Homework Assignment 1

Due in class, Wednesday Feb 7th

SDS 384-11 Theoretical Statistics

1. Given densities p_n and q_n with respect to some measure μ, let X be distributed according to the distribution with density p_n. Define the likelihood ratio $L_n(X)$ as $L_n(X) = q_n(X)/p_n(X)$ for $p_n(X) > 0$. $L_n(X) = 1$. if $p_n(X) = q_n(X) = 0$ and $L_n(X) = \infty$ otherwise. Show that the likelihood ratio is a uniformly tight sequence.
$E[|L_n(X)|] = E[L_n(X)] = \int_{x:p_n(x)\geq 0} \frac{q_n(x)}{p_n(x)} p_n(x)dx \leq 1$. So for $\epsilon > 0$, take $P(L_n \geq 1/\epsilon) \leq \epsilon$ for all n. So its UT.

2. Consider a sequence of iid random variables $\{X_n\}$ such that $X_i \sim Beta(\theta, 1)$, where $\theta > 0$. Let $\bar{X}_n$ denote the sample mean. The method of moments estimator of θ is $\hat{\theta}_n = \bar{X}_n/(1 - \bar{X}_n)$. Derive the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$. Recall that the expectation of a $beta(\beta, 1)$ random variable is $\theta/(1+\theta)$. So $\bar{X}_n \xrightarrow{P} \theta/(1+\theta)$ and variance $\sigma^2 = \frac{\theta}{(\theta+1)^2(\theta+2)}$. Now

$$\begin{aligned}\sqrt{n}(\hat{\theta}_n - \theta) &= \sqrt{n}\left(\frac{\bar{X}_n}{1-\bar{X}_n} - \theta\right)\\ &= \sqrt{n}(1+\theta)\frac{\bar{X} - \frac{\theta}{1+\theta}}{1-\bar{X}}\end{aligned}$$

Using CLT we have $\sqrt{n}(\bar{X} - \frac{\theta}{1+\theta}) \xrightarrow{d} N(0, \sigma^2)$. $1 - \bar{X} \xrightarrow{P} 1/(1+\theta)$. So $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \theta(\theta+1)^2/(\theta+2)$.

3. Derive the following one sided improvement of Chebyshev's inequality for a random variable X with variance σ^2 and any $t > 0$

$$P(X - E[X] \geq t) \leq \frac{\sigma^2}{\sigma^2 + t^2} \tag{1}$$

Take $E[X] = 0$ WLOG.

$$P(X \geq t) = P((X+u)^2 \geq (t+u)^2) \leq \frac{\sigma^2 + u^2}{(t+u)^2}$$

Minimizing the RHS w.r.t $u \geq 0$ gives the answer.

4. If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

   (a) $g(x) = 1(x \in (0, 10))$ $g(x)$ is discontinuous at 0 and 10, and $P(X \in \{0, 10\}) \neq 0$. So our theorem does not apply. We will create a counter-example. Let $X_n = X + 1/n$. $E[g(X_n)] = P(X \in (-1/n, 10 - 1/n)) \to P(X \in [0, 9])$. On the other hand $E[g(X)] = P(X \in [1, 9])$.

(b) $g(x) = e^{-x^2}$ e^{-x^2} is continuous and bounded, so the Portmanteau theorem proves that the expectation converges.

(c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.] $g(x)$ is discontinuous at $(\pi/2, 3\pi/2, \dots)$. However since X takes values in integers, we can safely say that $E[g(X_n)] \to E[g(X)]$

(d) $g(x) = x$ $g(x)$ is continuous but unbounded. So lets find a counter example. Let $X_n = X$ with probability $1 - 1/n$ and $X_n = n$ with probability $1/n$. So $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{d} X$. But $E[g(X_n)] = \lambda(1 - 1/n) + 1 \to \lambda + 1$, but $E[g(X)] = \lambda$.

5. Consider n i.i.d random variables $\{X_n\}$ uniformly distributed on the set of n points $\{1/n, 2/n, \dots, 1\}$. Show that $X_n \xrightarrow{d} X$ where $X \sim Uniform(0, 1)$. Does $X_n \xrightarrow{P} X$?

   WLOG let $x \in [i/n, (i+1)/n]. P(X_n \leq x) = i/n \to x$ as $n \to \infty$. So X_n is converging in distribution to a Uniform r.v. Remember that convergence in probability also implies convergence in the Cauchy sense, i.e. $P(|X_m - X_n| \geq \epsilon) \leq P(|X_m - X| + |X_n - X| \geq \epsilon) \leq P(|X_m - X| \geq \epsilon/2) + P(|X_n - X| \geq \epsilon/2) \to 0$ for $m, n \to \infty$ if $X_n \xrightarrow{P} X$.

   In this case however, $P(|X_m - X_n| \geq \epsilon) \not\to 0$ since X_m, X_n are independent.

   So tt is clear, that the sequence is non convergent in a Cauchy sense, since the sequence is iid. Another way to think about this is to look at a partial converse of convergence in probability. This states that a sequence converges in probability iff for every sequence, there is a subsequence that converges a.s. In our case, there is no such subsequence, and hence the sequence does not converge in probability.