THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 10: Expectation and Variance

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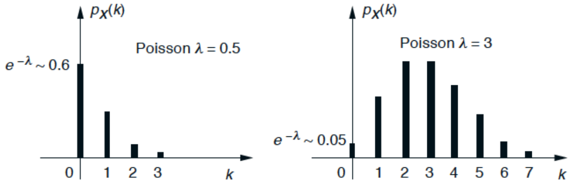
The Poisson random variable

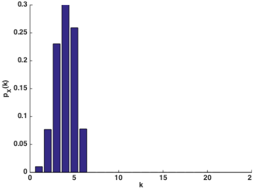
- A Poisson random variable takes non-negative integers as values. It has a nonnegative parameter λ.
- $P(X = k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, for $k = 0, 1, 2 \ldots.$
- $\sum_{k=0}^{\infty} P(X = k) = e^{-\lambda}(1 + \lambda + \dfrac{\lambda^2}{2!} + \dfrac{\lambda^3}{3!} + \ldots) = 1.$ (Exponential series!)

The PMF is monotonically decreasing for λ=0.5

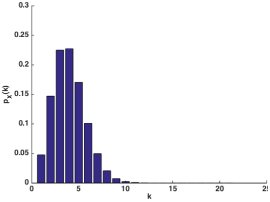


The PMF is increasing and then decreasing for λ=3

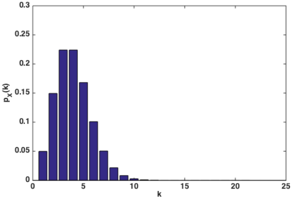
Poisson random variable



Binomial(5,0.6)



Binomial(100,0.03)



Poisson(3)

- When n is very large and p is very small, a binomial random variable can be well approximated by a Poisson with $\lambda = np$.
- In the above figure we increased n and decreased p so that $np = 3$.
- See how close the PMF's of the Binomial(100,0.03) and Poisson(3) are!
- More formally, we see that $\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{e^{-\lambda}\lambda^k}{k!}$ when n is large, k is fixed, and p is small and $\lambda = np$.

Example

Assume that on a given day 1000 cars are out in Austin. On an average three out of 1000 cars run into a traffic accident per day.

1. What is the probability that we see at least two accidents in a day?

4. If you know there is at least one accident, what is the probability that the total number of accidents is at least two?

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3. $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-3}(1 + 3) = 0.8$
4. If you know there is at least one accident, what is the probability that the total number of accidents is at least two?

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4. If you know there is at least one accident, what is the probability that the total number of accidents is at least two?
5. $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-3} = 0.950.$
   $P(X \geq 2 | X \geq 1) = P(X \geq 2)/P(X \geq 1) = 0.8/0.950 = 0.84$

Conditional PMF

- We have talked about conditional probabilities.
- We can also talk about conditional PMF's. Let A be an event with positive probability.
- The rules are the same. $P(X = x|A) = \dfrac{P(\{X = x\} \cap A)}{P(A)}$
- The conditional PMF is a valid PMF. $\sum_x P(X = x|A) = 1$

Conditional PMF-Example

- $X \sim Geometric(p)$
- What is $P(X = k | X > 1)$ for different values of k?

Conditional PMF-Example

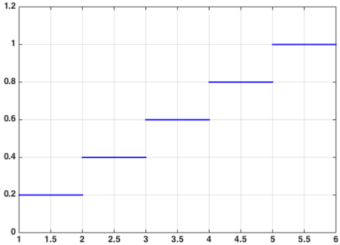
- $X \sim Geometric(p)$
- What is $P(X = k|X > 1)$ for different values of k?
- $$P(X = k|X > 1) = \begin{cases} 0 & \text{If } k = 1 \\ \frac{P(X=k)}{P(X>1)} & \text{Otherwise} \\ = \frac{(1-p)^{k-1}p}{(1-p)} & \\ = (1-p)^{k-2}p = P(X = k-1) & \end{cases}$$

Cumulative Distribution Functions

- For any random variable the cumulative distribution function is defined as:

$$F_X(a) = \sum_{x \leq a} p(x)$$

- Can you work out the PMF of the following random variable?



Function of a random variable

- A function of a random variable is also a random variable.
- Let X be the number of heads in 5 fair coin tosses.
- We know that X has the Binomial(5,1/2) distribution.
- Define $Y = X \mod 4$. Whats its PMF?

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- $Y = \begin{cases} 0 & X \in \{0, 4\} \\ 1 & X \in \{1, 5\} \\ 2 & X = 2 \\ 3 & X = 3 \end{cases}$
- Lets write down the PMF of Y.
  - $P(Y = 0) = P(X = 0) + P(X = 4) = (1/2)^5 + \binom{5}{4}(1/2)^5$
  - $P(Y = 1) = P(X = 1) + P(X = 5)$...and so on.

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  - $P(Y = 1) = P(X = 1) + P(X = 5)$...and so on.
  - More formally, if $Y = g(X)$ then we have:

$$p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x).$$

Function of a random variable-examples

- $X \sim Bernoulli(p)$.
  - What is the PMF of X^2?
  - What is the PMF of X^3?

- $X \sim Binomial(n, p)$.
  - What is the distribution of $n - X$?

Summing up

- Last time, we looked at the probability of a random variable taking on a given value:
$$p_X(x) = P(X = x)$$
- We also looked at plots of various PMFs of Uniform, Bernoulli, Binomial, Poisson and Geometric.
- Often, we want to make predictions for the value of a random variable
    - How many heads do I expect to get if I toss a fair coin 10 times?
    - How many lottery tickets should Alice expect buy until she wins the jackpot?
- We may also be interested in how far, on average, we expect our random variable to be from these predictions.

- Today we will talk about means and variances of these random variables.

Mean

You want to calculate average grade points from hw1. You know that 20 students got 30/30, 30 students got 25/30, and 50 students got 20/30. Whats the average?

- The average grade point is

$$\frac{30 \times 20 + 25 \times 30 + 20 \times 50}{100} = 30 \times 0.2 + 25 \times 0.3 + 20 \times 0.5$$

- Let X be a random variable which represents grade points of hw1.
- How will you calculate $P(X = 30)$?
  - See how many out of 100 students got 30 out of 30 points.
  - $P(X = 30) \approx 0.2$
  - $P(X = 25) \approx 0.3$
  - $P(X = 20) \approx 0.5$
- So roughly speaking,
  average grade $\approx 30 \times P(X = 30) + 25 \times P(X = 25) + 20 \times P(X = 20)$

Expectation

We define the expected value (or expectation or mean) of a discrete random variable X by

$$E[X] = \sum_x xP(X = x).$$

- X is a Bernoulli random variable with the following PMF:

$$P(X = x) = \begin{cases} p & X = 1 \\ 1 - p & X = 0 \end{cases}$$

So $E[X] =$.

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  So $E[X] = 1 \times p + 0 \times (1 - p) = p$.

- Expectation of a Bernoulli random variable is just the probability that it is one.
- You will also see notation like μ_X.

Expectation: example

You are tossing 4 fair coins independently. Let X denote the number of heads. What is $E[X]$?

- Any guesses? Well, on an average we should see about 2 coin tosses. No?
- Lets write down the PMF first.

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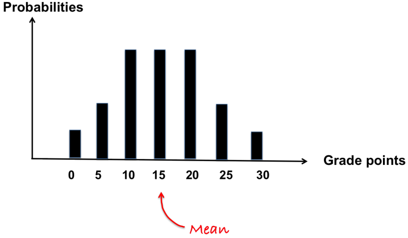
- So $E[X] = \frac{4}{2^4} + 2\frac{6}{2^4} + 3\frac{4}{2^4} + 4\frac{1}{2^4} = \frac{32}{16} = 2.$

Expectation of a function of a random variable

Lets say you want to compute $E[g(X)]$. Example, I know average temperature in Fahrenheit, but I now want it in Celsius.

- $E[g(X)] = \sum_x g(x)P(X = x)$.
- Follows from the definition of PMF of functions of random variables.
- Look at page 15 of Bersekas-Tsitsiklis and derive it at home!
- So $E[X^2] = \sum_x x^2 P(X = x)$. **Second moment of** X
- So $E[X^3] = \sum_x x^3 P(X = x)$. **Third moment of** X
- So $E[X^k] = \sum_x x^k P(X = x)$. k^{th} **moment of** X
- We are assuming "under the rugs" that all these expectations are well defined.

Expectation



- Think of expectation as center of gravity of the PMF or a representative value of X.
- How about the spread of the distribution? Is there a number for it?

Variance

Often, you may want to know the spread or variation of the grade points for homework1.

- If everyone got the same grade point, then variation is?
- If there is high variation, then we know that many students got grade points very different from the average grade point in class.
- Formally we measure this using variance of a random variable X.
- $\mathrm{var}(X) = E[(X - E[X])^2]$ or $E[(X - \mu)^2]$.
- The standard deviation of X is given by $\sigma_X = \sqrt{\mathrm{var} X}$.
- Its easier to think about σ_X, since its on the same scale.
- The grade points have average 20 out of 30 with a standard deviation of 5 grade points. Roughly this means, most of the students have grade points within $[20 - 5, 20 + 5]$.

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Computing the variance

- $\mathrm{var}(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 \, P(X=x)$
- Always remember! $E[X]$ or $E[g(X)]$ **do not depend on any particular value of** x. You can treat it as a constant. It only depends on the PMF of X.

- This can actually be made simpler.
- $\mathrm{var}(X) = E[X^2] - \mu^2$.

- So you can calculate $E[X^2]$ (second moment) and then subtract the square of $E[X]$ to get the variance!

A tiny bit of algebra

$$\mathrm{var}(X) =$$

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$$= \sum_x x^2 P(X=x) + \mu^2 - 2\mu^2 = E[X^2] - \mu^2$$

Some simple rules– Expectation

Say you are looking at a linear function (or transformation) of your random variable X.

- $Y = aX + b$. Remember celsius to fahrenheit conversions? They are linear too!
- $E[Y] = E[aX + b] = aE[X] + b$, as simple as that! why?
  - $$\begin{aligned} E[aX + b] &= \sum_x (ax + b)P(X = x) \\ &= a\sum_x xP(X = x) + b\sum_x P(X = x) \end{aligned}$$

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How about $E[Y]$ for $Y = aX^2 + bX + c$?

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- $Y = aX^3 + bX^2 + cX + d$. Can you guess what $E[Y]$ is?

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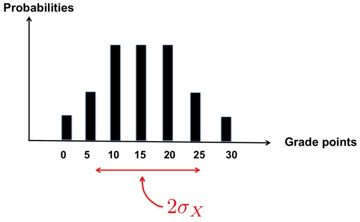
- $Y = aX^3 + bX^2 + cX + d$. Can you guess what $E[Y]$ is?

  $E[Y] = aE[X^3] + bE[X^2] + cE[X] + d$.

Some simple rules– Variance

Let $Y = X + b$. What is $\mathrm{var}(Y)$?



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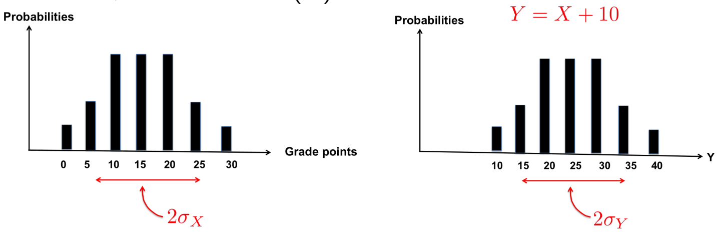

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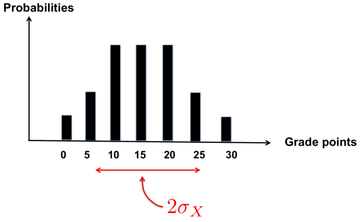

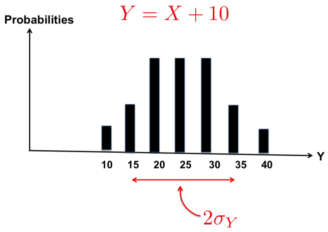

- Intuitively? Well you are just shifting everything by the same number.
- So? the spread of the numbers should stay the same!
- Prove it at home.

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- $$\begin{aligned}\mathrm{var}(X+b) &= E\left[(X+b)^2\right] - (E[X+b])^2 \\ &= E\left[X^2 + 2bX + b^2\right] - (E[X]+b)^2\end{aligned}$$

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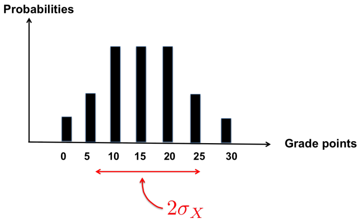

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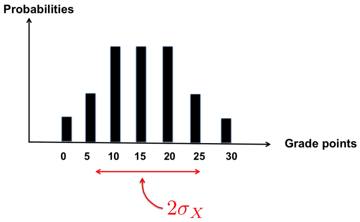

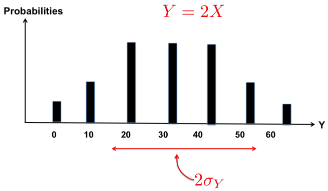

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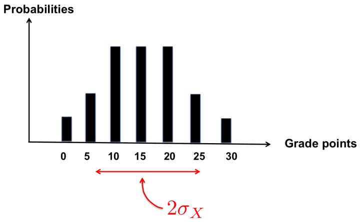

- Intuitively? Well you are just scaling everything by the same number.
- So? the spread should increase if $a > 1$!

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Let $Y = aX$. Turns out $\mathrm{var}(Y) = a^2 \mathrm{var}(X)$.

- Proof:
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- In general we can show that $\mathrm{var}(aX + b) = a^2\mathrm{var}(X)$.

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

- First lets get $E[X^2]$. This is

  $$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

  We see that $E[X^2] = E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values:

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$$E[X^2]=(1^2\times P(X=1)+0^2\times P(X=0))=p$$

  We see that $E[X^2]=E[X]$. Is this surprising?

- Well, what is the PMF of X^2?
  - X^2 can take two values: 0 and 1
  - $P(X^2=1)=P(X=1)=p$. $P(X^2=0)=P(X=0)=1-p$.
  - X and X^2 have identical PMF's! **They are identically distributed.**

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- $\mathrm{var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1 - p)$.

Mean and Variance of a Binomial

Let $X \sim Bin(n, p)$.

- $E[X] = np$ and $\mathrm{var}(X) = np(1 - p)$.
- We will derive these in the next class.

Mean and Variance of a Poisson

X has a Poisson(λ) distribution. What is its mean and variance?

- One can use algebra to show that $E[X] = \lambda$ and also $\mathrm{var}(X) = \lambda$.
- How do you remember this?
- Hint: mean and variance of the Binomial approach that of a Poisson when n is large and p is small, such that $np \approx \lambda$? Anything yet?

Mean and variance of a geometric

- The PMF of a geometric distribution is $P(X = k) = (1 - p)^{k-1} p$.
  - $E[X] = 1/p$
  - $\mathrm{var}(X) = (1 - p)/p^2$
  - We will also prove this later.

Conditional expectation

- We have done conditional probability and PMF.
- How about conditional expectation?
- Conditional expectation of random variable X conditioned on event A is written as $E[X|A]$
- $E[X|A] = \sum_{x} xP(X = x|A)$

Conditional expectation

- Say $X \sim \mathit{geometric}(p)$. What is $E[X|X > 1]$?

Conditional expectation

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- Remember $P(X = k|X > 1) = P(X = k - 1)$

Conditional expectation

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- Remember $P(X=k|X>1) = P(X=k-1)$
- So

$$
\begin{aligned}
E[X|X>1] &= \sum_{k=2}^{\infty} kP(X=k|X>1) \\
&= \sum_{k=2}^{\infty} kP(X=k-1) \\
&= \sum_{j=1}^{\infty} (j+1)P(X=j) \\
&= \sum_{j=1}^{\infty} jP(X=j) + 1 = E[X]+1
\end{aligned}
$$

The total expectation theorem

I am calculating the average combinatorics HW score in my class. I see that the average score of students who have taken combinatorics before is 90% whereas students who have not taken combinatorics before have an average of 70%. About 10% of the class has taken combinatorics before. How do I calculate the class average?

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- Should I do $(.9 + .7)/2$?

The total expectation theorem

- Consider disjoint events $\{A_1, \ldots, A_n\}$ which form a partition of the sample space.

- The total probability theorem says
$P(X = k) = \sum_i P(X = k | A_i) P(A_i)$.

- Similarly the total expectation theorem says:
$E[X] = \sum_i E[X | A_i] P(A_i)$.

- How?

The total expectation theorem

I am calculating the average combinatorics HW score in my class. I see that the average score of students who have taken combinatorics before is 90% whereas students who have not taken combinatorics before have an average of 75%. About 10% of the class has taken combinatorics before. How do I calculate the class average?

- $C = \{\text{A student has taken combinatorics}\}$. $P(C) = .1$.
- $E[X|C] =?$
- $E[X|C^c] =?$
- $E[X] = E[X|C]P(C) + E[X|C^c]P(C^c) =?$

Mean of $X \sim geometric(p)$

Last time we derived the mean using differentiation. But once we know the total expectation theorem we do it much more quickly!

- Define two disjoint events $\{X = 1\}$ (first trial is success) and $\{X > 1\}$
- We have: $E[X] = E[X|X = 1]P(X = 1) + E[X|X > 1]P(X > 1)$.
- What is $E[X|X = 1]$?

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- What is $E[X|X = 1]$? 1.
- What is $P(X = 1)$? p.
- What is $P(X > 1)$? $1 - p$.
- What is $E[X|X > 1]$? $E[X] + 1$.

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- So the whole thing is $E[X] = p + (1 + E[X])(1 - p)$

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- What is $E[X|X > 1]$? $E[X] + 1$.
- So the whole thing is $E[X] = p + (1 + E[X])(1 - p)$
- Move things around: $E[X] = 1/p$.

Variance of $X \sim Geometric(p)$

- $E[X^2] = \underbrace{E[X^2|X=1]P(X=1)}_{1\times p} + E[X^2|X>1]\underbrace{P(X>1)}_{1-p}$

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- Now,
$$E[X^2|X>1] = \sum_{k=2}^{\infty} k^2 P(X=k|X>1) = \sum_{k=2}^{\infty} k^2 \quad \underbrace{P(X=k-1)}_{\text{memoryless property}} \quad .$$

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  $E[(X+1)^2] = E[X^2+2X+1] = E[X^2] + 2E[X] + 1 = E[X^2] + \frac{2}{p} + 1.$

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  $E[X^2] = p + E[(1+X)^2](1-p) = p + (E[X^2] + \frac{2}{p} + 1)(1-p)$

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- So plugging everything in I have:
$$E[X^2] = p + E[(1+X)^2](1-p) = p + (E[X^2] + \frac{2}{p} + 1)(1-p)$$
- Solving, we get $E[X^2] = \dfrac{2}{p^2} - \dfrac{1}{p}$
- So $\mathrm{var}(X) = E[X^2] - (E[X])^2 = \dfrac{1-p}{p^2}$