

# **SDS 384 11: Theoretical Statistics**

## **Lecture 14: Uniform Law of Large Numbers- Covering number**

---

Purnamrita Sarkar  
Department of Statistics and Data Science  
The University of Texas at Austin

- Recall that a metric space  $(\mathcal{T}, \rho)$  consists of a nonempty set  $\mathcal{T}$  and a mapping  $\rho : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}$  that satisfies:

# Definitions

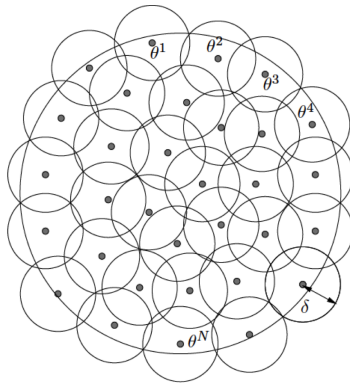
- Recall that a metric space  $(\mathcal{T}, \rho)$  consists of a nonempty set  $\mathcal{T}$  and a mapping  $\rho : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}$  that satisfies:
  - Non-negative:  $\rho(\theta, \theta') \geq 0$  for all  $(\theta, \theta')$  with equality iff  $\theta = \theta'$ .
  - Symmetric:  $\rho(\theta, \theta') = \rho(\theta', \theta)$  for all pairs  $(\theta', \theta)$ , and
  - Triangle ineq holds:  $\rho(\theta, \theta') + \rho(\theta', \theta'') \geq \rho(\theta, \theta'')$
- Examples:
  - $\mathcal{T} = \mathbb{R}^d$ ,  $\rho(\theta, \theta') = \|\theta - \theta'\|_2$
  - $\mathcal{T} = \{0, 1\}^d$  with  $\rho(\theta, \theta') = \frac{1}{d} \sum_i 1(\theta_i \neq \theta'_i)$

# Covering numbers

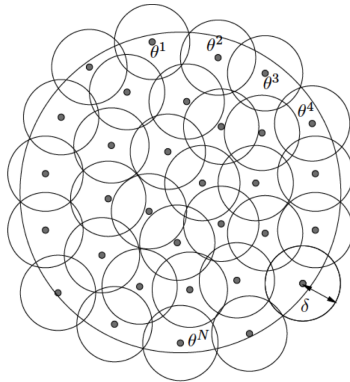
## Definition

A  $\delta$  cover of a set  $\mathcal{T}$  w.r.t to a metric  $\rho$  is a set  $\{\theta^1, \dots, \theta^N\}$  such that for every  $\theta \in \mathcal{T}$ ,  $\exists i \in [N]$ , s.t.  $\rho(\theta, \theta^i) \leq \delta$ . The  $\delta$  covering number  $N(\delta; \mathcal{T}, \rho)$  is the cardinality of the smallest  $\delta$  cover.

- We will consider metric spaces which are totally bounded, i.e.  $N(\delta; \mathcal{T}, \rho) < \infty$  for all  $\delta > 0$ .
- The covering number is non-increasing in  $\delta$ , i.e.  $N(\delta) \geq N(\delta')$  for all  $\delta < \delta'$
- We are interested in something called Metric entropy, which is the logarithm of the covering number.



**Figure 1:** [courtesy: Martin Wainwright's book]



**Figure 1:** [courtesy: Martin Wainwright's book]

- A  $\delta$  covering can be thought of as a union of balls with radius  $\delta$ .

# Covering number of a unit cube

## Example

Consider the interval  $[-1, 1]$  with  $\rho(\theta, \theta') = |\theta - \theta'|$ . We have

$$N(\delta; [-1, 1], |\cdot|) \leq \frac{1}{\delta} + 1$$

# Covering number of a unit cube

## Example

Consider the interval  $[-1, 1]$  with  $\rho(\theta, \theta') = |\theta - \theta'|$ . We have  
$$N(\delta; [-1, 1], |\cdot|) \leq \frac{1}{\delta} + 1$$

- Divide the interval into  $L$  sub-intervals centered at  $\theta^i := -1 + (2i - 1)\delta$  for  $i \in [L]$  and each of length at most  $2\delta$ .
- By construction this is a  $\delta$  covering.
- So  $L \leq 1 + 1/\delta$



# Covering the binary hypercube

## Example

Consider a  $d$  dimensional binary hypercube  $\mathcal{T} = \{0, 1\}^d$  with the Hamming metric defined before.

$$\frac{\log N(\delta; \mathcal{T}, \rho)}{\log 2} \leq \lceil d(1 - \delta) \rceil$$

# Covering the binary hypercube

## Example

Consider a  $d$  dimensional binary hypercube  $\mathcal{T} = \{0, 1\}^d$  with the Hamming metric defined before.

$$\frac{\log N(\delta; \mathcal{T}, \rho)}{\log 2} \leq \lceil d(1 - \delta) \rceil$$

- Let  $S = \{1, 2, \dots, \lceil \delta d \rceil\}$
- Consider the set of binary vectors  $\mathcal{S}(\delta) := \{\theta \in \mathcal{T} : \theta_j = 0, j \in S\}$ .
- By construction, for every binary vector  $\theta' \in \mathcal{T}$ , we can find a vector  $\theta \in \mathcal{S}(\delta)$  such that  $\rho(\theta, \theta') \leq \delta$
- $N(\delta; \mathcal{T}, \rho) \leq |\mathcal{S}(\delta)| = 2^{\lceil d(1-\delta) \rceil}$

# Lower bound on Covering number of the binary hypercube

## Example

Let  $\delta \in (0, 1/2)$ . Consider a  $d$  dimensional binary hypercube  $\mathcal{T} = \{0, 1\}^d$  with the Hamming metric defined before.

$$N \geq \exp\left(\frac{d}{2}(1/2 - \delta)^2\right)$$

# Lower bound on Covering number of the binary hypercube

## Example

Let  $\delta \in (0, 1/2)$ . Consider a  $d$  dimensional binary hypercube  $\mathcal{T} = \{0, 1\}^d$  with the Hamming metric defined before.

$$N \geq \exp\left(\frac{d}{2}(1/2 - \delta)^2\right)$$

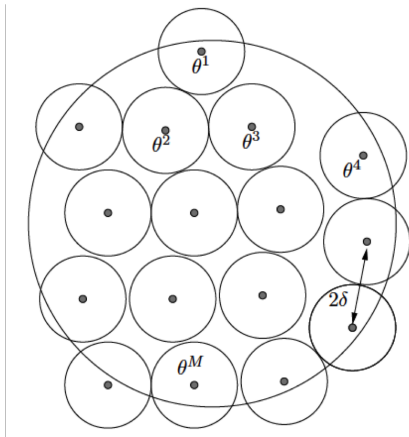
- If  $\{\theta^1, \dots, \theta^N\}$  is a  $\delta$  covering, then the (unrescaled) Hamming balls of radius  $s = \delta d$  around each  $\theta^\ell$  must contain all  $2^d$  vectors.
- Let  $s = \lfloor \delta d \rfloor$
- For each  $\theta^j$  there are exactly  $\sum_{j=0}^d \binom{d}{j}$  vectors within  $\delta d$  distance.
- So  $N \sum_{j=0}^d \binom{d}{j} \geq 2^d$

## Lower bound on Covering number of the binary hypercube

- Let  $\delta \in (0, 1/2)$
- So  $N \sum_{j=0}^s \binom{d}{j} \geq 2^d$
- Now take a Binomial  $(d, 1/2)$  random variable  $X$ .
- $P(X \leq \delta d) = \sum_{j=0}^s \binom{d}{j} / 2^d$
- So  $N \geq \frac{1}{P(X \leq \delta d)}$
- Using the Hoeffding bound gives:  $N \geq \exp(\frac{d}{2}(1/2 - \delta)^2)$
- Using more refined analysis gives:  $N \geq \exp(dKL(\delta||1/2))$

## Definition

An  $\delta$ -packing of  $\mathcal{T}$  w.r.t a metric  $\rho$  is a set  $\{\theta^1, \dots, \theta^M\}$  such that  $\rho(\theta^i, \theta^j) > \delta$  for every distinct pair  $i, j \in [M]$ . The  $\delta$  packing number  $M(\delta; \mathcal{T}, \rho)$  is the cardinality of the largest  $\delta$  packing.



**Figure 2:** [courtesy: Martin Wainwright's book]

- A  $2\delta$  covering can be thought of as a union of balls with radius  $\delta$  such that no two balls touch.

# Relationship between packing and covering numbers

## Theorem

*For all  $\delta > 0$ ,*

$$M(2\delta; \mathcal{T}, \rho) \leq N(\delta; \mathcal{T}, \rho) \leq M(\delta; \mathcal{T}, \rho)$$

- This is saying that packing and covering numbers exhibit the same scaling behavior as  $\delta \rightarrow 0$ .



- **Upper bound:** Let  $V = \{x_1, \dots, x_M\}$  be a  $\delta$  packing of  $\mathcal{T}$ . So for each  $y \in \mathcal{T} \setminus V$ ,  $\exists i, \|y - x_i\| \leq \delta$ . Otherwise we could have added this point and increased the packing number. So,  $V$  is also a  $\delta$  cover. But since the covering number is the size of the smallest  $\delta$  covering, the lower bound holds.
- **Lower bound:** Say there is a  $2\delta$  packing  $\{y_1, \dots, y_M\}$  and a  $\delta$  covering  $\{v_1, \dots, v_n\}$  with  $M > n$ . Now by pigeonhole, there must be two  $y_i, y_j$  who both are in the  $\delta$  ball around some  $v_k$ . But using triangle, we will have  $|y_i - y_j| \leq 2\delta$ , which is a contradiction. So we must have  $m \leq n$ .

# Covering and Packing numbers-example

## Theorem

*Let  $\rho$  be the Euclidean norm on  $\mathbb{R}^d$ . Let  $B_1(0)$  be the unit ball centered at the origin (WLOG).*

$$\frac{1}{\epsilon^d} \leq N(\epsilon, B_1, \rho) \leq (1 + 2/\epsilon)^d$$

# Covering and Packing numbers-example

## Theorem

Let  $\rho$  be the Euclidean norm on  $\mathbb{R}^d$ . Let  $B_1(0)$  be the unit ball centered at the origin (WLOG).

$$\frac{1}{\epsilon^d} \leq N(\epsilon, B_1, \rho) \leq (1 + 2/\epsilon)^d$$

- Consider an  $\epsilon$  cover  $\{\theta^1, \dots, \theta^N\}$ . Now,

$$B_1 \subseteq \bigcup_{i=1}^N B_\epsilon(\theta^i)$$

$$\text{vol}(B_1) \leq N \text{vol}(B_\epsilon(\theta^i)) = N \epsilon^d \text{vol}(B_1)$$

$$N \geq 1/\epsilon^d$$

## Proof-upper bound

- Consider a  $\epsilon$  packing  $\{\theta^1, \dots, \theta^M\}$
- This is an union of disjoint balls of radius  $\epsilon/2$

$$\bigcup_i B_{\epsilon/2}(\theta^i) \subseteq B_{1+\epsilon/2}$$

$$M \text{vol}(B_{\epsilon/2}(\theta^i)) \leq (1 + \epsilon/2) \text{vol}(B_{1+\epsilon/2})$$

$$M(\epsilon/2)^d \text{vol}(B_1) \leq (1 + \epsilon/2)^d \text{vol}(B_1)$$

$$M \leq (1 + 2/\epsilon)^d$$

# Suprema over an infinite space

## Theorem

*Consider a  $d$  dimensional vector of independent  $\text{subG}(\sigma^2)$  random variables. Let  $B_d$  be the unit ball in  $\|\cdot\|_2$  norm. Then the following holds:*

$$E\left[\sup_{\theta \in B_d} \theta^T X\right] \leq 4\sigma\sqrt{d}$$

*Also, for  $\delta \in (0, 1)$ , with probability  $1 - \delta$ ,*

$$\sup_{\theta \in B_d} \theta^T X \leq 4\sigma\sqrt{d} + \sqrt{2\sigma \log(1/\delta)}.$$

# Proof of first half

- Let  $\mathcal{N}_{1/2}$  be a half covering of  $B_d$ . So  $N(1/2, B_d, \|\cdot\|_2) \leq 5^d$
- So for each  $\theta \in B_d$ ,  $\exists z_\theta \in \mathcal{N}_{1/2}$  such that

$$\theta = z_\theta + x, \quad \|x\| = 1/2$$

- So,

$$Y := \sup_{\theta \in B_d} \theta^T X \leq \max_{z_\theta \in \mathcal{N}_{1/2}} z_\theta^T X + \underbrace{\sup_{x \in 1/2 B_d} x^T X}_{Y/2}$$

- Thus, we have:

$$EY \leq 2E \left[ \max_{z_\theta \in \mathcal{N}_{1/2}} z_\theta^T X \right] \leq 2\sigma \sqrt{2 \log |\mathcal{N}_{1/2}|} \leq \sigma \sqrt{8d \log 5} \leq 4\sigma \sqrt{d}$$

- We used the same result as last time.

## Proof of part 2

$$\begin{aligned} P(Y \geq t) &\leq P(\max_{z \in \mathcal{N}_{1/2}} z^T X \geq t/2) \\ &\leq |\mathcal{N}_{1/2}| P(z^T X \geq t/2) \\ &\leq 5^d \exp(-t^2/8\sigma^2 \|z\|^2) \leq 5^d \exp(-t^2/4\sigma^2) \leq \delta \end{aligned}$$

Solving for  $t$  gives,  $P\left(Y \geq 2\sigma\sqrt{\log 5} + 2\sigma\sqrt{\log(1/\delta)}\right) \leq \delta$

## Example-smoothly parametrized problems

- Consider the following function class parametrized by  $\theta \in \Theta$ .

$$\mathcal{F} := \{f_\theta(\cdot) : \theta \in \Theta\}$$

- Let  $\|\cdot\|_\Theta$  be the norm for  $\theta$  and  $\|\cdot\|_{\mathcal{F}}$  be the norm for  $\mathcal{F}$ .
- Say  $\|f_\theta(\cdot) - f_{\theta'}(\cdot)\|_{\mathcal{F}} \leq L\|\theta - \theta'\|_\Theta$
- Then  $N(\epsilon; \mathcal{F}, \|\cdot\|_{\mathcal{F}}) \leq N(\epsilon/L; \Theta, \|\cdot\|_\Theta)$



## Example-smoothly parametrized problems

- A Lipschitz parametrization allows us to go from cover of the  $\Theta$  space to cover of the  $f_\theta$  space with a loss of  $L$ .
- If  $\mathcal{F}$  is parametrized by a compact set of  $d$  parameters then  $N(\epsilon, \mathcal{F}) = O(1/\epsilon^d)$

## Example-Lipschitz functions on the unit interval

### Example

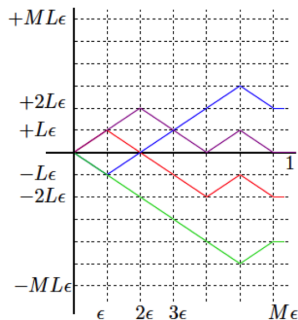
$$\mathcal{F}_L = \{g : [0, 1] \rightarrow \mathbb{R} \mid g(0) = 0, |g(x) - g(y)| \leq L|x - y|, \forall x, y \in [0, 1]\}$$

Metric entropy scales as  $\log N(\delta; \mathcal{F}_L, \|\cdot\|_\infty) \asymp L/\delta$  for small enough  $\delta > 0$ .

- Its sufficient to consider a sufficiently large packing of  $\mathcal{F}_L$
- For a given  $\epsilon$  define  $M = \lfloor \frac{1}{\epsilon} \rfloor$
- Let  $x_i = (i - 1)\epsilon$  for  $i = 1, \dots, M + 1$
- 

$$\phi(x) := \begin{cases} 0 & x < 0 \\ x & x \in [0, 1] \\ 1 & x > 1 \end{cases} \quad (1)$$

- Define  $f_\beta(y) = \sum_{i=1} \beta_i L \epsilon \phi\left(\frac{y - x_i}{\epsilon}\right)$  for  $\beta \in \{-1, 1\}^M$



**Figure 5-2.** The function class  $\{f_\beta, \beta \in \{-1, +1\}^M\}$  used to construct a packing of the Lipschitz class  $\mathcal{F}_L$ . Each function is piecewise linear over the intervals  $[0, \epsilon]$ ,  $[\epsilon, 2\epsilon]$ ,  $\dots$ ,  $[(M-1)\epsilon, M\epsilon]$  with slope either  $+L$  or  $-L$ . There are  $2^M$  functions in total, where  $M = \lceil 1/\epsilon \rceil$ .

- For any pair  $\beta \neq \beta' \in \{-1, 1\}^M$  there is at least one interval where they have the same starting point.
- So  $\|f_\beta(y) - f_{\beta'}(y)\|_\infty \geq 2L\epsilon$
- $f_\beta \in \mathcal{F}_L$  for all  $\beta \in \{-1, 1\}^M$
- So  $f_\beta$  forms a  $2L\epsilon$  packing.
- Making  $\epsilon L = \delta$  we see

$$N(\delta; \mathcal{F}_L, \|\cdot\|_\infty) \geq M(2L\epsilon; \mathcal{F}_L, \|\cdot\|_\infty) = 2^{\lfloor \frac{1}{\epsilon} \rfloor} = 2^{\lfloor \frac{L}{\delta} \rfloor}$$

- Also the set  $f_\beta$  also form a suitable covering of the original functions, and this gives the upper bound.

- The last example can be extended to Lipschitz functions on the Unit cube in higher dimensions, i.e.

$$|f(x) - f(y)| \leq \|x - y\|_\infty \quad \text{for all } x, y \in [0, 1]^d$$

- The same method can be used to show that the metric entropy for this class is the same order as  $(L/\delta)^d$

# Acknowledgment

This lecture was very much based on Martin Wainwright's book.

