

SDS 384 11: Theoretical Statistics

Lecture 10: U Statistics cont.

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- We will see many interesting examples of U statistics.
- Interesting properties
 - Unbiased (done)
 - Reduces variance (done)
 - Concentration (via McDiarmid) (done)
 - Asymptotic variance (done)
 - Asymptotic distribution (today)

Normal Convergence of U statistics-proof

- Trick: find some \hat{U} such that \hat{U} is asymptotically equivalent to U .
- Make sure \hat{U} is easy to analyze.

Theorem

If $X_n \xrightarrow{d} X$ and $|Y_n - X_n| \xrightarrow{P} 0$, then $Y_n \xrightarrow{d} X$.

- In our case we will use \hat{U} as a sum of functions of X_i
- Then use CLT on \hat{U}
- We will find the functions using Hájek projections.

Hájek Projections – Setup

- Let $\{X_1, \dots, X_n\}$ be independent random vectors.
- Consider a linear space \mathcal{S} of random variables.
 - E.g. \mathcal{S} can be the set of all random variables of the form

$$\sum_{i=1}^n g_i(X_i)$$

- g_i are arbitrary measurable functions $g_i : \mathbb{R}^d \rightarrow \mathbb{R}$ with $E[g_i(X_i)^2] < \infty$, for $i \in [n]$
- $ES^2 < \infty, \forall S \in \mathcal{S}$
- Consider a random variable T with $E[T^2] < \infty$

Hájek projections

- Define by the projection $\hat{S} = \arg \inf_{S \in \mathcal{S}} E[(T - S)^2]$

Theorem

\hat{S} is a projection of T onto a linear space \mathcal{S} with finite second moments, iff, $\hat{S} \in \mathcal{S}$ and

$$E[(T - \hat{S})S] = 0, \quad \text{For every } S \in \mathcal{S}. \quad \text{Orthogonality}$$

Every two projections of T onto \mathcal{S} are equal a.s. If \mathcal{S} contains the constant variables, then $E[T] = E[\hat{S}]$ and $\text{cov}(T - \hat{S}, S) = 0$ for every $S \in \mathcal{S}$.

Proof.

- First note that

$$E(T - S)^2 = E[(T - \hat{S})^2] + 2E[(T - \hat{S})(\hat{S} - S)] + E[(S - \hat{S})^2]$$

- If the orthogonality condition is satisfied, then the middle term is zero.
- So $E(T - S)^2 \geq E(T - \hat{S})^2$, and this inequality is strict unless $E(\hat{S} - S)^2 = 0$. This proves uniqueness.



Hájek projections-converse

Proof.

- For any number α

$$E(T - \hat{S} - \alpha S)^2 = E[(T - \hat{S})^2] - 2\alpha E[(T - \hat{S})S] + \alpha^2 E[S^2]$$

- If \hat{S} is the projection, then $\forall \alpha$ and $\forall S \in \mathcal{S}$,

$$\alpha^2 E[S^2] - 2\alpha E[(T - \hat{S})S] \geq 0$$

- So for $\alpha > 0$, $E[(T - \hat{S})S] \leq \alpha E[S^2]/2$
- for $\alpha < 0$, $E[(T - \hat{S})S] \geq -|\alpha| E[S^2]/2$
- So the orthogonality condition must hold.



Hájek projections-proof cont.

- If constants are in \mathcal{S} , then the orthogonality condition with $S = 1$ gives $E[T] = E[\hat{S}]$.
- So, $\text{cov}(T - \hat{S}, S) = E[(T - \hat{S})S] - E[T - \hat{S}]E[S] = 0$
- The first term is zero using orthogonality.
- The second term is zero because $E[T] = E[\hat{S}]$.
- Hájek projections do not always exist, i.e. the $\inf_{S \in \mathcal{S}}$ may not be achievable.
- However it is typically easy to establish existence directly

Projections and asymptotic equivalence

- By the orthogonality, we have $E[T^2] = E[(T - \hat{S})^2] + E[\hat{S}^2]$
- If \mathcal{S} contains constants, then $E[T] = E[\hat{S}]$
- So $\text{var}(T) = \text{var}(T - \hat{S}) + \text{var}(\hat{S})$
- So if \mathcal{S} has constants, and $\text{var}(T) = \text{var}(\hat{S})$, then $\hat{S} = T$ a.s.
- What if the variances are not equal, but almost (or asymptotically) equal?

Projections and asymptotic equivalence

Theorem

Consider linear spaces of random variables with finite second moment S_n that contains constants. Let T_n be random variables with projections \hat{S}_n onto S_n . If $\text{var}(T_n)/\text{var}(\hat{S}_n) \rightarrow 1$, then,

$$\frac{T_n - E[T_n]}{sd(T_n)} - \frac{\hat{S}_n - E[\hat{S}_n]}{sd(\hat{S}_n)} \xrightarrow{P} 0,$$

where $sd(X)$ is $\sqrt{\text{var}(X)}$.

Projections and asymptotic equivalence-proof

Proof.

- We will prove convergence in second mean.

- Let $D_n = \frac{T_n - E[T_n]}{\text{sd}(T_n)} - \frac{\hat{S}_n - E[\hat{S}_n]}{\text{sd}(\hat{S}_n)}$

- $E[D_n] = 0$

- So the variance calculation gives:

$$\begin{aligned}\text{var}(D_n) &= 2 - 2 \frac{\text{cov}(T_n, \hat{S}_n)}{\text{sd}(T_n)\text{sd}(\hat{S}_n)} \\ &= 2 - 2 \frac{\text{cov}(T_n - \hat{S}_n, \hat{S}_n) + \text{var}(\hat{S}_n)}{\text{sd}(T_n)\text{sd}(\hat{S}_n)} \\ &= 2 - 2 \frac{\text{var}(\hat{S}_n)}{\text{sd}(T_n)\text{sd}(\hat{S}_n)} \rightarrow 0\end{aligned}$$



How to get a Hájek projection

- Let $\{X_1, \dots, X_n\}$ be independent random vectors.
- Consider a linear space \mathcal{S} of random variables.
 - E.g. \mathcal{S} can be the set of all random variables of the form $\sum_{i=1}^n g_i(X_i)$.
 - g_i are arbitrary measurable functions $g_i : \mathbb{R}^d \rightarrow \mathbb{R}$ with $E[g_i(X_i)^2] < \infty$, for $i \in [n]$

Theorem

The Hájek projection of an arbitrary random variable $T(X_1, \dots, X_n)$ with finite second moment onto \mathcal{S} is given by

$$\hat{S} = \sum_{i=1}^n E[T|X_i] - (n-1)E[T].$$

How to get a Hájek projection

Proof.

- First note that $\hat{S} \in \mathcal{S}$
- All that remains is to check the orthogonality condition.

$$\begin{aligned} E[(T - \hat{S})S] &= E[(T - \hat{S}) \sum_i g_i(X_i)] \\ &= \sum_i E[(T - \hat{S})g_i(X_i)] \\ &= \sum_i E_{X_i} E[(T - \hat{S})g_i(X_i)|X_i] \\ &= \sum_i E g_i(X_i) E[T - \hat{S}|X_i] \end{aligned}$$

- But $E[\hat{S}|X_i] = E[\sum_j E[T|X_j]|X_i] - (n-1)E[T] = E[T|X_i]$. □

What if X_i 's are iid?

- If X_1, \dots, X_n are iid,
- So, in this case, as long as T is permutation invariant,
$$E[T|X_i = x] = E[T(X_1, \dots, X_{i-1}, x, X_i, \dots)]$$
$$= E[T(x, X_2, \dots, X_n)]$$
- Thus the Hájek projections can be computed by taking a projection on a smaller set $\mathcal{S}' \subset \mathcal{S}$
- \mathcal{S}' contains random variables of the form $\sum_{i=1}^n g(X_i)$ where g is some arbitrary measurable function with $E[g(X_i)^2] < \infty$

Normal Convergence of U statistics-proof

- Recall $U := \frac{1}{\binom{n}{r}} \sum_{S \in \mathcal{I}_r} h(X_S)$
- Define the Hájek projection as

$$\begin{aligned}\hat{U} &:= \sum_{i=1}^n E[U - \theta | X_i] \\ &= \frac{1}{\binom{n}{r}} \sum_{i=1}^n \sum_{S \in \mathcal{I}_r} E[h(X_S) - \theta | X_i]\end{aligned}$$

- Note that

$$E[h(X_S) - \theta | X_i = x] = \begin{cases} E[h(x, X_2, \dots, X_r)] - \theta =: g(x) & \text{When } i \in S \\ 0 & \text{o.w.} \end{cases}$$

Normal Convergence of U statistics-proof

- Define the Hájek projection as

$$\begin{aligned}\hat{U} &:= \sum_{i=1}^n E[U - \theta | X_i] \\&= \frac{1}{\binom{n}{r}} \sum_{i=1}^n \sum_{S \in \mathcal{I}_r} E[h(X_S) - \theta | X_i] \\&= \frac{1}{\binom{n}{r}} \sum_{i=1}^n \sum_{S \in \mathcal{I}_r: X_i \in S} E[h(X_S) - \theta | X_i] \\&= \frac{1}{\binom{n}{r}} \sum_{i=1}^n \binom{n-1}{r-1} g(X_i) \\&= \frac{r}{n} \sum_{i=1}^n g(X_i)\end{aligned}$$

Normal Convergence of U statistics-proof

- Ok. So we got a projection. Now we need to move to asymptotics
- So let us calculate the variance of \hat{U}

$$\begin{aligned}\text{var}(\hat{U}) &= \frac{r^2}{n} \text{var}(g(X_1)) \\ &= \frac{r^2}{n} \text{var}(E[h(X_S)|X_1]) = \frac{r^2}{n} \xi_1\end{aligned}$$

- Now CLT gives, $\sqrt{n}(\hat{U} - \theta) \xrightarrow{d} N(0, r^2 \xi_1)$

Normal Convergence of U statistics-proof

- $\sqrt{n}\hat{U} \xrightarrow{d} N(0, r^2\xi_1)$
- We already proved $\frac{\text{var}(U)}{\text{var}(\hat{U})} \rightarrow 1$
- So $\sqrt{n}(\hat{U} - (U - \theta)) \xrightarrow{P} 0$
- So $\sqrt{n}(U - \theta) \xrightarrow{d} N(0, r^2\xi_1)$