Homework Assignment 1

Due via canvas Feb 11th

SDS 384-11 Theoretical Statistics

Please **do not** add your name to the HW submission.

1. (1+3+(1+1+2) pts) We will do some examples of convergence in distribution and convergence in probability here.

   (a) Let $X_n \sim N(0, 1/n)$. Does $X_n \xrightarrow{d} 0$?

   Solution. Yes, first we show that X_n converges in probability to 0.

$$\begin{aligned}
\lim_{n\to\infty} \Pr(|X_n - 0| > \epsilon) &= \lim_{n\to\infty} \Pr(|X_n|^2 > \epsilon^2) \\
&\leq \lim_{n\to\infty} \frac{\text{var}(X_n)}{\epsilon^2} \\
&= \lim_{n\to\infty} \frac{1}{n\epsilon^2} \\
&= 0
\end{aligned}$$

   Thus $X_n \xrightarrow{p} 0$ which implies that $X_n \xrightarrow{d} 0$.

   (b) Let $\{X_n\}$ be independent r.v's such that $P(X_n = n^\alpha) = 1/n$ and $P(X_n = 0) = 1 - 1/n$ for $n \geq 1$, where $\alpha \in (-\infty, \infty)$ is a constant. For what values of α, will you have $X_n \xrightarrow{q.m} 0$? For what values will you have $X_n \xrightarrow{p} 0$?

   Solution.

   Convergence in quadratic mean:

$$\begin{aligned}
\text{E}[|X_n|^2] &= (n^\alpha)^2 \frac{1}{n} \\
&= \frac{n^{2\alpha}}{n}
\end{aligned}$$

   The above will converge to zero if $2\alpha < 1$, or $\alpha < \frac{1}{2}$.

   Convergence in Probability:

   For $\epsilon \geq n^\alpha$ we have $\Pr(|X_n| > \epsilon) = 0$. For $\epsilon < n^\alpha$ we have $\Pr(|X_n| > \epsilon) = \frac{1}{n}$. This probability converges to zero for all values of α.

   (c) Consider the average of n i.i.d random variables $X_1, \ldots, X_n$ with $E[X_1] = \mu$ and $E[|X_1|] < \infty$. Write true or false. Explain.

i. $\bar{X}_n = o_P(1)$ *Solution.* We know that $\bar{X}_n$ converges to μ in probability. If $\mu \neq 0$, $\bar{X}_n = o_P(1)$ is false.

ii. $\exp(\bar{X}_n - \mu) = o_P(1)$ *Solution.* We know that $\bar{X}_n - \mu$ converges to 0 in probability. By continuous mapping, If $\exp(\bar{X}_n - \mu) \xrightarrow{P} 1$. So false.

iii. $(\bar{X}_n - \mu)^2 = O_P(1/n)$

Solution. Fix $\epsilon > 0$. Now $P((\bar{X}_n - \mu)^2 \geq \underbrace{\frac{\sigma^2}{\epsilon}}_{M_\epsilon}) \leq \epsilon$. So, its true.

2. (8 pts) Consider random variables $X_1, \ldots, X_n$ be IID r.v's with mean μ and variance $\sigma^2 := \mathrm{var}(X_i)$. We will use the following statistic to estimate $\theta = \mu^2$.

$$\hat{\theta} = \frac{1}{\binom{n}{2}} \sum_{i<j} X_i X_j$$

(a) Find constants C_1, C_2 where

$$\hat{\theta} - \mu^2 = \frac{C_1}{\binom{n}{2}} \sum_{i<j} (X_i - \mu)(X_j - \mu) + \frac{C_2 \mu}{n} \sum_i (X_i - \mu)$$

Solution.

$C_1 = 1, C_2 = 2$.

(b) Show that the first term is $O_P(1/n)$ and the second term is $O_P(1/\sqrt{n})$.

Solution.

Calculate variance of the first part to note that

$$\frac{1}{n^4} \mathrm{var}\left(\sum_{i \neq j} (X_i - \mu)(X_j - \mu) \right) = \sum_{i \neq j} E[(X_i - \mu)^2] E[(X_j - \mu)^2] \leq \sigma^4 / n^2$$

Similarly compute the variance of the second part to see that it is c/n for some constant c. Note that a random var X with mean zero is always $O_P(\sqrt{\mathrm{var}(X)})$. To see this, note:

$$P(|X| \geq \sqrt{\mathrm{var}(X)/\epsilon}) \leq \epsilon$$

(c) Argue that $\hat{\theta} \xrightarrow{P} \mu^2$. *Solution.*

So, by this logic, $\mathrm{var}(\hat{\theta} - \mu^2) \leq C/n$, which shows $\hat{\theta} \xrightarrow{P} \mu^2$ via convergence in quadratic mean. To see this note that $\mathrm{var}(A + B) \leq 2 \max(\mathrm{var}(A), \mathrm{var}(B))$.

3. (8 pts) If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

(a) $g(x) = 1(x \in (0, 10))$

Solution. It is not necessarily true that $E[g(X_n)] \to E[g(X)]$ because there is a discontinuity in g at 0 and 10. Take the following counter example:

Let $X_n = X + \frac{1}{n}$. It is simple to show that $X_n \xrightarrow{P} X$, thus $X_n \xrightarrow{d} X$.

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = \lim_{n\to\infty} \Pr(|X + \frac{1}{n} - X| > \epsilon) = \lim_{n\to\infty} \Pr(\frac{1}{n} > \epsilon) = 0$$

Now we need to show that $E[g(X_n)] \not\to E[g(X)]$. We pick a convenient λ to show that the result doesn't hold. For simplicity let $\lambda \to 0$. Thus $E[g(X)] = \Pr(X \in (0,10)) = 0$ as $\lambda \to 0$, however

$$E[g(X_n)] = \Pr(X_n \in (0,10)) = 1$$

as $\lambda \to 0$, thus $E[g(X_n)] \not\to E[g(X)]$.

(b) $g(x) = e^{-x^2}$

Solution. Yes, from the Portmanteau Theorem it is true that $E[g(X_n)] \to E[g(X)]$ because $g(x)$ is bounded by 0 and 1 and continuous on the real line.

(c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

Solution. Yes it is true that $E[g(X_n)] \to E[g(X)]$. First the function $g(x)$ is bounded as it only takes on the values $(-1, 0, 1)$. Second the discontinuities only occur when $\cos(x) = 0$ which can be defined by the set $A = \{\pi(\frac{1}{2} + n)\}_{n=0}^{\infty}$. We define $C(g) = \mathbb{R} - A$ to be the set of values on which $g(x)$ is continuous. Since X only takes on integer values $\Pr(X \in C(g)) = 1$, thus by the Portmanteau theorem $E[g(X_n)] \to E[g(X)]$.

(d) $g(x) = x$ *Solution.* Since $g(x)$ is not bounded, it is not necessarily true that $E[g(X_n)] \to E[g(X)]$. Take the following counter example: Let

$$= \begin{cases} X & \text{w.pr. } \frac{n-1}{n} \\ n & \text{w.pr. } \frac{1}{n} \end{cases} \tag{1}$$

Clearly $X_n \xrightarrow{d} X$, however

$$\begin{aligned} \lim_{n\to\infty} E[g(X_n)] &= \lim_{n\to\infty} E[X_n] \\ &= \lim_{n\to\infty} (\frac{1}{n} n + E[X]\frac{n-1}{n}) \\ &= 1 + E[X] \\ &\neq E[X] \end{aligned}$$

Therefore $E[g(X_n)] \not\to E[g(X)]$.

4. (6 pts) Let $X_1, \ldots, X_n$ be independent r.v's with mean zero and variance $\sigma_i^2 := E[X_i^2]$ and $s_n^2 = \sum_i \sigma_i^2$. If $\exists \delta > 0$ s.t. as $n \to \infty$,

$$\frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0,$$

then $\sum_i X_i/s_n$ converges weakly to the standard normal.

Soln.

Proof. We want to show that

$$\frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}}$$

shows up in the upper bound on the quantity in the Lindeberg condition. If the above quantity converges to 0 as $n \to \infty$ then the Lindeberg must also be satisfied and thus $\sum_i X_i/s_n \xrightarrow{d} N(0,1)$.

$$\begin{aligned}
\frac{1}{s_n^2}\sum_{i=1}^{n} E[|X_i|^2 1(|X_i| \geq \epsilon s_n)] &= \frac{1}{s_n^2}\sum_{i=1}^{n} E\left[|X_i|^2 1(|X_i|^\delta \geq \epsilon^\delta s_n^\delta)\right] \\
&\leq \frac{1}{s_n^2}\sum_{i=1}^{n} E\left[|X_i|^2 \frac{|X_i|^\delta}{\epsilon^\delta s_n^\delta}\right] \\
&= \frac{1}{\epsilon^\delta}\frac{\sum_{i=1}^{n} E|X_i|^{2+\delta}}{s_n^{2+\delta}}
\end{aligned}$$

We used the fact that for a positive number X,

$$1(X \geq \epsilon) \leq X/\epsilon.$$

Thus, under the condition, the Lindeberg condition is also met. □