

# SDS 384 11: Theoretical Statistics

## Lecture 3: Concentration inequalities

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# Remember Markov's inequality?

## Theorem

For  $X \geq 0$ ,  $E[X] \leq \infty$ ,  $t > 0$ , we have:

$$P(X \geq t) \leq \frac{E[X]}{t}$$

Use total expectation theorem.

$$E[X] = E[X|X \geq t]P(X \geq t) + E[X|X < t]P(X < t)$$



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$$\begin{aligned} E[X] &= E[X|X \geq t]P(X \geq t) + E[X|X < t]P(X < t) \\ &\geq E[X|X \geq t]P(X \geq t) \end{aligned}$$



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$$P(X \geq t) \leq \frac{E[X]}{t}$$



# Higher order moments

## Theorem (Chebyshev's)

For  $t > 0$

$$P(|X - \mu| \geq t) = P((X - \mu)^2 \geq t^2) \leq \frac{E[(X - \mu)^2]}{t^2} = \frac{\text{var}(X)}{t^2}$$

## Theorem (Higher order markov)

For  $t > 0$

$$P(|X - \mu| \geq t) = P(|X - \mu|^k \geq t^k) \leq \frac{E[|X - \mu|^k]}{t^k}$$

# Chernoff bound

## Theorem (Chernoff bound for Bernoullis)

Let  $X_i \in \{0,1\}$  be independent random variables with  $E[X_i] = p_i$ . Let  $X := \sum_i X_i, \mu := \sum_i p_i$ . For  $0 < \delta < 1$ ,

$$P(X \geq \mu(1 + \delta)) \leq e^{-\delta^2 \mu / 3} \quad P(X \leq \mu(1 - \delta)) \leq e^{-\delta^2 \mu / 2}$$

**Proof.**

$$P(X \geq \mu(1 + \delta)) = \inf_{\lambda \geq 0} P(e^{\lambda X} \geq e^{\lambda \mu(1 + \delta)}) \leq \inf_{\lambda \geq 0} e^{-\lambda \mu(1 + \delta)} \underbrace{E[e^{\lambda X}]}_{\text{MGF of } X}$$

□

## Chernoff continued

$$\begin{aligned}\inf_{\lambda \geq 0} e^{-\lambda \mu(1+\delta)} E[e^{\lambda X}] &= \inf_{\lambda \geq 0} e^{-\lambda \mu(1+\delta)} \prod_i E[e^{\lambda X_i}] \\&= \inf_{\lambda \geq 0} e^{-\lambda \mu(1+\delta)} \prod_i (e^{\lambda} p_i + 1 - p_i) \\(\text{Since } 1 + x \leq e^x \text{ for } x \geq 0) &\leq \inf_{\lambda \geq 0} e^{-\lambda \mu(1+\delta)} \prod_i e^{p_i(e^{\lambda} - 1)} \\&= \inf_{\lambda \geq 0} e^{-\lambda \mu(1+\delta) + \mu(e^{\lambda} - 1)} \\(\text{minimized at } \lambda = \log(1 + \delta)) &= e^{\mu(\delta - (1+\delta) \log(1+\delta))} \\&\leq e^{-\mu \delta^2 / 3}\end{aligned}$$

The last line follows from the fact that  $\log(1 + x) \geq x/(1 + x/2)$

# Is it tight?

## Theorem (Chernoff bound for Gaussians)

Let  $X_i \sim N(\mu, \sigma^2)$  be independent random variables. Let  $X := \sum_i X_i$ .

$$P(X/n - \mu \geq t) \leq e^{-\frac{nt^2}{2\sigma^2}}$$

### Proof.

Following in the same lines:

$$P(X/n - \mu \geq t) \inf_{\lambda \geq 0} e^{-n\lambda t} E \left[ e^{\lambda(X - n\mu)} \right] = \inf_{\lambda \geq 0} e^{-n\lambda t} \prod_i E \left[ e^{\lambda(X_i - \mu)} \right]$$

$$\text{(Since } E[e^{\lambda X}] = e^{\lambda\mu + \sigma^2\lambda^2/2} \text{)} \quad = \inf_{\lambda \geq 0} e^{-n\lambda t + n\sigma^2\lambda^2/2}$$

$$\text{(Since } \lambda = t/\sigma^2 \text{ minimizes this)} \quad = e^{-\frac{nt^2}{2\sigma^2}}$$



# Is it tight?

- Let  $Z \sim N(0, 1)$ . We can show that for  $z > 0$ ,

$$\phi(z) \left( \frac{1}{z} - \frac{1}{z^3} \right) \leq P(Z \geq z) \leq \phi(z) \left( \frac{1}{z} - \frac{1}{z^3} + \frac{3}{z^5} \right),$$

where  $\phi(z)$  is the density of a standard normal.

- Since  $\bar{X}_n \sim N(\mu, \sigma^2/n)$ ,  $\lim_{n \rightarrow \infty} \log P(\bar{X}_n - \mu \geq t)/n = -\frac{t^2}{2\sigma^2}$
- So the Chernoff bound is asymptotically tight, in the sense that it gets the constant inside the exponent right.

# Hoeffding's lemma

## Theorem

For a random variable  $X \in [a, b]$  with  $E[X] = \mu$  and  $\lambda \in \mathbb{R}$ ,

$$M_{X-\mu}(\lambda) \leq e^{\frac{\lambda^2(b-a)^2}{8}}$$

- In comparison, for a Gaussian random variable  $X \sim N(\mu, \sigma^2)$ ,

$$M_{X-\mu}(\lambda) = e^{\frac{\lambda^2 \sigma^2}{2}}$$

- For a bounded random variable  $X \in [a, b]$ ,  $\text{var}(X) \leq (b-a)^2/4$  from Popoviciu's inequality.

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  - Recall that  $E[(X-t)^2]$  is minimized at  $t = E[X]$ .
  - So  $\text{var}(X) \leq E[(X - (a+b)/2)^2] \leq \frac{(b-a)^2}{4}$

## MGF of Rademacher variables

A Rademacher random variable  $\epsilon$  takes values in  $\{-1, 1\}$  equiprobable.

$$\begin{aligned} E[e^{\lambda\epsilon}] &= \frac{e^{\lambda} + e^{-\lambda}}{2} \\ &= \sum_i \frac{\lambda^{2i}}{(2i)!} \\ &\leq \sum_i \frac{\lambda^{2i}}{2^i i!} \\ &= e^{\lambda^2/2} \end{aligned}$$

# Hoeffding's Lemma: weaker version

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# Hoeffding's Lemma: weaker version

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$$M_{X-\mu}(\lambda) \leq e^{\frac{\lambda^2(b-a)^2}{2}}$$

- Consider an iid copy  $X'$  of  $X$ . Also consider a Radamacher random variable  $\epsilon$ .

$$\begin{aligned} E[e^{\lambda(X-E[X])}] &= E[e^{\lambda(X-E_{X'}[X'])}] = E_X[e^{\lambda E_{X'}(X-X')}] \\ &\leq E_{X,X'} e^{\lambda(X-X')} = E_{X,X'} E_{\epsilon} e^{\epsilon \lambda(X-X')} \\ &\leq E_{X,X'} e^{\frac{\lambda^2(X-X')^2}{2}} \leq e^{\frac{\lambda^2(b-a)^2}{2}} \end{aligned}$$

# Hoeffding's inequality

## Theorem

Consider i.i.d  $X_i \in [a_i, b_i]$ . Let  $X = \sum_i X_i$ .

$$P(X - E[X] \geq t) \leq e^{-\frac{2t^2}{\sum_i (b_i - a_i)^2}}$$

## Proof.

$$\begin{aligned} P(X - E[X] \geq t) &\leq \inf_{\lambda \geq 0} e^{-\lambda t} E[e^{\lambda(X - E[X])}] \\ &\leq \inf_{\lambda \geq 0} e^{-\lambda t} \prod_i E[e^{\lambda(X_i - E[X_i])}] \\ &\leq \inf_{\lambda \geq 0} e^{-\lambda t + \frac{\lambda^2 \sum_i (b_i - a_i)^2}{8}} = e^{-\frac{2t^2}{\sum_i (b_i - a_i)^2}} \end{aligned}$$



## How do we use this?

Consider  $n$  fair coins  $X_i \in \{0, 1\}$ . The Hoeffding inequality gives us

$$P(|\sum_i X_i - n/2| \geq t) \leq 2e^{-2t^2/n}$$

- How to pick  $t$ ?
- Set the failure probability at  $\delta$ .
- So  $t = \sqrt{\frac{n}{2} \log(1/\delta)}$ , i.e. we can also write the bound as

$$P\left(\left|\sum_i X_i - n/2\right| \geq \sqrt{\frac{n}{2} \log(1/\delta)}\right) \leq \delta$$

## Definition

$X$  is sub-gaussian with parameter  $\sigma^2$  if, for all  $\lambda \in \mathbb{R}$ ,

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- Gaussian random variables are also sub-gaussian.
- $X$  is sub-gaussian iff  $-X$  is also sub-gaussian

- $X$  is sub-gaussian implies that
  - $P(X - \mu \geq t) \leq e^{-t^2/2\sigma^2}$
  - $P(X - \mu \leq -t) \leq e^{-t^2/2\sigma^2}$
  - $P(|X - \mu| \geq t) \leq 2e^{-t^2/2\sigma^2}$

## Sub-gaussian r.v.'s – some properties

- Consider a R.V.  $X$  such that

$$E[\exp(\lambda X)] \leq \exp(\lambda\mu + \lambda^2\sigma^2/2)$$

- $E[X] = \mu$
- $\text{var}(X) \leq \sigma^2$
- If the smallest value of  $\sigma$  that satisfies the above equation is chosen, is it true that that will equal the variance?

## Sub-Gaussian random variables

- Let  $X_1, X_2$  be independent sub-gaussian random variables with parameters  $\sigma_1$  and  $\sigma_2$ . Then  $aX_1 + bX_2$  is sub-gaussian with parameter  $a^2\sigma_1^2 + b^2\sigma_2^2$ .

# Sub-Gaussian random variables

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$$\begin{aligned} M_{a(X_1-\mu_1)+b(X_2-\mu_2)}(\lambda) &= E[e^{\lambda(a(X_1-\mu_1)+b(X_2-\mu_2))}] \\ &= E[e^{\lambda a(X_1-\mu_1)}]E[e^{\lambda b(X_2-\mu_2)}] \\ &\leq e^{\frac{\lambda^2(a^2\sigma_1^2+b^2\sigma_2^2)}{2}} \end{aligned}$$