THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 8: End of counting and Discrete random variables

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Probability and counting: example 1b

- A bus with 5 passengers makes 10 stops. All configurations of discharging the passengers are equally likely.
- What is the probability p that no two passengers get down at the same stop?

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- $p = (10)_5 / 10^5$.

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- What is the probability if $r = 366$?
- Then at least two people must have the same birthday. This is also called the Pigeonhole Principle. So $p = 0$.
- How will you calculate this for large r, say $r = 30$?

Birthdays

- Problem... my calculator can't handle $365!$ or 365^{30}.
- Take logarithms! $365^{30} = 10^{76.8688} = 7.392 \times 10^{76}$.
- We can *approximate* factorials using Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

- The $\sim$ symbol means the ratio of the two sides tend to 1 as $n \to \infty$.

$$\begin{aligned}
\ln(365!) &\approx 1792.3 \\
\ln(335!) &\approx 1616.6 \\
\ln\left(\frac{365!}{335!}\right) &= \ln(365!) - \ln(335!) \approx 1792.3 - 1616.6 = 175.55 \\
\frac{365!}{335!} &\approx e^{175.55} = 2.1711 \times 10^{76}
\end{aligned}$$

- The actual value is 2.1710×10^{76} – not bad!

Birthdays

- So, we with 30 people, we have 7.392×10^{76} possible combinations of birthdays.
- 2.171×10^{76} of these possible combinations of birthdays have no repeats.
- So, the probability of no one having the same birthday is:

$$\frac{2.171 \times 10^{76}}{7.392 \times 10^{76}} \approx 0.296$$

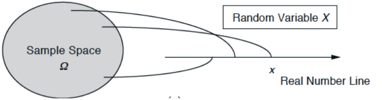
- Odds are, there's a shared birthday!

Random Variables

- So far we have talked about events and sample spaces.
- However for many experiments it is easier to use a summary variable.
- Say we are taking a opinion poll among 100 students about how understandable the lectures are.
- If "1" is used for understandable and "0" is used for not, then there are 2^{100} possible outcomes!!
- However, the thing that matters most is the number of students who think the class is understandable (or equivalently not).
- If we define a variable X to be that number, then the range of X is $\{0, 1, \ldots, 100\}$. **Much easier to handle that!**

Random Variable as a Mapping

Random Variable. A random variable is function from the sample space Ω into the real numbers.



Examples

- You toss a coin: is it head or tail?
- You roll a die: what number do you get?
- Number of heads in three coin tosses
- The sum of two rolls of die
- The number of die rolls it takes to get a six.

Discrete and continuous random variables

- A random variable is discrete if its range (the values it can take) is finite or at most countably finite.
  - $X =$ sum of two rolls of a die. $X \in \{2, \ldots, 12\}$.
  - $X =$ number of heads in 100 coin tosses. $X \in \{0, \ldots, 100\}$
  - $X =$ number of coin tosses to get a head. $X \in \{1, 2, 3, \ldots\}$
- Consider an experiment where you throw a dart which can fall anywhere between $[-1, 1]$.
- Let X be a the dart's position d. X is a random variable and X is not discrete.
- Now let Y be a random variable such that:

$$Y = \begin{cases} 1 & \text{If } d < 0 \\ 0 & \text{If } d \geq 0 \end{cases} \tag{2}$$

Now Y is a discrete random variable.

Probability Mass Functions

Earlier we talked about probability laws which assign probabilities to events in a sample space. One can play the same game with random variables.

- $X =$ number of heads in two fair coin tosses
- X can take values in $\{0, 1, 2\}$.
- $P(X = 0) =$
- $P(X = 1) =$
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- $P(X = 1) = P(\{HT, TH\}) = 1/2$.
- $P(X = 2) = P(\{HH\}) = 1/4$.
- In general, the probability that a random variable X takes up a value x is written as $p_X(x)$, or $P_X(x)$ or $P_X(X = x)$ etc.
- A random variable is always written with upper-case and the numerical value we are trying to evaluate the probability for is written with a lower case.

Properties of P.M.F's

- Perhaps not so surprisingly, $\sum_{x} P(X = x) = 1$
- $P(X \in S) = \sum_{x \in S} P(X = x)$
- To compute $P(X = x)$
  - Collect all possible outcomes that give $\{X = x\}$.
  - Add their probabilities to get $P(X = x)$.
- You are throwing 2 fair coins. What is the probability that you see at least one head?
  - Well at least one head is $\{HT, TH, HH\}$. So probability of this is $3/4$.

The Uniform random variable

Consider the roll of a fair die. You are interested in the number on the roll.

- X can take how many different values?
- Now what are the probabilities of taking on those values?
- $P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = 1/6$
- For an **uniform** random variable X, each value has equal probability mass.
- If an **uniform** random variable X takes on k different values, then the probability mass at each of those values are $1/k$.

The Bernoulli random variable

Consider the toss of a biased coin, which gives a head with probability p.

- A **Bernoulli** random variable X takes two values: 1 if a head comes up and 0 if not. $X = \begin{cases} 1 & \text{If head} \\ 0 & \text{If tail} \end{cases}$
- The PMF is given by: $p_X(x) = \begin{cases} p & \text{If } x = 1 \\ 1 - p & \text{If } x = 0 \end{cases}$
- Examples of a Bernoulli
  - A person can be healthy or sick with a certain disease.
  - A test result for a disease can be positive or negative.
  - It may rain one day or not.

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What if the coin is biased? Probability of H is 0.8?

A biased coin

You are throwing 10 biased coins with $P(\{H\}) = p$. What is the probability that the sum X equals 5?

- Probability of a length 10 binary sequence with 5 1's is
- So $P(X = 5) =$
- What about $P(X = 8)$?

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- So $P(X=5) = \binom{10}{5} p^5(1-p)^5$.
- What about $P(X=8)$? $\binom{10}{8} p^8(1-p)^2$.
- In general $P(X=k) = \binom{n}{k} p^k(1-p)^{n-k}$

The Binomial random variable

A biased coin ($P(\{H\}) = p$) is tossed n times. All tosses are mutually independent.

- Let X_i be a Bernoulli random variable which is 1 if the i^{th} toss gave a head. Then $\{X_i, i = 1, \ldots, n\}$ are independent Bernoullis.
- Let Y be the number of heads we see at the end of all n tosses.
- Whats the relationship of Y to the X_i's?

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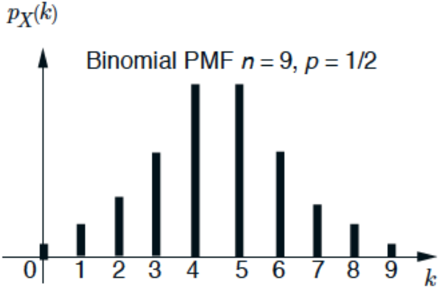
- $P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k}$ for $k \in \{0, 2, \ldots n\}$
- $\sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} = 1$ (Binomial expansion)!

The Binomial random variable

- Sum of independent Bernoullis give a Binomial!
- We will denote the Binomial PMF by $Binomial(n, p)$ or $Bin(n, p)$.
- We will write $X \sim Binomial(n, p)$ to indicate that X is **distributed** as a Binomial random variable with parameters n and p.
- Quick look at the histogram of $X \sim Binomial(9, 1/2)$.



Urn example

You have a urn with an 10 blue and 20 red balls. You pick 9 balls with replacement. Let X be the number of blue balls. What is $P(\text{X}=3)$?

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- So the probability $P(X = 3) = \begin{pmatrix} 9 \\ 3 \end{pmatrix} (1/3)^3(2/3)^6$!
- X is a Binomial(9,3) random variable! $X \sim \mathit{Binomial}(9, 3)$

The Geometric random variable

- The Bernoulli PMF describes the probability of success/failure in a single trial.
- The Binomial PMF describes the probability of k successes out of n trials.
- Sometimes we may also be interested in doing trials until we see a success.

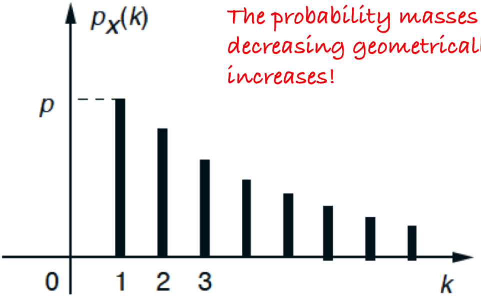
- Alice resolves to keep buying lottery tickets until he wins a hundred million dollars. She is interested in the random variable "number of lottery tickets bought until he wins the 100M$ lottery".
- Annie is trying to catch a taxi. How many occupied taxis will drive past before she finds one that is taking passengers?

- The number of trials required to get a single success is a **Geometric Random Variable**

The geometric random variable

We repeatedly toss a biased coin $(P(\{H\}) = p)$. The geometric random variable is the number X of tosses to get a head.

- X can take any integral value.
- $P(X = k) = P(\{\underbrace{TT\ldots T}_{k-1}H\}) = (1-p)^{k-1}p.$
- $\sum_k P(X = k) = 1$ (why?)



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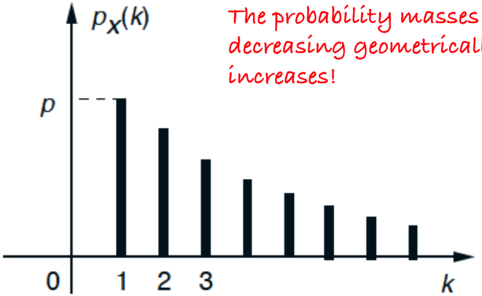

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- $P(X \geq k) = \sum_{i=k}^{\infty} p(1-p)^{i-1} = (1-p)^{k-1}$
- Intuitively, this is asking for the probability that the first $k-1$ tosses are tails.
- This probability is $P(X \geq k) = (1-p)^{k-1}$

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- Intuitively, this is asking for the probability that the first $k-1$ tosses are tails.
- This probability is $P(X \geq k) = (1-p)^{k-1}$
- $X > k$ is the event that $X \geq k+1$, and so $P(X > k) = (1-p)^{k}$

The memoryless property

What is $P(X > a + b | X > a)$?

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What is $P(X > a + b | X > a)$?

- $$P(X > a + b | X > a) = \frac{P(X > a + b)}{P(X > a)}$$
  $$= \frac{(1-p)^{a+b}}{(1-p)^a} = (1-p)^b$$
  $$= P(X > b)$$
- You forgot about $X > a$ and started the clock afresh!

The Poisson random variable

I have a book with 10000 words. Probability that a word has a typo is $1/1000$. I am interested in how many misprints can be there on average? So a Poisson often shows up when you have a Binomial random variable with very large n and very small p but $n \times p$ is moderate. Here $np = 10$.

Our random variable might be:

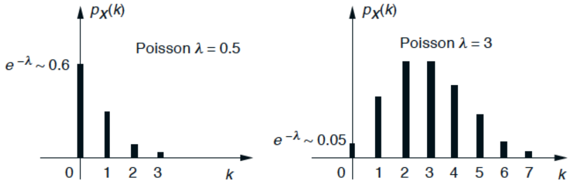
- The number of car crashes in a given day.
- The number of buses arriving within a given time period.
- The number of mutations on a strand of DNA.

We can describe such situations using a **Poisson random variable**.

The Poisson random variable

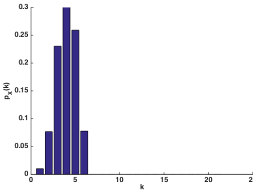
- A Poisson random variable takes non-negative integers as values. It has a nonnegative parameter λ.
- $P(X=k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, for $k = 0, 1, 2 \dots$.
- $\displaystyle\sum_{k=0}^{\infty} P(X=k) = e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots) = 1.$ (Exponential series!)

The PMF is monotonically decreasing for λ=0.5

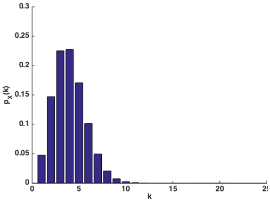


The PMF is increasing and then decreasing for λ=3

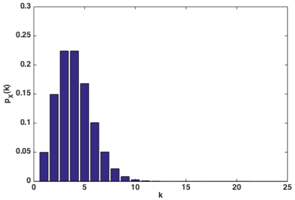
Poisson random variable



Binomial(5,0.6)



Binomial(100,0.03)



Poisson(3)

- When n is very large and p is very small, a binomial random variable can be well approximated by a Poisson with $\lambda = np$.
- In the above figure we increased n and decreased p so that $np = 3$.
- See how close the PMF's of the Binomial(100,0.03) and Poisson(3) are!
- More formally, we see that $\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{e^{-\lambda}\lambda^k}{k!}$ when n is large, k is fixed, and p is small and $\lambda = np$.

Example

Assume that on a given day 1000 cars are out in Austin. On an average three out of 1000 cars run into a traffic accident per day.

1. What is the probability that we see at least two accidents in a day?

4. If you know there is at least one accident, what is the probability that the total number of accidents is at least two?

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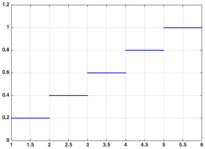
5. $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-3} = 0.950.$
   $P(X \geq 2 | X \geq 1) = P(X \geq 2)/P(X \geq 1) = 0.8/0.950 = 0.84$

Cumulative Distribution Functions

- For any random variable the cumulative distribution function is defined as:

$$F_X(a) = \sum_{x \leq a} p(x)$$

- Can you work out the PMF of the following random variable?



Function of a random variable

- A function of a random variable is also a random variable.
- Let X be the number of heads in 5 fair coin tosses.
- We know that X has the Binomial(5,1/2) distribution.
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- Lets write down the PMF of Y.
  - $P(Y=0) = P(X=0) + P(X=4) = (1/2)^5 + \binom{5}{4}(1/2)^5$
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  - $P(Y = 1) = P(X = 1) + P(X = 5)$...and so on.
  - More formally, if $Y = g(X)$ then we have:

$$p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x).$$