THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 5: Independence

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Statistical Independence

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  - $P(H_1|H_2) = \frac{P(H_1 \cap H_2)}{P(H_2)}$.

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- Say two fair coins are tossed together. We say H_i for $i \in \{1, 2\}$ is the event that the i^{th} coin gives H. Similarly define T_1 and T_2.
  - What is $P(H_1|H_2)$?
  - $P(H_1|H_2) = \dfrac{P(H_1 \cap H_2)}{P(H_2)} = \dfrac{1/4}{1/2} = 1/2 = P(H_1)$.
  - Knowing H_2 does not give me additional information about H_1.

Pairwise Independence

- If $P(A|B) = P(A)$, we say the events A and B are **independent**.
- In other words, knowing B tells us nothing about the probability of event A!
- We can rewrite our definition by writing $P(A|B) = P(A \cap B)/P(B)$:

$$P(A \cap B) = P(A)P(B)$$

- We generally prefer this definition... why?
- We know that $P(A \cap B) = P(B \cap A)$... so if A is independent of B, then B is independent of A.
- **Definition** Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

The gambler

A gambler is rolling 4 fair dice. What is the probability that there is at least one 6 in 4 rolls.

- Each roll is independent. Let X_i denote the event that there is no six in the i^{th} roll.
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Some ground rules

Theorem. If A and B are independent ($A \perp\!\!\!\perp B$), then so are

- A and B^c
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  - $$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) = P(A)P(B^c) \end{aligned}$$
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Mutual independence

Q: I have three events A, B and C, s.t. $P(A \cap B \cap C) = P(A)P(B)P(C)$. Are A, B, C mutually independent?

- You are tossing two fair dice. $A = \{\text{First roll is odd}\}$, $B = \{\text{First roll is} \leq 3\}$ and $C = \{\text{The sum is } 9\}$.

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- $A = \{\text{First roll is } 1, 3, 5\}$. $P(A) = 1/2$.

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- $B = \{\text{First roll is } 1, 2, 3\}$. $P(B) = 1/2$.
- $C = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$. $P(C) = 4/36$.

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- $A \cap B \cap C = \{(3,6)\}$. $P(A \cap B \cap C) = 1/36$.

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- $A \cap B \cap C = \{(3, 6)\}$. $P(A \cap B \cap C) = 1/36$.
- $P(A \cap B \cap C) = P(A)P(B)P(C)$.

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- $A \cap B \cap C = \{(3, 6)\}$. $P(A \cap B \cap C) = 1/36$.
- $P(A \cap B \cap C) = P(A)P(B)P(C)$.
- How about $P(A \cap B)$?

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- $A \cap B \cap C = \{(3, 6)\}$. $P(A \cap B \cap C) = 1/36$.
- $P(A \cap B \cap C) = P(A)P(B)P(C)$.
- How about $P(A \cap B)$?
  - $P(A \cap B) = P(\{\text{First roll is 1 or 3}\}) =$

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- $A \cap B \cap C = \{(3, 6)\}$. $P(A \cap B \cap C) = 1/36$.
- $P(A \cap B \cap C) = P(A)P(B)P(C)$.
- How about $P(A \cap B)$?
  - $P(A \cap B) = P(\{\text{First roll is 1 or 3}\}) = 12/36 = 1/3$.
  - But $P(A) \times P(B) = 1/4$. So $P(A \cap B) \neq P(A)P(B)$.

Mutual independence

$P(A \cap B \cap C) = P(A)P(B)P(C)$ **is too weak for mutual independence.**

- **Definition.** Events $A_1, \ldots A_n$ are mutually independent if for any subset S of $\{1, \ldots, n\}$ we have $P(\cap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$.
- Also, any combination of a set of events and the complements of each the remaining events are mutually independent too. i.e. if A, B, C are mutually independent, then so are A^c, B^c, C and A, B^c, C or A^c, B, C^c etc.
- Mutual independence implies pairwise dependence.
- Does the converse hold?

Pairwise independence $\not\to$ mutual independence

- You are tossing two fair coins. $A = \{\text{First toss is head}\}$, $B = \{\text{Second toss is head}\}$, and $C = \{\text{The outcomes are same}\}$

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- $P(A) = 1/2$, $P(B) = 1/2$ and $P(C) = 1/2$.
- $P(A \cap B) = 1/4 = P(A)P(B)$. So $A \perp\!\!\!\perp B$

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- $P(A) = 1/2$, $P(B) = 1/2$ and $P(C) = 1/2$.
- $P(A \cap B) = 1/4 = P(A)P(B)$. So $A \perp\!\!\!\perp B$
- $P(C \cap B) = 1/4 = P(C)P(B)$. So $C \perp\!\!\!\perp B$

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- $P(A \cap C) = 1/4 = P(A)P(C)$. So $A \perp\!\!\!\perp C$

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- $P(A) = 1/2$, $P(B) = 1/2$ and $P(C) = 1/2$.
- $P(A \cap B) = 1/4 = P(A)P(B)$. So $A \perp\!\!\!\perp B$
- $P(C \cap B) = 1/4 = P(C)P(B)$. So $C \perp\!\!\!\perp B$
- $P(A \cap C) = 1/4 = P(A)P(C)$. So $A \perp\!\!\!\perp C$
- $P(A \cap B \cap C) = 1/4 \neq P(A)P(B)P(C)$

Mutual independence

You are tossing a coin 4 times. Let H_i indicate the event that the i^{th} toss is a head.

- We know that these are mutually independent.
- How about H_1, $H_2 \cap H_3$ and H_4? Are these mutually independent as well?
- How about $H_1 \cap H_2 \cap H_3$ and H_4?
- How about $H_1 \cap H_2 \cap H_3$ and $H_2 \cup H_4$?

Practice problems

Q1. Bob and Alice are playing a game. They alternatively keep tossing a coin and the first one to get a H wins 5\$. The coin is fair.

- What is the sample space? *Hint: the game stops the moment someone gets a head.*
- What are the associated probabilities of the elements in the sample space? Do they sum to one?
- Alice insists she should toss first. Why?

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Bob and Alice are playing a game. They alternatively keep tossing a coin and the first one to get a H wins 5$. The coin is fair.

- Hint 1: the game stops the moment someone gets a head.
- All outcomes have exactly one H, and stops with a H
- $\Omega = \{H, TH, TTH, TTTH, \dots\}$. Its countably infinite.
- What are the probabilities of these events?
  - $P(H) = 1/2$. $P(TH) = (1/2)^2 . P(TTH) = (1/2)^3, \dots$
  - But do they sum to one? $\sum_{i=1}^{\infty} (1/2)^i = 1$.
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Practice problem: solution

Bob and Alice are playing a game. They alternatively keep tossing a coin and the first one to get a H wins 5\$. What is the sample space? The coin is fair.

- Who should play first? Does the person who plays first have a better chance at winning?

Practice problem: solution

Bob and Alice are playing a game. They alternatively keep tossing a coin and the first one to get a H wins 5$. What is the sample space? The coin is fair.

- Who should play first? Does the person who plays first have a better chance at winning?
- When does the first person win?
  - The sequences are $H, TTH, TTTTH \dots$
  - The probability that the first person wins is
  $$(1/2) + (1/2)^3 + (1/2)^5 + \dots = 1/2 \sum_{i=1}^{\infty} (1/4)^i = 2/3.$$
  - So 2 out of 3 times this game is played, Alice will win.
  - We used $\sum_{i=0}^{\infty} p^i = 1/(1-p)$ for some $p < 1$.

Conditional Independence

Bob, and Alice mostly go to their 9am probability class when the weather is sunny. Are the events {Bob goes to class} and {Alice goes to class} independent events?

- **No.** If I know Bob went to class. Then its likely that its sunny. This makes it likely that Alice goes too.
- Given the event {its sunny}, {Bob went to class} does not give us any information about {Alice went to class}.
- So {Bob goes to class} and {Alice goes to class} are **conditionally independent** given {its sunny}.
- This brings us to conditional independence.
- **Definition** Two events A and B are conditionally independent given another event C if $P(A|B \cap C) = P(A|C)$
- We write this as $A \perp\!\!\!\perp B|C$

Conditional Independence

- Recall, we said two events A and B were independent if
  - $P(A|B) = P(A)$ - knowing B tells us nothing about the probability of A.
  - This means that $P(A \cap B) = P(A)P(B)$.

- We can extend this definition to conditional probabilities. We say two events A and B are *conditionally* independent *given* some event C if
  - $P(A|B \cap C) = P(A|C)$.
  - We write this as $A \perp\!\!\!\perp B|C$.
  - Like before, this boils down to: $P(A|B \cap C) = P(A|C)P(B|C)$
  - Can you prove it?

Conditional Independence: Urn example

- Consider two urns, each containing 100 balls.
- The first urn contains all red balls.
- The second urn contains all blue balls.
- We select an urn at random. Let A be the event that the first urn is chosen.
- We select a ball from the urn, note it's color, and put it back. We then select another ball from the urn, note it's color, and put it back.
- Let A be the event that the first urn was chosen, let R_1 be the event that the first ball was red, and let R_2 be the event that the second ball was red.
- *Are R_1 and R_2 independent?*

Conditional Independence: Urn example

Think about it intuitively first.

- I tell you that the first ball is red.

Conditional Independence: Urn example

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- Knowing about the first ball tells you a lot about the color of the second ball.

Conditional Independence: Urn example

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- I tell you that the first ball is red.
- Then you know for sure that the first urn is picked.
- So the second ball has to be red as well.
- Knowing about the first ball tells you a lot about the color of the second ball.
- Clearly they are not independent!

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- We need to compare $P(R_1 \cap R_2)$ with $P(R_1)P(R_2)$
- By the law of total probability,

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$$P(R_1) = P(R_1|A)P(A) + P(R_1|A^c)P(A^c)$$

Conditional Independence: Urn example

- We need to compare $P(R_1 \cap R_2)$ with $P(R_1)P(R_2)$
- By the law of total probability,

$$\begin{aligned} P(R_1) =& P(R_1|A)P(A) + P(R_1|A^c)P(A^c) \\ =& 1 \times 0.5 + 0 \times 0.5 = 0.5 \end{aligned}$$

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- What is $P(R_2)$? Its also 0.5.

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- What is $P(R_2)$? Its also 0.5.
  - Again, by the law of total probability,

$$\begin{aligned} P(R_1 \cap R_2) =& P(R_1 \cap R_2|A)P(A) + P(R_1 \cap R_2|A^c)P(A^c) \\ =& 1 \times 1 \times 0.5 + 0 \times 0 \times 0.5 = 0.5 \end{aligned}$$

  - Let's compare: $P(R_1)P(R_2) = 0.5 \times 0.5 = 0.25$

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- What is $P(R_2)$? Its also 0.5.
  - Again, by the law of total probability,

$$\begin{aligned} P(R_1 \cap R_2) =& P(R_1 \cap R_2|A)P(A) + P(R_1 \cap R_2|A^c)P(A^c) \\ =& 1 \times 1 \times 0.5 + 0 \times 0 \times 0.5 = 0.5 \end{aligned}$$

  - Let's compare: $P(R_1)P(R_2) = 0.5 \times 0.5 = 0.25$
  - So, $P(R_1 \cap R_2) \neq P(R_1)P(R_2)$ – i.e. $R_1 \not\perp R_2$.

Conditional Independence: Urn example

- What if we condition on the chosen urn? Are the two colors now independent?

Conditional Independence: Urn example

- What if we condition on the chosen urn? Are the two colors now independent?
- Our gut says yes... let's just double check the math.

$$P(R_1|A) = P(R_2|A) = 1$$

$$P(R_1 \cap R_2|A) = 1 \times 1 = 1 = P(R_1|A)P(R_2|A)$$

- So, $R_1 \perp\!\!\!\perp R_2|A$
- Knowing which urn was used tells us something about how likely it is that they are both red!
- **Conditional independence does not imply independence!**

Conditional Independence: Urn example

- What if we condition on the chosen urn? Are the two colors now independent?
- Our gut says yes... let's just double check the math.

$$P(R_1|A) = P(R_2|A) = 1$$

$$P(R_1 \cap R_2|A) = 1 \times 1 = 1 = P(R_1|A)P(R_2|A)$$

- So, $R_1 \perp\!\!\!\perp R_2|A$
- Knowing which urn was used tells us something about how likely it is that they are both red!
- **Conditional independence does not imply independence!**
- What if both urns were identical?

Mutual independence $\overset{?}{\rightarrow}$ Conditional Independence

We know that two events which are conditionally independent given another event, need not be independent.

- I toss a fair coin twice. $H_i = \{i^{th}\text{toss is H}\}$. $H_1 \perp\!\!\!\perp H_2$.
- Now I tell you that the two tosses have different outcomes. Call this event E. Is $H_1 \perp\!\!\!\perp H_2 | E$ true?

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- Conditioned on H_1 and E, you know that the tosses are different and the first toss is a H. Does this tell you anything about the second toss?
- Of course! The second toss **has to** be a T! So intuitively, H_1 and H_2 should not be independent given E.

Announcements

- HW2 is now available.
- Please turn in HW1 by 4pm today.
- Today we will practice some problems in class!