Endterm

SDS321

Spring 2016 PART 2

You may use two pages (4 sides) of notes, and you may use a calculator. You must hand in your notes, plus any rough work, after the exam. There are 6 short questions each of 3 points and one long question with 6 points.

Please answer all problems in the space provided on the exam. Use extra pages if needed. Of course, please put your name on extra pages.

Read each question carefully, show your work and clearly present your answers. Note, the exam is printed two-sided - please don't forget the problems on the even pages!

Good Luck!

Name: ____

UTeid: ____

1. (3 pts) If a fair coin is tossed a sequence of times (infinitely many times), what is the probability that the first head will occur after the 10^{th} toss, given that it has not occurred in the first 5 tosses? In your solution let $A = \{\text{first head after 10th toss}\}$; and $B = \{\text{no head in first 5 tosses}\}$.

   Solution: $p(AB)/p(B) = p(A)/p(B) = .5^{10}/.5^5 = .5^5$

   Grading: 1 pt for $p(AB) = p(A)$. 1 pt for p(A) and 1 pt for p(B)

2. In $n = 10,000$ independent tosses of a coin, the coin landed on heads 5126 times. Let X denote the number of heads in n tosses, and let p denote the probability of head.

   (a) (1pt) Assuming that the coin is a fair coin, that is $p = 0.5$, find the expected value $\mu = E(X)$ and standard deviation σ of X.

   Solution: $\mu = np = 5000$ $\sigma = \sqrt{10000 \times .25} = 50$

   Grading: 1/2 for mean, 1/2 for standard deviation. If they dont take square root take 1/2 pt off.

   (b) (2pts) Find $P(X > 5126)$ if the coin is fair. Your answer may not be a bound obtained using Markov or Chebyshev's inequalities, but may be an approximation. Clearly state which approximation or result you are using and why it applies in this case.

Solution: *Note that* $np(1-p) > 10$. *We can use the normal approximation, using a normal distribution* $N(\mu, \sigma)$ *with* $\mu = np = 5000$ *and* $\sigma = 50$. *We find* $p(X > 5126) = p\left(\frac{X-\mu}{\sigma} > \frac{5126.5-5000}{50}\right) \approx p(Z > 2.53) = 0.0057$ *for a standard normal r.v.* $Z \sim N(0, 1)$. *Note the continuity correction in* 5126.5.

Grading: [1pt] for the correct Z-transform; 1/2 for the continuity correction. -1/2 for looking up the wrong tail (or any other wrong table lookup).

3. (3 pts) Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and let $Z = 1 - X/2$. What is the PDF of Z? You should explicitly state the support of $f_Z(z)$ (i.e. values of z where $f_Z(z) \neq 0$).

Solution: $F_Z(z) = P(1 - X/2 \leq z) = P(X \geq 2(1-z)) = e^{-2\lambda(1-z)}$.
$f_Z(z) = 2\lambda e^{-2\lambda} e^{2\lambda z}$ *when* $z \in (-\infty, 1]$

Grading: 1/2 pt for correct CDF setup. 1 pt for correct CDF. 1 pt for derivative. 1/2 pt for range.

4. (3 pts) The joint distribution of three random variables X, Y and Z is given by:

$$f_{X,Y,Z}(x,y,z) = \begin{cases} ce^{-x/2-3y-2z} & x \geq 0, y \geq 0, z \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Find c.

Solution: $c = 1/2 \times 3 \times 2 = 3$.

Grading: 1 pt for getting the normalization constant for each of the pdf's.

5. (3 pts) What is the probability that among a set of 10 people, at least two have their birthdays in the same month of the year (assuming the months are equally likely for birthdays).

   In your solution let $A = \{$at least two matching birthday months.$\}$

   > **Solution:** *We find $p(A^c)$ – that is easier to think about, that is, $A^c =$ "no matching b-day months." First count the number of outcomes in A^c. It is $n(A^c) = 12 \cdot 11 \ldots (12-(n-1)) = 12!/(12-n)!$. The number of possible n-tuples of birthday monts is 12^n. Therefore $p(A^c) = \frac{12!/(12-n)!}{12^n}$ and $p(A) = 1 - p(A^c) = 1 - \frac{12!/(12-n)!}{12^n}$*
   >
   > *Grading:* [1pt] for using A^c; [1pt] for $n(A^c)$; [1pt] for the normalizer.

6. (3 pts) The number of 110 degree days X in a good year is Poisson(5) whereas in a bad year it is Poisson(10). Probability that a year will be a bad year is 0.6. Calculate $\mathrm{var}(X)$.

   Hint: Introduce D as a Bernoulli random variable which is 1 when its a bad year. You can write:

$$X \sim Poisson(5 + 5D)$$
$$D \sim Bernoulli(0.6)$$

Solution: $var(X) = E[var(X|D)] + var(E[X|D]) = E[5+5D] + var(5+5D) = 5 + 3 + 25 \times .24 = 8 + 6 = 14$

Grading: 1/2 pt for correct formula. 1/2 pt for $E[X|D]$, 1/2 pt for var(E), 1/2 pt for var($X|D$), 1/2 pt for E[var]. 1/2 pt for var(D).

7. (7 pts) A continuous r.v. X has density function

$$f_X(x) = c\,e^{-|x|} = \begin{cases} c\,e^{-x} & x \geq 0 \\ c\,e^{x} & x < 0 \end{cases} \tag{1}$$

(a) (2 pts) Find c.

Solution: *e^x integrates to 1 over negative x. So does e^{-x} over positive x. So $2c = 1$. $c = 1/2$.*

Grading: 1/2 pt for understanding this is to do with normalization. 1/2 pt for each part of the integration. 1/2 part for correct answe.

(b) (2 pts) Find $P(|X| < 2)$.

Solution: $p(-2 < X < 2) = 2\int_0^2 f(x)dx = 2 \times .5\ \int_0^2 e^{-x}dx = 1 - e^{-2} = 0.86$

Grading: [1pt] for correct integration limits; [1pt] for recognizing $p(X \in A) = \int_A f(x)dx$;
numerical answer is not required.

(c) (2 pts) Find $E[X|X \geq 0]$.

Solution: *The conditional pdf $f_{X|X\geq 0}$ is $Exp(1)$. So the expectation is 1.*

Grading: 1 pt for correct conditional density. 1 pt for correct expectation.

(d) (1 pt) Find $E[X|X < 0]$.

Solution: *E[X]=0 by symmetry. So this must be -1*

Grading: 1 pt for correct answer

Useful distributions

All PDFs/PMFs are zero outside the range specified.

	PDF/PMF	$E[X]$	$\mathrm{var}(X)$
Bernoulli	$p^x(1-p)^{1-x}, x = 0, 1$	p	$p(1-p)$
Binomial	$\binom{n}{k}p^k(1-p)^{n-k}$, k=0,1,...,n	np	$np(1-p)$
Geometric	$p(1-p)^{k-1}, k = 1, 2, 3, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$\frac{\lambda^k e^{-\lambda}}{k!}, k = 1, 2, 3, \ldots$	λ	λ
Normal	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Exponential	$\lambda e^{-\lambda x}, x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Uniform	$\frac{1}{b-a} \qquad a \leq x \leq b, b > a$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$

Standard normal table

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990