Homework Assignment 2

Due Friday March 4th midnight

SDS 384-11 Theoretical Statistics

1. Remember Hoeffding's Lemma? We proved it with a weaker constant in class using a symmetrization type argument. Now we will prove the original version. Let X be a bounded r.v. in $[a, b]$ such that $E[X] = \mu$. Let $f(\lambda) = \log E[e^{\lambda(X-\mu)}]$. Show that $f''(\lambda) \leq (b-a)^2/4$. Now use the fundamental theorem of calculus to write $f(\lambda)$ in terms of $f''(\lambda)$ and finish the argument.

2. Consider a r.v. X such that for all $\lambda \in \Re$

$$E[e^{\lambda X}] \leq e^{\frac{\lambda^2\sigma^2}{2}+\lambda\mu} \tag{1}$$

   Prove that:

   (a) $E[X] = \mu$.

   (b) $\mathrm{var}(X) \leq \sigma^2$.

   (c) If the smallest value of σ satisfying the above equation is chosen, is it true that $\mathrm{var}(X) = \sigma^2$? Prove or give a counter example.

3. Given a positive semidefinite matrix $Q \in \mathbb{R}^{n\times n}$, consider $Z = \sum_{i,j} Q_{ij}X_iX_j$. When $X_i \sim N(0, 1)$, prove the Hanson-Wright inequality.

$$P\left(Z \geq \mathrm{trace}(Q) + t\right) \leq \exp\left(-\min\left\{c_1 t/\|Q\|_{op}, c_2 t^2/\|Q\|_F^2\right\}\right),$$

   where $\|Q\|_{op}$ and $\|Q\|_F$ denote the operator and frobenius norms respectively. *Hint: The rotation-invariance of the Gaussian distribution and sub-exponential nature of χ^2-variables could be useful.*

4. We will prove properties of subgaussian random variables here. Prove that:

   (a) Moments of a mean zero subgaussian r.v. X with variance proxy σ^2 satisfy:

$$E[|X^k|] \leq k2^{k/2}\sigma^k\Gamma(k/2), \tag{2}$$

   where Γ is the gamma function.

   (b) If X is a mean 0 subgaussian r.v. with variance proxy σ^2, prove that, $X^2 - E[X^2]$ is a subexponential $(c_1\sigma^2, c_2\sigma^2)$ (we are using the (ν, b) parametrization of subexponentials we did in class, so ν^2 is the variance proxy). Here c_1, c_2 are positive constants.

   (c) Consider two independent mean zero subgaussian r.v.s X_1 and X_2 with variance proxies σ_1^2 and σ_2^2 respectively. Show that X_1X_2 is a subexponential r.v. with parameters $(d_1\sigma_1\sigma_2, d_2\sigma_1\sigma_2)$. Here d_1, d_2 are positive constants.