Homework Assignment 1

Due via canvas Feb 12th

SDS 384-11 Theoretical Statistics

1. We will do some examples of convergence in distribution and convergence in probability here.

   (a) Let $X_n \sim N(0, 1/n)$. Does $X_n \xrightarrow{d} 0$?

   (b) Let $\{X_n\}$ be independent r.v's such that $P(X_n = n^\alpha) = 1/n$ and $P(X_n = 0) = 1 - 1/n$ for $n \geq 1$, where $\alpha \in (-\infty, \infty)$ is a constant. For what values of α, will you have $X_n \xrightarrow{q.m} 0$? For what values will you have $X_n \xrightarrow{p} 0$?

   (c) Consider the average of n i.i.d random variables $X_1, \ldots, X_n$ with $E[X_1] = \mu$ and $E[|X_1|] < \infty$. Write true or false. Explain.

      i. $\bar{X}_n = o_P(1)$

      ii. $\exp(\bar{X}_n - \mu) = o_P(1)$

      iii. $(\bar{X}_n - \mu)^2 = O_P(1/n)$

2. If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

   (a) $g(x) = 1(x \in (0, 10))$

   (b) $g(x) = e^{-x^2}$

   (c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

   (d) $g(x) = x$

3. Let $X_1, \ldots, X_n$ be independent r.v's with mean zero and variance $\sigma_i^2 := E[X_i^2]$ and $s_n^2 = \sum_i \sigma_i^2$. If $\exists \delta > 0$ s.t. as $n \to \infty$,

$$\frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0,$$

   then $\sum_i X_i / s_n$ converges weakly to the standard normal.

4. The following inequality bounds the worst case error that may be made using a Poisson Approximation. It is also known as Le Cam's inequality. Let $X_1, \ldots, X_n$ be i.i.d Bernoulli R.V.'s with $P(X_i = 1) = p_i$. Let $S_n = \sum_i X_i$ and let $\lambda = \sum_i p_i$, and let Z be an R.V. with the Poisson(λ) distribution, i.e. $\mathcal{P}(\lambda)$. Show that for all sets A,

$$|P(S_n \in A) - P(Z \in A)| \leq \sum_i p_i^2.$$

   Hint: We will prove this using a coupling argument, i.e. we will use a construction which defines S_n and Z to be on the same probability space, so that they are close. Let $U \sim Uniform(0, 1)$ be i.i.d uniform R.V.'s. Now let $X_i = 1(U_i \geq 1 - p_i)$. Now let $Y_i = 0$ if $U_i < e^{-p_i}$. Construct the rest of Y_i's PMF using U_i such that $Y_i \sim \mathcal{P}(p_i)$. Now show $|P(S_n \in A) - P(Z \in A)| \leq \sum_i P(X_i \neq Y_i)$. Finish the rest of the proof.