Homework Assignment 6

Due on Friday 10th by midnight via canvas

SDS 321 Intro to Probability and Statistics

1. (7pts) The CDF of a random variable is given by:

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{1}{2} & 0 \leq b < 1 \\ \frac{3}{5} & 1 \leq b < 2 \\ \frac{4}{5} & 2 \leq b < 3 \\ \frac{9}{10} & 3 \leq b < 3.5 \\ 1 & b \geq 3.5 \end{cases}$$

   (a) (3 pts) Calculate the PMF of X.

$$P(X = b) = \begin{cases} \frac{1}{2} & b = 0 \\ \frac{1}{10} & b = 1 \\ \frac{1}{5} & b = 2 \\ \frac{1}{10} & b = 3 \\ \frac{1}{10} & b = 3.5 \\ 0 & b < 0 \text{ or } b > 3.5 \end{cases}$$

   (b) (2 pts) Calculate $E[X]$.

$$E[X] = 1/10 + 2/5 + 3/10 + 3.5/10 = 1.15$$

   (c) (2 pts) Calculate the variance of X.

$$E[X^2] = 1/10 + 4/5 + 9/10 + 3.5^2/10 = 3.025 \text{var}(X) \quad = E[X^2] - (E[X])^2 = 3.025 - 1.15^2 = 1.7025$$

2. (2+2 pts) Suppose that, in flight, airplane engines will fail with probability $1 - p$, independently from engine to engine. An airplane needs at least half of its engines operative to complete a successful flight.

   (a) If $p = 3/4$, which is preferable, a four-engine plane or a two-engine plane? Let X be number of engines which have not failed. For 4 engine plane, $X \sim Bin(4, p)$ we want $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - (1-p)^4 - 4p(1-p)^3 = .9492$. For 2 engine plane, $X \sim Bin(2, p)$ we want $P(X \geq 1) = 1 - P(X = 0) = 1 - (1 - p)^2 = .9375$. So the 4 engine plane has higher chance of being in flight.

(b) What about if $p = 1/2$?

Repeat the former exercise for $p = 1/2$. For 4 engine plane, $X \sim Bin(4, p)$ we want $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - (1-p)^4 - 4p(1-p)^3 = .6875$. For 2 engine plane, $X \sim Bin(2, p)$ we want $P(X \geq 1) = 1 - P(X = 0) = 1 - (1-p)^2 = .75$. So the 2 engine plane has higher chance of being in flight. Bottom line: its not always the best thing to go for plane with more engines.

3. (2+2 pts) The covariance between two random variables X and Y is defined as $\text{cov}(X, Y) := E[(X - E[X])(Y - E[Y])]$.

(a) (2 pts) Show that if X and Y are independent, then $\text{cov}(X, Y) = 0$. For X, Y independent, $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$. Let $f(X) = X - E[X]$ and $g(Y) = Y - E[Y]$. Note that $E[f(X)] = E[g(Y)] = 0$.

$$\text{cov}(X, Y) = E[f(X)g(Y)] = E[f(X)]E[g(Y)] = 0 \tag{1}$$

(b) (2 pts) Consider $X \sim Binomial(n, p)$. Let Y denote $n - X$. Calculate $\text{cov}(X, Y)$. For $Y = n - X$ independent, $Y - E[Y] = -(X - E[X])$.

$$\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[-(X - E[X])^2] = -\text{var}(X) = -np(1-p) \tag{2}$$

4. (2+2+1 = 5pts) Let the number of cars in the UT campus roads on a given day be denoted by X. On a rainy day $X \sim Poisson(100)$, whereas on a sunny day $X \sim Poisson(60)$. Denote the event of rain by R. $P(R) = 0.1$.

(a) Calculate $E[X]$. $E[X] = E[X|R]P(R) + E[X|R^c]P(R^c) = 10 + 54 = 64$.

(b) Calculate $E[X^2]$. For any random variable, $\text{var}(X) = E[X^2] - E[X]^2$ and so $E[X^2] = \text{var}(X) + E[X]^2$. For $X \sim Poi(\lambda)$, $E[X^2] = \lambda + \lambda^2$. $E[X^2] = E[X^2|R]P(R) + E[X^2|R^c]P(R^c) = (100^2 + 100).1 + (60^2 + 60).9 = 4304$.

(c) Calculate $\text{var}[X]$. $\text{var}(X) = E[X^2] - E[X]^2 = 208$.