Homework Assignment 1

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SDS 384-11 Theoretical Statistics

1. Solution:

(a)

$$\text{Let} \quad z_n = \frac{X_n - E(X_n)}{\sqrt{var(X_n)}} \xrightarrow{d} X \qquad w_n = \frac{Y_n - E(Y_n)}{\sqrt{var(Y_n)}}$$

$$corr(X_n, Y_n) = corr(z_n, w_n) = cov(z_n, w_n) = E(z_n w_n) \to 1$$

$$E[(z_n - w_n)^2] = E(z_n^2 + w_n^2 - 2z_n w_n) = 2 - 2E(z_n w_n) \to 0$$

$$\text{So} \quad (w_n - z_n) \xrightarrow{q.m.} 0 \qquad \Rightarrow \qquad (w_n - z_n) \xrightarrow{d} 0$$

$$\text{We have} \quad z_n \xrightarrow{d} X, \quad (w_n - z_n) \xrightarrow{d} 0$$

So according to Slutsky's theorem:

$$(z_n + w_n - z_n) \xrightarrow{d} (X + 0) \qquad \Rightarrow \qquad w_n \xrightarrow{d} X$$

$$\text{So} \quad \frac{Y - E(Y_n)}{\sqrt{var(Y_n)}} \xrightarrow{d} X$$

(b) It's not true.

A counterexample:

$$\text{Let} \quad u = \begin{cases} 1 & w.p.\ \frac{1}{2} \\ -1 & w.p.\ \frac{1}{2} \end{cases}$$

$$X,\ Y \text{ be independent } r.v. \quad X \sim N(0,1) \quad Y \sim U(-\sqrt{3}, \sqrt{3})$$

$$\Rightarrow \quad E(u) = E(X) = E(Y) = 0 \qquad var(u) = var(X) = var(Y) = 1$$

$$\text{Let} \quad (X_n, Y_n) = \begin{cases} (nu, nu) & w.p.\ \frac{1}{n} \\ (X, Y) & w.p.\ 1 - \frac{1}{n} \end{cases}$$

$$\Rightarrow \quad var(X_n) = var(Y_n) = E(n^2u^2)P(X_n = nu) + E(X^2)P(X_n = X) = n + 1 - \frac{1}{n}$$

$$corr(X_n, Y_n) = \frac{E(X_nY_n)}{var(X_n)} = \frac{\frac{1}{n}E(n^2u^2) + (1 - \frac{1}{n})E(XY)}{n + 1 - \frac{1}{n}} = \frac{n}{n + 1 - \frac{1}{n}} \to 1$$

$$P(X_n \leq x) = P(un \leq x)\frac{1}{n} + P(X \leq x)(1 - \frac{1}{n}) \to P(X \leq x) \quad as\ n \to \infty$$

So, $X_n \xrightarrow{d} X$, similarly we can show that $Y_n \xrightarrow{d} Y$

2.

$$|P(S_n \in A) - P(Z \in A)| \leq \sum_i p_i^2.$$

Solution:

$$\text{Let} \quad U_i \stackrel{iid}{\sim} U(0,1) \quad X_i = I(U_i > 1 - p_i) \quad S_n = \sum_{i=1}^{n} X_i$$

$$\text{Let} \quad Y_i \sim Poisson(p_i) \quad P(Y_i = y) = \frac{e^{-p_i}p_i^y}{y!} \quad for\ y = 0, 1, 2, ...$$

$$P(Y_i = y|U_i) = I[\sum_{k=0}^{y-1} P(Y_i = k) \leq U_i < \sum_{k=0}^{y} P(Y_i = k)]$$

$$\text{Let} \ \ Z = \sum_{i=1}^{n} Y_i \sim Poisson(\lambda)$$

(1)

when $P(S_n \in A) \leq P(Z \in A)$

$$\begin{aligned} |P(S_n \in A) - P(Z \in A)| &= P(Z \in A) - P(S_n \in A) \\ &\leq P(Z \in A) - P(S_n \in A, Z \in A, S_n = Z) \\ &= P(z \in A) - P(z \in A, S_n = Z) \\ &= P(Z \in A, S_n \neq Z) \\ &\leq P(S_n \neq Z) \end{aligned}$$

Similarly, we can show that when $P(S_n \in A) \geq P(Z \in A)$

$$|P(S_n \in A) - P(Z \in A)| \leq P(S_n \neq Z)$$

(2)

$$\sum_{i=1}^{n} P(X_i \neq Y_i) \geq P(\bigcup_{i=1}^{n} X_i \neq Y_i) = 1 - P(\bigcap_{i=1}^{n} X_i = Y_i)$$
$$\geq 1 - P(S_n = Z) = P(S_n \neq Z)$$

$$\text{So} \quad |P(S_n \in A) - P(Z \in A)| \leq P(S_n \neq Z) \leq \sum_{i=1}^{n} P(X_i \neq Y_i)$$

(3)

$$P(X_i \neq Y_i) = P(X_i = 0, Y_i > 0) + P(X_i = 1, Y_i = 0) + P(X_i = 1, Y_i > 1)$$
$$= P(e^{-p_i} \leq U_i \leq 1 - p_i) + P(1 - p_i < U_i < e^{-p_i}) + P(U_i > e^{-p_i} + e^{-1} p_i)$$
$$= 0 + e^{-p_i} - (1 - p_i) + 1 - (e^{-p_i} + e^{-p_i} p_i)$$
$$= p_i(1 - e^{-p_i}) \leq p_i^2$$

$$\text{So} \quad |P(S_n \in A) - P(Z \in A)| \leq \sum_{i=1}^{n} P(X_i \neq Y_i) \leq \sum_{i=1}^{n} p_i^2$$

3. Solution:

$$T_n - \mu_n = \sum_{j=1}^{n} z_{nj} X_j - \sum_{j=1}^{n} z_{nj} E(X_j) = \sum_{j=1}^{n} z_{nj}(X_j - \mu)$$

$$\text{Let} \quad Y_j = z_{nj}(X_j - \mu) \qquad E(Y_j) = 0$$

$$\text{Let} \quad \sigma_{nj}^2 = var(Y_j) = var[z_{nj}(X_j - \mu)] = z_{nj}^2 \sigma^2$$

$$\sigma_n^2 = var(T_n) = var(T_n - \mu_n) = \sum_{j=1}^{n} \sigma_{nj}^2 = \sum_{j=1}^{n} z_{nj}^2 \sigma^2$$

So $T_n - \mu_n = \sum_{j=1}^{n} Y_j$, where Y_j are independent $r.v.$ with mean 0, variance σ_{nj}^2

Therefore $\frac{T_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1)$, as long as the Lindeberg condition holds.

$$\frac{1}{\sigma_n^2}\sum_{j=1}^{n} E[Y_j^2 I(|Y_j| \geq \epsilon\sigma_n)]$$

$$= \frac{1}{\sigma_n^2}\sum_{j=1}^{n} E[z_{nj}^2 (X_j-\mu)^2 I(z_{nj}^2 (X_j-\mu)^2 \geq \epsilon^2\sigma_n^2)]$$

$$= \frac{1}{\sigma_n^2}\sum_{j=1}^{n} z_{nj}^2 E[(X_j-\mu)^2 I((X_j-\mu)^2 \geq \frac{1}{z_{nj}^2}\epsilon^2\sigma_n^2)]$$

$$\leq \frac{1}{\sigma_n^2}\sum_{j=1}^{n} z_{nj}^2 E[(X_j-\mu)^2 I((X_j-\mu)^2 \geq \frac{1}{max\ z_{nj}^2}\epsilon^2\sigma_n^2)]$$

$$= \frac{\sum z_{nj}^2}{\sigma^2 \sum z_{nj}^2} E[(X_j-\mu)^2 I((X_j-\mu)^2 \geq \frac{1}{max\ z_{nj}^2}\epsilon^2\sigma_n^2)]$$

$$= \frac{1}{\sigma^2} E[(X_j-\mu)^2 I((X_j-\mu)^2 \geq \frac{1}{max\ z_{nj}^2}\epsilon^2\sigma_n^2)]$$

$$\text{As} \quad n \to \infty \quad \frac{max\ z_{nj}^2}{\sum z_{nj}^2} \to 0 \quad \Rightarrow \quad \frac{\sum z_{nj}^2}{max\ z_{nj}^2} \to \infty$$

$$\Rightarrow \quad I((X_j-\mu)^2 \geq \frac{1}{max\ z_{nj}^2}\epsilon^2\sigma_n^2) \to 0$$

$$\Rightarrow \quad E[(X_j-\mu)^2 I((X_j-\mu)^2 \geq \frac{1}{max\ z_{nj}^2}\epsilon^2\sigma_n^2)] \to 0$$

$$\text{So} \quad \frac{1}{\sigma_n^2}\sum_{j=1}^{n} E[Y_j^2 I(|Y_j| \geq \epsilon\sigma_n)] \to 0$$

The Lindeberg condition holds, so $\frac{T_n-\mu_n}{\sigma_n} \xrightarrow{d} N(0,1)$

4. Solution:

(a) $g(x) = I(0 < x < 10)$

It is bounded, but not continues at x=0 and x=10, so it's not necessarily true that $E[g(X_n)] \to E[g(x)]$

Counterexample: let $X_n = X + \frac{1}{n}$

$$P(X_n \leq x) = P(X + \frac{1}{n} \leq x) \to P(X \leq x) \qquad \text{so} \quad X_n = X + \frac{1}{n} \to X$$

$$\text{But} \quad E[g(X_n)] = P(0 < X_n < 10) = P(0 \leq X \leq 9)$$
$$E[g(x)] = P(0 < X < 10) = P(1 \leq X \leq 9)$$

$$\text{So} \;\; E[g(X_n)] \;\; \text{doesn't converge to} \;\; E[g(x)]$$

(b) $g(x) = e^{-x^2}$

$0 < e^{-x^2} \leq 1$, so $g(x)$ is bounded and continuous, so according to the Portmanteau Theorem, $E[g(X_n)] \to E[g(x)]$

(c) $g(x) = sgn(cos(x))$

It's bounded and continuous at all integers, so according to the Portmanteau Theorem, $E[g(X_n)] \to E[g(x)]$

(d) $g(x) = x$ It is not bounded, so it's not necessarily true that $E[g(X_n)] \to E[g(x)]$

Counterexample:

$$\text{Let} \quad X_n = \begin{cases} n & w.p. \ \frac{1}{n} \\ X & w.p. \ 1 - \frac{1}{n} \end{cases}$$

$$P(X_n \leq X) = \frac{1}{n}P(n \leq x) + (1 - \frac{1}{n})P(X \leq x) \to P(X \leq x) \quad as \quad n \to \infty$$

$$\text{So} \quad X_n \xrightarrow{d} X$$

$$\text{But} \quad E[g(X_n)] = E[X_n] = 1 + (1 - \frac{1}{n})E(X) \to 1 + E[g(x)]$$

$$\text{So} \;\; E[g(X_n)] \;\; \text{doesn't converge to} \;\; E[g(x)]$$

5. Solution:

$$X_n \xrightarrow{d} X \qquad \Rightarrow \qquad P(X_n \le x) \to P(X \le x)$$

$$Y_n \xrightarrow{d} Y \qquad \Rightarrow \qquad P(Y_n \le y) \to P(Y \le y)$$

X_n and Y_n are independent, so

$$P(X_n \le x, Y_n \le y) = P(X_n \le x)P(Y_n \le y) \to P(X \le x)P(Y \le y) = P(X \le x, Y \le y)$$

$$\Rightarrow \qquad P(X_n \le x, Y_n \le y) \to P(X \le x, Y \le y)$$

$$\text{So} \quad (X_n, Y_n) \xrightarrow{d} (X, Y)$$