Theoretical Statistics : Homework 2

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1. Define

$$f(\lambda) := \exp\left(\lambda\mu + \frac{\lambda^2\sigma^2}{2}\right) - \mathbb{E}\left[\exp\left(\lambda X\right)\right]$$

Then, we note that f is a differentiable function and $f(\lambda) \geq 0 \;\; \forall \lambda \in \mathbb{R}$ and $f(0) = 0$.

(a)

$$f'(\lambda) = \left(\mu + \lambda\sigma^2\right)\exp\left(\lambda\mu + \frac{\lambda^2\sigma^2}{2}\right) - \mathbb{E}\left[\exp\left(\lambda X\right)X\right]$$

Therefore,

$$f'(0) = \mu - \mathbb{E}\left[X\right]$$

Since $f(0) = 0$ and $f(\lambda) \geq 0$, therefore $f'(0) = 0$. We can prove it via contradiction. Without loss of generality, let us consider the case of $f'(0) < 0$. Then, $\exists\ \alpha > 0$ such that $f(\alpha) < f(0)$ since f is a decreasing function at $\lambda = 0$, which is not possible since $f(\lambda) \geq 0$. The case for $f'(0) > 0$ can be handled similarly.

(b)

$$f''(\lambda) = \exp\left(\lambda\mu + \frac{\lambda^2\sigma^2}{2}\right)\left[\left(\mu + \sigma^2\lambda\right)^2 + \sigma^2\right] - \mathbb{E}\left[\exp\left(\lambda X\right)X^2\right]$$

Therefore,

$$f''(0) = \sigma^2 + \mu^2 - \mathbb{E}\left[X^2\right] = \sigma^2 - Var\left[X\right]$$

Using the result from part (a), we have that

$$f(0) = 0, f'(0) = 0, f(\lambda) \geq 0 \;\; \forall \lambda \in \mathbb{R}$$

Therefore, $\lambda = 0$ is a local minima of f. Hence, $f''(0) \geq 0$.

(c) Consider a bernoulli random variable X, with mean p specified later. We know that it is a sub-gaussian random variable via Hoeffding's lemma. The MGF of X is given as

$$\mathbb{E}\left[\exp\left(\lambda X\right)\right] = 1 - p + e^{\lambda}$$

and the variance is given as $Var\left(X\right) = p\left(1-p\right)$. To disprove the statement given in the problem, we show that

$$\exists \alpha \in \mathbb{R}, \text{ such that } \underbrace{1 - p + e^{\alpha}}_{LHS} > \underbrace{e^{p\alpha + \frac{p(1-p)\alpha^2}{2}}}_{RHS}$$

For $p = 0.1, \alpha = 10$, we have

$$LHS = 0.9 + e^{10}, \;\; RHS = e^{1+\frac{9}{2}} = e^{5.5}$$

Therefore, $LHS > RHS$ which gives us a counterexample for the claim given in the problem.

2. Consider the random variable cY^2 for a constant c and $Y \sim N(0,1)$. Then the MGF can be written as,

$$\begin{aligned}
\mathbb{E}\left[\exp\left(c\lambda\left(Y^2-1\right)\right)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{c\lambda\left(y^2-1\right)} e^{\frac{-y^2}{2}}\, dy \\
&= \frac{e^{-c\lambda}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2\left(\frac{1-2c\lambda}{2}\right)}\, dy \\
&= \frac{e^{-c\lambda}}{\sqrt{1-2c\lambda}}, \quad |\lambda c| < \frac{1}{2} \\
&\leq e^{2c^2\lambda^2}, |\lambda c| < \frac{1}{4}
\end{aligned}$$

Therefore, cY^2 is a sub-exponential random variable with parameters $(2c, 4c)$.

Now, let $v = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \in \mathbb{R}^n$. Let the eigendecomposition of Q be denoted as $Q = P\Lambda P^T$. Then,

$$\begin{aligned}
Z &= v^T Q v \\
&= v^T P \Lambda P^T v \\
&= \left(P^T v\right)^T \Lambda P^T v \\
&= \sum_{i=1}^{n} \lambda\left(Q\right)_i \left(P^T v\right)_i^2
\end{aligned}$$

Note that $\left(P^T v\right)_i \sim N(0,1)$ since P is a rotation matrix and the gaussian distribution is rotation invariant. Therefore,

$$mean\left(Z\right) = \sum_{i=1}^{n} \lambda\left(Q\right)_i = trace\left(Q\right)$$

and $\lambda\left(Q\right)_i \left(P^T v\right)_i^2$ is a sub-exponential random variable with parameters $\left(2\lambda\left(Q\right)_i, 4\lambda\left(Q\right)_i\right)$. Further, since the X_i's are independent, $\left(P^T v\right)_i$'s are also independent since their covariance matrix can be written as

$$\begin{aligned}
P^T v v^T P &= P^T P \text{ since } vv^T = I \\
&= I \text{ since } P \text{ is a rotation matrix}
\end{aligned}$$

and the being uncorrelated is necessary and sufficient for the independence of gaussian random variables.

Therefore, by the sum property of independent sub-exponential random variables, Z is a sub-exponential random variable with parameters p such that

$$\begin{aligned}
p &:= \left(2\sqrt{\sum_{i=1}^{n} \lambda\left(Q\right)_i^2}, 4\max_{i=1}^{n} \lambda\left(Q\right)_i\right) \\
&= \left(2\left\|Q\right\|_F, 4\left\|Q\right\|_{op}\right) \text{ since } Q \text{ is p.s.d}
\end{aligned}$$

Therefore, using the properties of the tail-bounds of sub-exponential random variables, we have,

$$\begin{aligned}
\mathbb{P}\left(Z \geq mean\left(Z\right) + t\right) &\leq \max\left(\exp\left(-\frac{t^2}{8\left\|Q\right\|_F^2}\right), \exp\left(-\frac{t}{8\left\|Q\right\|_{op}}\right)\right) \\
&= \exp\left(-\min\left\{\frac{t^2}{8\left\|Q\right\|_F^2}, \frac{t}{8\left\|Q\right\|_{op}}\right\}\right)
\end{aligned}$$

Hence proved.

3. (a) We have that

$$\mathbb{E}\left[e^{\lambda X}\right] \leq e^{\frac{\lambda^2\sigma^2}{2}} \quad \forall \lambda \in \mathbb{R}$$

Therefore, using Markov's inequality, we have,

$$\begin{aligned}\mathbb{P}(X \geq t) &= \mathbb{P}\left(e^{\lambda X} > e^{\lambda t}\right) \\ &\leq \frac{e^{\frac{\lambda^2\sigma^2}{2}}}{e^{\lambda t}}\end{aligned}$$

Optimizing for λ we have, $\lambda = \frac{t}{\sigma^2}$. Therefore,

$$\mathbb{P}(X \geq t) \leq e^{-\frac{t^2}{2\sigma^2}}$$

Therefore, following a similar argument for $\mathbb{P}(X \leq -t)$, we have,

$$\mathbb{P}(|X| \geq t) \leq 2e^{-\frac{t^2}{2\sigma^2}}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[|X|^p\right] &= \int_{t=0}^{\infty} \mathbb{P}\left(|X|^p > t\right)\, dt \\ &= \int_{t=0}^{\infty} \mathbb{P}\left(|X| > t^{\frac{1}{p}}\right)\, dt \\ &\leq 2\int_{t=0}^{\infty} e^{-\frac{t^{\frac{2}{p}}}{2\sigma^2}}\, dt\end{aligned}$$

Let

$$\begin{aligned}u &:= \frac{t^{\frac{2}{p}}}{2\sigma^2}, \text{ then} \\ du &= \frac{1}{2\sigma^2}\frac{2}{p}t^{\frac{2}{p}-1}\, dt \\ &= \frac{1}{p\sigma^2}t^{\frac{2}{p}-1}\, dt \\ &= \frac{1}{p\sigma^2}\left(2\sigma^2 u\right)^{1-\frac{p}{2}}\, dt\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[|X|^p\right] &\leq 2\int_{u=0}^{\infty} e^{-u} p\sigma^2 \left(2\sigma^2 u\right)^{\frac{p}{2}-1}\, du \\ &= \frac{2p\sigma^2}{2\sigma^2}\left(2\sigma^2\right)^{\frac{p}{2}} \int_{u=0}^{\infty} e^{-u} u^{\frac{p}{2}-1} du \\ &= p\left(2\sigma^2\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right)\end{aligned}$$

Hence proved.

(b) Using the result of part (a), we have,

$$\begin{aligned}\mathbb{E}\left[\left|X^{2k}\right|\right] &\leq 2k2^k\left(\sigma^2\right)^k \Gamma(k) \\ &= 2k\left(2\sigma^2\right)^k (k-1)\,! \\ &= 2\left(2\sigma^2\right)^k k! \qquad (1)\end{aligned}$$

Then,

$$
\begin{aligned}
\mathbb{E}\left[e^{\lambda X^2}\right] &= \sum_{k=0}^{\infty} \frac{\lambda^k \mathbb{E}\left[X^{2k}\right]}{k!} \\
&= 1 + \lambda \mathbb{E}\left[X^2\right] + \frac{\lambda^2 \mathbb{E}\left[X^4\right]}{2} + \sum_{k=3}^{\infty} \frac{\lambda^k \mathbb{E}\left[X^{2k}\right]}{k!} \\
&\leq 1 + \lambda \mathbb{E}\left[X^2\right] + 8\lambda^2\sigma^4 + 2\sum_{k=3}^{\infty} \lambda^k 2^k \left(\sigma^2\right)^k \quad \text{using 1} \\
&\leq 1 + \lambda \mathbb{E}\left[X^2\right] + 8\lambda^2\sigma^4 \left(1 + \sum_{k=3}^{\infty} \lambda^{k-2} 2^{k-2} \left(\sigma^2\right)^{k-2}\right) \\
&= 1 + \lambda \mathbb{E}\left[X^2\right] + 8\lambda^2\sigma^4 \left(1 + \sum_{k=1}^{\infty} \left(2\sigma^2\lambda\right)^k\right) \\
&= 1 + \lambda \mathbb{E}\left[X^2\right] + 8\lambda^2\sigma^4 \left(1 + \frac{2\sigma^2\lambda}{1-2\sigma^2\lambda}\right) \\
&= 1 + \lambda \mathbb{E}\left[X^2\right] + \frac{8\lambda^2\sigma^4}{1-2\sigma^2\lambda} \\
&\leq 1 + \lambda \mathbb{E}\left[X^2\right] + 16\lambda^2\sigma^4, \;\; 2\sigma^2\left|\lambda\right| \leq \frac{1}{2} \\
&\leq \exp\left(\lambda \mathbb{E}\left[X^2\right] + 16\lambda^2\sigma^4\right) \text{ using } 1 + x \leq e^x
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\left[e^{\lambda\left(X^2 - \mathbb{E}\left[X^2\right]\right)}\right] \leq \exp\left(16\lambda^2\sigma^4\right), 2\sigma^2\left|\lambda\right| \leq \frac{1}{2}
$$

which completes our proof.

(c) We have $\forall\ \lambda \in \mathbb{R}$,

$$
\begin{aligned}
\mathbb{E}\left[e^{\lambda X_1}\right] &\leq e^{\frac{\lambda^2\sigma_1^2}{2}} \\
\mathbb{E}\left[e^{\lambda X_2}\right] &\leq e^{\frac{\lambda^2\sigma_2^2}{2}}
\end{aligned}
$$

Using the result from part (a), we have,

$$
\begin{aligned}
\mathbb{E}\left[\left|X_1^k\right|\right] &\leq k2^{\frac{k}{2}}\sigma_1^k\Gamma\left(\frac{k}{2}\right) \\
\mathbb{E}\left[\left|X_2^k\right|\right] &\leq k2^{\frac{k}{2}}\sigma_2^k\Gamma\left(\frac{k}{2}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\left(X_1X_2\right)^k\right] &\leq \mathbb{E}\left[\left|X_1^k\right|\right]\mathbb{E}\left[\left|X_2^k\right|\right] \\
&= k^2 2^k \left(\sigma_1\sigma_2\right)^k \left(\Gamma\left(\frac{k}{2}\right)\right)^2
\end{aligned} \tag{2}
$$

We now prove that $f(k) := \frac{\left(k\Gamma\left(\frac{k}{2}\right)\right)^2}{2\times k!} \leq 1$ for $k \geq 3$. To show this claim, we consider the case of even and odd values of k separately.

Let $k = 2l, l \geq 2$. Then,

$$\begin{aligned}
f(2l) &= \frac{4l^2\left((l-1)!\right)^2}{2(2l)!} \\
&= \frac{4l^2\left((l-1)!\right)^2}{2.2l.(2l-1)(2l-2)!} \\
&= \frac{l}{(2l-1)\binom{2l-2}{l-1}} \\
&\leq 1 \text{ since } \frac{l}{2l-1} < 1 \text{ and } \binom{2l-2}{l-1} \geq 1
\end{aligned}$$

Next, let $k = 2l+1, l \geq 1$. Then,

$$\begin{aligned}
f(2l+1) &= \frac{(2l+1)!^2}{2(2l+1)!}\left(\frac{(2l)!\sqrt{\pi}}{4^l l!}\right)^2 \\
&= \frac{(2l+1)!^2\,\pi}{2(2l+1)!\,4^{2l}\left(l!\right)^2} \\
&= \frac{(2l+1)!\,\pi}{2\times 16^l\left(l!\right)^2} \\
&= \frac{(2l+1)(2l)!\,\pi}{2\times 16^l\left(l!\right)^2} \\
&= \frac{(2l+1)\,\pi\binom{2l}{l}}{2\times 16^l} \\
&\leq \frac{(2l+1)\,\pi}{2\times 16^l}\frac{4^l}{\sqrt{\pi l}} \text{ since } \binom{2l}{l} \leq \frac{4^l}{\sqrt{\pi l}} \\
&= \frac{2l+1}{4^l\sqrt{l}}\frac{\sqrt{\pi}}{2} \\
&\leq 1
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda X_1 X_2}\right] &= 1 + \lambda\mathbb{E}\left[X_1X_2\right] + \frac{\lambda^2\mathbb{E}\left[X_1^2X_2^2\right]}{2} + \sum_{k=3}^{\infty}\frac{\lambda^k\mathbb{E}\left[X_1^kX_2^k\right]}{k!} \\
&\leq 1 + 8\lambda^2\left(\sigma_1\sigma_2\right)^2\left[1 + \sum_{k=3}^{\infty}\frac{\left(k\Gamma\left(\frac{k}{2}\right)\right)^2}{2\times k!}\left(2\lambda\sigma_1\sigma_2\right)^{k-2}\right] \quad \text{using 2 along with } \mathbb{E}\left[X_1X_2\right] = \mathbb{E}\left[X_1\right]\mathbb{E}\left[X_2\right] = 0 \\
&= 1 + 8\lambda^2\left(\sigma_1\sigma_2\right)^2\left[1 + \sum_{k=1}^{\infty} f(k)\left(2\lambda\sigma_1\sigma_2\right)^k\right] \\
&\leq 1 + 8\lambda^2\left(\sigma_1\sigma_2\right)^2\left[1 + \sum_{k=1}^{\infty}\left(2\lambda\sigma_1\sigma_2\right)^k\right] \\
&= 1 + 8\lambda^2\left(\sigma_1\sigma_2\right)^2\left(1 + \frac{2\lambda\sigma_1\sigma_2}{1-2\lambda\sigma_1\sigma_2}\right) \\
&= 1 + \frac{8\lambda^2\left(\sigma_1\sigma_2\right)^2}{1-2\lambda\sigma_1\sigma_2} \\
&\leq 1 + 16\lambda^2\left(\sigma_1\sigma_2\right)^2 \quad 2\lambda\sigma_1\sigma_2 \leq \frac{1}{2} \\
&\leq \exp\left(16\lambda^2\left(\sigma_1\sigma_2\right)^2\right)
\end{aligned}$$

which completes our proof.

4. For ($\rightarrow$), we have

$$
\begin{aligned}
\mathbb{E}\left[e^{\lambda X^2}\right] &= \sum_{k=0}^{\infty} \frac{\lambda^k \mathbb{E}\left[X^{2k}\right]}{k|} \\
&\leq \sum_{k=0}^{\infty} \frac{\lambda^k .2k.2^k.\sigma^{2k}\,(k-1)\,!}{k!} \quad \text{using the result from part (a)} \\
&= 2\sum_{k=0}^{\infty} \left(2\lambda\sigma^2\right) k \\
&= \frac{2}{1-2\lambda\sigma^2} \text{ for } \left|2\lambda\sigma^2\right| \leq 1 \\
&\leq 4 \text{ for } \left|2\lambda\sigma^2\right| \leq \frac{1}{2}
\end{aligned}
$$

For ($\leftarrow$), let D be such that, without loss of generality, $\mathbb{E}\left[\exp\left(\frac{X^2}{D^2}\right)\right] \leq 2$. Then we have,

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(tX\right)\right] &= \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{tX^k}{k!}\right] \\
&= 1 + \mathbb{E}\left[\sum_{k=2}^{\infty} \frac{tX^k}{k!}\right] \text{ since } \mathbb{E}\left[X\right] = 0 \\
&\leq 1 + \frac{t^2}{2}\mathbb{E}\left[X^2 \exp\left(|tX|\right)\right] \quad \text{since } \frac{(tX)^k}{k!} \leq \frac{t^2X^2}{2}\frac{|tX|^{k-2}}{(k-2)\,!} \text{ for } k \geq 2
\end{aligned}
$$

Now, using the $AM - GM$ inequality, we have,

$$
2D^2t^2 + \frac{X^2}{2D^2} \geq |tX|
$$

Therefore, we have,

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(tX\right)\right] &\leq 1 + \frac{t^2}{2}\exp\left(2D^2t^2\right)\mathbb{E}\left[X^2\exp\left(\frac{X^2}{2D^2}\right)\right] \\
&\leq 1 + \frac{2D^2t^2}{2}\exp\left(D^2t^2\right)\mathbb{E}\left[\exp\left(\frac{X^2}{2D^2}\right)\exp\left(\frac{X^2}{2D^2}\right)\right] \\
&= 1 + D^2t^2\exp\left(D^2t^2\right)\mathbb{E}\left[\exp\left(\frac{X^2}{D^2}\right)\right] \\
&\leq 1 + 2D^2t^2\exp\left(D^2t^2\right) \text{ since } \mathbb{E}\left[\exp\left(\frac{X^2}{D^2}\right)\right] \leq 1 \\
&\leq \left(1 + 2D^2t^2\right)\exp\left(D^2t^2\right) \\
&\leq \exp\left(2D^2t^2\right)\exp\left(D^2t^2\right) \text{ since } 1 + x \leq e^x \\
&= \exp\left(3D^2t^2\right)
\end{aligned}
$$

Therefore, X is sub-gaussian.