SDS384 HOMEWORK 1

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1 PROBLEM 1

1.1 (a)

Using $|\cdot|$ as the distance metric on $\mathbb{R}$, $\forall\ \epsilon > 0$, by the Chebyshev's inequality,

$$\Pr\left(|X_n - 0| \geq \epsilon\right) \leq \frac{1}{n\epsilon^2} \to 0 \text{ as } n \to \infty. \tag{1}$$

Therefore, $X_n \xrightarrow{\mathcal{P}} 0 \implies X_n \xrightarrow{\mathcal{D}} 0$.

1.2 (b)

1.2.1 $\xrightarrow{q.m.}$

We have,

$$\mathbb{E}\left(X_n - 0\right)^2 = \mathbb{E}\left(X_n\right)^2 = n^{2\alpha} \cdot \frac{1}{n} = n^{2\alpha-1} \to 0 \text{ if } 2\alpha - 1 < 0 \iff \alpha < \frac{1}{2}. \tag{2}$$

Thus, $X_n \xrightarrow{q.m.} 0 \iff \alpha < \frac{1}{2}$.

1.2.2 $\xrightarrow{\mathcal{P}}$

$\forall\ \epsilon > 0$, we have,

$$1 - \Pr\left(|X_n| \geq \epsilon\right) = \Pr\left(|X_n| < \epsilon\right) \geq \Pr(X_n = 0) = 1 - \frac{1}{n} \to 1 \text{ as } n \to \infty. \tag{3}$$

Thus, $\Pr\left(|X_n| \geq \epsilon\right) \to 0$ as $n \to \infty \iff X_n \xrightarrow{\mathcal{P}} 0$, for all $\alpha \in \mathbb{R}$.

1.3 (c)

Lemma 1. If $X_n \xrightarrow{\mathcal{P}} X, X_n \xrightarrow{\mathcal{P}} Y$, then $\Pr(X = Y) = 1$.

Proof. $\Pr(X = Y) = 1 \iff \Pr(X \neq Y) = 0 \iff \Pr(|X - Y| > 0) = 0 \iff \forall\ \epsilon > 0, \Pr(|X - Y| >$

$\epsilon) = 0$. This is true because $\forall\ \epsilon > 0$,

$$\begin{aligned}\Pr(|X-Y|>\epsilon) &= \Pr(|X-X_n+X_n-Y|>\epsilon)\\ &\le \Pr(|X-X_n|+|X_n-Y|>\epsilon)\\ &= \Pr\left(|X-X_n|>\frac{\epsilon}{2}\cup|X_n-Y|>\frac{\epsilon}{2}\right)\\ &\le \Pr\left(|X-X_n|>\frac{\epsilon}{2}\right)+\Pr\left(|X_n-Y|>\frac{\epsilon}{2}\right)\end{aligned} \tag{4}$$

Thus, $\Pr(|X-Y|>\epsilon) = 0$, since,

$$\Pr(|X-Y|>\epsilon) = \lim_{n\to\infty}\Pr(|X-Y|>\epsilon) \le \lim_{n\to\infty}\Pr\left(|X-X_n|>\frac{\epsilon}{2}\right)+\Pr\left(|X_n-Y|>\frac{\epsilon}{2}\right) = 0 \tag{5}$$

□

1.3.1 (i)

$\bar{X}_n = o_p(1) \iff \bar{X}_n \xrightarrow{\mathcal{P}} 0$. Consider the following two cases,

If $\mu \neq 0$, $\bar{X}_n \xrightarrow{a.s.} \mu \neq 0 \implies \bar{X}_n \xrightarrow{\mathcal{P}} \mu \neq 0$. From Lemma 1, $\bar{X}_n$ will not converge to 0 in probability and hence $\bar{X}_n \neq o_p(1)$.

If $\mu = 0$, $\bar{X}_n \xrightarrow{a.s.} 0 \implies \bar{X}_n \xrightarrow{\mathcal{P}} 0 \implies \bar{X}_n = o_p(1)$.

Therefore, $\bar{X}_n = o_p(1)$ only when $\mu = 0$.

1.3.2 (ii)

No. By Strong Law of Large Number, $\bar{X}_n \xrightarrow{a.s.} \mu \iff \bar{X}_n - \mu \xrightarrow{a.s.} 0$. Since $\exp(\cdot)$ is continuous on $\mathbb{R}$, by Continuous Mapping Theorem, $\exp(\bar{X}_n-\mu) \xrightarrow{a.s.} \exp(0) = 1 \implies \exp(\bar{X}_n-\mu) \xrightarrow{\mathcal{P}} 1$ and hence, $\exp(\bar{X}_n-\mu)$ will not converge in probability to $0 \iff \exp(\bar{X}_n-\mu) \neq o_p(1)$.

1.3.3 (iii)

True when the second moment exists. Assume $\mathrm{Var}(X_1) = \sigma^2$, since $X_1, \ldots, X_n$ are iid, by the Chebyshev's inequality,

$$\begin{aligned}\Pr\left(n\left(\bar{X}_n-\mu\right)^2 > M\right) &= \Pr\left(\left(\bar{X}_n-\mu\right)^2 > \frac{M}{n}\right)\\ &= \Pr\left(\left|\bar{X}_n-\mu\right| > \sqrt{\frac{M}{n}}\right)\\ &\le \frac{\sigma^2}{n}\cdot\frac{n}{M} = \frac{\sigma^2}{M}.\end{aligned} \tag{6}$$

Therefore $\forall\ \epsilon > 0$, for choosing M, we have,

$$\sup_n \Pr\left(n\left(\bar{X}_n - \mu\right)^2 > M\right) \le \sup_n \frac{\sigma^2}{M} = \frac{\sigma^2}{M} < \epsilon \implies M > \frac{\sigma^2}{\epsilon}. \tag{7}$$

Therefore,

$$n\left(\bar{X}_n - \mu\right)^2 = O_p(1) \iff \left(\bar{X}_n - \mu\right)^2 = O_p\left(\frac{1}{n}\right). \tag{8}$$

2 PROBLEM 2

2.1 (a)

No. $g(x)$ is discontinuous at $\{0, 10\}$ and $\Pr(X \in \{0, 10\}) \neq 0$.

Let $X_n = X + \frac{1}{n}$, then $\forall\ \epsilon > 0, \Pr(|X_n - X| > \epsilon) = \Pr\left(\frac{1}{n} > \epsilon\right) \xrightarrow{n\to\infty} 0 \implies X_n \xrightarrow{\mathcal{P}} X \implies X_n \xrightarrow{\mathcal{D}} X$. Then $\mathbb{E}g(X) = \Pr(X \in (0, 10)) = e^{\lambda}\sum_{k=1}^{9}\frac{\lambda^k}{k!}$, and $\mathbb{E}g(X_n) = \Pr\left(X + \frac{1}{n} \in (0, 10)\right) = 1, \forall\ \lambda \implies \lim_{n\to\infty}\mathbb{E}g(X_n) = 1$. Take $\lambda = 1, \mathbb{E}g(X) \approx 0.6321 \implies \lim_{n\to\infty}\mathbb{E}g(X_n) = 1 \neq 0.6321 = \mathbb{E}g(X)$.

2.2 (b)

Yes.

$g(x) \in (0, 1] \implies |g(x)| \le 1$ and $g(x)$ is continuous on $\mathbb{R} \implies \mathbb{E}g(X_n) \xrightarrow{n\to\infty} \mathbb{E}g(X)$.

2.3 (c)

Yes. Denote $\lambda(\cdot)$ as the Lebesgue measure on $\mathbb{R}$, $D(g)$ as the set of discontinuous points of function g, and $C(g)$ the set of continuous points

$\operatorname{sgn}(\cdot)$ is discontinuous only at 0. $\forall\ x\ s.t.\ y = \cos(x) \neq 0$, cos is continuous at x, sgn is continuous at y, and thus g is continuous at x. Then $D(g) \subseteq \{x : \cos(x) = 0\} = \{\pi\left(k + \frac{1}{2}\right) : k \in \mathbb{Z}\} \implies D(g) \cap \mathbb{N}_0 = \emptyset$ and $\lambda(D(g)) = 0$. Then $\Pr(X \in D(g)) = 0 \iff \Pr(X \in C(g)) = 1$.

Furthermore, $|g| \le 1$ and g is continuous *almost everywhere*, which implies that g is measurable, and hence, $\mathbb{E}g(X_n) \xrightarrow{n\to\infty} \mathbb{E}g(X)$.

2.4 (d)

No. $g(x)$ is unbounded.

Take

$$X_n = \begin{cases} X & \text{with } p = 1 - \frac{1}{n} \\ n & \text{with } p = \frac{1}{n} \end{cases} = X \cdot Y + n \cdot (1 - Y) \tag{9}$$

where $Y \sim Bernoulli\left(1-\frac{1}{n}\right)$. The for any function f bounded and continuous,

$$\begin{aligned}\mathbb{E}f(X_n) &= \mathbb{E}(f(X_n) \mid Y=1)\cdot\frac{n-1}{n} + \mathbb{E}(f(X_n) \mid Y=0)\cdot\frac{1}{n} \\ &= \mathbb{E}(f(X))\cdot\frac{n-1}{n} + f(n)\cdot\frac{1}{n} \xrightarrow{n\to\infty} \mathbb{E}(f(X))\end{aligned} \tag{10}$$

since $f(n)$ is bounded. Therefore $X_n \xrightarrow{\mathcal{D}} X$.

But $\mathbb{E}g(X_n) = \frac{n-1}{n}\mathbb{E}X + 1 \xrightarrow{n\to\infty} \mathbb{E}X + 1 \neq \mathbb{E}X$.

3 PROBLEM 3

Lemma 2. $\forall x \geq 0, \epsilon > 0, \mathbb{1}\left(x \geq \epsilon\right) \leq \frac{x}{\epsilon}$.

Proof. $\forall x \geq 0, \epsilon > 0$,

$$\mathbb{1}\left(x \geq \epsilon\right) = \begin{cases} 1, & x \geq \epsilon \\ 0, & x \leq \epsilon \end{cases}, \quad \frac{x}{\epsilon} \geq \begin{cases} 1 = \mathbb{1}\left(x \geq \epsilon\right), & x \geq \epsilon \\ 0 = \mathbb{1}\left(x \geq \epsilon\right), & x \leq \epsilon \end{cases} \tag{11}$$

□

Need to check that the Linderberg Condition holds. $\forall\ \epsilon > 0$, since $\delta, \epsilon, s_n > 0$,

$$\begin{aligned}\frac{1}{s_n^2}\sum_{i=1}^{n}\mathbb{E}\left[X_i^2\mathbb{1}\left(|X_i| \geq \epsilon s_n\right)\right] &= \frac{1}{s_n^2}\sum_{i=1}^{n}\mathbb{E}\left[X_i^2\mathbb{1}\left(|X_i|^\delta \geq \epsilon^\delta s_n^\delta\right)\right] \\ &\leq \frac{1}{s_n^2}\sum_{i=1}^{n}\mathbb{E}\left[X_i^2\cdot\frac{|X_i|^\delta}{\epsilon^\delta s_n^\delta}\right] \\ &= \frac{1}{\epsilon^\delta}\cdot\frac{\sum_{i=1}^n \mathbb{E}\left[|X_i|^{2+\delta}\right]}{s_n^{2+\delta}} \\ &\xrightarrow{n\to\infty} \frac{1}{\epsilon^\delta}\cdot 0 = 0.\end{aligned} \tag{12}$$

Thus Lindeberg Condition holds. By Lindeberg-feller CLT, $\frac{\sum_i X_i}{s_n}$ converges weakly to the standard normal.

4 PROBLEM 4

For $k \in \mathbb{N}$, let

$$Y_i = \begin{cases} 0, & \text{if } U_i < e^{-p_i} \\ k, & \text{if } e^{-p_i}\sum_{j=0}^{k-1}\frac{p_i^j}{j!} \leq U_i < e^{-p_i}\sum_{j=0}^{k}\frac{p_i^j}{j!}. \end{cases} \tag{13}$$

$$
\begin{aligned}
&\implies \Pr(Y_i = k) = \begin{cases} e^{-p_i}, & \text{if } k = 0 \\ \frac{p_i^k e^{-p_i}}{k!}, & \text{if } k \in \mathbb{N} \end{cases} = \frac{p_i^k e^{-p_i}}{k!} \overset{d}{=} \text{Poi}(p_i) \\
&\implies Z = \sum_{i=1}^{n} Y_i \sim \text{Poi}(\lambda).
\end{aligned} \tag{14}
$$

For S_n and Z, we have,

$$
\begin{aligned}
&\left|\Pr\left(S_n \in A\right) - \Pr(Z \in A)\right| \\
=& \left|\Pr\left(S_n \in A \cap Z \in A\right) + \Pr\left(S_n \in A \cap Z \notin A\right) - \Pr(Z \in A \cap S_n \in A) - \Pr(Z \in A \cap S_n \notin A)\right| \\
=& \left|\Pr\left(S_n \in A \cap Z \notin A\right) - \Pr(Z \in A \cap S_n \notin A)\right| \\
\leq& \Pr\left(\bigcup_{i=1}^{n} \{X_i \neq Y_i\}\right) && (15) \\
\leq& \sum_{i=1}^{n} \Pr\left(X_i \neq Y_i\right) && (16) \\
\leq& \sum_{i=1}^{n} p_i^2. && (17)
\end{aligned}
$$

Equation 15 is from the following fact,

$$
\begin{aligned}
&\{S_n \in A \cap Z \notin A\} \cup \{S_n \notin A \cap Z \in A\} \\
&= \left\{\sum_i X_i \in A \cap \sum_i Y_i \notin A\right\} \cup \left\{\sum_i X_i \notin A \cap \sum_i Y_i \in A\right\} \\
&\subseteq \{\exists\, i \; s.t. X_i \neq Y_i\}
\end{aligned} \tag{18}
$$

which is true since,

$$
\{\forall\, i,\; X_i = Y_i\} \implies \sum_i X_i = \sum_i Y_i \implies \{S_n \in A \cap Z \in A\} \cup \{S_n \notin A \cap Z \notin A\}. \tag{19}
$$

From Equation 18 we have,

$$
\begin{aligned}
&\Pr\left(\{S_n \in A \cap Z \notin A\} \cup \{S_n \notin A \cap Z \in A\}\right) \leq \Pr\left(\bigcup_{i=1}^{n} \{X_i \neq Y_i\}\right) \\
&\implies 0 \leq \Pr\left(\{S_n \in A \cap Z \notin A\}\right) \leq \Pr\left(\bigcup_{i=1}^{n} \{X_i \neq Y_i\}\right) \\
&\qquad 0 \leq \Pr\left(\{S_n \notin A \cap Z \in A\}\right) \leq \Pr\left(\bigcup_{i=1}^{n} \{X_i \neq Y_i\}\right)
\end{aligned} \tag{20}
$$

Using the fact that $\forall\, a, b, c \in \mathbb{R}\,, 0 \leq a \leq c, 0 \leq b \leq c \implies -c \leq a - b \leq c \implies |a - b| \leq c$,

$$0 \leq |\Pr(\{S_n \in A \cap Z \notin A\}) - \Pr(\{S_n \notin A \cap Z \in A\})| \leq \Pr\left(\bigcup_{i=1}^{n} \{X_i \neq Y_i\}\right). \tag{21}$$

This justifies Equation 15.

Equation 16 is from the fact that for events $A, B \subseteq \Omega, \Pr(A \cup B) \leq \Pr(A) + \Pr(B)$.

Equation 17 is from the following fact.

$$X_i = \begin{cases} 0, & \text{if } U_i \leq 1 - p_i \\ 1, & \text{if } U_i \geq 1 - p_i \end{cases}, Y_i = \begin{cases} 0, & \text{if } U_i < e^{-p_i} \\ 1, & \text{if } e^{-p_i} < U_i < e^{-p_i}(1 + p_i) \\ \geq 2, & \text{if } \cdots \end{cases} \tag{22}$$

Note also that $e^{-p_i} \geq 1 - p_i \iff p_i \geq 1 - e^{-p_i}$, we have,

$$\begin{aligned} & \Pr(X_i = Y_i) = \Pr(X_i = Y_i = 0) + \Pr(X_i = Y_i = 1) = 1 - p_i + p_i e^{-p_i} \\ \implies & \Pr(X_i \neq Y_i) = p_i \underbrace{\left(1 - e^{-p_i}\right)}_{\leq p_i} \leq p_i^2 \end{aligned} \tag{23}$$

Sum over both side with respect to i, we get Equation 17.