THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 385: Stat Models for Big Data

Lecture 6: Support Vector Machines

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

https://psarkar.github.io/teaching

Support Vector Machines

- Given training data $(x_i, y_i)_{i=1}^n \in \mathbb{R}^p \times \{-1, 1\}$, we want to minimize:

$$\min_w \frac{w^T w}{2} + C \sum_i \max(0, 1 - y_i w^T x_i)$$

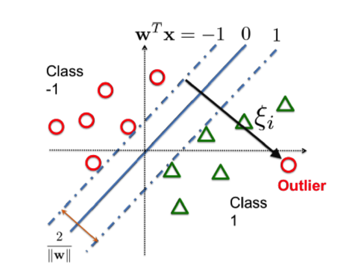


Figure 1: Courtesy Cho-Jui Hsieh's class

SGD for SVM

- Define:

$$f(w) = \frac{1}{n}\sum_i \left(\underbrace{\frac{w^T w}{2} + nC\max(0, 1 - y_i w^T x_i)}_{f_i(w)} \right)$$

- For $t = 1 \ldots$
  - Pick j uniformly at random.
  - Compute $\nabla f_j(w)$
  - Update $w = w - \eta_t \nabla f_j(w)$

SGD for SVM

- In this case, the hinge loss is not differentiable.
- A subgradient of the hinge loss $\max(0, 1 - y_i w^T x_i)$

$$\begin{cases} -y_i x_i & \text{if } 1 - y_i w^T x_i > 0 \\ 0 & \text{if } 1 - y_i w^T x_i < 0 \\ 0 & \text{if } 1 - y_i w^T x_i = 0 \end{cases}$$

SGD for SVM

- For $t = 1 \dots$
  - Pick j uniformly at random.
  - If $y_j w^T x_j < 1$
    - $w_{t+1} = w_t(1 - \eta_t) + \eta_t C n y_i x_i$
  - Else update $w_{t+1} = w_t(1 - \eta_t)$
    - If you store w as a scalar vector pair $w = \gamma v$, then just updating γ leads to $O(1)$ computation.
- This is in "Pegasos: primal estimated subgradient solver for SVM", ICML 2007, Shalev-Schwartz et al.

Dual

- Remember the primal problem?

$$\min_{w,\xi} \frac{w^T w}{2} + C\sum_i \xi_i$$
$$s.t. \quad y_i w^T x_i - 1 + \xi_i \geq 0, \xi_i \geq 0, i = 1, \dots, n$$

- Add lagrange multipliers:

$$\min_{w,\xi} \max_{\alpha \geq 0, \beta \geq 0} \frac{w^T w}{2} + C\sum_i \xi_i - \sum_i \alpha_i (y_i w^T x_i - 1) - \sum_i \beta_i \xi_i$$
$$s.t. \quad y_i w^T x_i - 1 + \xi_i \geq 0, \xi_i \geq 0, i = 1, \dots, n$$

- Under Slater's condition, exchanging min and max does not change the optimal solution.

Dual

- The dual is:

$$\max_{\alpha\geq 0,\beta\geq 0}\min_{w,\xi}\frac{w^Tw}{2}+C\sum_i \xi_i-\sum_i \alpha_i(y_iw^Tx_i-1)-\sum_i \beta_i\xi_i$$
$$s.t.\quad y_iw^Tx_i-1+\xi_i\geq 0,\xi_i\geq 0, i=1,\ldots,n$$

- Differentiate w.r.t w.

$$w^*=\sum_i \alpha_i^* y_i x_i$$

- Differentiate w.r.t ξ_i.

$$C=\alpha_i+\beta_i$$

- Substituting

$$\max_{0\leq\alpha\leq C}-\frac{1}{2}\sum_{ij}\alpha_i\alpha_j y_i y_j x_i^T x_j+\sum_i \alpha_i$$

SVM: the dual problem

- The dual of SVM is given by:

$$\min_{\alpha} \frac{1}{2}\alpha^T Q\alpha - \sum_i \alpha_i$$

$$\text{s.t.}\alpha_i \in [0, C]$$

Where $Q_{ij} = y_i y_j x_i^T x_j$.

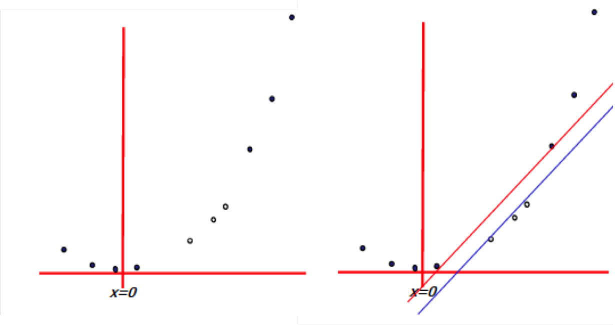
- The primal solution can be written in terms of the dual solution as:

$$w^* = \sum_i y_i \alpha_i^* x_i$$

SVM: the kernel trick

- How do we use an SVM here?
- What if we use more than one dimensions?
- Say $z = (x, x^2)$

SVM: the kernel trick



- So using a different mapping to a higher dimensional space helped.
- So if I want to map my features to a quadratic space, I will have coefficients: $(1, x_1, x_2, \ldots, x_1^2, x_2^2, \ldots, x_1x_2, x_1x_3 \ldots,)$
- So a total of $O(p^2)$ terms. If it is a cubic, then $O(p^3)$ terms. Wow! storage increases exponentially with the dimensionality.

SVM: the kernel trick

$$
\begin{pmatrix} 1 \\ \sqrt{2}a_1 \\ \sqrt{2}a_2 \\ \vdots \\ \sqrt{2}a_m \\ a_1^2 \\ a_2^2 \\ \vdots \\ a_m^2 \\ \sqrt{2}a_1a_2 \\ \sqrt{2}a_1a_3 \\ \vdots \\ \sqrt{2}a_1a_m \\ \sqrt{2}a_2a_3 \\ \vdots \\ \sqrt{2}a_1a_m \\ \vdots \\ \sqrt{2}a_{m-1}a_m \end{pmatrix}
\bullet
\begin{pmatrix} 1 \\ \sqrt{2}b_1 \\ \sqrt{2}b_2 \\ \vdots \\ \sqrt{2}b_m \\ b_1^2 \\ b_2^2 \\ \vdots \\ b_m^2 \\ \sqrt{2}b_1b_2 \\ \sqrt{2}b_1b_3 \\ \vdots \\ \sqrt{2}b_1b_m \\ \sqrt{2}b_2b_3 \\ \vdots \\ \sqrt{2}b_1b_m \\ \vdots \\ \sqrt{2}b_{m-1}b_m \end{pmatrix}
$$

$$
1 + \sum_{i=1}^{m} 2a_ib_i + \sum_{i=1}^{m} a_i^2b_i^2 + \sum_{i=1}^{m}\sum_{j=i+1}^{m} 2a_ia_jb_ib_j
$$

- But this dot product is the same as $(a^T b + 1)^2$!
- Same for cubic maps. So instead of doing $O(p^k)$ computation to compute a dot product in degree k polynomial, you can compute it in $O(p)$ time!

SVM: the kernel trick

- But, how do you predict the class of a new example?
  - You would need to compute $w^T\phi(x)$, where $\phi(x)$ is the high dimensional mapping.
  - This is proportional to the length of $\phi(x)$
  - But remember the form of w?
  - $w^T\phi(x) = \sum_i \alpha_i y_i \phi(x_i)^T \phi(x) = \sum_i \alpha_i y_i K(x_i, x)$

Stochastic Dual coordinate ascent

- For $t = 1 \dots$
  - Compute
    $$\delta^* = \arg \min_{0 < \alpha_i + \delta < C} f(\alpha + \delta e_i)$$
  - Update $\alpha_i = \alpha_i + \delta^*$
  - Update $w = w + \delta^* y_i x_i$ (time complexity $O(nnz(x_i))$)
  - After convergence this gives $w^* = \sum_i \alpha_i^* y_i x_i$

Stochastic Dual coordinate ascent

- Consider the one variable problem:

$$
\begin{aligned}
f(\alpha+\delta e_i) &= \frac{1}{2}(\alpha+\delta e_i)^T Q(\alpha+\delta e_i) - \sum_i \alpha_i - \delta \\
&= \frac{1}{2}\alpha^T Q\alpha + \delta\alpha^T Q e_i + \delta^2 \frac{Q_{ii}}{2} - \sum_i \alpha_i - \delta
\end{aligned}
$$

- Set the gradient to zero:

$$
(Q\alpha)_i + Q_{ii}\delta^* - 1 = 0 \rightarrow \delta^* = \frac{1-(Q\alpha)_i}{Q_{ii}}
$$

- But we have the constraint $0 \leq \alpha_i + \delta \leq C$, so we have:

$$
\alpha_i + \delta^* = \begin{cases} \alpha_i + \dfrac{1-(Q\alpha)_i}{Q_{ii}} & \text{If } \alpha_i + \delta \in [0, C] \\ 0 & \text{If } \alpha_i + \delta < 0 \\ C & \text{If } \alpha_i + \delta > C \end{cases}
$$

Fast computation

- Main computational bottleneck $Q\alpha$
- Write $Q = \underbrace{\text{diag}(y)X}_{R} \underbrace{X^T\text{diag}(y)}_{R^T}$
- Note that:

$$(Q\alpha)_i = R_i \underbrace{R^T\alpha}_{w} = y_i x_i^T w$$

- If you maintain w through the steps, computational complexity becomes $O(nnz(x_i))$
- After each $\alpha_i \leftarrow \alpha_i + \delta^* e_i$ update, you have:

$$w \leftarrow w + \delta^* Q e_i = w + \delta^* Q(:, i)$$

Acknowledgment

Cho-Jui Hsieh's class notes at UC Davis. Andrew Moore's notes on SVM.