will do some examples of convergence in distribution and convergence in probability here.

   (a) Let $X_n \sim N(0, 1/n)$. Does $X_n \xrightarrow{d} 0$?

   (b) Let $\{X_n\}$ be independent r.v's such that $P(X_n = n^\alpha) = 1/n$ and $P(X_n = 0) = 1 - 1/n$ for $n \geq 1$, where $\alpha \in (-\infty, infty)$ is a constant. For what values of α, will you have $X_n \xrightarrow{q.m} 0$? For what values will you have $X_n \xrightarrow{p} 0$?

3. If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

   (a) $g(x) = 1(x \in (0, 10))$

   (b) $g(x) = e^{-x^2}$

   (c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

   (d) $g(x) = x$

4. Let $X_1, \ldots, X_n$ be independent r.v's with mean zero and variance $\sigma_i^2 := E[X_i^2]$ and $s_n^2 = \sum_i \sigma_i^2$. If $\exists \delta > 0$ s.t. as $n \to \infty$,

$$\frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0,$$

   then $\sum_i X_i/s_n$ converges weakly to the standard normal.

5. Recall the converse of the Lindeberg Feller theorem. We will gather some intuition about that here. Let $X_1, \ldots, X_n$ be independent r.v's with mean zero and variance $\sigma_i^2 := E[X_i^2]$ and $s_n^2 = \sum_i \sigma_i^2$.

   (a) If $\max_i \sigma_i^2/s_n^2$ does not converge to zero as $n \to \infty$, then the Lindeberg condition does not hold.

   (b) Construct an example where the above is true, but still we have $\sum_i X_i/s_n$ converges weakly to $N(0, 1)$. This shows that the Lindeberg condition is not necessary. You can show this by showing that the moment generating function converges to that of a standard normal.