THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 384 11: Theoretical Statistics

Lecture 2: Stochastic Convergence

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Convergence of expectations: exchanging limit and integral

Theorem (Monotone convergence theorem)
If $0 \leq X_1 \leq X_2 \leq \cdots \leq X_n \uparrow X$, *a.s. then*

$$E[X_n] \to E[X]$$

Lemma (Fatou's lemma)
If $X_n \geq Y \ \forall n$ *for some random variable* Y *with* $E|Y| < \infty$ *then*

$$\liminf_{n\to\infty} E[X_n] \geq E[\liminf_n X_n]$$

Theorem (Dominated convergence theorem)
If $X_n \stackrel{a.s.}{\to} X$ *and* $|X_n| \leq Y$ *with* $E[|Y|] < \infty$, *then*

$$E[X_n] \to E[X]$$

Convergence of expectations: exchanging limit and integral

- Another version of MCT requires $X_i > Y$ s.t. $EY > -\infty$.
- Consider $Z \sim U([0,1])$
- $X_n = -\frac{1}{z} 1[z \in (0, 1/n)]$
- This is an increasing sequence, $X_n \overset{a.s.}{\to} 0$
- But $EX_n = -\int_0^{1/n} 1/z dz = -\infty$

Remember liminf and limsup

- $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m$
- $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m$

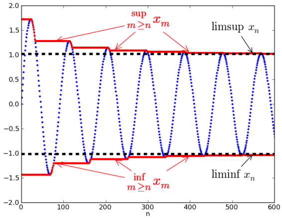

Figure 1: limsup and liminf always exist even though the sequence x_n is not converging. Courtest: wikipedia.
`https://en.wikipedia.org/wiki/Limit_inferior_and_limit_superior`

MCT$\to$ Fatou

Proof.

- Consider the random variable $X_n - Y$. These are positive. For all $m \geq n$,

$$
\begin{aligned}
\inf_{k\geq n}(X_k - Y) &\leq X_m - Y \\
E\left[\inf_{k\geq n}(X_k - Y)\right] &\leq E[X_m - Y] \quad \text{Take } E[] \text{ of both sides} \\
&\leq \inf_{m\geq n} E[X_m - Y] \\
\lim_{n\to\infty} E\left[\inf_{k\geq n}(X_k - Y)\right] &\leq \lim_{n\to\infty}\inf_{m\geq n} E[X_m - Y] = \liminf_{n\to\infty} E[X_n - Y]
\end{aligned}
$$

  - All that is left, is to exchange limit and integral on LHS.
  - Note that $\inf_{k\geq n}(X_k - Y)$ is an increasing positive sequence. This converges to $\liminf_{n\to\infty}(X_n - Y)$. Apply MCT.

□

We never used $E|Y| < \infty$

- Well, we are saying $E[Y]$ exists.
- Unless $E|Y| < \infty$, $E[Y]$ is not very well defined.
- Consider $1 - 1 + 1/2 - 1/2 + 1/3 - 1/3 + 1/4 - 1/4 + 1/5 - 1/5 \ldots$
- This is clearly zero, right?
- But what if I permute it to have first two +ve and first -ve
- $1 + 1/2 - 1 + 1/3 + 1/4 - 1/2 + 1/5 + 1/6 - 1/3 \ldots$—this is $\ln 2$
- If you take a sum of the absolute values then that diverges.

Fatou $\rightarrow$ DCT

Proof.

- Note that random variable $-Y \leq X_n \leq Y$.
- Also note that $E[\liminf_{n\to\infty} X_n] = E[X] = E[\limsup_{n\to\infty} X_n]$.
- Apply Fatou on X_n and $-X_n$.

$$E[\liminf_{n\to\infty} X_n] \leq \liminf_{n\to\infty} E[X_n] \leq \limsup_{n\to\infty} E[X_n] \leq E[\limsup_{n\to\infty} X_n]$$

- But both ends equal $E[X]$ and so the middle two quantities must be equal and hence proved.

$\square$

Things you should know

Consider n i.i.d. random variables $X_i \sim F$.

Definition (Empirical distribution function)
The empirical distribution function is defined as:

$$F_n(x) = \frac{1}{n} \sum_i 1(X_i \le x).$$

Theorem (Glivenko-Cantelli)
The random variable $\sup_x |F_n(x) - F(x)|$ *almost surely converges to zero.*

$$P\left(\sup_x |F_n(x) - F(x)| \to 0\right) = 1$$

Things you should know

Let $X_1, \dots X_n$ be i.i.d random variables with $E[|X_1|] \leq \infty$, mean μ.

Theorem (Weak law of large numbers)

$$\bar{X}_n \xrightarrow{P} \mu$$

Theorem (Strong law of large numbers)

$$\bar{X}_n \xrightarrow{a.s.} \mu$$

Theorem (Central limit theorem)
If $E[X_i^2] = \sigma^2$, $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$.

Things you should know

Let $X_1, \ldots X_n$ be i.i.d random variables with mean μ.

Theorem (Berry Esseen)
If $E[X_i^2] = \sigma^2$, and $E[|X_i|^3] = \rho < \infty$,

$$\sup_x \left| P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq x \right) - \Phi(x) \right| \leq \frac{C\rho}{\sigma^3 \sqrt{n}} \qquad \forall n,$$

where $\Phi(x)$ is the CDF of the standard normal and c is an universal constant known to be greater than 0.4097 and less that 0.7975.

Lindeberg-feller CLT for triangular arrays

$$X_{11}$$
$$X_{21}, X_{22}$$
$$X_{31}, X_{32}, X_{33}$$
$$\dots$$

Theorem
For each n let $(X_{ni})_{i=1}^{n}$ be independent random variables with mean zero and variance σ_{ni}^2. Let $Z_n = \sum_{i=1}^{n} X_{ni}$ and $B_n^2 = var(Z_n)$. Then $Z_n/B_n \xrightarrow{d} N(0,1)$, as long as the **Lindeberg condition** *holds.*

The Lindeberg condition

Definition (Lindeberg condition)
For every $\epsilon > 0$,

$$\frac{1}{B_n^2} \sum_{j=1}^{n} E[X_{nj}^2 1(|X_{nj}| \geq \epsilon B_n)] \to 0 \text{ as } n \to \infty \tag{1}$$

Converse: If $\frac{\sigma_{nj}^2}{B_n^2} \to 0$ as $n \to \infty$, i.e. no one variance plays a significant role in the limit, and if $Z_n/B_n \xrightarrow{d} N(0,1)$, then the Lindeberg condition holds.

Necessary and Sufficient: If $\frac{\sigma_{nj}^2}{B_n^2} \to 0$, the the Lindeberg condition is necessary and sufficient to show the CLT.

Example

Let $X_1, \ldots, X_n$ be independent random variables with mean zero and variance one. Do you think $\sqrt{n}\bar{X}_n \xrightarrow{d} N(0,1)$?

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- $$X_{nj} = \begin{cases} 2j & \text{w.p. } \dfrac{1}{8j^2} \\ 0 & \text{w.p. } 1 - \dfrac{1}{4j^2} \\ -2j & \text{w.p. } \dfrac{1}{8j^2} \end{cases}$$

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- $E[X_{nj}] = 0$ and $\mathrm{var}(X_{nj}) = 1$. $B_n^2 = n$.

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- $E[X_{nj}] = 0$ and $\mathrm{var}(X_{nj}) = 1$. $B_n^2 = n$.
- Lets check the Lindeberg condition with $\epsilon = 1$.

$$\frac{1}{n}\sum_j E[X_{nj}^2 1(|X_{nj}| \geq \sqrt{n})] = \frac{1}{n}\sum_j 2 \times 4j^2 1(2j \geq \sqrt{n})\frac{1}{8j^2} = \frac{1}{n}\sum_{j \geq \sqrt{n}/2} 1 \to 1$$

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- Since $\sigma_{nj}^2 / B_n^2 = 1/n \to 0$, this implies that the CLT does not hold for the sum.

Paired experiment example

Consider $2n$ paired experimental units with measurement $(X_i, Y_i)_{i=1}^n$ in which X_j is the result of the treatment and Y_j is the result of control.

- H_0 is that the treatment has had no effect, i.e. $Z_j = X_j - Y_j$ conditioned on the magnitude $|Z_j|$ is symmetric, i.e. $P(Z_j = |z_j|) = P(Z_j = -|z_j|) = 1/2$.

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- Thus, under H_0, $(Z_1, \dots, Z_n)$ has 2^n possible values $(\pm|z_1|, \dots, \pm|z_n|)$.

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- Thus, under H_0, $(Z_1, \ldots, Z_n)$ has 2^n possible values $(\pm|z_1|, \ldots, \pm|z_n|)$.
- Conditioned on the magnitudes of the differences, $B_n^2 = \sum_i z_i^2$.

  Assume that $\max_i z_i^2 / B_n^2 \to 0$. Then $\sum_i Z_i / B_n \xrightarrow{d} N(0, 1)$ using the Lindeberg-feller theorem.

Paired experiment example: proof

Proof.

- Lets check the Lindeberg condition:

$$\frac{\sum_{j=1}^{n} E[Z_j^2 1(|Z_j| \geq \epsilon B_n) ||Z_1| = z_1, \ldots, |Z_n| = z_n]}{B_n^2} = \frac{\sum_j z_j^2 1(z_j \geq \epsilon B_n)}{B_n^2}$$

$$\leq \frac{(\sum_j z_j^2) 1(\max_j z_j \geq \epsilon B_n)}{B_n^2}$$

$$= 1(\max_j z_j \geq \epsilon B_n)$$

- Since $\max_i z_i^2 / B_n^2 \to 0$, the above is zero for all sufficiently large n.

□

Getting the regular Lindeberg-Levy CLT from Lindeberg Feller

Proof.

- In this case, $B_n^2 = n\sigma^2$.
- The L.C. condition boils down to checking if $\forall \epsilon$

$$\frac{1}{\sigma^2}\mathbb{E}[|X_1|1(|X_1| \geq \sqrt{n}\sigma\epsilon)] \to 0$$

- How will you show this?

$\square$

Lyapunov's CLT

Theorem
Let $X_1, \ldots, X_n$ are mean zero independent random variables with $s_n^2 = \sum_i E[X_i^2]$. As long as, for some $\delta > 0$, Lyapunov's condition holds, i.e.

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}[X_i^{2+\delta}] \to 0,$$

we have

$$\frac{1}{s_n} \sum_{i=1}^{n} X_i \xrightarrow{d} N(0,1)$$

- Prove this condition holds if L.C. holds.

Tail probabilities

- Most often, we are interested in the question, how far is an empirical quantity from its "population variant"?
- This empirical quantity can be an eigenvalue of a matrix, or the weight vector learned using linear regression and so on.
- For rest of today, and a few more lectures we will brush up on tail inequalities.
- Lets start with the mean?

Concentration inequalities

- How will you bound $P(|\bar{X}_n - \mu| \geq t)$? Central limit theorem works under regularity conditions, but its only asymptotic.
- We will look at three methods:
    - Moment based:

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  - Martingale based methods:

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  - Martingale based methods: Azuma-Hoeffding, McDiarmid