

# Homework Assignment 1

Due in class, Friday Feb 15th midnight via Canvas

SDS 384-11 Theoretical Statistics

1. We will examine asymptotic equivalence in this question.
  - (a) Show that two sequences of normalized R.V.'s (mean 0 and variance 1) are asymptotically equivalent if their correlation converges to one. Conclude that if  $(X_n - E[X_n])/\sqrt{\text{var}(X_n)} \xrightarrow{d} X$  and if  $\text{corr}(X_n, Y_n) \rightarrow 1$ , then  $(Y_n - EY_n)/\sqrt{\text{var}(Y_n)} \xrightarrow{d} X$ .
  - (b) Suppose  $X_n, Y_n$  have zero mean and equal variance. Is it true that if  $X_n \xrightarrow{d} X$ ?
2. The following inequality bounds the worst case error that may be made using a Poisson Approximation. It is also known as Le Cam's inequality. Let  $X_1, \dots, X_n$  be i.i.d Bernoulli R.V.'s with  $P(X_i = 1) = p_i$ . Let  $S_n = \sum_i X_i$  and let  $\lambda = \sum_i p_i$ , and let  $Z$  be an R.V. with the  $\text{Poisson}(\lambda)$  distribution, i.e.  $\mathcal{P}(\lambda)$ . Show that for all sets  $A$ ,

$$|P(S_n \in A) - P(Z \in A)| \leq \sum_i p_i^2.$$

*Hint: We will prove this using a coupling argument, i.e. we will use a construction which defines  $S_n$  and  $Z$  to be on the same probability space, so that they are close. Let  $U \sim \text{Uniform}(0, 1)$  be i.i.d uniform R.V.'s. Now let  $X_i = 1(U_i \geq 1 - p_i)$ . Now let  $Y_i = 0$  if  $U_i < e^{-p_i}$ . Construct the rest of  $Y_i$ 's PMF using  $U_i$  such that  $Y_i \sim \mathcal{P}(p_i)$ . Now show  $|P(S_n \in A) - P(Z \in A)| \leq \sum_i P(X_i \neq Y_i)$ . Finish the rest of the proof.*

3. Suppose  $X_1, \dots, X_n$  are i.i.d random variables with mean  $\mu$  and variance  $\sigma^2$ . Let  $T_n = \sum_i X_i$ ,  $\mu_n = E[T_n]$  and  $\sigma_n^2 = \text{var}(T_n)$ . Using the Lindeberg-Feller theorem show that

$$\frac{T_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0, 1)$$

4. If  $X_n \xrightarrow{d} X \sim \text{Poisson}(\lambda)$ , is it necessarily true that  $E[g(X_n)] \rightarrow E[g(X)]$ ?
  - (a)  $g(x) = 1(x \in (0, 10))$
  - (b)  $g(x) = e^{-x^2}$
  - (c)  $g(x) = \text{sgn}(\cos(x))$  [ $\text{sgn}(x) = 1$  if  $x > 0$ ,  $-1$  if  $x < 0$  and  $0$  if  $x = 0$ .]
  - (d)  $g(x) = x$
5. Show that if  $\{X_n\}$  and  $\{Y_n\}$  are independent, and if  $X_n \xrightarrow{d} X$  and  $Y_n \xrightarrow{d} Y$ , then  $(X_n, Y_n) \xrightarrow{d} (X, Y)$ , where  $X$  and  $Y$  are taken to be independent.