

SDS 385: Stat Models for Big Data

Lecture 5: Proximal methods

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Proximal methods

- You want to minimize functions of the form

$$f(x) = \underbrace{g(x)}_{\text{convex, differentiable}} + \underbrace{h(x)}_{\text{convex, nonsmooth}}$$

- If h was differentiable, we would use

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$

- Here we would use:

$$x_{k+1} = \arg \min_z \frac{1}{2\alpha} \underbrace{\|z - (x_t - \alpha \nabla f(x_t))\|^2}_{\text{Stay close to the gradient}} + \underbrace{h(z)}_{\text{minimize } h}$$

Proximal mapping

- Define:

$$\text{prox}_{\alpha}(x) = \arg \min_z \frac{1}{2\alpha} \|x - z\|^2 + h(z)$$

- Proximal GD:

- Choose initial $x^{(0)}$
- Repeat, for $k = 1, 2, 3$

$$x_{k+1} = \text{prox}_{\alpha_k}(x_k - \alpha_k \nabla g(x_k))$$

- But, we just turned one minimization into another. And both has h which is the troublesome part.

Example: Lasso

$$f(\beta) = \frac{1}{2} \|y - X\beta\|^2 + \lambda \|\beta\|_1$$

- The proximal map is:

$$\begin{aligned} \text{prox}_\alpha(\beta) &= \arg \min_z \left(\frac{1}{2\alpha} \|\beta - z\|^2 + \lambda \|\beta\|_1 \right) \\ &= S_{\lambda\alpha}(\beta) \\ [\text{prox}_\alpha(\beta)]_i &= \begin{cases} \beta_i - \lambda\alpha & \text{If } \beta_i > \lambda\alpha \\ 0 & \text{If } |\beta_i| \leq \lambda\alpha \\ \beta_i + \lambda\alpha & \text{If } \beta_i < -\lambda\alpha \end{cases} \end{aligned}$$

Lasso

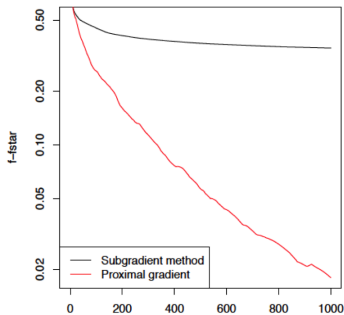
- In this case, the gradient is

$$\nabla g(\beta) = -X^T(y - X\beta)$$

- So the update step for Lasso becomes:

$$\beta_{k+1} = S_{\lambda\alpha}(\beta_k + \alpha X^T(y - X\beta))$$

- This is the Iterative Soft Thresholding Algorithm.



Example: matrix completion

Given a matrix $Y \in \mathbb{R}^{m \times n}$ and observed entries $(i, j) \in \Omega$, you want to fill missing entries by solving:

$$\min_{B \in \mathbb{R}^{m \times n}} \frac{1}{2} \sum_{ij \in \Omega} (Y_{ij} - B_{ij})^2 + \lambda \|B\|_*$$

- $\|B\|_*$ is the nuclear norm of B , defined as:

$$\|B\|_* = \sum_{i=1}^k \sigma_i(B),$$

where k is the rank of B and $\sigma_1(B) \geq \sigma_2(B) \dots$ are the singular values.

- Nuclear norm is a convex approximation of rank, think how you cannot easily minimize ℓ_0 norm, aka the number of nonzero entries, an instead minimize the ℓ_1 norm to induce sparsity in regression problems.

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$$[P_{\Omega}(B)]_{ij} = B_{ij}1((ij) \in \Omega)$$

- So the optimization can also be written as:

$$\min \frac{1}{2} \|P_{\Omega}(Y) - P_{\Omega}(B)\|_F^2 + \lambda \|B\|_*$$

- Gradient of smooth first part: $-(P_{\Omega}(Y) - P_{\Omega}(B))$
- Prox function:

$$\text{prox}_{\alpha}(B) = \arg \min_{Z \in \mathbb{R}^{m \times n}} \frac{1}{2\alpha} \|B - Z\|_F^2 + \lambda \|Z\|_*$$

- We will show that $\text{prox}_\alpha(B) = S_\alpha(B)$, where
- $S_\alpha(B)$ is $U\Sigma_\alpha V^T$, where $B = U\Sigma V^T$ and

$$\Sigma_\alpha(i, i) = \max(\Sigma_{ii} - \lambda, 0)$$

- First, it is known that the subdifferential of the nuclear norm is given by: $\partial\|Z\|_* = \{UV^T + W : \|W\| \leq 1, U^T W = 0, W V = 0\}$, where $U \in \mathbb{R}^{m \times r}$, $V \in \mathbb{R}^{n \times r}$ where $Z = U\Sigma V^T$ where Σ contains the nonzero singular values of Z .
- Now we will show that $0 \in S_{\alpha\lambda}(B) - B + \lambda\alpha\partial\|S_{\alpha\lambda}(B)\|_*$

- Take U_0, V_0 as the singular vectors corresponding to $\sigma_i(B) > \lambda\alpha =: t$.
- Take the remaining singular vectors as U_\perp, V_\perp and the corresponding singular value matrix as Σ_\perp
- $S_t(B) - B = -tU_0V_0^T - U_\perp\Sigma_\perp V_\perp^T$
- $S_t(B) - B + t(U_0V_0^T + W) = tW - U_\perp\Sigma_\perp V_\perp^T$
- Taking $W = U_\perp\Sigma_\perp V_\perp^T/t$, we see that
 - $U^T W = 0$
 - $WV = 0$
 - $\|W\| \leq 1$

- $B_{k+1} = S_{\lambda\alpha}(B + t(P_{\Omega}(Y) - P_{\Omega}(B)))$
- This is called the Soft Impute algorithm.
 - Cai et al, "A Singular Value Thresholding Algorithm for Matrix Completion", 2010.
 - Mazumdar et al 2011, Spectral regularization algorithms for learning large incomplete matrices

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