

SDS 384 11: Theoretical Statistics

Lecture 15: Uniform Law of Large Numbers-

Applications

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Application-Random matrix singular value

Theorem

Consider a random matrix $M = (\xi_{ij})_{i,j \in [n]}$ where ξ_{ij} are standard normal random variables.

$$P(\|M\|_{op} \ge A\sqrt{n}) \le C \exp(-cAn)$$

where c, C are absolute constants and $A \ge C$.

 This works for symmetric wigner ensembles and hermitian matrices as well.

Operator norm

- Let $S_n := \{x \in \mathbb{R}^n : ||x||_2 = 1\}$
- $\bullet \ \|M\|_{op} := \sup_{x \in \mathbb{R}^n} \|Mx\|$
- First note that we have

$$P(\|Mx\| \ge A\sqrt{n}) \le C \exp(-cAn)$$

• This is because for each row M_i , we have

$$\textit{M}_{i}^{T} \times \sim \textit{Subgaussian}(1), (\textit{M}_{i}^{T} \times)^{2} - 1 \sim \textit{Subexponential}(2, 4)$$

• $||Mx||^2 - n \sim Subexponential(2\sqrt{n}, 4)$

Recall sub-exponential random variables?

Theorem

Let X be a sub-exponential random variable with parameters (ν, b) . Then,

$$P(X \ge \mu + t) \le \begin{cases} e^{-\frac{t^2}{2\nu^2}} & \text{if } 0 \le t \le \frac{\nu^2}{b} \\ e^{-\frac{t}{2b}} & \text{if } t \ge \frac{\nu^2}{b} \end{cases}$$

• $P(\|Mx\|^2 - n \ge Cn) \le e^{-Cn/8}, C > 1.$

Can I just use an Union bound?

- Not really.
- But I can form a 1/2 cover of S_n .
- Find $C = \{x^1, \dots, x^N\}$ such that for all $x \in S_n$, $\exists x^i \in S$ $||x x^i|| \le 1/2$.
- Consider $y \in S$ such that $||My|| = ||M||_{op}$. Let x^i be a member of the 1/2 cover s.t. $||y x^i|| \le 1/2$
- So $||M(y x^i)|| \le ||M||_{op}/2$ and $||M(y x^i)|| \ge ||My|| ||Mx^i|| \ge ||M||_{op} ||Mx^i||$.
- Hence $||Mx^{i}|| \ge ||M||_{op}/2$

Using the covering number

$$P(\|M\|_{op} \ge \sqrt{(C+1)n}) \le P(\exists x^i \in \mathcal{C}, \|Mx^i\| \ge \sqrt{(C+1)n}/2)$$

$$\le |\mathcal{C}|P(\|Mx^i\| \ge \sqrt{(C+1)n}/4)$$

$$\le |\mathcal{C}|P(\|Mx^i\|^2 - n \ge (C-3)n/4)$$

$$C > 7 \text{ gives } (C-3)n/4 \ge \nu^2/b \qquad \le |\mathcal{C}| \exp(-(C-3)n/32)$$

• ϵ covering number of the unit ball in n dimensions is bounded by $(1+2/\epsilon)^n$

$$P(\|M\|_{op} \ge \sqrt{(C+1)n}) \le 5^n \exp(-(C-3)n/32)$$

 $\le \exp(-n((C-3)/32-1.6))$

• So C will have to be something like 55!!

Kernel density estimation

Let X_1, X_2, \ldots, X_n be i.i.d. samples of random variable with density f on the real line with support [0,1]. A standard estimate of f is the kernel density estimate

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$$

where $K: \mathbb{R} \to [0,\infty]$ is a kernel function satisfying $\int_{-\infty}^{\infty} K(t)dt = 1$, and h is a bandwidth parameter. Also assume that $|K(x) - K(y)| \le L|x - y|$. Let $K(x) \le K(0)$.

We are interested in the quantity $\sup_{x \in [0,1]} |\hat{f}(x) - E[\hat{f}(x)]|$

Kernel Density Estimation

- First do a ϵ cover of x by $\mathcal{C} := \{x^1, \dots, x^N\}$.
- Let $\tilde{K}((x-X_i)/h) = K(.) EK(.)$
- Similarly $\tilde{f}(.) = \hat{f}(.) E[\hat{f}(.)]$
- The Lipschitz condition gives $\left| \tilde{K} \left(\frac{x X_i}{h} \right) \tilde{K} \left(\frac{y X_i}{h} \right) \right| \le \frac{2L|x y|}{h}$
- So $|\tilde{f}(x) \tilde{f}(x^i)| \le \frac{2L|x x^i|}{h^2}$
- So this gives a $2L\epsilon/h^2$ cover for the \tilde{f} values.

Kernel Density Estimation

- Let y be the point where $\sup_{x \in [0,1]} |\tilde{f}(x)|$ is achieved.
- There exists a *i* such that $|\tilde{f}(y) \tilde{f}(x^i)| \leq 2L\epsilon/h^2$
- So $\exists i, |\tilde{f}(x^i)| \ge \sup_{x \in [0,1]} |\tilde{f}(x)| 2L\epsilon/h^2$
- Finally

$$P\left(\sup_{x\in[0,1]}|\tilde{f}(x)|\geq\delta\right)\leq P(\exists i\in\mathcal{C},|\tilde{f}(x^i)|\geq\sup_{x\in[0,1]}|\tilde{f}(x)|-2L\epsilon/h^2)$$
$$\leq |\mathcal{C}|P\left(|\tilde{f}(x^i)|\geq\delta-2L\epsilon/h^2\right)$$

• Set $\delta = 4L\epsilon/h^2$, the RHS can be obtained using Hoeffding.

Kernel Density Estimation

Hoeffding bound gives:

$$P(|\tilde{f}(x^i)| \ge \delta/2) \le 2 \exp\left(-\frac{nh^2\delta^2}{2}\right)$$

- Also, the covering number of a d dimensional unit sphere is upper bounded by $(1+2/\epsilon)^d$.
- Now plug in $\epsilon = \delta h^2/4L$

$$P\left(\sum_{x\in[0,1]}|\hat{f}(x)-E[\hat{f}(x)]|\geq\delta\right)\leq 2\left(1+\frac{8L}{\delta h^2}\right)^d\exp\left(-\frac{nh^2\delta^2}{2}\right)$$