Homework Assignment 5

Due in class, Monday April 23rd

SDS 384-11 Theoretical Statistics

1. (VC dimension) Compute the VC dimension of the following function classes
   (a) Circles in $\mathcal{R}^2$
   (b) Axis aligned rectangles in $\mathcal{R}^2$
   (c) Axis aligned squares in $\mathcal{R}^2$
2. In class, you upper bounded the Rademacher complexity of a function class. Now you will derive a lower bound.
   (a) For function classes $\mathcal{F}$ with function values in $[0, 1]$, prove that $E\|\hat{P}_n - P\|_{\mathcal{F}} \geq \frac{\mathcal{R}_{\mathcal{F}}}{2} - \sqrt{\frac{\log 2}{2n}}$.
   (b) Now prove that $\|P - \hat{P}_n\|_{\mathcal{F}} \geq E\|P - \hat{P}_n\|_{\mathcal{F}} - \epsilon$ with probability at least $1 - \exp(-cn\epsilon^2)$ for some constant c.
   (c) Recall the class of all subsets with finite size in $[0, 1]$? Prove that then Rademacher complexity of this class is at least $1/2$. What does this imply?
3. In this exercise, we explore the connection between VC dimension and metric entropy. Given a set class $\mathcal{S}$ with finite VC dimension ν, we show that the function class $\mathcal{F}_S := 1_S, S \in \mathcal{S}$ of indicator functions has metric entropy at most

$$N(\delta; \mathcal{F}_{\mathcal{S}}, L^1(P)) \leq \left(\frac{K \log(3e/\delta)}{\delta}\right)^{\nu} \qquad \text{For a constant } K \tag{1}$$

   Let $\{1_{S^1}, \ldots, 1_{S^N}\}$ be a maximal delta packing in the $L^1(P)$ norm, so that:

$$\|1_{S_i} - 1_{S_j}\|_1 = E[|1_{S_i}(X) - 1_{S_j}(X)|] > \delta \qquad \text{for all } i \neq j$$

   This is an upper bound on the δ covering number.

   (a) Suppose that we generate n samples $X_i, i = 1, \ldots, n$ drawn i.i.d. from P. Show that the probability that every set S_i picks out a different subset of $\{X_1, \ldots, X_n\}$ is at least $1 - \binom{N}{2}(1 - \delta)^n$.
   (b) Using part (a), show that for $N \geq 2$ and $n = \lceil 2 \log N/\delta \rceil$, there exists a set of n points from which $\mathcal{S}$ picks out at least N subsets, and conclude that $N \leq \left(\frac{3e \log N}{\nu\delta}\right)^{\nu}$.
   (c) Use part (b) to show that Eq (1) holds with $K := 3e^2/(e - 1)$.