THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 385: Stat Models for Big Data

Lecture 5a: Duality

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

https://psarkar.github.io/teaching

Duality

- So far we were doing unconstrained optimization:

$$\min_{x} f_0(x)$$

- Often you will need to add constraints:

$$\min_{x} f_0(x) \qquad \text{s.t.} \quad f_i(x) \leq 0, i = 1, \ldots, m$$

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- Idea: turn this into an unconstrained optimization – how about optimizing the following instead:

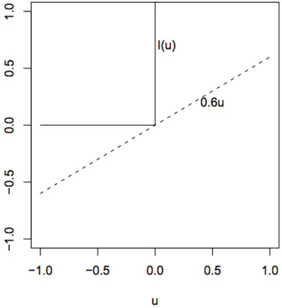
$$J(x) = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0, i = 1, \ldots, m \\ \infty & \text{otherwise} \end{cases} = f_0(x) + \sum_i I(f_i(u))$$

Penalty

- $I(u)$ basically gives infinite penalty if $u > 0$

$$I(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases}$$

- Really messy formulation, non differentiable and discontinuous.



Proximal methods

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- Recall I wanted to minimize $J(x)$, so the problem becomes

$$\min_x \max_\lambda L(x, \lambda)$$

- Still tricky, but in many instances gets easier if we switch the order.

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- $g(\lambda)$ is the dual function.
- the maximization over λ is known as the dual problem
- Note that $g(\lambda)$ is concave, why?
- Since it is a point wise maximum over affine functions.
    - For a fixed x $L(x,\lambda)$ is essentially a linear function of the $\lambda's$

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- So $d^* = \max_{\lambda \geq 0} g(\lambda) \leq p^*$
- So, solving the dual is like finding the tightest lower bound on p^*
- Strong duality: $d^* = p^*$
    - Holds if the optimization problem is convex, and a strictly feasible point exists, i.e. all constraints are satisfied and the inequality constraints are satisfied with strict inequalities.

Example – thanks to Vasko Lalkov and Jingya Li

We are presented with the following linear program.

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

Let us use Lagrangian multipliers to obtain the dual problem. The Lagrangian is:

$$\Lambda(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(4 - x_1 + x_2).$$

The Lagrangian dual function is of the following form:

$$g(\lambda) = \min_{x \in \mathcal{D}} \Lambda(x_1, x_2, \lambda) = 4\lambda + \min_{x_1 \geq 0}\{x_1^2 - \lambda x_1\} + \min_{x_2 \geq 0}\{x_2^2 - \lambda x_2\}.$$

Taking derivatives with respect to x_1 and x_2, we obtain $x_1^* = x_2^* = \frac{\lambda}{2}$, and thus $g(\lambda) = 4\lambda - \frac{\lambda^2}{2}$. Now, we can maximize with respect to λ.

$$\frac{dg(\lambda)}{d\lambda} = 4 - \lambda = 0 \Rightarrow \lambda = 4 \Rightarrow \boxed{x_1^* = x_2^* = 2}$$

Reading

Look at the fantastic writeup by David Knowles on "Lagrangian Duality for Dummies". I have linked this from the class website.