

SDS 384 11: Theoretical Statistics

Lecture 1: Introduction

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Manegerial Stuff

- Instructor- Purnamrita Sarkar
- Course material and homeworks will be posted under <https://psarkar.github.io/teaching/sds384.html>
- Office hours: TBD
- Homeworks are due Biweekly
- Grading - 4-5 homeworks (55%), class participation (10%) Final Exam (35%)
- Books
 - Asymptotic Statistics, Aad van der Vaart. Cambridge. 1998.
 - Martin Wainwright's High dimensional statistics: A non-asymptotic view point

Why do theory?

- Say you have estimated $\hat{\theta}_n$ from data X_1, \dots, X_n . How do we know we have a “good” estimation method?
 - Does $\hat{\theta}_n \rightarrow \theta$? This brings us to **Stochastic Convergence**.
- How about the rate of convergence?
 - Can we give any guarantees on how quickly our estimate converges?

$$P(|\hat{\theta}_n - \theta| = \text{large}) = \text{small}$$

This class

Your instructor “hopes to cover”:

- Consistency of parameter estimates
 - Stochastic Convergence
 - Concentration inequalities
 - Asymptotic normality of estimators
- Empirical processes, VC classes, covering numbers
- Examples of network clustering with a bit of random matrix theory
- Bootstrap, Nonparametric regression and density estimation

Stochastic Convergence

Assume that $X_n, n \geq 1$ and X are elements of a separable metric space (S, d) .

Definition (Weak Convergence)

A sequence of random variables converge in “law” or in “distribution” to a random variable X , i.e. $X_n \xrightarrow{d} X$ if $P(X_n \leq x) \rightarrow P(X \leq x) \forall x$ at which $P(X \leq x)$ is continuous.

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Definition (Convergence in Probability)

A sequence of random variables converge in “probability” to a random variable X , i.e. $X_n \xrightarrow{P} X$ if $\forall \epsilon > 0, P(d(X_n, X) \geq \epsilon) \rightarrow 0$.

Stochastic Convergence

Assume that $X_n, n \geq 1$ and X are elements of a separable metric space (S, d) .

Definition (Almost Sure Convergence)

A sequence of random variables converges almost surely to a random variable X , i.e. $X_n \xrightarrow{a.s.} X$ if $P\left(\lim_{n \rightarrow \infty} d(X_n, X) = 0\right) = 1$.

- If you think about a (scalar) random variable as a function that maps events to a real number, almost sure convergence means
$$P(\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)) = 1$$

Definition (Convergence in quadratic mean)

A sequence of random variables converges in quadratic mean to a random variable X , i.e. $X_n \xrightarrow{q.m.} X$ if $E\left[d(X_n, X)^2\right] \rightarrow 0$.

Unwinding a.s. convergence

- $X_n \xrightarrow{\text{a.s.}} X$ implies $P(\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)) = 1$
- What does convergence mean for a sequence of real numbers?

Unwinding a.s. convergence

- $X_n \xrightarrow{a.s.} X$ implies $P(\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)) = 1$
- What does convergence mean for a sequence of real numbers?
- $\forall \epsilon > 0, \exists n, \forall m \geq n, |X_n(\omega) - X(\omega)| < \epsilon$
 - Consider a sequence of events $A_1, \dots, A_n,$
 $A_n = \{|X_n(\omega) - X(\omega)| < \epsilon\}$

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 - $\forall \epsilon > 0, \exists n$, s.t. $\forall m \geq n, |X_n(\omega) - X(\omega)| < \epsilon$, boils down to:

$$\bigcup_{i=1}^n \bigcap_{m \geq n} A_m$$

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- Another way of saying this is, A_n^c happens finitely often. (f.o.)
- $X_n \xrightarrow{a.s.} X$ implies $\forall \epsilon > 0, P(\{|X_n - X| \geq \epsilon \text{ f.o.}\}) = 1$

Theorem

$$X_n \xrightarrow{a.s.} X, X_n \xrightarrow{q.m.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$$
$$X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$$

Converses: $X_n \xrightarrow{d} X \not\Rightarrow X_n \xrightarrow{P} X$

- Convergence in law needs no knowledge of the joint distribution of X_n and the limiting random variable X .
- Convergence in probability does.

Example

Consider $X \sim N(0, 1)$, $X_n = -X$. $X_n \xrightarrow{d} X$. But how about $X_n \xrightarrow{P} X$?

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- $P(|X_n - X| \geq \epsilon) = P(2|X| \geq \epsilon) \not\rightarrow 0 \forall \epsilon > 0$. So X_n does not converge in probability to X .

Example

Example

Let $Z \sim U(0, 1)$ and for $n = 2^k + m$ for $k \geq 0, 0 \leq m < 2^k$
 $X_n = 1(Z \in [m2^{-k}, (m+1)2^{-k}])$, i.e. $X_1 = 1$, $X_2 = 1(Z \in [0, 1/2))$,
 $X_3 = 1(Z \in [1/2, 1))$, $X_4 = 1(Z \in [0, 1/4))$, $X_5 = 1(Z \in [1/4, 1/2))$.

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- For any $Z \in (0, 1)$, the sequence $\{X_n(Z)\}$ does not converge. So $X_n \not\overset{a.s.}{\rightarrow} 0$.

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- For any $Z \in (0, 1)$, the sequence $\{X_n(Z)\}$ does not converge. So $X_n \not\stackrel{a.s.}{\rightarrow} 0$.
- For any $\epsilon > 0$, $P(\{|X_n| > \epsilon\} \text{ i.o.})$
- X_n are a sequence of bernoulli's with probabilities $p_n = 1/2^k$ where $k = \lfloor \log n \rfloor$.

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- X_n are a sequence of bernoulli's with probabilities $p_n = 1/2^k$ where $k = \lfloor \log n \rfloor$.
- So $X_n \stackrel{P}{\rightarrow} 0$ and $X_n \stackrel{qm}{\rightarrow} 0$

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- $P(\lim_{n \rightarrow \infty} X_n = X) = P(Z > 0) = 1$. So $X_n \xrightarrow{a.s.} X$.
- $E|X_n|^2 = 2^{2n}/n \rightarrow \infty$. So $X_n \not\xrightarrow{qm} 0$

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- $E|X_n|^2 = 2^{2n}/n \rightarrow \infty$. So $X_n \not\xrightarrow{qm} 0$
- $P(|X_n| \geq \epsilon) = P(X_n = 2^n) = P(Z \in [0, 1/n]) = 1/n \rightarrow 0$

- $X_n \xrightarrow{a.s.} X$ implies $\exists \epsilon > 0, P(\{|X_n - X| \geq \epsilon \text{ i.o.}\}) = 0$

- $X_n \xrightarrow{a.s.} X$ implies $\exists \epsilon > 0, P(\{|X_n - X| \geq \epsilon \text{ i.o.}\}) = 0$
- Consider a sequence of events A_1, \dots, A_n .
- Infinitely often means $\forall n, \exists m \geq n$, s.t. A_m occurs.
- More concretely

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

Borel Cantelli Lemma (I)

Theorem

If $\sum_i P(A_i) < \infty$, then $P(\{A_n \text{ i.o.}\}) = 0$.

Example

Let $X_n \sim \text{Bernoulli}(2^{-n})$. Then $X_n \xrightarrow{\text{a.s.}} 0$.

Check if $X_n = 1$ infinitely often.

Borel Cantelli Lemma (I)

Theorem

If $\sum_i P(A_i) < \infty$, then $P(\{A_n \text{ i.o.}\}) = 0$.

- Recall that $\{A_n \text{ i.o.}\}$ is equivalent to $\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m=n}^{\infty} A_m}_{B_n}$

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Theorem

If $\sum_i P(A_i) < \infty$, then $P(\{A_n \text{ i.o.}\}) = 0$.

- Recall that $\{A_n \text{ i.o.}\}$ is equivalent to $\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m=n}^{\infty} A_m}_{B_n}$
- Note that $B_{n+1} \subseteq B_n$, and so we have $B_n \downarrow B := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$, hence using monotone convergence we have:

$$\lim_{n \rightarrow \infty} P(B_n) = P(B)$$

Borel Cantelli Lemma (I)

Theorem

If $\sum_i P(A_i) < \infty$, then $P(\{A_n \text{ i.o.}\}) = 0$.

$$P(A_i \text{ i.o.}) = \lim_{n \rightarrow \infty} P(B_n) \leq \lim_{n \rightarrow \infty} \sum_{m \geq n} P(A_m) = 0$$

Borel Cantelli Lemma (II)

Theorem

If $\sum_i P(A_i) = \infty$ and $\{A_n\}$ are independent then $P(\{A_n \text{ i.o.}\}) = 1$.

Example

Consider $Z \sim U[0, 1]$, $A_n := \{Z \leq 1/n\}$, and $X_n = 1(A_n)$. $\sum_i P(A_n) \rightarrow \infty$.

But we know that $X_n \xrightarrow{a.s.} 0$.

- Does BC II apply?
- If not, how do you prove it?

Borel Cantelli Lemma (II)

- Start with the complement – we will show $P((A_i \text{ i.o.})^c) = 0$.

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$$\begin{aligned} P((A_i \text{ i.o.})^c) &= P\left(\bigcup_n \bigcap_{m \geq n} A_m^c\right) \\ &= \lim_{n \rightarrow \infty} P\left(\bigcap_{m \geq n} A_m^c\right) \\ &= \lim_{n \rightarrow \infty} \prod_{m \geq n} P(A_m^c) \\ &= \lim_{n \rightarrow \infty} \prod_{m \geq n} (1 - P(A_m)) \\ &\leq \lim_{n \rightarrow \infty} \exp\left(-\sum_{m \geq n} P(A_m)\right) = 0 \end{aligned}$$

Continuous Mapping Theorem

Theorem

Let g be continuous on a set C where $P(X \in C) = 1$. Then,

$$X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$$

$$X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$$

$$X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$$

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- Use $X^2 \sim \chi_1^2$.
- So $X_n^2 \xrightarrow{d} \chi_1^2$

Example-continuity points

Let X_1, \dots, X_n be i.i.d. with mean μ and variance σ^2 . We have $\bar{X}_n - \mu \xrightarrow{d} 0$. Consider $g(x) = 1_{x>0}$. Then $g((\bar{X}_n - \mu)^2) \xrightarrow{d} ?$

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- Using Continuous Mapping Theorem, $(\bar{X}_n - \mu)^2 \xrightarrow{d} 0$
- Can we use Continuous Mapping Theorem to claim that $g(\bar{X}_n - \mu)^2 \xrightarrow{d} 0$?
- NO. Because, 0 is a random variable whose mass is at 0, where g is discontinuous.

How about convergence in q.m.?

- If $X_n \xrightarrow{qm} X$, then is it true that for continuous f (discontinuous only at a measure zero set), $f(X_n) \xrightarrow{qm} f(X)$?

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- If $X_n \xrightarrow{qm} X$, then is it true that for continuous f (discontinuous only at a measure zero set), $f(X_n) \xrightarrow{qm} f(X)$?
- Consider an L -Lipschitz function $f(X)$. $|f(x) - f(y)| \leq L|x - y|$.
- $E[|f(X_n) - f(X)|^2] \leq L^2 E[|X_n - X|^2] \rightarrow 0$. So for Lipschitz functions quadratic mean convergence goes through.
- Can you come up with a non-Lipschitz function and a sequence $\{X_n\}$ where $f(X_n) \not\xrightarrow{qm} 0$?

Portmanteau Theorem

Theorem

The following are equivalent.

- $X_n \xrightarrow{d} X$
- $E[f(X_n)] \rightarrow E[f(X)]$ for all continuous f that vanish outside a compact set.
- $E[f(X_n)] \rightarrow E[f(X)]$ for all bounded and continuous f .
- $E[f(X_n)] \rightarrow E[f(X)]$ for all bounded measurable functions f s.t. $P(X \in C(f)) = 1$, where $C(f) = \{x : f \text{ is continuous at } x\}$ is called the continuity set of f .

Example-bounded

Consider $f(x) = x$ and

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- $E[X_n] = 1$. What went wrong?

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- $X_n \xrightarrow{d} 0$, but $E[X_n] \rightarrow ?$
- $E[X_n] = 1$. What went wrong?
- $f(x) = x$ is not bounded.

Putting everything together

Theorem

$$X_n \xrightarrow{d} X \text{ and } d(X_n, Y_n) \xrightarrow{P} 0 \Rightarrow Y_n \xrightarrow{d} X \quad (1)$$

$$X_n \xrightarrow{d} X \text{ and } Y_n \xrightarrow{d} c \Rightarrow (X_n, Y_n) \xrightarrow{d} (X, c) \quad (2)$$

$$X_n \xrightarrow{P} X \text{ and } Y_n \xrightarrow{P} Y \Rightarrow (X_n, Y_n) \xrightarrow{P} (X, Y) \quad (3)$$

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- Eq 3 does not hold if we replace convergence in probability by convergence in distribution.

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- Example: $X_n \sim N(0, 1)$, $Y_n = -X_n$. $X \perp Y$ and X, Y are independent standard normal random variables.

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- Eq 3 does not hold if we replace convergence in probability by convergence in distribution.
- Example: $X_n \sim N(0, 1)$, $Y_n = -X_n$. $X \perp Y$ and X, Y are independent standard normal random variables.
- Then $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$. But $(X_n, Y_n) \xrightarrow{d} (X, -X)$, not $(X_n, Y_n) \xrightarrow{d} (X, Y)$.

Putting everything together

Theorem (Slutsky's theorem)

$X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} c$ imply that

$$X_n + Y_n \xrightarrow{d} X + c$$

$$X_n Y_n \xrightarrow{d} cX$$

$$X_n / Y_n \xrightarrow{d} X / c$$

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- Does $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ imply $X_n + Y_n \xrightarrow{d} X + Y$?

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- Does $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ imply $X_n + Y_n \xrightarrow{d} X + Y$?
- Take $Y_n = -X_n$, and X, Y as independent standard normal random variables. $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ but $X_n + Y_n \xrightarrow{d} 0$.

Using all this

If X_1, \dots, X_n are i.i.d. random variables with mean μ and variance σ^2 ,
prove that $\sqrt{n} \frac{\bar{X}_n - \mu}{S_n} \xrightarrow{d} N(0, 1)$.

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- First note that $S_n^2 = \frac{1}{n} \sum_i X_i^2 - \bar{X}_n^2$

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- First note that $S_n^2 = \frac{1}{n} \sum_i X_i^2 - \bar{X}_n^2$
- Law of large numbers give $\frac{\sum_i X_i^2}{n} \xrightarrow{P} E[X^2]$ and $\bar{X}_n \xrightarrow{P} \mu$.

Using all this

If X_1, \dots, X_n are i.i.d. random variables with mean μ and variance σ^2 , prove that $\sqrt{n} \frac{\bar{X}_n - \mu}{S_n} \xrightarrow{d} N(0, 1)$.

- First note that $S_n^2 = \frac{1}{n} \sum_i X_i^2 - \bar{X}_n^2$
- Law of large numbers give $\frac{\sum_i X_i^2}{n} \xrightarrow{P} E[X^2]$ and $\bar{X}_n \xrightarrow{P} \mu$.
- So $(\frac{\sum_i X_i^2}{n}, \bar{X}_n) \xrightarrow{P} (E[X^2], \mu)$ and now using the continuous mapping theorem, $S_n^2 \xrightarrow{P} \sigma^2$.

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- Finally, $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$ using CLT.
- Now using Slutsky's lemma, $\sqrt{n}(\bar{X}_n - \mu)/S_n \xrightarrow{d} N(0, 1)$ using CLT.

Definition

X is defined to be “tight” if $\forall \epsilon > 0 \exists M$ for which,

$$P(\|X\| > M) < \epsilon$$

$\{X_n\}$ is defined to uniformly tight if $\forall \epsilon > 0 \exists M$ for which,

$$\sup_n P(\|X_n\| > M) < \epsilon$$

Uniformly tight

- Give an example of a sequence that is **Not** UT
- $X_n = \text{Uniform}([-n, n])$
- $P(|X_n| > n(1 - \epsilon/2)) = \epsilon$, so you cannot find an ϵ such that $P(|X_n| > M) \leq \epsilon$ for all n

Theorem

- $X_n \xrightarrow{d} X \Rightarrow \{X_n\}$ is UT.
- $\{X_n\}$ is UT implies that, there exists a subsequence $\{n_j\}$ such that $X_{n_j} \xrightarrow{d} X$.

Notation for rates, big O and big O-pea

Definition

- Big O. Let $g(\cdot)$ be a positive function.

$$f(x) = O(g(x)) \text{ as } x \rightarrow \infty$$
$$\exists M, x_0, \quad |f(x)| \leq Mg(x) \quad \text{For } x \geq x_0$$

For large x , $f(x)$ is bounded by $g(x)$ up-to a multiplicative constant

- The big O_P :

$$X_n = O_P(1) \Leftrightarrow \{X_n\} \text{ is UT}$$
$$X_n = O_P(R_n) \Leftrightarrow X_n = Y_n R_n \text{ and } Y_n = O_P(1)$$

X_n is likely to lie within a ball of finite radius

Definition

- The small o :

$$f(x) = o(g(x)) \Leftrightarrow f(x)/g(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

- The small o_P :

$$X_n = o_P(1) \Leftrightarrow X_n \xrightarrow{P} 0$$

$$X_n = o_P(R_n) \Leftrightarrow X_n = Y_n R_n \text{ and } Y_n = o_P(1)$$

X_n is vanishing in probability

How do they interact

Lemma

Let $R : \mathbb{R}^k \rightarrow \mathbb{R}$ be a function with $R(0) = 0$. Let $X_n = o_P(1)$ be a sequence of random variables defined on the domain of \mathbb{R} . Then as $\|h\| \rightarrow 0$, $\forall q > 0$

$$R(h) = o(\|h\|^q) \text{ implies } R(X_n) = o_P(\|X_n\|^q)$$

$$R(h) = O(\|h\|^q) \text{ implies } R(X_n) = O_P(\|X_n\|^q)$$

- Work out the proof at home.
- Hint: apply continuous mapping to $R(h)/\|h\|^q$.

How do they interact

$$o_P(1) + o_P(1) = o_P(1).$$

$$o_P(1) + O_P(1) = O_P(1).$$

$$O_P(1)o_P(1) = o_P(1).$$

$$1 + O_P(1) = O_P(1).$$

$$(1 + o_P(1))^{-1} = 1 + o_P(1).$$

Be careful:

$$e^{o_P(1)} \neq o_P(1)$$

$O_P(1) + O_P(1)$ Can actually be $o_P(1)$ because of cancellation.