

SDS 384 11: Theoretical Statistics

Lecture 5: Martingale inequalities

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- Now $f(X) - E[f(X)] = \sum_{i=0}^{n-1} \underbrace{(Y_{i+1} - Y_i)}_{D_i}$
- This forms a Martingale difference sequence.

Martingales

Definition

A sequence of random variables $\{Y_i\}$ adapted to a filtration \mathcal{F}_i is a martingale if, for all i ,

$$E|Y_i| < \infty \quad E[Y_{i+1}|\mathcal{F}_i] = Y_i$$

- A filtration $\{\mathcal{F}_i\}$ is a sequence of nested σ -fields, i.e. $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$.
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Example-partial sums of i.i.d sequences

Example

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d random variables with $E[X_1] = \mu$. Let $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$. Then $\{Y_i = \sum_{k=1}^i X_k - i\mu\}$ is a martingale sequence w.r.t $\{X_i\}$.

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- Y_i is measurable w.r.t \mathcal{F}_i .
- Finally,

$$\begin{aligned} E[Y_{i+1} | \mathcal{F}_i] &= E[X_{i+1} + \sum_{k=1}^i X_k - (i+1)\mu | \mathcal{F}_i] \\ &= \mu + \sum_{k=1}^i X_k - (i+1)\mu = Y_i \end{aligned}$$

Example

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d random variables. Let $Y_i = E[f(X)|X_1, \dots, X_i]$ and assume that $E[|f(X)|] < \infty$. Then $\{Y_i\}_{i=0}^n$ is a martingale sequence w.r.t $\{X_i\}_{i=1}^n$.

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- $E[|Y_i|] = E[|E[f(X)|X_1, \dots, X_i]|] \leq E[|f(X)|] < \infty$. (Use Jensen on $|(\cdot)|$)

Doob construction

Example

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- $E[|Y_i|] = E[|E[f(X)|X_1, \dots, X_i|]|] \leq E[|f(X)|] < \infty$. (Use Jensen on $|(\cdot)|$)
- Furthermore,

$$\begin{aligned} E[Y_{i+1}|X_1, \dots, X_i] &= E[E[f(X)|X_1, \dots, X_{i+1}]|X_1, \dots, X_i] \\ &= E[f(X)|X_1, \dots, X_i] = Y_i \end{aligned} \quad \text{The tower property}$$

Likelihood ratio

Example

Let f, g be two densities such that g is absolutely continuous w.r.t f . Suppose $\{X_i\}_{i=1}^{\infty} \stackrel{iid}{\sim} f$ and Y_n is the likelihood ratio $\prod_{i=1}^n \frac{g(X_i)}{f(X_i)}$ for the first n datapoints. Then $\{Y_n\}$ forms a Martingale sequence w.r.t $\{X_n\}$.

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- First recall that $E[|Y_n|] = E[Y_n] = 1$

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$$\begin{aligned} E[Y_{n+1}|X_1, \dots, X_n] &= E \left[\prod_{i=1}^{n+1} \frac{g(X_i)}{f(X_i)} \middle| X_1, \dots, X_n \right] \\ &= \prod_{i=1}^n \frac{g(X_i)}{f(X_i)} E \left[\frac{g(X_{n+1})}{f(X_{n+1})} \right] = Y_n \end{aligned}$$

Martingale Difference Sequence

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- Let $\{Y_i\}$ be a martingale sequence.
- Then $D_{i+1} = Y_{i+1} - Y_i$ define a Martingale Difference Sequence.
- $E[D_{i+1}|\mathcal{F}_i] = E[Y_{i+1}|\mathcal{F}_i] - E[Y_i|\mathcal{F}_i] = Y_i - Y_i = 0$.

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- $E[D_{i+1}|\mathcal{F}_i] = E[Y_{i+1}|\mathcal{F}_i] - E[Y_i|\mathcal{F}_i] = Y_i - Y_i = 0$.
 - $E[Y_{i+1}|\mathcal{F}_i] = Y_i$ because of the martingale property,
 - $E[Y_i|\mathcal{F}_i] = Y_i$ since Y_i is measurable w.r.t the filtration \mathcal{F}_i .

Concentration inequalities

Theorem

Consider a Martingale sequence $\{D_i\}$ (adapted to a filtration $\{\mathcal{F}_i\}$) that satisfies $E[e^{\lambda D_i} | \mathcal{F}_{i-1}] \leq e^{\lambda^2 \nu_i^2 / 2}$ a.s. for any $|\lambda| < 1/b_i$.

- The sum $\sum_i D_i$ is sub-exponential with parameters $(\sqrt{\sum_k \nu_k^2}, b_*)$ where $b_* := \max_i b_i$.
- Hence for all $t \geq 0$,

$$P \left[\left| \sum_{i=1}^n D_i \right| \geq t \right] \leq \begin{cases} 2e^{-\frac{t^2}{2 \sum_k \nu_k^2}} & \text{If } 0 \leq t \leq \frac{\sum_k \nu_k^2}{b_*} \\ 2e^{-\frac{t}{2b_*}} & \text{If } t > \frac{\sum_k \nu_k^2}{b_*} \end{cases}$$

Proof.

$$\text{Let } X := \sum_{i=1}^n D_i.$$

$$\begin{aligned} E[e^{\lambda \sum_i D_i}] &= E[E[e^{\lambda \sum_i D_i} | \mathcal{F}_{n-1}]] = E[e^{\lambda \sum_{i=1}^{n-1} D_i} E[e^{\lambda D_n} | \mathcal{F}_{n-1}]] \\ &\leq E[e^{\lambda \sum_{i=1}^{n-1} D_i}] e^{\lambda^2 \nu_n^2 / 2} \quad \text{If } |\lambda| < 1/b_n \\ &\leq E[e^{\lambda \sum_{i=1}^{n-2} D_i}] e^{\lambda^2 (\nu_{n-1}^2 + \nu_n^2) / 2} \quad \text{If } |\lambda| < 1/b_n, 1/b_{n-1} \\ &\leq e^{\sum_i \lambda^2 \nu_i^2 / 2} \quad \text{If } |\lambda| < \min_i 1/b_i \end{aligned}$$

Using our previous theorem on sub-exponential random variables, the result is proven in one direction. The other direction is identical leading to the factor of 2. \square

Corollary (Azuma-Hoeffding)

Let $\{D_k\}$ be a Martingale Difference Sequence adapted to the filtration $\{\mathcal{F}_k\}$ and suppose $|D_k| \leq b_k$ a.s. for all $k \geq 1$. Then $\forall t \geq 0$,

$$P \left[\left| \sum_{k=1}^n D_k \right| \geq t \right] \leq 2e^{-\frac{t^2}{2 \sum_{k=1}^n b_k^2}}$$

Proof.

- We can rework the last proof. We need $|E[e^{\lambda D_n} | \mathcal{F}_{n-1}]|$.
- This is bounded by $e^{\lambda^2 b_n^2 / 2}$, since D_n is mean zero sub-gaussian with $\sigma = b_n$.

□

McDiarmid's inequality

Theorem

Let $f : \mathcal{X}^n \rightarrow \mathbb{R}$ satisfy the following bounded difference condition

$\forall x_1, \dots, x_n, x'_i \in \mathcal{X}$:

$$|f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq B_i,$$

then, $P(|f(X) - E[f(X)]| \geq t) \leq 2 \exp\left(-\frac{2t^2}{\sum_i B_i^2}\right)$

- Note that this boils down to Hoeffding's when f is the sum of bounded random variables.

Proof.

- Define $Y_i = E[f(X)|\mathcal{F}_i]$ and $D_i = Y_i - Y_{i-1}$.



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- Define $Y_i = E[f(X)|\mathcal{F}_i]$ and $D_i = Y_i - Y_{i-1}$.
- Since $\{Y_i\}$ is a Martingale sequence w.r.t $\{X_i\}$, $\{D_i\}$ is a Martingale difference sequence.
- We have:

$$\begin{aligned} D_i &= E[f(X)|\mathcal{F}_i] - E[f(X)|\mathcal{F}_{i-1}] \\ &= E[f(X)|X_1, \dots, X_i] - E[f(X)|X_1, \dots, X_{i-1}] \\ &\leq \sup_x (E[f(X)|X_1, \dots, x] - E[f(X)|X_1, \dots, X_{i-1}]) =: U_i \\ D_i &\geq \inf_x (E[f(X)|X_1, \dots, x] - E[f(X)|X_1, \dots, X_{i-1}]) =: L_i \end{aligned}$$

$$U_i - L_i \leq B_i$$



Proof.

- We also have:

$$U_i - L_i \leq B_i$$

- How?

$$\begin{aligned} U_i - L_i &= \sup_x E[f(X)|X_1, \dots, x] - \inf_y E[f(X)|X_1, \dots, y] \\ &= \sup_{x,y} (E[f(X)|X_1, \dots, x] - E[f(X)|X_1, \dots, y]) \\ &= \sup_{x,y} \int (f(x_{1:i-1}, x, X_{i+1:n}) - f(x_{1:i-1}, y, X_{i+1:n})) dP(X_{i+1:n}) \\ &\leq \sup_{x,y} \int |f(x_{1:i-1}, x, X_{i+1:n}) - f(x_{1:i-1}, y, X_{i+1:n})| dP(X_{i+1:n}) \\ &\leq B_i \end{aligned}$$

- Now apply Azuma-Hoeffding.

Example: Mean absolute deviation

Example

Consider an i.i.d random variable sequence $\{X_k\}_{k=1}^{\infty}$ with $|X_k| \leq b$. Define the mean absolute deviation:

$$U = \frac{1}{\binom{n}{2}} \sum_{j \neq k} |X_j - X_k|$$

As we will see later, the above is a type of a pairwise U-Statistics. We want to bound $|U - E[U]|$.

- Note that the summands are not independent.

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- Also note that $||X_i - X_j| - |X_i - X'_j|| \leq |X_j - X'_j| \leq 2b$
- So $|U(x_1, \dots, x_i, \dots, x_n) - U(x_1, \dots, x'_i, \dots, x_n)| \leq \frac{(n-1)2b}{\binom{n}{2}} = \frac{4b}{n}$

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- So $|U(x_1, \dots, x_i, \dots, x_n) - U(x_1, \dots, x'_i, \dots, x_n)| \leq \frac{(n-1)2b}{\binom{n}{2}} = \frac{4b}{n}$
- Use McDiarmid's inequality, $P(|U - E[U]| \geq t) \leq 2 \exp\left(\frac{-nt^2}{8b^2}\right)$

Example: Number of triangles in an Erdos Renyi graph

Example

Consider an Erdős Rényi ($ER(p)$) random graph. What can we say about the number of triangles Δ ?

- Let n be the number of nodes. $m = \binom{n}{2}$ be the number of ordered pairs. Call this set E .
- An $ER(p)$ graph chooses the edges randomly as iid Bernoulli r.v.s $\{X_e : e \in E\}$ with $P(X_e = 1) = p$.
- Let $\mathcal{T} \subset E^3$ be the set of 3-tuples of node pairs which can form a triangle. e.g. $\{(i, j), (j, k), (k, i)\} \in \mathcal{T}$. $|\mathcal{T}| = \binom{n}{3}$.
- We have
$$f(X) = \sum_{\{e_1, e_2, e_3\} \in \mathcal{T}} X_{e_1} X_{e_2} X_{e_3}.$$

Example: Number of triangles in an Erdos Renyi graph–Cont.

Example

Consider an Erdős Rényi (ER(p)) random graph. What can we say about the number of triangles Δ ?

- If I switch $X_e = 1$ to 0 how much can $f(X)$ change?
- It changes by all triangles incident on that edge. The maximum number of such triangles is $n - 2$. So $L = n - 2$.
- Hence $P(|f(X) - E[f(X)]| \geq t) \leq 2e^{-\frac{2t^2}{m(n-2)^2}}$
- $E[f(X)] = \binom{n}{3} p^3$. If we set $t = \Theta(n^2 \log n)$, then the error probability goes to zero.
- But in order for this to give concentration we need, $t/n^3 p^3 \rightarrow 0$, i.e. $np \gg n^{2/3}$

Example: Number of triangles in an Erdos Renyi graph–Cont.

Example

Consider an Erdős Rényi (ER(p)) random graph. What can we say about the number of triangles Δ ?

- One can however use Chen-Stein method to show that $f(X)$ is approximately *Poisson* $\left(\binom{n}{3} p^3\right)$.
- So the above should hold as long as $np \rightarrow \infty$. But McDiarmid requires a much stronger condition!
- What if we could plug in the expected value of the Lipschitz constant, i.e. np^2 ?
- Then the exponent would be $e^{-2t^2/n^4 p^4}$. Taking $t = n^2 p^2$, we see that concentration would amount to having $np \gg \log n$ which matches with the Poisson limit argument.

Lipschitz functions of Gaussian random variables

Definition

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is L -Lipschitz w.r.t the Euclidean norm if

$$|f(x) - f(y)| \leq L\|x - y\|_2 \quad \forall x, y \in \mathbb{R}^n$$

Theorem

Let (X_1, \dots, X_n) be a vector of iid $N(0, 1)$ random variables. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz w.r.t the Euclidean norm. Then $f(X) - E[f(X)]$ is sub-gaussian with parameter at most L , i.e. $\forall t \geq 0$,

$$P(|f(X) - E[f(X)]| \geq t) \leq e^{-\frac{t^2}{2L^2}}$$

- A L Lipschitz function of a vector of i.i.d $N(0, 1)$ random variables concentrate like a $N(0, L^2)$ random variable, irrespective of how long the vector is.