

# **SDS 384 11: Theoretical Statistics**

## **Lecture 16: Uniform Law of Large Numbers- Dudley's chaining Introduction**

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# Rademacher complexity of function classes

## Example

Suppose  $\mathcal{F}$  is a class parametric functions  $\mathcal{F} := \{f(\theta, \cdot) : \theta \in B_2\}$ , where  $B_2$  is the unit  $L_2$  ball in  $\mathbb{R}^d$ . Assume that  $\mathcal{F}$  is closed under negation.  $f$  is  $L$  Lipschitz w.r.t. the Euclidean distance on  $\Theta$ , i.e.

$$|f(\theta, \cdot) - f(\theta', \cdot)| \leq L\|\theta - \theta'\|_2.$$

$$\mathcal{R}_n(\mathcal{F}) = O\left(L\sqrt{\frac{d \log(Ln)}{n}}\right)$$

- How do we do this?
- Using covering numbers. But we need to define a bunch of stuff first.

# A Stochastic Process

- Consider a set  $\mathcal{T} \subseteq \mathcal{R}^d$ .
- The family of random variables  $\{X_\theta : \theta \in \mathcal{T}\}$  define a Stochastic process indexed by  $\mathcal{T}$ .
- We are often interested in the behavior of this process given its dependence on the structure of the set  $\mathcal{T}$ .
- In the other direction, we want to know the structure of  $\mathcal{T}$  given the behavior of this process.

# Gaussian and Rademacher processes

## Definition

A canonical Gaussian process indexed by  $\mathcal{T}$  is defined as:

$$G_\theta := \langle z, \theta \rangle = \sum_k z_k \theta_k,$$

where  $z_k \stackrel{\text{iid}}{\sim} N(0, 1)$ . The supremum  $\mathcal{G}(\mathcal{T}) := E_z[\sup_{\theta \in \mathcal{T}} G_\theta]$  is the Gaussian complexity of  $\mathcal{T}$ .

# Rademacher complexity

- Replacing the iid standard normal variables by iid Rademacher random variables gives a Rademacher process  $\{R_\theta, \theta \in \mathcal{T}\}$ , where

$$R_\theta := \langle \epsilon, \theta \rangle = \sum_k \epsilon_k \theta_k, \quad \text{where } \epsilon_k \stackrel{\text{iid}}{\sim} \text{Uniform}\{-1, 1\}$$

- $\mathcal{R}(\mathcal{T}) := E_\epsilon[\sup_{\theta \in \mathcal{T}} R_\theta]$  is called the Rademacher complexity of  $\mathcal{T}$ .

# How does this relate to the former notions of Rademacher complexity?

- Recall that

$$\mathcal{R}_{\mathcal{F}} := E\left[\sup_{f \in \mathcal{F}} \left| \sum_i \epsilon_i f(X_i) \right| \right] = E\left[E\left[\sup_{f \in \mathcal{F}} \left| \sum_i \epsilon_i f(X_i) \right| \middle| X_1, \dots, X_n\right]\right]$$

- Now the inner expectation can be upper bounded by

$$E_{\epsilon} \sup_{\theta \in \mathcal{T} \cup -\mathcal{T}} \sum_i \epsilon_i \theta_i, \text{ where } \mathcal{T} \subseteq \mathbb{R}^n \text{ can be written as}$$

$$\mathcal{T} = \{(f(X_1), \dots, f(X_n)) \mid f \in \mathcal{F}\}$$

## Theorem

For  $\mathcal{T} \in \mathbb{R}^d$ ,

$$\mathcal{R}(\mathcal{T}) \leq \sqrt{\frac{\pi}{2}} \mathcal{G}(\mathcal{T}) \leq c \sqrt{\log d} \mathcal{R}(\mathcal{T})$$

- This is showing that there can be there are some sets where the Gaussian complexity can be substantially larger than the Rademacher complexity.
- We will in fact give an example.

## Proof (of first inequality)

$$\begin{aligned}\mathcal{G}(\mathcal{T}) &= E \sup_{\theta \in \mathcal{T}} \sum_i z_i \theta_i \\ &= E \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i |z_i| \theta_i \\ &= E_{\epsilon} E_Z \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i |z_i| \theta_i \\ &\geq E_{\epsilon} \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i E |z_i| \theta_i \\ &= \sqrt{\frac{2}{\pi}} \mathcal{R}(\mathcal{T})\end{aligned}$$



# Proof (of second inequality)

## Theorem (Ledoux-Talagrand contraction (simple form))

Consider  $n$  1-Lipschitz functions  $\phi_i$ .

$$E \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i \phi_i(\theta_i) \leq E \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i \theta_i$$

## Proof (of second inequality)

**Proof.**

$$\begin{aligned}\mathcal{G}(\mathcal{T}) &= E \sup_{\theta \in \mathcal{T}} \sum_i z_i \theta_i = E \sup_{\theta \in \mathcal{T}} \|z\|_\infty \sum_i \frac{z_i}{\|z\|_\infty} \theta_i \\ &= E_{Z, \epsilon} \sup_{\theta \in \mathcal{T}} \|z\|_\infty \sum_i \epsilon_i \frac{|z_i|}{\|z\|_\infty} \theta_i \\ &\leq E_{Z, \epsilon} \sup_{\theta \in \mathcal{T}} \|z\|_\infty \sum_i \epsilon_i \theta_i\end{aligned}$$

Last step follows from the contraction argument. □

# Proof of Ledoux-Talagrand contraction

Proof.

$$\begin{aligned} E \sup_{\theta \in \mathcal{T}} \underbrace{\sum_i \epsilon_i \phi_i(\theta_i)}_{h_n(\theta)} &= E_\epsilon \sup_{\theta} \left( \sum_{i=1}^{n-1} \epsilon_i \phi_i(\theta_i) + \epsilon_n \phi_n(\theta_n) \right) \\ &= E_{\epsilon_1} E_{\epsilon_n} \sup_{\theta} (h_{n-1}(\theta) + \epsilon_n \phi_n(\theta_n)) \\ &= E_{\epsilon_1} E_{\epsilon_n} \left( \frac{1}{2} \sup_{\theta} (h_{n-1}(\theta) + \phi_n(\theta_n)) + \frac{1}{2} \sup_{\theta} (h_{n-1}(\theta) - \phi_n(\theta_n)) \right) \\ &= E_{\epsilon_1} E_{\epsilon_n} \left( \frac{1}{2} (h_{n-1}(\theta^*) + \phi_n(\theta_n^*)) + \frac{1}{2} (h_{n-1}(\tilde{\theta}) - \phi_n(\tilde{\theta}_n)) \right) \\ &= E_{\epsilon_1} E_{\epsilon_n} \left( \frac{1}{2} (h_{n-1}(\theta^*) + h_{n-1}(\tilde{\theta})) + \phi_n(\theta_n^*) - \phi_n(\tilde{\theta}_n) \right) \\ &\leq E_{\epsilon_1} E_{\epsilon_n} \left( \frac{1}{2} (h_{n-1}(\theta^*) + h_{n-1}(\tilde{\theta})) + s(\theta_n^* - \tilde{\theta}_n) \right) \end{aligned}$$

# Proof of Ledoux-Talagrand contraction

Proof.

$$\begin{aligned} E \sup_{\theta \in \mathcal{T}} \underbrace{\sum_i \epsilon_i \phi_i(\theta_i)}_{h_n(\theta)} &= E_{\epsilon} \sup_{\theta} \left( \sum_{i=1}^{n-1} \epsilon_i \phi_i(\theta_i) + \epsilon_n \phi_n(\theta_n) \right) \\ &\leq E_{\epsilon_1^{n-1}} \left( \frac{1}{2} (h_{n-1}(\theta^*) + h_{n-1}(\tilde{\theta})) + s(\theta_n^* - \tilde{\theta}_n) \right) \\ &= E_{\epsilon_1^{n-1}} \left( \frac{1}{2} (h_{n-1}(\theta^*) + s\theta_n^*) + \frac{1}{2} (h_{n-1}(\tilde{\theta}) - s\tilde{\theta}_n) \right) \\ &\leq E_{\epsilon_1^{n-1}} \left( \frac{1}{2} \sup_{\theta} (h_{n-1}(\theta) + s\theta_n) + \frac{1}{2} \sup_{\theta} (h_{n-1}(\theta) - s\theta_n) \right) \\ &\leq E_{\epsilon_1^{n-1}} E_{\epsilon_n} \left( \sup_{\theta} (h_{n-1}(\theta) + \epsilon_n \theta_n) \right) \end{aligned}$$

# Example

## Example

Consider the  $L_1$  ball in  $\mathcal{R}^d$  denoted by  $B_1^d$ .

$$\mathcal{R}(B_1^d) = 1, \mathcal{G}(B_1^d) \leq \sqrt{2 \log d}$$

- $\mathcal{R}(B_1^d) = E\left[\sup_{\|\theta\|_1 \leq 1} \sum_i \theta_i \epsilon_i\right] = E[\|\epsilon\|_\infty] = 1$
- Similarly,  $\mathcal{G}(B_1^d) = E[\|z\|_\infty]$

## Recall the finite class lemma?

### Theorem

*Consider  $z$  with independent standard normal components.*

$$E \max_{a \in A} \langle z, a \rangle \leq \max_{a \in A} \|a\| \sqrt{2 \log |A|}$$

- In our case,  $A = \{e_i, i \in [d]\}$ ,  $e_i(j) = \pm 1(j = i)$ ,  $|A| = 2d$  and  $\max_{a \in A} \|a\| = 1$ .
- This gives a weaker bound on the Gaussian complexity.

# A sub-gaussian process

## Definition

A stochastic process  $\theta \rightarrow X_\theta$  with indexing set  $T$  is sub-Gaussian w.r.t a metric  $d_X$  if  $\forall \theta, \theta' \in T$  and  $\lambda \in \mathbb{R}$ ,

$$E \exp(\lambda(X_\theta - X_{\theta'})) \leq \exp\left(\frac{\lambda^2 d_X(\theta, \theta')^2}{2}\right)$$

- This immediately implies the following tail bound.

$$P(|X_\theta - X_{\theta'}| \geq t) \leq 2 \exp\left(-\frac{t^2}{2d_X(\theta, \theta')^2}\right)$$

# Upper bound by 1 step discretization

## Theorem

(1-step discretization bound). Let  $\{X_\theta, \theta \in \mathcal{T}\}$  be a zero-mean sub-Gaussian process with respect to the metric  $d_X$ . Then for any  $\delta > 0$ , we have

$$E \left[ \sup_{\theta, \theta' \in \mathcal{T}} (X_\theta - X_{\theta'}) \right] \leq 2E \left[ \sup_{\substack{\theta, \theta' \in \mathcal{T} \\ d_X(\theta, \theta') \leq \delta}} (X_\theta - X_{\theta'}) \right] + 2D \sqrt{\log N(\delta; \mathcal{T}, d_X)},$$

where  $D := \max_{\theta, \theta' \in \Theta} d_X(\theta, \theta')$ .

- The mean zero condition gives us:

$$E \left[ \sup_{\theta \in \mathcal{T}} X_\theta \right] = E \left[ \sup_{\theta \in \mathcal{T}} (X_\theta - X_{\theta_0}) \right] \leq E \left[ \sup_{\theta, \theta' \in \mathcal{T}} (X_\theta - X_{\theta'}) \right]$$



$$E \left[ \sup_{\theta, \theta' \in \mathcal{T}} (X_\theta - X_{\theta'}) \right] \leq \underbrace{2 E \left[ \sup_{\substack{\theta, \theta' \in \mathcal{T} \\ d_X(\theta, \theta') \leq \delta}} (X_\theta - X_{\theta'}) \right]}_{\text{Approximation error}} + \underbrace{4 \sqrt{D^2 \log N(\delta; \mathcal{T}, d_X)}}_{\text{Estimation error}}$$

- As  $\delta \rightarrow 0$ , the cover becomes more refined, and so the approximation error decays to zero.
- But the estimation error grows.
- In practice the  $\delta$  can be chosen to achieve the optimal trade-off between two terms.

- Choose a  $\delta$  cover  $T$ .
- For  $\theta, \theta' \in \mathcal{T}$ , let  $\theta^1, \theta^2 \in T$  such that  $d_X(\theta, \theta^1) \leq \delta$  and  $d_X(\theta', \theta^2) \leq \delta$ .

$$\begin{aligned} X_\theta - X_{\theta'} &= (X_\theta - X_{\theta^1}) + (X_{\theta^1} - X_{\theta^2}) + (X_{\theta^2} - X_{\theta'}) \\ &\leq 2 \sup_{\substack{\theta, \theta' \in \mathcal{T} \\ d_X(\theta, \theta') \leq \delta}} (X_\theta - X_{\theta'}) + \sup_{\theta^1, \theta^2 \in T} (X_{\theta^1} - X_{\theta^2}) \end{aligned}$$

- But note that  $X_{\theta^1} - X_{\theta^2} \sim \text{Subgaussian}(d_X(\theta^1, \theta^2))$ .

# Finite class lemma for subgaussian processes

## Theorem

Consider  $X_\theta$  sub-gaussian w.r.t  $d$  on  $\mathcal{T}$  and  $A$  is a set of pairs from  $\mathcal{T}$ .

$$E \max_{(\theta, \theta') \in A} (X_\theta - X_{\theta'}) \leq D \sqrt{2 \log |A|},$$

where  $D := \max_{(\theta, \theta') \in A} d_X(\theta, \theta')$ .

## Finite class lemma

$$\begin{aligned}\exp\left(\lambda E \max_{(\theta, \theta') \in A} (X_\theta - X_{\theta'})\right) &\leq E \exp\left(\lambda \max_{(\theta, \theta') \in A} (X_\theta - X_{\theta'})\right) \\&= \max_{(\theta, \theta') \in A} E \exp(\lambda(X_\theta - X_{\theta'})) \\&\leq \sum_{(\theta, \theta') \in A} \exp\left(\frac{\lambda^2 d_X(\theta, \theta')^2}{2}\right) \\&\leq |A| \exp\left(\frac{\lambda^2 D^2}{2}\right)\end{aligned}$$

- Now optimize over  $\lambda$ .

## Finishing the proof

$$X_\theta - X_{\theta'} \leq 2 \sup_{\substack{\theta, \theta' \in \mathcal{T} \\ d_X(\theta, \theta') \leq \delta}} (X_\theta - X_{\theta'}) + \sup_{\theta^i, \theta^j \in \mathcal{T}} (X_{\theta^1} - X_{\theta^2})$$

$$\begin{aligned} E \left[ \sup_{\theta, \theta' \in \mathcal{T}} (X_\theta - X_{\theta'}) \right] &\leq 2E \left[ \sup_{\substack{\theta, \theta' \in \mathcal{T} \\ d_X(\theta, \theta') \leq \delta}} (X_\theta - X_{\theta'}) \right] + E \left[ \sup_{\theta^i, \theta^j \in \mathcal{T}} (X_{\theta^1} - X_{\theta^2}) \right] \\ &\leq 2E \left[ \sup_{\substack{\theta, \theta' \in \mathcal{T} \\ d_X(\theta, \theta') \leq \delta}} (X_\theta - X_{\theta'}) \right] + D \sqrt{2 \log N(\delta; \mathcal{T}, d_X)^2} \end{aligned}$$

## Revisiting: smoothly parametrized class

### Example

Suppose  $\mathcal{F}$  is a class parametric functions  $\mathcal{F} := \{f(\theta, \cdot) : \theta \in B_2\}$ , where  $B_2$  is the unit  $L_2$  ball in  $\mathbb{R}^d$ . Assume that  $\mathcal{F}$  is closed under negation.  $f$  is  $L$  Lipschitz w.r.t. the Euclidean distance on  $\Theta$ , i.e.

$$|f(\theta, \cdot) - f(\theta', \cdot)| \leq L\|\theta - \theta'\|_2.$$

$$\mathcal{R}_n(\mathcal{F}) = O\left(L\sqrt{\frac{d \log(Ln)}{n}}\right)$$

- Denote  $f(\theta, X_1^n)$  as the vector  $(f(\theta, X_1), \dots, f(\theta, X_n))$ .
- Recall that  $n\mathcal{R}_n(\mathcal{F}) = E \left[ \sup_{f \in \mathcal{F}} \langle \epsilon, f(\theta, X_1^n) \rangle \right] = E \left[ \sup_{\theta \in \Theta} \langle \epsilon, f(\theta, X_1^n) \rangle \right]$
- The process  $f(\theta, X_1^n) \rightarrow \langle \epsilon, f(\theta, X_1^n) \rangle =: Y_\theta$  is mean zero subgaussian.
- Note that  $Y_\theta - Y_{\theta'} \sim \text{Subgaussian}(d_X(\theta, \theta'))$
- We have:

$$d_X(\theta, \theta')^2 = \|f(\theta, X_1^n) - f(\theta', X_1^n)\|^2 \leq nL^2 \|\theta - \theta'\|_2^2$$

- So it is  $L\sqrt{n}$  Lipschitz.

- Also,

$$n\mathcal{R}_n(\mathcal{F}) = E[\sup_{\theta \in \Theta} (Y_\theta - Y_{\theta'})] \leq E[\sup_{\theta, \theta' \in \Theta} (Y_\theta - Y_{\theta'})]$$

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$$n\mathcal{R}_n(\mathcal{F}) \leq \underbrace{2E \sup_{\substack{d_X(\theta, \theta') \leq \delta \\ \theta, \theta' \in \Theta}} (Y_\theta - Y_{\theta'})}_A + 2D\sqrt{\log N(\delta; \mathcal{F}(\Theta, X_1^n), d_X)}$$

- $A \leq \delta E \left[ \sup_{\|v\|_2=1} \langle \epsilon, v \rangle \right] \leq \delta \sqrt{n}$
- $D = \sup_{\theta, \theta'} d_X(\theta, \theta') = 2L\sqrt{n}$



- $N(\delta; \mathcal{F}, d_X) \leq N(\delta/L\sqrt{n}, \Theta, \|\cdot\|_2) \leq \left(1 + \frac{L\sqrt{n}}{\delta}\right)^d$
- Finally,

$$\mathcal{R}_n(\mathcal{F}) \leq \frac{4\delta}{\sqrt{n}} + 4L\sqrt{\frac{d \log(1 + L\sqrt{n}/\delta)}{n}}$$

- Setting  $\delta = 1$  gives:

$$\mathcal{R}_n(\mathcal{F}) \leq \frac{4L}{\sqrt{n}} + 4L\sqrt{\frac{d \log(1 + L\sqrt{n})}{n}}$$

## Examples: Nonparametric functions

### Example

Suppose  $\mathcal{F}$  is a class of  $L$  Lipschitz functions which are supported on  $[0, 1]$  and  $f(0) = 0$ . Note that  $\mathcal{F}$  is closed under negation.  $f$  is  $L$  Lipschitz i.e.  $|f(x) - f(x')| \leq L|x - x'| \forall x, x' \in [0, 1]$ .

$$\mathcal{R}_n(\mathcal{F}) = O\left(\frac{L}{n}\right)^{1/3}$$

## Examples: Nonparametric functions

- Consider the process  $f(X_1^n) \rightarrow \langle \epsilon, f(X_1^n) \rangle = Y_f$
- $Y_f - Y_{f'} \sim \text{subGaussian}(\|f(X_1^n) - f'(X_1^n)\|_2)$
- So  $d_Y(f, f') = \|f(X_1^n) - f'(X_1^n)\|_2 \leq \sqrt{n} \|f - f'\|_\infty$
- The diameter is  $D = \sup_{f, f' \in \mathcal{F}(X_1^n)} d_X(f, f') \leq 2L\sqrt{n}$
- So,  $N(\delta, \mathcal{F}(X_1^n), \|\cdot\|_2) \leq N(\delta/\sqrt{n}, \mathcal{F}(X_1^n), \|\cdot\|_\infty)$

$$\begin{aligned} n\mathcal{R}_n(\mathcal{F}) &\leq E\left[\sup_{f \in \mathcal{F}(X_1^n)} Y_f\right] \leq E\left[\sup_{f, f' \in \mathcal{F}(X_1^n)} (Y_f - Y_{f'})\right] \\ &\leq 2E\left[\sup_{d_Y(f, f') \leq \delta} (Y_f - Y_{f'})\right] + 2D\sqrt{\log N(\delta, \mathcal{F}, \|\cdot\|_2)} \\ &\leq 2E\left[\sup_{d_Y(f, f') \leq \delta} (Y_f - Y_{f'})\right] + 2D\sqrt{\log N(\delta/\sqrt{n}, \mathcal{F}, \|\cdot\|_\infty)} \\ &\leq 2\delta\sqrt{n} + 4L\sqrt{n(L\sqrt{n})/\delta} \\ &\leq 2\delta\sqrt{n} + 4L^{3/2}\sqrt{n^{3/2}/\delta} \end{aligned}$$

- Set  $\delta^{3/2} = CL^{3/2}n^{1/4}$ , i.e.  $\delta = C'Ln^{1/6}$  to get  $\mathcal{R}_n = O(n^{-1/3})$