Homework Assignment 1

Due via canvas Feb 11th

SDS 384-11 Theoretical Statistics
Please **do not** add your name to the HW submission.

1. (1+3+(1+1+2) pts) We will do some examples of convergence in distribution and convergence in probability here.

   (a) Let $X_n \sim N(0, 1/n)$. Does $X_n \xrightarrow{d} 0$?

   (b) Let $\{X_n\}$ be independent r.v's such that $P(X_n = n^\alpha) = 1/n$ and $P(X_n = 0) = 1 - 1/n$ for $n \geq 1$, where $\alpha \in (-\infty, \infty)$ is a constant. For what values of α, will you have $X_n \xrightarrow{q.m} 0$? For what values will you have $X_n \xrightarrow{p} 0$?

   (c) Consider the average of n i.i.d random variables $X_1, \ldots, X_n$ with $E[X_1] = \mu$ and $E[|X_1|] < \infty$. Write true or false. Explain.

      i. $\bar{X}_n = o_P(1)$

      ii. $\exp(\bar{X}_n - \mu) = o_P(1)$

      iii. $(\bar{X}_n - \mu)^2 = O_P(1/n)$

2. (8 pts) Consider random variables $X_1, \ldots, X_n$ be IID r.v's with mean μ and variance $\sigma_i^2 := \text{var}(X_i)$. We will use the following statistic to estimate $\theta = \mu^2$.

$$\hat{\theta} = \frac{1}{\binom{n}{2}} \sum_{i<j} X_i X_j$$

   (a) Find constants C_1, C_2 where

$$\hat{\theta} - \mu^2 = \frac{C_1}{\binom{n}{2}} \sum_{i<j} (X_i - \mu)(X_j - \mu) + \frac{C_2 \mu}{n} \sum_i (X_i - \mu)$$

   (b) Show that the first term is $O_P(1/n)$ and the second term is $O_P(1/\sqrt{n})$.

   (c) Argue that $\hat{\theta} \xrightarrow{P} \mu^2$.

3. (8 pts) If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$?

   (a) $g(x) = 1(x \in (0, 10))$

   (b) $g(x) = e^{-x^2}$

   (c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

   (d) $g(x) = x$

4. (6 pts) Let $X_1, \ldots, X_n$ be independent r.v's with mean zero and variance $\sigma_i^2 := E[X_i^2]$ and $s_n^2 = \sum_i \sigma_i^2$. If $\exists \delta > 0$ s.t. as $n \to \infty$,

$$\frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0,$$

then $\sum_i X_i / s_n$ converges weakly to the standard normal.