

# SDS 384 11: Theoretical Statistics

## Lecture 5: Martingale inequalities

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- This forms a Martingale difference sequence.



## Definition

A sequence of random variables  $\{Y_i\}$  adapted to a filtration  $\mathcal{F}_i$  is a martingale if, for all  $i$ ,

$$E|Y_i| < \infty \quad E[Y_{i+1}|\mathcal{F}_i] = Y_i$$

- A filtration  $\{\mathcal{F}_i\}$  is a sequence of nested  $\sigma$ -fields, i.e.  $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ .

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- If  $\mathcal{F}_i = \sigma(Y_1, \dots, Y_i)$ , then we say that  $\{Y_i\}$  forms a martingale sequence.

## Example-partial sums of i.i.d sequences

### Example

Let  $\{X_i\}_{i=1}^{\infty}$  be a sequence of i.i.d random variables with  $E[X_1] = \mu$ . Let  $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$ . Then  $\{Y_i = \sum_{k=1}^i X_k - i\mu\}$  is a martingale sequence w.r.t  $\{X_i\}$ .

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- $Y_i$  is measurable w.r.t  $\mathcal{F}_i$ .
- Finally,

$$\begin{aligned} E[Y_{i+1}|\mathcal{F}_i] &= E[X_{i+1} + \sum_{k=1}^i X_k - (i+1)\mu|\mathcal{F}_i] \\ &= \mu + \sum_{k=1}^i X_k - (i+1)\mu = Y_i \end{aligned}$$

# Doob construction

## Example

Let  $\{X_i\}_{i=1}^{\infty}$  be a sequence of random variables. Let  $Y_i = E[f(X)|X_1, \dots, X_i]$  and assume that  $E[|f(X)|] < \infty$ . Then  $\{Y_i\}_{i=0}^n$  is a martingale sequence w.r.t  $\{X_i\}_{i=1}^n$ .



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- $E[|Y_i|] = E[|E[f(X)|X_1, \dots, X_i]|] \leq E[|f(X)|] < \infty$ . (Use Jensen on  $|(\cdot)|$ )

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- $E[|Y_i|] = E[|E[f(X)|X_1, \dots, X_i]|] \leq E[|f(X)|] < \infty$ . (Use Jensen on  $|(\cdot)|$ )
- Furthermore,

$$\begin{aligned} E[Y_{i+1}|X_1, \dots, X_i] &= E[E[f(X)|X_1, \dots, X_{i+1}]|X_1, \dots, X_i] \\ &= E[f(X)|X_1, \dots, X_i] = Y_i \end{aligned} \quad \text{The tower property}$$

# Doob construction - examples

## Example

We are throwing  $m$  balls into  $n$  bins. At step  $i$  we place ball  $i$  into a bin chosen uniformly at random. Call the index of the bin  $X_i$ . Let  $Z$  denote the number of empty bins.  $E[Z|X_1, \dots, X_i]$  is a martingale.

- What's the big deal, just write  $Y_i = 1(\text{Bin } i \text{ is empty})$
- $Z = \sum_i Y_i$ , and so I can compute expectation easily.
- Can we use traditional concentration arguments to say  $Z - EZ$  is small?

# Doob construction - examples

## Example

Consider a random graph  $G(n, p)$  where the edge between  $i, j$  is added with probability  $p$ , independent of any other edges. We are interested in the Chromatic number of this graph ( $\chi$ ), i.e. the minimum number of colors to “properly” color this graph, i.e. no two nodes connected by an edge should have the same color.

- Let the vertices be labeled as  $1, \dots, n$
- Let  $G_i$  denote the graph induced by nodes  $1, \dots, i$ .
- Set  $E[\chi | G_1, \dots, G_i]$  is a martingale.
- This is also called a vertex exposure filtration.
- See “Sharp concentration of the chromatic number on random graphs  $G_{n,p}$ ” by Shamir and Spencer

# Likelihood ratio

## Example

Let  $f, g$  be two densities such that  $g$  is absolutely continuous w.r.t  $f$ .

Suppose  $\{X_i\}_{i=1}^{\infty} \stackrel{iid}{\sim} f$  and  $Y_n$  is the likelihood ratio  $\prod_{i=1}^n \frac{g(X_i)}{f(X_i)}$  for the first  $n$  datapoints. Then  $\{Y_n\}$  forms a Martingale sequence w.r.t  $\{X_n\}$ .

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- First recall that  $E[|Y_n|] = E[Y_n] = 1$

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$$\begin{aligned} E[Y_{n+1}|X_1, \dots, X_n] &= E \left[ \prod_{i=1}^{n+1} \frac{g(X_i)}{f(X_i)} \middle| X_1, \dots, X_n \right] \\ &= \prod_{i=1}^n \frac{g(X_i)}{f(X_i)} E \left[ \frac{g(X_{n+1})}{f(X_{n+1})} \right] = Y_n \end{aligned}$$

# Martingale Difference Sequence

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A sequence  $\{D_i\}$  of random variables adapted to a filtration  $\{\mathcal{F}_i\}$  is a Martingale Difference Sequence if,

$$E[|D_i|] < \infty \quad E[D_{i+1}|\mathcal{F}_i] = 0$$



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- Let  $\{Y_i\}$  be a martingale sequence.
- Then  $D_{i+1} = Y_{i+1} - Y_i$  define a Martingale Difference Sequence.
- $E[D_{i+1}|\mathcal{F}_i] = E[Y_{i+1}|\mathcal{F}_i] - E[Y_i|\mathcal{F}_i] = Y_i - Y_i = 0$ .

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- $E[D_{i+1}|\mathcal{F}_i] = E[Y_{i+1}|\mathcal{F}_i] - E[Y_i|\mathcal{F}_i] = Y_i - Y_i = 0$ .
  - $E[Y_{i+1}|\mathcal{F}_i] = Y_i$  because of the martingale property,
  - $E[Y_i|\mathcal{F}_i] = Y_i$  since  $Y_i$  is measurable w.r.t the filtration  $\mathcal{F}_i$ .

# Concentration inequalities

## Theorem

Consider a Martingale sequence  $\{D_i\}$  (adapted to a filtration  $\{\mathcal{F}_i\}$ ) that satisfies  $E[e^{\lambda D_i} | \mathcal{F}_{i-1}] \leq e^{\lambda^2 \nu_i^2 / 2}$  a.s. for any  $|\lambda| < 1/b_i$ .

- The sum  $\sum_i D_i$  is sub-exponential with parameters  $(\sqrt{\sum_k \nu_k^2}, b_*)$  where  $b_* := \max_i b_i$ .
- Hence for all  $t \geq 0$ ,

$$P \left[ \left| \sum_{i=1}^n D_i \right| \geq t \right] \leq \begin{cases} 2e^{-\frac{t^2}{2 \sum_k \nu_k^2}} & \text{If } 0 \leq t \leq \frac{\sum_k \nu_k^2}{b_*} \\ 2e^{-\frac{t}{2b_*}} & \text{If } t > \frac{\sum_k \nu_k^2}{b_*} \end{cases}$$

## Proof.

$$\text{Let } X := \sum_{i=1}^n D_i.$$

$$\begin{aligned} E[e^{\lambda \sum_i D_i}] &= E[E[e^{\lambda \sum_i D_i} | \mathcal{F}_{n-1}]] = E[e^{\lambda \sum_{i=1}^{n-1} D_i} E[e^{\lambda D_n} | \mathcal{F}_{n-1}]] \\ &\leq E[e^{\lambda \sum_{i=1}^{n-1} D_i}] e^{\lambda^2 \nu_n^2 / 2} \quad \text{If } |\lambda| < 1/b_n \\ &\leq E[e^{\lambda \sum_{i=1}^{n-2} D_i}] e^{\lambda^2 (\nu_{n-1}^2 + \nu_n^2) / 2} \quad \text{If } |\lambda| < 1/b_n, 1/b_{n-1} \\ &\leq e^{\sum_i \lambda^2 \nu_i^2 / 2} \quad \text{If } |\lambda| < \min_i 1/b_i \end{aligned}$$

Using our previous theorem on sub-exponential random variables, the result is proven in one direction. The other direction is identical leading to the factor of 2.  $\square$

## Corollary (Azuma-Hoeffding)

Let  $\{D_k\}$  be a Martingale Difference Sequence adapted to the filtration  $\{\mathcal{F}_k\}$  and suppose  $|D_k| \leq b_k$  a.s. for all  $k \geq 1$ . Then  $\forall t \geq 0$ ,

$$P \left[ \left| \sum_{k=1}^n D_k \right| \geq t \right] \leq 2e^{-\frac{t^2}{2 \sum_{k=1}^n b_k^2}}$$

## Proof.

- We can rework the last proof. We need  $|E[e^{\lambda D_n} | \mathcal{F}_{n-1}]|$ .
- This is bounded by  $e^{\lambda^2 b_n^2 / 2}$ , since  $D_n$  is mean zero sub-gaussian with  $\sigma = b_n$ .



# McDiarmid's inequality

## Theorem

Let  $f : \mathcal{X}^n \rightarrow \mathbb{R}$  satisfy the following bounded difference condition

$\forall x_1, \dots, x_n, x'_i \in \mathcal{X}$ :

$$|f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq B_i,$$

then,  $P(|f(X) - E[f(X)]| \geq t) \leq 2 \exp\left(-\frac{2t^2}{\sum_i B_i^2}\right)$

- Note that this boils down to Hoeffding's when  $f$  is the sum of bounded random variables.

## Proof.

- Define  $Y_i = E[f(X)|\mathcal{F}_i]$  and  $D_i = Y_i - Y_{i-1}$ .





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- Define  $Y_i = E[f(X)|\mathcal{F}_i]$  and  $D_i = Y_i - Y_{i-1}$ .
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- Since  $\{Y_i\}$  is a Martingale sequence w.r.t  $\{X_i\}$ ,  $\{D_i\}$  is a Martingale difference sequence.
- We have:

$$\begin{aligned} D_i &= E[f(X)|\mathcal{F}_i] - E[f(X)|\mathcal{F}_{i-1}] \\ &= E[f(X)|X_1, \dots, X_i] - E[f(X)|X_1, \dots, X_{i-1}] \\ &\leq \sup_x (E[f(X)|X_1, \dots, x] - E[f(X)|X_1, \dots, X_{i-1}]) =: U_i \end{aligned}$$

$$D_i \geq \inf_x (E[f(X)|X_1, \dots, x] - E[f(X)|X_1, \dots, X_{i-1}]) =: L_i$$

$$U_i - L_i \leq B_i$$



## Proof.

- We also have:

$$U_i - L_i \leq B_i$$

- How?

$$\begin{aligned} U_i - L_i &= \sup_x E[f(X)|X_1, \dots, x] - \inf_y E[f(X)|X_1, \dots, y] \\ &= \sup_{x,y} (E[f(X)|X_1, \dots, x] - E[f(X)|X_1, \dots, y]) \\ &= \sup_{x,y} \int (f(x_{1:i-1}, x, X_{i+1:n}) - f(x_{1:i-1}, y, X_{i+1:n})) dP(X_{i+1:n}) \\ &\leq \sup_{x,y} \int |f(x_{1:i-1}, x, X_{i+1:n}) - f(x_{1:i-1}, y, X_{i+1:n})| dP(X_{i+1:n}) \\ &\leq B_i \end{aligned}$$

- Now apply Azuma-Hoeffding.
- So, where is independence being used?

## Example: Mean absolute deviation

### Example

Consider an i.i.d random variable sequence  $\{X_k\}_{k=1}^{\infty}$  with  $|X_k| \leq b$ . Define the mean absolute deviation:

$$U = \frac{1}{\binom{n}{2}} \sum_{j < k} |X_j - X_k|$$

As we will see later, the above is a type of a pairwise U-Statistics. We want to bound  $|U - E[U]|$ .

- Note that the summands are not independent.

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- Note that the summands are not independent.
- Also note that  $||X_i - X_j| - |X_i - X'_j|| \leq |X_j - X'_j| \leq 2b$
- So  $|U(x_1, \dots, x_i, \dots, x_n) - U(x_1, \dots, x'_i, \dots, x_n)| \leq \frac{(n-1)2b}{\binom{n}{2}} = \frac{4b}{n}$

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- So  $|U(x_1, \dots, x_i, \dots, x_n) - U(x_1, \dots, x'_i, \dots, x_n)| \leq \frac{(n-1)2b}{\binom{n}{2}} = \frac{4b}{n}$
- Use McDiarmid's inequality,  $P(|U - E[U]| \geq t) \leq 2 \exp\left(\frac{-nt^2}{8b^2}\right)$

## Example: Number of triangles in an Erdos Renyi graph

### Example

Consider an Erdős Rényi ( $ER(p)$ ) random graph. What can we say about the number of triangles  $\Delta$ ?

- Let  $n$  be the number of nodes.  $m = \binom{n}{2}$  be the number of ordered pairs. Call this set  $E$ .
- An  $ER(p)$  graph chooses the edges randomly as iid Bernoulli r.v.s  $\{X_e : e \in E\}$  with  $P(X_e = 1) = p$ .
- Let  $\mathcal{T} \subset E^3$  be the set of 3-tuples of node pairs which can form a triangle. e.g.  $\{(i, j), (j, k), (k, i)\} \in \mathcal{T}$ .  $|\mathcal{T}| = \binom{n}{3}$ .
- We have 
$$f(X) = \sum_{\{e_1, e_2, e_3\} \in \mathcal{T}} X_{e_1} X_{e_2} X_{e_3}.$$



## Example: Number of triangles in an Erdos Renyi graph–Cont.

### Example

Consider an Erdős Rényi (ER( $p$ )) random graph. What can we say about the number of triangles  $\Delta$ ?

- If I switch  $X_e = 1$  to 0 how much can  $f(X)$  change?
- It changes by all triangles incident on that edge. The maximum number of such triangles is  $n - 2$ . So  $L = n - 2$ .
- Hence  $P(|f(X) - E[f(X)]| \geq t) \leq 2e^{-\frac{2t^2}{m(n-2)^2}}$
- $E[f(X)] = \binom{n}{3} p^3$ . If we set  $t = \Theta(n^2 \log n)$ , then the error probability goes to zero.
- But in order for this to give concentration we need,  $t/n^3 p^3 \rightarrow 0$ , i.e.  $np \gg n^{2/3}$

## Example: Number of triangles in an Erdos Renyi graph–Cont.

### Example

Consider an Erdős Rényi (ER( $p$ )) random graph. What can we say about the number of triangles  $\Delta$ ?

- One can however use Chen-Stein method to show that  $f(X)$  is approximately *Poisson*  $\left(\binom{n}{3} p^3\right)$ .
- So the above should hold as long as  $np \rightarrow \infty$ . But McDiarmid requires a much stronger condition!
- What if we could plug in the expected value of the Lipschitz constant, i.e.  $np^2$ ?
- Then the exponent would be  $e^{-2t^2/n^4 p^4}$ . Taking  $t = n^2 p^2$ , we see that concentration would amount to having  $np \gg \log n$  which matches with the Poisson limit argument.

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### Example

Consider a random graph  $G(n, p)$  where the edge between  $i, j$  is added with probability  $p$ , independent of any other edges. We are interested in the Chromatic number of this graph ( $\chi$ ), i.e. the minimum number of colors to “properly” color this graph, i.e. no two nodes connected by an edge should have the same color.

- We need independent RVs  $Z_1, \dots, Z_i$  so that we can construct a Doob martingale  $E[\chi | Z_1, \dots, Z_i]$  and apply McDiarmid’s inequality.
- Let  $Z_i$  be the edges from node  $i$  to nodes  $1, \dots, i-1$ .
- $\chi$  cannot decrease by more than 1, because if the graph without node  $i$  can be colored by  $k$  colors, then the graph with node  $i$  can be colored using  $k+1$  colors.
- Similarly, it can’t increase by more than 1, because you can just color node  $i$  with a new color, thereby increasing the chromatic number by 1.

# Lipschitz functions of Gaussian random variables

## Definition

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $L$ -Lipschitz w.r.t the Euclidean norm if

$$|f(x) - f(y)| \leq L\|x - y\|_2 \quad \forall x, y \in \mathbb{R}^n$$

## Theorem

*Let  $(X_1, \dots, X_n)$  be a vector of iid  $N(0, 1)$  random variables. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be  $L$ -Lipschitz w.r.t the Euclidean norm. Then  $f(X) - E[f(X)]$  is sub-gaussian with parameter at most  $L$ , i.e.  $\forall t \geq 0$ ,*

$$P(|f(X) - E[f(X)]| \geq t) \leq e^{-\frac{t^2}{2L^2}}$$

- A  $L$  Lipschitz function of a vector of i.i.d  $N(0, 1)$  random variables concentrate like a  $N(0, L^2)$  random variable, irrespective of how long the vector is.