

SDS 384 11: Theoretical Statistics

Lecture 12: Uniform Law of Large Numbers- Rademacher Complexity

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

Proof of the GC theorem

- We will work on a proof that can handle general function classes \mathcal{F} with bounded functions. WLOG let $|f(X_i)| \leq 1$ for $f \in \mathcal{F}$.
- Recall that we want to bound $\|\hat{P}_n - P\|_{\mathcal{F}}$
 $(:= \sup_{f \in \mathcal{F}} |\frac{1}{n} \sum_i f(X_i) - E[f]|)$
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 - Concentration inequality to bound $\|\hat{P}_n - P\|_{\mathcal{F}} - E[\|\hat{P}_n - P\|_{\mathcal{F}}]$

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 - Symmetrization to relate $E[\|\hat{P}_n - P\|_{\mathcal{F}}]$ to Rademacher complexity
 - Bound this complexity using the effective “size” of the function class.

Concentration

- First note that we cannot apply Hoeffding/Chernoff here.
- Let $X := \{X_1, \dots, X_n\}$
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- Let f_1 maximize $g(X)$, and f_2 maximize $g(Y)$
-

$$\begin{aligned} g(X) - g(Y) &= \left| \frac{\sum_i f_1(X_i)}{n} - Ef_1[X_1] \right| - \left| \frac{\sum_i f_2(Y_i)}{n} - Ef_2[X_1] \right| \\ &\leq \left| \frac{\sum_i f_1(X_i)}{n} - Ef_1[X_1] \right| - \left| \frac{\sum_i f_1(Y_i)}{n} - Ef_1[X_1] \right| \\ &\leq \frac{2}{n} \end{aligned}$$

- Using McDiarmid's inequality, we get:

$$P(g(X) - E[g(X)] \geq \epsilon) \leq \exp(-\epsilon^2 n/2)$$

- So, with probability $1 - \exp(-\epsilon^2 n/2)$,

$$\|\hat{P}_n - P\|_{\mathcal{F}} \leq E[\|\hat{P}_n - P\|_{\mathcal{F}}] + \epsilon.$$

- So, we need to bound $E[\|\hat{P}_n - P\|_{\mathcal{F}}]$.

Symmetrization

- Consider an iid copy of X' of X

$$\begin{aligned} E\|\hat{P}_n - P\|_{\mathcal{F}} &= E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - E[f(X_i)]) \right| \\ &= E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - E[f(X'_i)]) \right| \\ &= E_X \sup_{f \in \mathcal{F}} \left| E_{X'} \left[\frac{1}{n} \sum_i (f(X_i) - f(X'_i)) \right] \right| \\ &\leq E_{X, X'} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - f(X'_i)) \right| \\ &= E_{X, X'} \|\hat{P}_n - \hat{P}'_n\|_{\mathcal{F}} \end{aligned}$$

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Symmetrize again

- Let $\epsilon_i \in \{1, -1\}$.
- Note that $f(X_i) - f(X'_i)$ is symmetric
- For a symmetric random variable R , and a random variable $\epsilon \in \{-1, 1\}$ (independent of R)

$$\begin{aligned}P(\epsilon R \leq t) &= P(R \leq t)P(\epsilon = 1) + P(R \geq -t)P(\epsilon = -1) \\ &= P(R \leq t)\end{aligned}$$

- Hence $\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - f(X'_i)) \right|$ and $\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i (f(X_i) - f(X'_i)) \right|$ have the same distribution, and expectation
- We will choose ϵ_i 's uniformly, i.e. we will consider Rademacher random variables.

Rademacher complexity

$$\begin{aligned} E\|\hat{P}_n - P\|_{\mathcal{F}} &\leq E_{X, X'} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i (f(X_i) - f(X'_i)) \right| \\ &= E_{X, X', \epsilon} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i (f(X_i) - f(X'_i)) \right| \\ &\leq E_{X, \epsilon} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i f(X_i) \right| + E_{X', \epsilon} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i f(X'_i) \right| \\ &= 2E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i f(X_i) \right| =: 2\mathcal{R}_{\mathcal{F}} \end{aligned}$$

- $\mathcal{R}_{\mathcal{F}}$ is also called the Rademacher complexity of the function class.

Why the Rademacher complexity?

- We have now shown that $\|\hat{P}_n - P\|_{\mathcal{F}} \leq 2\mathcal{R}_{\mathcal{F}} + \epsilon$ with prob. $1 - e^{-n\epsilon^2/2}$.
- $\mathcal{R}_{\mathcal{F}}$ measures the maximum possible correlation (over all $f \in \mathcal{F}$) between the vector $(f(X_1), \dots, f(X_n))$ and the “noise vector” $(\epsilon_1, \dots, \epsilon_n)$.
- If a function class has some function which has a high correlation with a random noise vector, then we should not expect concentration.
- If $\mathcal{R}_{\mathcal{F}}$ is $o(1)$ then the Borel Cantelli lemma gives $\|\hat{P}_n - P\|_{\mathcal{F}} \xrightarrow{a.s.} 0$.

Size of a function class \mathcal{F}

- Let $\mathcal{F}(X) = \{(f(X_1), \dots, f(X_n)) : f \in \mathcal{F}\}$
- $\mathcal{R}_{\mathcal{F}} = E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i f(X_i) \right| = E \left[E \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_i \epsilon_i f(X_i) \right| \middle| X_1, \dots, X_n \right]$
- In the next slide we will bound this using the cardinality of $\mathcal{F}(X)$

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Theorem

Let $A \subseteq \mathbb{R}^n$, $R = \max_{a \in A} \|a\|$,

$$E \sup_{a \in A} \langle \epsilon, a \rangle \leq \sqrt{2R^2 \log |A|}.$$

And,

$$E \sup_{a \in A} |\langle \epsilon, a \rangle| \leq \sqrt{2R^2 \log |2A|}.$$

Proof.

$$\begin{aligned}\exp \left(\lambda E \sup_{a \in A} \langle \epsilon, a \rangle \right) &\leq E \exp \left(\lambda \sup_{a \in A} \langle \epsilon, a \rangle \right) \\ &= E \sup_{a \in A} \exp (\lambda \langle \epsilon, a \rangle) \\ &\leq \sum_{a \in A} E \exp (\lambda \langle \epsilon, a \rangle) \\ (\langle \epsilon, a \rangle &\sim \text{Subgaussian}(\|a\|_2^2)) \leq \sum_{a \in A} \exp \left(\frac{\lambda^2 \|a\|_2^2}{2} \right) \\ &\leq |A| \exp \left(\frac{\lambda^2 R^2}{2} \right)\end{aligned}$$

Take $\lambda = 2 \log |A|/R^2$.

□

Size of a function class \mathcal{F}

- Note that in this case \mathcal{A} contains of vectors $(f(X_1)/n, \dots, f(X_n)/n)$, where f is a indicator function, i.e. $f(X_i) = 1(X_i \leq t)$.
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- So $R^2 = 1/n$.
- The question is for a given dataset X_1, \dots, X_n , how many distinct points are there in \mathcal{A} ?

$$\begin{aligned} |\mathcal{A}| &= |\mathcal{F}(X)| = |\{(f(X_{(1)}), \dots, f(X_{(n)})) : f \in \mathcal{F}\}| \\ &= |\{(1(X_{(1)} \leq t), \dots, 1(X_{(n)} \leq t)) : t \in \mathbb{R}\}| \\ &\leq n + 1 \quad (\text{HUH!!}) \end{aligned}$$

Proof.

If \mathcal{F} is the set of one sided indicator functions, then

$$\begin{aligned}\|\hat{P}_n - P\|_{\mathcal{F}} &\leq 2\mathcal{R}_{\mathcal{F}} + \epsilon = 2E[E[\sup_{f \in \mathcal{F}} \sum_i \epsilon_i f(X_i)/n] | X] + \epsilon \\ &\leq \sqrt{8R^2 \log(n+1)} + \epsilon \\ &\leq \sqrt{\frac{8 \log(n+1)}{n}} + \epsilon\end{aligned}$$

By Borel Cantelli, $\|\hat{P}_n - P\|_{\mathcal{F}} \xrightarrow{a.s.} 0$

□

