Homework Assignment 1

Due via canvas Feb 9th

SDS 384-11 Theoretical Statistics

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1. (2 pt) Given densities f_n and g_n with respect to some measure μ, let X be distributed according to the distribution with density f_n. Define the likelihood ratio $L_n(X)$ as $L_n(X) = g_n(X)/f_n(X)$ for $f_n(X) > 0$, and $L_n(X) = 1$, if $f_n(X) = g_n(X) = 0$ and $L_n(X) = \infty$ otherwise. Show that the likelihood ratio is a uniformly tight sequence. First note that $EL_n(X) = 1$, and hence by Markov's inequality, for any ϵ, $P(L_n(X) > 1/\epsilon) \leq \epsilon$. This establishes tightness.

2. (1+2+3) We will do some examples of convergence in distribution and convergence in probability here.

   (a) Let $X_n \sim N(0, n)$. Prove that $X_n = O_p(\sqrt{n})$ and $o_P(n)$. Since $X_n/\sqrt{n} \xrightarrow{d} N(0,1)$, we see that $X_n/\sqrt{n} = O_P(1)$ and hence $X_n = O_P(\sqrt{n})$. As for the last part, $P(X_n/n \geq t) \leq 1/nt^2$ and hence $X_n/n = o_P(1)$.

   (b) Let $\{X_n\}$ be independent r.v's such that $P(X_n = n^\alpha) = 1/n$ and $P(X_n = 0) = 1 - 1/n$ for $n \geq 1$, where $\alpha \in (-\infty, \infty)$ is a constant. For what values of α, will you have $X_n \xrightarrow{q.m} 0$? For what values will you have $X_n \xrightarrow{p} 0$?

   Convergence in quadratic mean:

   $$\mathrm{E}[|X_n|^2] = \frac{n^{2\alpha}}{n}$$

   The above will converge to zero if $2\alpha < 1$, or $\alpha < \frac{1}{2}$.

   Convergence in Probability:

   For $\epsilon \geq n^\alpha$ we have $\Pr(|X_n| > \epsilon) = 0$. For $\epsilon < n^\alpha$ we have $\Pr(|X_n| > \epsilon) = \frac{1}{n}$. This probability converges to zero for all values of α.

   (c) Consider the average of n i.i.d random variables $X_1, \ldots, X_n$ with $E[X_1] = \mu$ and $E[|X_1|] < \infty$. Write true or false.

      i. $\bar{X}_n = o_P(1)$

      We know that $\bar{X}_n$ converges to μ in probability. If $\mu \neq 0$, $\bar{X}_n = o_P(1)$ is false.

      ii. $\exp(\bar{X}_n - \mu) = o_P(1)$

      Solution. We know that $\bar{X}_n - \mu$ converges to 0 in probability. By continuous mapping, If $\exp(\bar{X}_n - \mu) \xrightarrow{P} 1$. So false.

      iii. $(\bar{X}_n - \mu)^2 = O_P(1/n)$

      Fix $\epsilon > 0$. Now $P((\bar{X}_n - \mu)^2 \geq \underbrace{\frac{\sigma^2}{\epsilon}}_{M_\epsilon}) \leq \epsilon$. So, its true.

3. (2+4+1) Consider random variables $X_1, \ldots, X_n$ be IID r.v's with mean μ and variance $\sigma^2 := \mathrm{var}(X_i)$. We will use the following statistic to estimate $\theta = \mu^2$.

$$\hat{\theta} = \frac{1}{\binom{n}{2}} \sum_{i<j} X_i X_j$$

(a) Find constants C_1, C_2 where

$$\hat{\theta} - \mu^2 = \frac{C_1}{\binom{n}{2}} \sum_{i<j} (X_i - \mu)(X_j - \mu) + \frac{C_2 \mu}{n} \sum_i (X_i - \mu)$$

We have,

$$\begin{aligned}
\frac{1}{\binom{n}{2}} \sum_{i<j} X_i X_j - \mu^2 &= \frac{1}{n(n-1)} \sum_{i \neq j} X_i X_j - \mu^2 \\
&= \frac{1}{n(n-1)} \sum_{i \neq j} ((X_i - \mu)(X_j - \mu) + \mu(X_i - \mu) + \mu(X_j - \mu)) \\
&= \underbrace{\frac{1}{n(n-1)} \sum_{i \neq j} (X_i - \mu)(X_j - \mu)}_{T_1} + \underbrace{\frac{2}{n} \mu \sum_i (X_i - \mu)}_{T_2}
\end{aligned}$$

Thus, $C_1 = 1, C_2 = 2$.

(b) Show that the first term is $O_P(1/n)$ and the second term is $O_P(1/\sqrt{n})$.
Observe that,

$$\mathrm{var}(T_1) = \frac{1}{n^2(n-1)^2} \left(\sum_{i \neq j, k \neq \ell} E(X_i - \mu)(X_j - \mu)(X_k - \mu)(X_\ell - \mu) \right)$$

But in the above sum, all tuples with $i \neq j \neq k \neq \ell$ are zero. All tuples with $i \neq j = k \neq \ell$ are also zero. The only nonzero terms arise from $i = k \neq j = \ell$ or $i = \ell \neq j = k$. And there are $O(n^2)$ such terms all with expectation σ^4. Thus the variance of T_1 is $O(1/n^2)$. We also see that

$$\mathrm{var}(T_2) = O(1/n)$$

Now note that for any sequence of mean zero random variables X_n, $Y_n = X_n/\sqrt{\mathrm{var}(X_n)} = O_P(1)$. This is because,

$$\sup_n P(|Y_n| \geq 1/\sqrt{\epsilon}) \leq \epsilon$$

Therefore, $T_1 = O_P(1/n)$ and $T_2 = O_P(1/\sqrt{n})$.

(c) Argue that $\hat{\theta} \xrightarrow{P} \mu^2$.
Since $\hat{\theta} - \mu^2 = o_P(1)$, this is proved.

4. (3+2+2+3) If $X_n \xrightarrow{d} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$? Prove your answer when you believe the answer is true. When you believe it is "not necessarily true", provide a counter-example.

(a) $g(x) = 1(x \in (0, 10))$

This is not necessarily true since $g(x)$ is not continuous at $x = 0$. Consider the sequence of random variables

$$X_n = X + \frac{1}{n}$$

Clearly, $X_n \xrightarrow{p} X$ (and consequently $X_n \xrightarrow{d} X$). However, since $X \sim \mathcal{P}(\lambda)$, therefore $X \geq 0$. Therefore, $\forall n \geq 1$, $X_n > 0$. Therefore,

$$g(X_n) = \begin{cases} 1 \text{ for } X < 10 - \frac{1}{n} \\ 0 \text{ otherwise} \end{cases}$$

and

$$g(X) = \begin{cases} 1 \text{ for } X < 10 \text{ and } X \geq 1 \\ 0 \text{ otherwise} \end{cases}$$

Therefore,

$$Eg(X_n) = P(X < 10 - \frac{1}{n}) \to P(X < 10)$$

but

$$Eg(X) = P(X < 10) - P(X = 0) = P(X < 10) - e^{-\lambda}$$

(b) $g(x) = e^{-x^2}$

True by Portmanteau thm.

(c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

Also true by Portmanteau thm, since $g(x)$ is bounded and the discontintuity points are all at odd multiples of $\pi/2$, which are not intergers, and hence the limiting random variable has zero probability mass on this set.

(d) $g(x) = x$

Not necessarily true since $g(x)$ is not bounded. Consider a counter example:

$$X_n = \begin{cases} X & \text{with probability } 1 - 1/n \\ n & \text{with probability } 1/n \end{cases}$$

But $EX_n = EX(1 - 1/n) + 1 \to EX + 1$.

5. (1+4) Consider X_n Uniform on $\{1/n, 2/n, \ldots, 1\}$. Let $X \sim$ Uniform($[0, 1]$). For the questions below, either give a proof or a counter-example.

(a) Does $X_n \xrightarrow{d} X$?

Yes. If $t \leq 1$, $P(X_n \leq t) = \frac{\lfloor \min(tn, n) \rfloor}{n} \to t$.

(b) Does $X_n \xrightarrow{P} X$?

No, first, we need to define X_n and X on the same probability space to even start thinking about convergence in probability. But we will show with a counter example that even with such a construction we can couple X_n and X such that $X_n \xrightarrow{d} X$ but X_n does not converge in probability to X.

First define $Y_n = \lceil nX \rceil / n$. Note that Y_n is a discrete Uniform. Now define $X_n = 1 + 1/n - Y_n$. Clearly, this is also a discrete uniform, and hence converges in distribution to X, but what about convergence in probability?

$$
\begin{aligned}
&P(X_n - X \geq 1/2) = P(1 + 1/n - Y_n - X \geq 1/2) \\
&= P(Y_n + X \leq 1/2 + 1/n) = P(\lceil nX \rceil + nX \leq n/2 + 1) \geq P(X \leq 1/4)
\end{aligned}
$$

which does not converge to zero.