Homework Assignment 2

Due Wednesday March 17th midnight

SDS 384-11 Theoretical Statistics

1. Remember Hoeffding's Lemma? We proved it with a weaker constant in class using a symmetrization type argument. Now we will prove the original version. Let X be a bounded r.v. in $[a, b]$ such that $E[X] = \mu$. Let $f(\lambda) = \log E[e^{\lambda(X-\mu)}]$. Show that $f''(\lambda) \leq (b-a)^2/4$. Now use the fundamental theorem of calculus to write $f(\lambda)$ in terms of $f''(\lambda)$ and finish the argument.

2. Given a positive semidefinite matrix $Q \in \mathbb{R}^{n\times n}$, consider $Z = \sum_{i,j} Q_{ij} X_i X_j$. When $X_i \sim N(0,1)$, prove the Hanson-Wright inequality.

$$P\left(Z \geq \text{trace}(Q) + t\right) \leq \exp\left(-\min\left\{c_1 t/\|Q\|_{op}, c_2 t^2/\|Q\|_F^2\right\}\right),$$

where $\|Q\|_{op}$ and $\|Q\|_F$ denote the operator and frobenius norms respectively. *Hint: The rotation-invariance of the Gaussian distribution and sub-exponential nature of χ^2-variables could be useful.*

3. We will prove properties of subgaussian random variables here. Prove that:

   (a) Moments of a mean zero subgaussian r.v. X with variance proxy σ^2 satisfy:

   $$E[|X^k|] \leq k 2^{k/2} \sigma^k \Gamma(k/2), \tag{1}$$

   where Γ is the gamma function.

   (b) If X is a mean 0 subgaussian r.v. with variance proxy σ^2, prove that, $X^2 - E[X^2]$ is a subexponential $(c_1\sigma^2, c_2\sigma^2)$ (we are using the (ν, b) parametrization of subexponentials we did in class, so ν^2 is the variance proxy). Here c_1, c_2 are positive constants.

   (c) Consider two independent mean zero subgaussian r.v.s X_1 and X_2 with variance proxies σ_1^2 and σ_2^2 respectively. Show that $X_1 X_2$ is a subexponential r.v. with parameters $(d_1\sigma_1\sigma_2, d_2\sigma_1\sigma_2)$. Here d_1, d_2 are positive constants. *Hint: You may need to prove that $(k\Gamma(k/2))^2 \leq k!$. In order to prove that you may need to use the fact that $\Gamma(1/2 + n) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$.*

4. In class we proved McDiarmid's inequality for bounded random variables. But now we will look at extensions for unbounded R.V's. Take a look at "Concentration in unbounded metric spaces and algorithmic stability" by Aryeh Kontorovich, `https://arxiv.org/pdf/1309.1007.pdf`. Reproduce the proof of theorem 1. The steps of this proof is very similar to the martingale based inequalities we looked at in class.