Two provably consistent divide and conquer clustering algorithms for large networks

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Based on joint work with

Peter Bickel (UC Berkeley) and Purnamrita Sarkar (UT Austin)

IISA Conference 2017, Hyderabad

December 28, 2017

Community detection

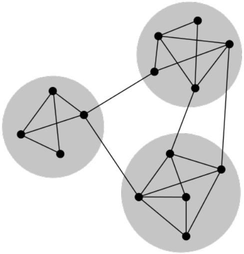

Image source: Wikipedia

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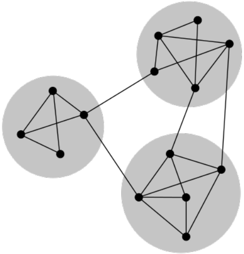

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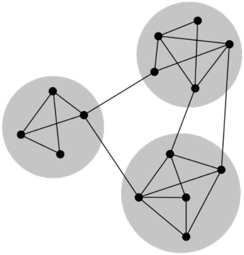

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- **political orientation** of blogs,
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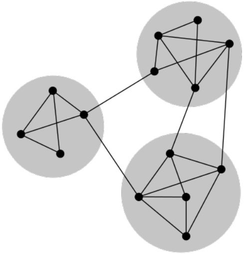

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- **Stochastic Block Model (SBM)**
  K : number of communities/blocks
  σ : latent block membership vector
  B : block connection probabilities
- $\mathbb{P}_{\sigma(i)=a,\sigma(j)=b}(A_{ij} = 1) = B_{ab}$.

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- **Idea of divide and conquer:** take small subgraphs, apply clustering on them and finally put together the community structures.
- Added advantage: **parallelizable**.

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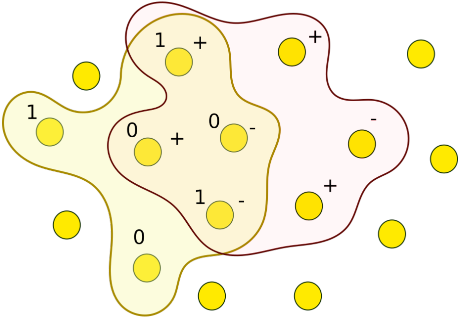

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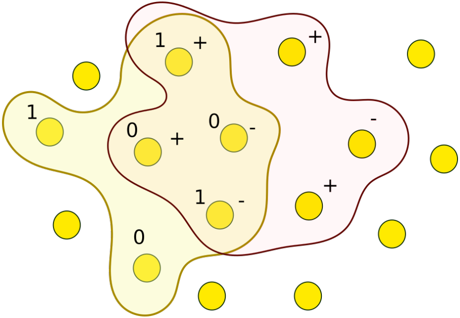

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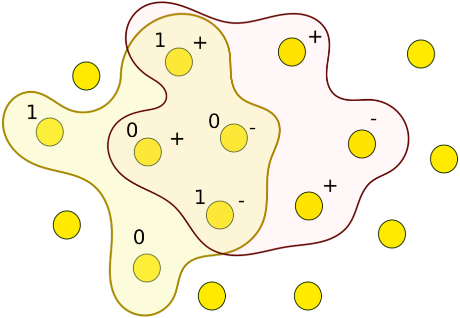

- We propose two such techniques: PACE and GALE.
- Will talk about PACE only.

PACE: Piecewise Averaged Community Estimation

Let $\sigma : [n] \to [K]$ be the unknown community assignment. An equivalent description is given by the $n \times K$ matrix Z with

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Our main idea is to concentrate on estimating **whether two individuals belong to the same community or not**, instead of estimating Z directly, i.e. we focus on

$$C = ZZ^\top,$$

the so-called clustering matrix. However, this is equivalent to estimating the community structure.

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Let A_{S_ℓ} : the adjacency matrix of the network induced by S_ℓ.

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3. Extend $\hat{C}_{S_\ell}$ to an $n \times n$ matrix $\hat{C}^{(\ell)}$:

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Can use any standard algorithm in the first step, e.g. profile likelihood (PL), mean field method (MF), spectral clustering (SC), semi-definite programming (SDP) etc.

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Here $1 \leq \tau \leq T$ is an integer tuning parameter. We will call $\hat{C}_\tau$ as Piecewise Averaged Community Estimator (PACE).

Simulations

Algorithm	ME(%)	Time (seconds)
SDP	0	1588
SDP + PACE + spectral	0	**288**
SDP + PACE + RPKmeans	9.1	**281**

Table: PACE with SDP. $n = 5000$, $d_n = 128$, $m = 500$, 4 equal sized clusters, $T = 100$, 20 parallel workers in Matlab.

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Algorithm	ME(%)	Time (seconds)
RSC	39.6	87
RSC + PACE + spectral	**3.4**	**21**
RSC + PACE + RPKmeans	**34.2**	**14**

Table: PACE with RSC. $n = 5000$, $d_n = 7$, cluster proportions $\boldsymbol{\pi} = (0.2, 0.8)$, $T = 100$, 20 parallel workers in Matlab, 3-hop neighbourhoods were used as subgraphs

Consistency

Measure of distance between two clustering matrices $C = ZZ^T$ and $C' = Z'Z'^T$:

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Theorem (informal)

Suppose $\tilde{\delta}_S$ denotes the misclustering error on a randomly selected subgraph S. Then

$$\mathbb{E}\tilde{\delta}_{\text{PACE}} \preceq \mathbb{E}\tilde{\delta}_S + \textit{ coverage error.}$$

Coverage error is $O(\exp(-O(Tm^2/n^2)))$ for random m-subgraphs.

A concrete application

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Using results of Lei and Rinaldo (2015), Guédon and Vershynin (2015) can show

1. one needs subgraphs of size $m \gg n d_n^{-1}$, and $T \gg \frac{n^2}{m^2}$ for consistency.
2. Complexity becomes $O_\star(\frac{n^3}{d_n})$, from $O(n^3)$.
3. Also a significant boost comes from parallelizability.

Thank You!

Recovering $\hat{Z}$ from $\hat{C}$

- Note that

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- Thus we expect that any distance based clustering algorithm working on rows of $\hat{C}$ will do well.
- Computing distance for n-dimensional vectors is expensive, so we do a random projection of the rows to $\Omega(\log n)$ dimensions (**Johnson-Lindenstrauss**) first, and then do clustering of those projected rows.