Homework Assignment 1

Due in class, Wednesday January 31st

SDS 384-11 Theoretical Statistics

1. Given densities p_n and q_n with respect to some measure μ, let X be distributed according to the distribution with density p_n. Define the likelihood ratio $L_n(X)$ as $L_n(X) = q_n(X)/p_n(X)$. For $p_n(X) > 0, L_n(X) = 1$, if $p_n(X) = q_n(X) = 0$ and $L_n(X) = \infty$ otherwise. Show that the likelihood ratio is a uniformly tight sequence.

2. Consider a sequence of iid random variables $\{X_n\}$ such that $X_i \sim Beta(\theta, 1)$, where $\theta > 0$. Let $\bar{X}_n$ denote the sample mean. The method of moments estimator of θ is $\hat{\theta}_n = \bar{X}_n/(1 - \bar{X}_n)$. Derive the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$.

3. Derive the following one sided improvement of Chebyshev's inequality for a random variable X with variance σ^2.

$$P(X - E[X] \geq t) \leq \frac{\sigma^2}{\sigma^2 + t^2} \tag{1}$$

4. If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

   (a) $g(x) = 1(x \in (0, 10))$

   (b) $g(x) = e^{-x^2}$

   (c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

   (d) $g(x) = x$

5. Consider n i.i.d random variables $\{X_n\}$ uniformly distributed on the set of n points $\{1/n, 2/n, \ldots, 1\}$. Show that $X_n \xrightarrow{d} X$ where $X \sim Uniform(0, 1)$. Does $X_n \xrightarrow{P} X$?