Homework Assignment 1

Due in class, Friday Feb 15th midnight via Canvas

SDS 384-11 Theoretical Statistics

1. We will examine asymptotic equivalence in this question.

   (a) Show that two sequences of normalized R.V.'s (mean 0 and variance 1) are asymptotically equivalent if their correlation converges to one. Conclude that if $(X_n - E[X_n])/\sqrt{\text{var}(X_n)} \xrightarrow{d} X$ and if $corr(X_n, Y_n) \to 1$, then $(Y_n - EY_n)/\sqrt{\text{var}(Y_n)} \xrightarrow{d} X$.

   (b) Suppose X_n, Y_n have zero mean and equal variance. If $X_n \xrightarrow{d} X$ and $corr(X_n, Y_n) \to 1$, is it true that $Y_n \xrightarrow{d} X$?

2. The following inequality bounds the worst case error that may be made using a Poisson Approximation. It is also known as Le Cam's inequality. Let $X_1, \dots, X_n$ be i.i.d Bernoulli R.V.'s with $P(X_i = 1) = p_i$. Let $S_n = \sum_i X_i$ and let $\lambda = \sum_i p_i$, and let Z be an R.V. with the Poisson(λ) distribution, i.e. $\mathcal{P}(\lambda)$. Show that for all sets A,

$$|P(S_n \in A) - P(Z \in A)| \leq \sum_i p_i^2.$$

   Hint: We will prove this using a coupling argument, i.e. we will use a construction which defines S_n and Z to be on the same probability space, so that they are close. Let $U \sim Uniform(0,1)$ be i.i.d uniform R.V.'s. Now let $X_i = 1(U_i \geq 1 - p_i)$. Now let $Y_i = 0$ if $U_i < e^{-p_i}$. Construct the rest of Y_i's PMF using U_i such that $Y_i \sim \mathcal{P}(p_i)$. Now show $|P(S_n \in A) - P(Z \in A)| \leq \sum_i P(X_i \neq Y_i)$. Finish the rest of the proof.

3. Suppose $X_1, \dots, X_n$ are i.i.d random variables with mean μ and variance σ^2. Let $T_n = \sum_i z_{ni} X_i$, $\mu_n = E[T_n]$ and $\sigma_n^2 = \text{var}(T_n)$. Using the Lindeberg-Feller theorem show that

$$\frac{T_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1),$$

   provided $\max_{j \leq n} z_{nj}^2 / \sum_j z_{nj}^2 \to 0$.

4. If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

   (a) $g(x) = 1(x \in (0, 10))$

   (b) $g(x) = e^{-x^2}$

   (c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

   (d) $g(x) = x$

5. Show that if $\{X_n\}$ and $\{Y_n\}$ are independent, and if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (X, Y)$, where X and Y are taken to be independent.