9/01/23
Monday

UNIT – II

Infinite Series :-

⇒ Consider an infinite sequence $\{a_n\}_{n=1}^{\infty}$ an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} a_n$.

⇒ **Partial Sums :-**

The sequence $\{S_n\}$ defined as

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

is the sequence of partial sums of the series, $\sum_{n=1}^{\infty} a_n$.

Convergence of series :-

Let Σa_n be a series and $\{S_n\}$ be its corresponding sequence of partial sums.
then,

① Σa_n is convergent [Cgt] if $\lim_{n\to\infty} S_n =$ finite

② Σa_n is divergent [dgt] if $\lim_{n\to\infty} S_n =$ DNE (does not exist)

Geometric Series:

A series of the form $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \ldots$ is called Geometric series with first term $= a$ and common ratio $[C.r] = r$. It is denoted by $\sum_{n=1}^{\infty} ar^{n-1}$

NOTE:

① Sum of a finite G.S $= S = a\left(\frac{1-r^n}{1-r}\right)$, if $r < 1$.

② Sum of infinite G.S $= S_\infty = \frac{a}{1-r}$

Geometric Series Test :-

A G.S $\sum_{n=1}^{\infty} ar^{n-1}$ is

① Cgt if $|r| < 1$, i.e; $-1 < r < 1$

② dgt if $|r| \geq 1$; ie: $r \leq -1$ (or) $r \geq 1$

Q 2 (1) $3 + 2 + \frac{4}{3} + \frac{8}{9} + \ldots\ldots$

$\Rightarrow$ Given Series is a G.S with $a = 3$ & $r = \frac{2}{3}$ (<1)

By G.S.T given series is cgt.

$\therefore S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{2}{3}} = \frac{3}{\left(\frac{1}{3}\right)} = 9$

(ii) $1 + 0.4 + 0.16 + 0.064 + \ldots\ldots$

$\Rightarrow$ Given Series is a G.S with

$a = 1 \quad , \quad r = 0.4 \ (< 1)$

By G.S.T it is cgt

$S_\infty = \frac{a}{1-r} = \frac{1}{1-0.4} = 1.66.$

(iii) $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{6^n}$.

$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{3^n - 2^n}{6^n}\right) = \sum_{n=0}^{\infty} \left(\frac{3^n}{6^n} - \frac{2^n}{6^n}\right)$

$= \sum_{n=0}^{\infty} \left[\left(\frac{3}{6}\right)^n - \left(\frac{2}{6}\right)^n\right]$

$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right]$

$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

$= S_1 - S_2$

$S_1 = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots$

which is a G.S with $a = 1$ & $r = \frac{1}{2}$ (<1)

$\therefore S_1$ is cgt

$\Rightarrow \boxed{(S_1)_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2}$

Also, $S_2 = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \cdots$

which is a G.S with $a=1$ & $r = \frac{1}{3}$ (<1)

$\therefore S_2$ = convergent

$\boxed{(S_2)_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}}$

$\therefore$ from Eq^n (1)

$\sum_{n=0}^{\infty} \left(\frac{3^n - 2^n}{6^n}\right) = \cancel{2+3}\ 2 - \frac{3}{2} = \frac{1}{2}$

(iv) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

$\Rightarrow \sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} 4^n 3^1 . 3^{-n}$

$= \sum_{n=1}^{\infty} 4^n \cdot 3 \cdot \frac{1}{3^n}$

$= 3 \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$

$= 3\left[\left(\frac{4}{3}\right) + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \cdots\cdots\right]$

$= 3$ [G.S with $a = \left(\frac{4}{3}\right)$ & $r = \frac{4}{3}$ (>1)]

$\therefore$ by GST given series is dgt.

P-Series :-

A series of the type $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots$ is called p-series.

P-Series Test :

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

① Convergent (cgt) if $p > 1$

② dgt if $p \leqslant 1$
↓
divergent

Example:

① $\sum \frac{1}{n^3}$ is cgt.

$\rightarrow 1+ 1/2 + 1/3 + 1/4 \cdots$

② $\sum 1/n$ is dgt (harmonic series) $\xrightarrow{\sum 1/n}$ dgt

③ $\sum \frac{1}{n^{1/2}}$ is dgt

④ $\sum \frac{1}{n^{5/2}}$ is cgt.

10/01/23
Tuesday

Limit Comparison Test :-

Let $\sum a_n$ & $\sum b_n$ be two series of +ve
if $\lim_{n\to\infty} \frac{a_n}{b_n}$ = finite $(\neq 0)$ then $\sum a_n$ &
behave alike.

NOTE :-

① If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ cgs then $\sum a$

② If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ dgs then $\sum a$

Ratio Test :-

Ratio Test :-

Let Σa_n be a series of +ve terms.

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, then

① Σa_n cgs if $L < 1$

② Σa_n dgs if $L > 1$

③ Ratio test fails if $L = 1$

NOTE :-

Series that involves factorials (or) Constant raised to the power of n are often tested using Ratio Test.

Comparison Test :-

Let Σa_n & Σb_n be two series of +ve terms

① If $a_n \leq b_n$, $\forall n$ & Σb_n is cgt then Σa_n is cgt.

② If $a_n \leq b_n$, $\forall n$ and Σa_n is dgt., then Σb_n is dgt.

Q4 (i) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

Ans Let $a_n = \frac{n+1}{n^2}$

$$b_n = \frac{\text{Highest degree term of Numerator of } a_n}{\text{Highest degree term of Denominator of } a_n}$$

$$b_n = \frac{n}{n^2} = \frac{1}{n}$$

$\Rightarrow \sum b_n = \sum \frac{1}{n}$ is divergent by p series test

$$\lim \frac{a_n}{b_n} = \text{finite } (\neq 0)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{n+1}{\cancel{n^2}}}{1/\cancel{n}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) = 1 \ (\neq 0)$$

$\therefore$ by LCT $\sum a_n$ is also dgt.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$

Ans Let $a_n = \dfrac{\cos^2 n}{n^2+1}$

WKT, $\cos^2 n < 1$, $\forall n$

$$\Rightarrow \frac{\cos^2 n}{n^2+1} < \frac{1}{n^2+1}, \forall n \quad \text{——— (1)}$$

But $n^2 < n^2+1$, $\forall n$

$$\Rightarrow \frac{1}{n^2} > \frac{1}{n^2+1}$$

$$\Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2}, \forall n \quad \text{——— (2)}$$

$$\Rightarrow \frac{\cos^2 n}{n^2+1} < \frac{1}{n^2}, \forall n$$

[from (1) & (2)]

But $\sum \frac{1}{n^2}$ is cgt by p-series test.

$\therefore$ by comparison test.

$\sum \frac{\cos^2 n}{n^2+1}$ is also cgt.

(iii) $\sum \frac{n^2-5}{n^3+n+2}$

Ans let, $a_n = \frac{n^2-5}{n^3+n+2}$

$b_n = \frac{\text{highest degree term of Numerator of } a_n}{\text{highest " " " denominator of } a_n}$

$$b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

$\Sigma b_n = \Sigma \frac{1}{n}$ is dgt by p series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{n^2-5}{n^3+n+2}}{\frac{1}{n}} \right)$$

$$= \frac{(n^2-5)n}{n^3+n+2}$$

$$= \frac{n^3-5n}{n^3+n+2}$$

$$= \frac{n^3(1-5/n^2)}{n^3 + }$$

$$= 1 (\neq 0)$$

11/02/23
Wednesday

(iv) $\Sigma \sqrt{\frac{n+1}{n^3+1}}$

$\Rightarrow a_n = \sqrt{\frac{n+1}{n^3+1}}$

$$b_n = \sqrt{\frac{n}{n^3}}$$

$$b_n = \sqrt{\frac{1}{n^2}} = \frac{1}{n}$$

$\sum b_n = \sum \frac{1}{n}$ is dgt by p-series test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^3+1}}}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} n \sqrt{\frac{n+1}{n^3+1}}$$

$$= \lim_{n\to\infty} n \sqrt{\frac{n(1+1/n)}{n^3(1+1/n^3)}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}} \sqrt{\frac{(1+1/n)}{(1+1/n^3)}}$$

$$= \sqrt{\frac{1+\frac{1}{\infty}}{1+\frac{1}{\infty}}}$$

$$= 1 \ (\neq 0)$$

by LCT $\sum a_n = \sum \sqrt{\frac{n+1}{n^3+1}}$ is also dgt.

(v) $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots\dots$

$\Rightarrow \quad a_n = \frac{n^2}{n!}$

$a_{n+1} = \frac{(n+1)^2}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)^2}{(n+1)!}\right)}{\left(\frac{n^2}{n!}\right)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \, n!}{(n+1)! \, n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{\cancel{2}} \cdot \cancel{n!}}{\cancel{(n+1)}\cancel{n!} \cdot n^2} \qquad (4! = 4 \cdot 3!)$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n}{n^2} + \frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right)$$

$$= 0 \; (<1)$$

by RT Σa_n cgs.

Ex: $\sum \frac{1}{n+3^n}$

$\Rightarrow \quad a_n = \frac{1}{n+3^n}$

$a_{n+1} = \frac{1}{(n+1)+3^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)+3^{n+1}}\right)}{\left(\frac{1}{n+3^n}\right)}$$

$$= \lim_{n\to\infty} \frac{n+3^n}{(n+1)+3^n\cdot 3}$$

$$= \lim_{n\to\infty} \frac{n.3^n\left[\frac{1}{3^n}+\frac{1}{n}\right]}{n.3^n\left[\frac{(1+1/n)}{3^n}+\frac{3}{n}\right]}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{3^\infty}+\frac{1}{\infty}}{\frac{(1+1/\infty)}{3^\infty}+\frac{3}{\infty}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty} \frac{1+3^n\log 3}{1+3\cdot 3^n\log 3}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n}\left[1/3^n+\log 3\right]}{\cancel{3^n}\left[\frac{1}{3^n}+3\log 3\right]}$$

$$= \frac{\log 3}{3 \log 3}$$

$$= \frac{1}{3} \quad (< 1)$$

by RT Σa_n is cgt.

Eg: $\sum \frac{1}{\sqrt{n^2+1}}$

$\Rightarrow \quad a_n = \frac{1}{\sqrt{n^2+1}}$

$b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$

$\Sigma b_n = \sum \frac{1}{n}$ is dgt by p-series test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n^2+1}}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} n\left(\frac{1}{\sqrt{n^2+1}}\right)$$

$$= \lim_{n\to\infty} \cancel{n} \times \frac{1}{\cancel{n}\sqrt{\left(\frac{1}{n^2}+1\right)}}$$

$$= \frac{1}{1} \Rightarrow (1 \neq 0)$$

$= 1 \; (\neq 0)$

by L.C.T. $\therefore \sum a_n = \sum \frac{(1+n)}{\sqrt{n^2+1}}$ is also dgt

Alternating Series :-

A.S is a series whose terms are alternately +ve and -ve.

i.e,

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

or

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=2}^{\infty} (-1)^{n} a_{n-1}$$

Leibnitz's Test :-

An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ cgs if

(1) $a_n - a_{n+1} \geq 0 \;. \forall n$ or $a_n \geq a_{n+1} \;, \forall n$

(2) $\lim_{n \to \infty} a_n = 0$

Q(5) (i) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n-1}$

$\Rightarrow$ Here, $a_n = \frac{n}{2n-1}$

$\Rightarrow a_n = \frac{n}{2n-1}$

$\Rightarrow a_{n+1} = \frac{n+1}{2(n+1)-1} = \frac{n+1}{2n+1}$

① $a_n - a_{n+1} = \frac{n}{(2n-1)} - \frac{n+1}{(2n+1)}$

$= \frac{\cancel{2n^2} + \cancel{n} - \cancel{2n^2} - 2n + \cancel{n} + 1}{(2n-1)(2n+1)}$

$= \frac{1}{(2n-1)(2n+1)} \geq 0$

1st condition of LT satisfied.

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n-1}$

$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(2-\frac{1}{n}\right)}$

$= \frac{1}{2} \quad (\neq 0)$

$\therefore$ by LT Σa_n is not cgt.

(ii) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots\ldots$

$\Rightarrow \Sigma a_n = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\log n}$

$a_n = \frac{1}{\log n}$

$$a_{n+1} = \frac{1}{\log(n+1)}$$

(1) WKT. $\log n < \log(n+1)$, $\forall n$

$$\Rightarrow \frac{1}{\log n} > \frac{1}{\log(n+1)}, \forall n$$

$$a_n > a_{n+1}, \forall n$$

1^{st} condⁿ is satisfied.

(2) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{\log n}\right)$$

$$= \frac{1}{\log \infty} = \frac{1}{\infty} = 0$$

by LT Σa_n is cgt.

(iii) $$\sum_{n=0}^{\infty} \frac{\cos n\pi}{n^2+1} \quad (-1)^n$$

$\Rightarrow$ WKT: $\cos n\pi = (-1)^n$

$$\therefore \sum_{n=0}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore a_n = \frac{1}{n^2+1}$$

$$a_{n+1} = \frac{1}{(n+1)^2+1}$$

① WKT: $n^2 < (n+1)^2$; $\forall n$

$$n^2+1 < (n+1)^2+1 , \forall n$$

$$\frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} , \forall n$$

$$a_n > a_{n+1} , \forall n$$

② $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^2+1} = 0$

$\therefore$ by LT Σa_n is cgt.

13/01/23
Friday (iv) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\dots$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$\Rightarrow a_n = \frac{1}{n}$$

$$= a_{n+1} = \frac{1}{n+1}$$

① WKT: $n < n+1$, $\forall n$

$$\frac{1}{n} > \frac{1}{n+1} , \forall n$$

$$\boxed{a_n > a_{n+1} , \forall n}$$

③ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$

$$\boxed{\lim_{n\to\infty} a_n = \frac{1}{\infty} = 0}$$

∴ by LT given series is cgt.

Eg: $\sum \frac{(-1)^n}{\sqrt{n+1}}$ cgt

Eg: $\sum \frac{(-1)^n}{\sqrt{n}}$ cgt

Power Series :-

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \quad \text{— ①}$$

where $c_0, c_1, c_2, \ldots$ are constants.

a = centre of power series ①

NOTE

① Power series ① is called power series about the ~~series~~ point $x=a$.

② Power series about a point $x=0$ is

$$\sum_{n=0}^{\infty} c_n \cdot x^n = c_0 + c_1 x + c_2 x^2 + \cdots \quad \text{— ②}$$

③ Centre of power series ② is 0 (zero).

④ Every power series cgs at its centre.

Interval of Convergence of a Power Series (IOC)

Defn :- IOC is the interval that consists of all the values of x for which the power series cgs.

Radius of Convergence (ROC) :-

Defn: ROC is the radius of IOC.

Method of computing ROC & IOC :-

Given: $\sum_{n=0}^{\infty} C_n (x-a)^n$

We use ratio test to find ROC & IOC.

① $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \Rightarrow$ IOC = $(-\infty, \infty)$ (lim = 0, ROC = ∞)
ROC = ∞

② $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty \Rightarrow$ IOC = $\{a\}$ $\boxed{\text{lim} = \infty,\ \text{ROC} = 0}$
ROC = 0

③ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = K|x-a| \Rightarrow$ ROC = $1/K$

$K|x-a| < 1$
$|x-a| < 1/K$
$-1/K < x-a < 1/K$
$a - 1/K < x < \frac{1}{K} + a$

IOC = $(a-1/K, a+1/K)$ or $[a-1/K, a+1/K]$ or $[a-1/K, a+1/K)$ or $(a-1/K, a+1/K]$.

6] (i) $\sum_{n=0}^{\infty}$ ...
$\Rightarrow a_n$ ...
$\Rightarrow a_n$ ...
$\lim_{n\to\infty}$ ...

6] (i) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

$\Rightarrow a_n = \frac{(-3)^n \cdot x^n}{\sqrt{n+1}}$

$\Rightarrow a_{n+1} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$

$= \lim_{n\to\infty} \left|\frac{\cancel{(-3)^n}(-3)\cancel{x^n} \cdot x}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{(-3)^n}\cancel{x^n}}\right|$

$= 3|x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}}$

$= 3|x| \; (1)$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3|x|$

By RT series is cgt if $3|x| < 1$
$\Rightarrow |x| < \frac{1}{3}$ & $\boxed{ROC = \frac{1}{3}}$

$= -\frac{1}{3} < x < \frac{1}{3}$

At $x = 1/3$:- $a_n = \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}}$

$$a_n = \frac{(-1)^n}{\sqrt{n+1}}$$

$\sum (-1)^n \frac{1}{\sqrt{n+1}}$ is an alternating series.

with $b_n = \frac{1}{\sqrt{n+1}}$

① WKT :- $\sqrt{n+1} < \sqrt{n+2}$

$$\frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{n+2}}$$

$$b_n > b_{n+1} \quad \forall n$$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

by LT series is cgt.

At $x = -1/3$:- $a_n = \frac{(-3)^n (-1/3)^n}{\sqrt{n+1}}$

$$a_n = \frac{1}{\sqrt{n+1}}$$

$\therefore \sum a_n = \sum \frac{1}{\sqrt{n+1}}$

$\sum b_n = \sum \frac{1}{\sqrt{n}}$ is dgt by P-series test.

16/01/23 Monday

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \sqrt{n} = 1 \; (\neq 0)$$

by LCT Σa_n is also dgt.

$$\therefore IOC = \left(\frac{-1}{3}, \frac{1}{3}\right]$$

16/01/23
Monday

(ii) $$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$$

$$\Rightarrow a_n = \frac{(x-2)^n}{n^2+1}$$

$$\Rightarrow a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(n+2)^2+1} \cdot \frac{n^2+1}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\cancel{(x-2)^n}(x-2)}{(n+1)^2+1} \cdot \frac{n^2+1}{\cancel{(x-2)^n}}\right|$$

$$= |x-2| \lim_{n\to\infty} \left[\frac{n^2+1}{(n+1)^2+1}\right]$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-2| \; [1]$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-2|$$

By RT series cgs if $|x-2| < 1$

$\Rightarrow |x-2| < 1$ & $\boxed{ROC = 1}$

$\Rightarrow -1 < x-2 < 1$

$\Rightarrow -1+2 < x < 1+2$ $\Rightarrow$ $\boxed{1 < x < 3}$

→ At $x = 1$:

$$a_n = \frac{(-1)^n}{n^2+1}$$

Which is an A.S with $b_n = \frac{1}{n^2+1}$

(i) WKT:- $\frac{1}{n^2+1} > \frac{1}{(n+1)^2+1}$

$\Rightarrow$ $\boxed{b_n > b_{n+1}, \forall n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$

by LT $\sum \frac{(-1)^n}{n^2+1}$ is cgt.

→ At $x = 3$:

$$a_n = \frac{1}{n^2+1}$$

WKT : $n^2 < n^2+1$

$$\frac{1}{n^2} > \frac{1}{n^2+1}$$

$$\sum \frac{1}{n^2} > \sum \frac{1}{n^2+1}$$

$\sum \frac{1}{n^2}$ is cgt

by CT $\sum \frac{1}{n^2+1}$ is also cgt.

$\therefore$ $\boxed{IOC = [1, 3]}$

(iii) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

$\Rightarrow$ $a_n = \frac{x^n}{\sqrt{n}}$

$\Rightarrow$ $a_{n+1} = \frac{x^{n+1}}{\sqrt{n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^n \cdot x}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}$$

classmate

Date
Page

(iv)

$= |x| \, (1)$

$= |x|$

by RT serie cgt if $|x| < 1$ & $\boxed{ROC = 1}$

$|x| < 1$

$\Rightarrow \boxed{-1 < x < 1}$

At $x = -1$:- $a_n = \frac{(-1)^n}{\sqrt{n}}$

$\sum a_n = \sum \frac{(-1)^n}{\sqrt{n}}$ is an A.S with $b_n = \frac{1}{\sqrt{n}}$

(i) WKT : $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$, $\forall n$

$b_n > b_{n+1}$, $\forall n$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

by LT $\sum a_n$ is cgt.

At $x = 1$:- $a_n = \frac{1}{\sqrt{n}}$

$\sum a_n = \sum \frac{1}{\sqrt{n}}$ is dgt by p-serie test

$\therefore \boxed{IOC = [-1, 1)}$

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

$\Rightarrow \quad a_n = \dfrac{(x-2)^n}{n^n}$

$$a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\cancel{(x-2)^n}\,(x-2)}{(n+1)^n\,(n+1)} \cdot \frac{n^n}{\cancel{(x-2)^n}}\right|$$

$$= |x-2| \lim_{n\to\infty} \frac{n^n}{(n+1)^n\,(n+1)}$$

$$= |x-2| \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n (n+1)}$$

$$= |x-2| \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{(n+1)}$$

<u>WKT:</u> $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

$$= |x-2| \; \frac{1}{e} \cdot \frac{1}{\infty}$$

$$= 0$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

$\Rightarrow$ ROC $= \infty$

IOC $= (-\infty, \infty)$.

(V) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow a_n = \frac{x^n}{n!}$

$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x}{(n+1)!} \cdot \frac{n!}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x \cdot \cancel{n!}}{(n+1)\cancel{n!}}\right|$$

$$= |x| \lim_{n\to\infty}\left(\frac{1}{n+1}\right)$$

$$= |x| \cdot 0$$

$$= 0$$

$\therefore$ ROC $= \infty$

IOC $= (-\infty, \infty)$ //

(vi:) $\sum_{n=0}^{\infty} n!\, x^n$

$\Rightarrow a_n = n!\, x^n$

$a_{n+1} = (n+1)!\, x^{n+1}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)\cancel{n!} \cdot \cancel{x^n} \cdot x}{\cancel{n!}\, \cancel{x^n}}\right|$$

$$= |x| \lim_{n\to\infty} (n+1)$$

$$= |x| (\infty)$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$$

ROC = 0

IOC = $\{0\}$.

Eg:

$$\sum_{n=1}^{\infty} \frac{(-2)^n\, x^n}{\sqrt[4]{n}}$$

$a_n = \dfrac{(-2)^n\, x^n}{\sqrt[4]{n}}$ $\qquad$ $a_{n+1} = \dfrac{(-2)^{n+1}\, x^{n+1}}{\sqrt[4]{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}\ x^{n+1}}{4\sqrt{n+1}} \cdot \frac{4\sqrt{n}}{(-2)^n\ x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(-2)^n}(-2)\ \cancel{x^n}\ x}{4\sqrt{n+1}} \cdot \frac{4\sqrt{n}}{\cancel{(-2)^n}\ \cancel{x^n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{-2x\ .\ 4\sqrt{n}}{4\sqrt{n+1}}\right|$$

17/01/23
Tuesday

Power Series Representation :-

① $\dfrac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots, \; |x| < 1$

$S_\infty = \dfrac{a}{1-r}$

② $\dfrac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots, \; |x| < 1$ $\quad S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1+x}$

Differentiation & Integration of Power Series :-

If the Power series $\sum c_n (x-a)^n$ has ROC $= R > 0$ then the function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is differentia

and

1. $f'x = \sum_{n=1}^{\infty} c_n \, n (x-a)^{n-1}$ ——— (1)

2. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ ——— (2)

Radius of convergence of (1) & (2) is R.

Q 8] (iii) $f(x) = \ln(5-x)$

$$\Rightarrow f(x) = \int\left[\frac{d}{dx}\ln(5-x)\right]dx$$

$$f(x) = \int \frac{-1}{5-x}\,dx$$

$$f(x) = -\frac{1}{5}\int\left(\frac{1}{1-\frac{x}{5}}\right)dx \qquad \boxed{\frac{1}{1-x} = 1+x+x^2+x^3+\cdots}$$

$$f(x) = -\frac{1}{5}\int\left[1+\frac{x}{5}+\left(\frac{x}{5}\right)^2+\left(\frac{x}{5}\right)^3+\cdots\right]dx$$

$$f(x) = -\frac{1}{5}\int\left[1+\frac{x}{5}+\frac{x^2}{5^2}+\frac{x^3}{5^3}+\cdots\right]dx$$

$$f(x) = -\frac{1}{5}\left[x+\frac{x^2}{5.2}+\frac{x^3}{5^2.3}+\cdots\right]+C \quad \text{—— (1)}$$

centre of series $= 0$ $\left[1+\frac{x-0}{5}+\frac{(x-0)^2}{5^2}+\cdots\right]$

put $x=0$ in Eqⁿ (1)

$$f(0) = 0+C$$
$$\ln(5-0) = C$$
$$\boxed{\ln 5 = C}$$

Eqⁿ (1) reduces to

$$\ln(5-x) = -\frac{1}{5}\left[x+\frac{x^2}{5.2}+\frac{x^3}{5^2.3}+\cdots\right]+\ln 5$$

$$= -\left[\frac{x}{5}+\frac{x^2}{5^2.2}+\frac{x^3}{5^3.3}+\cdots\right]+\ln 5$$

$$\boxed{\ln(5-x) = -\sum_{n=1}^{\infty} \frac{x^n}{5^n \cdot n} + \ln 5}$$

(iv) $f(x) = \tan^{-1}(x)$

$$\Rightarrow \tan^{-1} x = \int \left(\frac{d}{dx} [\tan^{-1} x] \right) dx$$

$$= \int \left[\frac{1}{1+x^2} \right] dx$$

WKT: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

$$= \int \left[1 - x^2 + x^4 - x^6 + \cdots \right] dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right] + C \quad \text{—— (1)}$$

put $x = 0$ in Eqn —— (1)

$0 = [0] + C \Rightarrow \boxed{C = 0}$

Eq (1) reduces to

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\boxed{\tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}}$$ (odd funⁿ)

Q7] $f(x) = \ln(1+x)$

To find :- ① $f(x) = x\ln(1+x)$
② $f(x) = \ln(1+x^2)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ —— ①

① Multiply Eqn ① by x.

$$\boxed{x\ln(1+x) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots}$$

② change x to x^2 in ①

$$\boxed{\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots}$$

18/01/23
Wednesday

Taylor Series :-

Taylor series of $f(x)$ abt $x=a$ is,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots\cdots$$

Maclaurin Series :-

Maclaurin series of $f(x)$ about $x=0$ is,

$$f(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f'''(0) + \cdots$$

$$= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Q 9] (i) $f(x) = \cos x$

$\Rightarrow$ Maclaurin Series is :-

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad (1)$$

$f(x) = \cos x \quad ; \quad f(0) = 1$

$f'(x) = -\sin x \quad ; \quad f'(0) = 0$

$f''(x) = -\cos x \quad ; \quad f''(0) = -1$

$f'''(x) = +\sin x \quad ; \quad f'''(0) = 0$

Eqⁿ (1) reduces to :

$$\cos x = 1 + x[0] + \frac{x^2}{2!}[-1] + \frac{x^3}{3!}[0] + \cdots$$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + \cdots}$$

(ii) $f(x) = \sin(2x)$

$\Rightarrow$ Maclaurin Series is:-

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$f(x) = \sin(2x) \quad ; \quad f(0) = 0$

$f'(x) = 2\cos(2x) \quad ; \quad f'(0) = 2$

$f''(x) = -4\sin(2x)$; $f''(0) = 0$

$f'''(x) = -8\cos(2x)$; $f'''(0) = -8$

Eqⁿ (1) reduces to :

$$\sin(2x) = 0 + x[2] + \frac{x^2}{2!}[0] + \frac{x^3}{3!}[-8] + \cdots$$

$$\boxed{\sin(2x) = 2x - \frac{4}{3}x^3 + \cdots}$$

(iii) $f(x) = (1+x)^{-3}$

⇒ Maclaurin series is :-

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$f(x) = (1+x)^{-3}$; $f(0) = 1$

$f'(x) = -3(1+x)^{-4}$; $f'(0) = -3$

$f''(x) = (-3)(-4)(1+x)^{-5}$; $f''(0) = +12$

$f'''(x) = (-3)(-4)(-5)(1+x)^{-6}$; $f'''(0) = -60$

Eqⁿ (1) reduces to :

$$(1+x)^{-3} = 1 + x[-3] + \frac{x^2}{2!}[12] + \frac{x^3}{3!}[60]$$

$$\boxed{(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots}$$

classmate

Date
Page

(iv) $f(x) = xe^x$

$\Rightarrow$ Maclaurin Series is :-

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$f(x) = xe^x$; $f(0) = 0$

$f'(x) = xe^x + e^x = (x+1)e^x$; $f'(0) = 1$

$f''(x) = (x+1)e^x + e^x = (x+2)e^x$; $f''(0) = 2$

$f'''(x) = (x+2)e^x + e^x =$; $f'''(0) = 3$

Eqn (1) reduces to :-

$$xe^x = 0 + x[1] + \frac{x^2}{2!}[2] + \frac{x^3}{3!}[3] + \cdots$$

$$\boxed{xe^x = x + x^2 + \frac{x^3}{2} + \cdots}$$

Q] 10(i) $f(x) = \ln x$, at $a = 2$

$\Rightarrow$ Taylor series is:-

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3}f'''$$

$f(x) = \ln x$ ——— (1)

$f'(x) = \frac{1}{x} = x^{-1}$; $f(2) = \ln 2 = 0.6931$

; $f'(2) = 0.5$

$f''(x) = \frac{-1}{x^2} = -x^{-2}$; $f''(2) = -0.25$

$f'''(x) = +2x^{-3}$; $f'''(2) = +0.25$

Eq^n (1) reduces to :

$$\ln x = (0.6931) + (x-2)[0.5] + \frac{(x-2)^2}{2!}[-0.25] + \frac{(x-2)^3}{3!}(0.25) + \cdots$$

$$\ln x = (0.6931) + [x-2](0.5) - (x-2)^2[0.125] + (x-2)^3(0.0417) + \cdots$$

(ii) $f(x) = \frac{1}{\sqrt{x}}$ at $a = 9$

Taylor series :-

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{3}f'''(a) + \cdots \longrightarrow (1)$$

$f(x) = 1/\sqrt{x} = x^{-1/2}$; $f(9) = \frac{1}{3} = 0.3333$

$f'(x) = \frac{-1}{2}x^{-3/2}$; $f'(9) = -0.0185$

$f''(x) = \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{-5/2}$; $f''(9) = 0.0031$

$f'''(x) = \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)x^{-7/2}$; $f'''(9) = -0.0009$

Eq^n (1) reduces to :-

$$\frac{1}{\sqrt{x}} = (0.3333) + (x-9)[-0.00185] + \frac{(x-9)^2}{2!}[0.0031] + \frac{(x-9)^3}{3!}[-0.0009] + \cdots$$

classmate

Date
Page

(iii) $f(x) = 1 + x + x^2$ at $a = -2$.

$\Rightarrow$ Taylor Series :-

$f(x) = 1 + x + x^2$; $f(-2) = 3$
$f'(x) = 1 + 2x$; $f'(-2) = -3$
$f''(x) = 2$; $f''(-2) = 2$
$f'''(x) = 0$; $f'''(-2) = 0$

Eq^n (1) reduces to :

$$1 + x + x^2 = 3 + (x+2)[-3] + \frac{(x+2)^2}{2!}[2] + \frac{(x+2)^3}{3!}[0] + \dots$$

$$\boxed{1 + x + x^2 = 3 - 3(x+2) + (x+2)^2}$$

Q 12] To find :- $\sin 62^\circ$

(FM) Taylor Series is :-

Here, $f(x) = \sin x$; $a = 60^\circ = \dfrac{\pi}{3}$

$\frac{(x+2)^3}{3!}[0]+\ldots$

$f(x) = \sin x$; $f(\pi/3) = \sqrt{3}/2$

$f'(x) = +\cos x$; $f'(\pi/3) = 1/2$

$f''(x) = -\sin x$; $f''(\pi/3) = -\sqrt{3}/2$

$f'''(x) = -\cos x$; $f'''(\pi/3) = -1/2$

<u>Eq^n ① reduces to :</u>

$\pi/3 + \pi/90$

$$\sin x = \frac{\sqrt{3}}{2} + \left(x - \frac{\pi}{3}\right)\left[\frac{1}{2}\right] + \left(x-\frac{\pi}{3}\right)^2 \frac{1}{2}\left[\frac{-\sqrt{3}}{2}\right] + \left(x - \frac{\pi}{3}\right)^3 \frac{1}{6}\left[\frac{-1}{2}\right] + \cdots \rightarrow ②$$

But $62^\circ = 60^\circ + 2^\circ$

$$62^\circ = \frac{\pi}{3} + \frac{\pi}{90} \qquad \left(1^\circ = \frac{\pi}{180}\right)$$

put $x = \frac{\pi}{3} + \frac{\pi}{90}$ in RHS of ②

$$\sin 62^\circ = \frac{\sqrt{3}}{2} + \left(\frac{\pi}{90}\right)\left(\frac{1}{2}\right) + \left(\frac{\pi}{90}\right)^2\left[\frac{-\sqrt{3}}{4}\right] + \left(\frac{\pi}{90}\right)^3\left[\frac{-1}{12}\right] + \cdots$$

$$\sin 62^\circ = 0.8660 + 0.01745 + -0.00052 + -0.00000354$$

$$= 0.8829$$

<u>Q11</u> <u>To find : $\tan 46^\circ$</u>

FM

<u>Taylor Series :-</u>

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

classmate
Date
Page

Q 3))

Here, $f(x) = \tan x$ & $a = 45^\circ = \frac{\pi}{4}$

$f(x) = \tan x$; $f(\pi/4) = 1$
$f'(x) = \sec^2 x = 1 + \tan^2 x = 1 + f^2$; $f'(\pi/4) = 2$
$f''(x) = 2ff'$; $f''(\pi/4) = 4$
$f'''(x) = 2\left[ff'' + (f')^2\right]$; $f'''(\pi/4) = 16$

Eq^n (1) reduces to :

$$\tan x = 1 + \left(x - \frac{\pi}{4}\right)[2] + \left(x - \frac{\pi}{4}\right)^2 \frac{1}{2}(4) + \left(\frac{x - \pi}{16}\right)^3 \left(\frac{1}{6}\right)[16] + \dots \quad (2)$$

$(\pi/4 + \pi/180)$

But , $46^\circ = 45^\circ + 1^\circ$

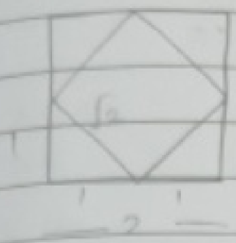
$$46^\circ = \frac{\pi}{4} + \frac{\pi}{180}$$

put $x = \frac{\pi}{4} + \frac{\pi}{180}$ in RHS of (2)

$$\tan 46^\circ = 1 + \frac{\pi}{180}[2] + \left(\frac{\pi}{180}\right)^2 [2] + \left(\frac{\pi}{180}\right)^3 \left[\frac{8}{3}\right] + \dots$$

$$\boxed{\tan 46^\circ = 1.0355}$$

Q3))



Area of outer square $= 4\,m^2$,
so its side is $= 2\,m$
By pythagorous theorem, we get the side of the 2nd square as $\sqrt{2}$ m. and so on...

Taking the sum of the areas of all the squares we get,

$$A = (2)^2 + (\sqrt{2})^2 + (1)^2 + \cdots\cdots$$

$$A = 4 + 2 + 1 + \cdots$$

which is geometric series with $a = 4$, $r = 1/2$

$$r = \frac{1}{2} \; (< 1)$$

Therefore, $A = \dfrac{a}{1-r} = \dfrac{4}{1-1/2} = \dfrac{4}{1/2} = 8\,m^2$ //.

19/01/23 Thursday

Unit - III

Chapter - 5

Integral Calculus ...

Beta Functions :-

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx,$$

$m, n > 0$

Example :-

① $B(5,8) = \int_0^1 x^4 (1-x)^7\, dx$

② $\int_0^1 x^{1/2} (1-x)^{3/2}\, dx = B\left(3/2, 5/2\right)$

③ $\int_0^1 x^{-1/2} (1-x)^{-3/2}\, dx = B\left(1/2, -1/2\right)$ × not Beta funtion $(m, n > 0)$

Gamma Functions :-

Γ = Gamma

γ = Small Gamma

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n > 0$$

Example :-

① $\Gamma 5 = \int_0^\infty e^{-x} x^4\, dx$

② $\int_0^\infty e^{-x} x^{3/2}\, dx = \Gamma 5/2$

③ $\int_0^\infty e^{-x} x^{-3/2}\, dx = \Gamma -3/2$ × $(n > 0)$

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

put $1-x = t \Rightarrow 1-t = x \Rightarrow -dt = dx$

$x = 0$; $t = 1$

$x = 1$; $t = 0$

$$B(m,n) = -\int_1^0 (1-t)^{m-1}\, t^{n-1}\, dt \Rightarrow +\int_0^1 t^{n-1} (1-t)^{m-1}\, dt$$

$$\boxed{B(m,n) \Rightarrow B(n,m)}$$

Properties :-

① $B(m,n) = B(n,m)$

② Relation b/w Beta & Gamma

$$\boxed{B(m,n) = \frac{\Gamma m \; \Gamma n}{\Gamma m+n}}$$

③ $\Gamma(n+1) = n\,\Gamma n$

④ $\Gamma n+1 = n!$

⑤ $\Gamma \frac{1}{2} = \sqrt{\pi} = (\pi)^{1/2}$

$\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \Gamma\frac{1}{4}\,\Gamma 1-\frac{1}{4} = \frac{\pi}{\sin(\frac{1}{4}\pi)} = \frac{\pi}{(\frac{1}{\sqrt{2}})}$

⑥ $\Gamma n \; \Gamma 1-n = \frac{\pi}{\sin(n\pi)}$

⑦ ** $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{(p+1)}{2} \; \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+1}{2}\right) + \left(\frac{q+1}{2}\right)}$$

Beta
tion
o)

$-\frac{3}{2}$ X

$$\Gamma\frac{9}{2} = \frac{7}{2}\Gamma\frac{7}{2} = \frac{7}{2}\cdot\frac{5}{2}\Gamma\frac{5}{2} = \frac{7}{2}\frac{5}{2}\frac{3}{2}\Gamma\frac{3}{2}$$

$$= \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}$$

Note :-

① $\Gamma\frac{9}{2} = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

② $\Gamma\frac{11}{3} = \frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}\Gamma\frac{2}{3}$

Q 3] (a) $\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta$

$\Rightarrow$ Let $I = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

there $p = 6$, $q = 7$

WKT : $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$

$$I = \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\frac{7+1}{2}}{2\,\Gamma\frac{7}{2}+\frac{8}{2}} = \frac{\Gamma 7/2\,\Gamma 4}{2\,\Gamma\frac{15}{2}}$$

$$= \frac{\cancel{\Gamma\frac{7}{2}}\,(3!)}{\cancel{2}\cdot\frac{13}{\cancel{2}}\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cancel{\Gamma\frac{7}{2}}}$$

$$= \frac{{}^{2}\cancel{6}\,\cancel{a}.8}{13.11.7.\cancel{9}_{3}}$$

$$I = \frac{16}{3003}$$

⑥ $\displaystyle\int_0^{\pi/2} \sqrt{\tan x}\; dx$

$\Rightarrow$ Let $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}}\, dx$

$$= \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x \, dx$$

Here, $P = 1/2$ & $q = -1/2$

WKT: $\displaystyle\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$

$$I = \frac{\Gamma\frac{1/2+1}{2} \cdot \Gamma\frac{-1/2+1}{2}}{2\,\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$I = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\,\Gamma 1}$$

$$I = \frac{\Gamma\left(1-\frac{1}{4}\right)\Gamma\frac{1}{4}}{2(0!)}$$

classmate

Date ______
Page ______

$$I = \frac{1}{2(1)} \Gamma\frac{1}{4} \Gamma 1-\frac{1}{4}$$

$$I = \frac{1}{2} \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)}$$

$$I = \frac{1}{2} \frac{\pi}{(1/\sqrt{2})}$$

$$I = \frac{\pi}{\sqrt{2}}$$

(c) $\int_0^{\pi/2} \frac{\sqrt[3]{\sin^8 x}}{\sqrt{\cos x}} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^{8/3} x}{\cos^{1/2} x} dx$$

$$I = \int_0^{\pi/2} \sin^{8/3} x \cos^{-1/2} x \, dx$$

wkt: $I = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \Gamma\frac{q+1}{2}}{2\Gamma\left(\frac{p+1}{2}\right)+\left(\frac{q+1}{2}\right)}$

here, $p = 8/3$, $q = -1/2$

$$I = \frac{\Gamma\frac{8/3+1}{2} \Gamma\frac{-1/2+1}{2}}{2\Gamma\frac{11}{6}+\frac{1}{4}}$$

$$I = \frac{\left(\Gamma \frac{11}{6}\right) \cdot \Gamma \frac{1}{4}}{2\,\Gamma \frac{25}{12}}$$

$$I = \frac{\left(\frac{5}{6}\,\Gamma \frac{5}{6}\right)\left(\Gamma \frac{1}{4}\right)}{\cancel{2} \cdot \frac{13}{\cancel{12}} \cdot \frac{1}{12}\,\Gamma \frac{1}{12}}$$

$$I = \frac{60}{13}\,\frac{\Gamma 5/6\ \Gamma 1/4}{\Gamma \frac{1}{12}}$$

(h) $\int_0^1 \log \frac{1}{y}\, dy$

$\Rightarrow$ put $\boxed{\log \frac{1}{y} = t}$

$\Rightarrow \frac{1}{y} = e^t$

$\Rightarrow y = e^{-t}$

$\boxed{dy = -e^{-t}\, dt}$

$\log \infty = \infty$

$\log 0 = -\infty$

$\log 1 = 0$

$y=0$; $t=\infty$

$y=1$; $t=0$

$$I = -\int_{\infty}^{0} t \cdot e^{-t}\, dt$$

$$I = +\int_0^{\infty} e^{-t}\, t^1\, dt$$

wkt: $\int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma n$

$I = \Gamma 2$

$\boxed{I = 1! = 1}$

20/01/23
Friday

(d) $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx.$

$\Rightarrow \quad I = \int_0^\infty \sqrt{x}\, e^{-x^2}\, dx$

put $x^2 = t$

$\Rightarrow x = t^{1/2} = \sqrt{t}$

$\Rightarrow \boxed{dx = \frac{1}{2}\frac{1}{\sqrt{t}}\, dt}$

$x = 0 \quad ; \quad t = 0$

$x = \infty \quad ; \quad t = \infty$

$\Rightarrow I = \int_0^\infty \sqrt{t^{1/2}}\, e^{-t}\, \frac{1}{2}\frac{1}{\sqrt{t}}\, dt$

$I = \frac{1}{2}\int_0^\infty t^{1/4}\, t^{-1/2}\, e^{-t}\, dt$

$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/4}\, dt$

$\boxed{I = \frac{1}{2}\Gamma\frac{3}{4}}$

(f) $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$

$\Rightarrow$ $I = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$

let $\boxed{I = I_1 \times I_2}$ —— (1)

$I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$

put $x^4 = t$

$\Rightarrow x = t^{1/4}$

$\Rightarrow dx = \frac{1}{4} t^{-3/4}\,dt$

$x=0 \; ; \; t=0$

$x=1 \; ; \; t=1$

$I_1 = \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}} \; \frac{1}{4} t^{-3/4}\,dt$

$I_1 = \int_0^1 t^{-1/4}(1-t)^{-1/2}\,dt$

$I_1 = \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right)$

$I_1 = \frac{1}{4} \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} = \boxed{\frac{1}{4} \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{4}} = I_1}$

$\left(B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right)$

classmate

Date
Page

$$I_2 = \int_{\boxed{0}}^{\boxed{1}} \frac{1}{\sqrt{1+x^4}} \, dx$$

put $x^4 = \tan^2\theta$

$\Rightarrow x = \tan^{1/2}\theta$

$dx = \frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$

$x = 0 \; ; \; \theta = 0$

$x = 1 \; ; \; \theta = \frac{\pi}{4}$

$$I_2 = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \frac{1}{2} \tan^{-1/2}\theta \sec^2\theta \, d\theta$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cancel{\sec\theta}} \tan^{-1/2}\theta \sec^{\cancel{2}}\theta \, d\theta$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta} \cdot \frac{1}{\cos\theta} \, d\theta.$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sin^{1/2}\cos^{1/2}\theta} \, d\theta.$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{2^{1/2}}{2^{1/2}\sin^{1/2}\theta\cos^{1/2}\theta} \, d\theta$$

multiply & divide with $\sqrt{2} = 2^{1/2}$ on both sides.

$$I_2 = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{2\sin\theta\cos\theta}} \, d\theta$$

$$I_2 = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{\sin(2\theta)}} \, d\theta$$

$\theta = \frac{\pi}{4}$; $t = \frac{\pi}{2}$

$$I_2 = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\sqrt{\sin t}} \frac{dt}{2}$$

$$I_2 = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \; dt$$

$$I_2 = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \cos^0 t \; dt$$

$$p = -1/2 \quad , \quad q = 0$$

$$I_2 = \frac{1}{2\sqrt{2}} \frac{\Gamma\frac{-1/2+1}{2} \, \Gamma\frac{0+1}{2}}{2\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$I_2 = \frac{1}{2\sqrt{2}} \frac{\Gamma 1/4 \; \Gamma 1/2}{2\,\Gamma 3/4}$$

$$\boxed{I_2 = \frac{1}{4\sqrt{2}} \frac{\Gamma 1/4 \; \Gamma 1/2}{\Gamma 3/4}}$$

Eq^n (1) reduces to,

$$I = \left(\frac{1}{4}\frac{\Gamma 3/4\,\Gamma 1/2}{\Gamma 5/4}\right)\left(\frac{1}{4\sqrt{2}}\frac{\Gamma 1/4\,\Gamma 1/2}{\Gamma 3/4}\right)$$

$$I = \frac{\pi}{16\sqrt{2}}\frac{\Gamma 1/4}{\Gamma 5/4}$$

$$= \frac{\pi}{16\sqrt{2}}\frac{\cancel{\Gamma 1/4}}{1/4\,\cancel{\Gamma 1/4}}$$

$$\boxed{I = \frac{\pi}{4\sqrt{2}}}$$

$$\int_0^2 x\sqrt[3]{8-x^3}\,dx$$

$$\Rightarrow I = \int_0^2 x(8-x^3)^{1/3}\,dx$$

put $x^3 = 8t$

$x = 2t^{1/3}$

$$\boxed{dx = \frac{2}{3}t^{-2/3}\,dt}$$

$x=0$; $t=0$

$x=2$; $t=1$

$$I = \int_0^1 2t^{1/3} (8-8t)^{1/3} \, \frac{2}{3} t^{-2/3} \, dt$$

$$I = \frac{4}{3} (8^{1/3}) \int_0^1 t^{-1/3} (1-t)^{1/3} \, dt$$

$$I = \frac{8}{3} \int_0^1 t^{-1/3} (1-t)^{1/3} \, dt \qquad x^{m+1} (1-x)^{n-1} \, dt$$

$$I = \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right)$$

$$I = \frac{8}{3} \frac{\Gamma(2/3)\,\Gamma(4/3)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} \qquad B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

$$I = \frac{8}{3} \frac{\Gamma(2/3) \, \frac{1}{3}\Gamma(1/3)}{\Gamma 2} \qquad \Gamma(1-n)\,\Gamma n = \frac{\pi}{\sin n\pi}$$

$$I = \frac{8}{3} \cdot \frac{1}{3} \cdot \frac{\Gamma\left(1-\frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right)}{1!}$$

$$I = \frac{8}{9} \frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = I = \frac{8}{9} \frac{\pi}{\left(\frac{\sqrt{3}}{2}\right)}$$

$$\boxed{I = \frac{16\pi}{9\sqrt{3}}}$$

classmate

Date

Page

(9) $\int_0^1 (x \log x)^4 \, dx$

$\Rightarrow \quad I = \int_0^1 x^4 (\log x)^4 \, dx$

put $\log x = -t$

$\Rightarrow \quad x = e^{-t}$

$dx = -e^{-t} \, dt$

$x = 0 \; ; \; t = \infty$

$x = 1 \; ; \; t = 0$

$I = \int_\infty^0 e^{-4t} \, t^4 \left[-e^{-t} \, dt\right]$

$I = \int_\infty^0 t^4 \, e^{-5t} \, dt$

put $5t = z \Rightarrow (t = z/5)$

$5 \, dt = dz$

$\boxed{dt = \frac{dz}{5}}$

$t = 0 \; ; \; z = 0$

$t = \infty \; ; \; z = \infty$

$I = \int_0^\infty \left(\frac{z}{5}\right)^4 e^{-z} \frac{dz}{5}$

$I = \frac{1}{5} \int_0^\infty \frac{z^4}{5^4} e^{-z} \, dz$

$e^{-x} \; x^{n-1}$

$$I = \frac{1}{5^5}\int_0^\infty e^{-z} z^{+4}\, dz$$

$$I = \frac{1}{5^5}\,\Gamma 5$$

$$I = \frac{4!}{5^5} = \frac{24}{5^5}$$

$$I = \frac{24}{5^5}$$

23/01/23 Monday

Tracing of curves :-

The process of determining the Geometrical shape of a curve on the basis of its equation is called Tracing of curve.

I] Tracing of cartesian curves :-

Given :- $y = f(x)$

① Domain

② Intercepts : x-int ; put $y = 0$
y-int ; put $x = 0$

③ Symmetry :

(i) About x-axis, if only even powers of y occur in its equation.

[Example : $y^2 = x$]

(ii) About y-axis, if only even powers of x occurs in its equation.
[Example : $y = x^2$]

(4) Origin :
(a) A curve passes through origin if $x=0$ gives $y=0$
If it does then find the equation of tangents at the origin by equating the lowest degree term to zero.

(i) Origin is node if the tangents are real & distinct.
(ii) origin is cusp, if the tangents are real & coincident

(5) Asymptote :-

(i) Horizontal Asymptote :- $\lim_{x \to \infty} f(x) = L$

(ii) Vertical Asymptote :- $\lim_{x \to 0} f(x) = \infty$

Q1] (iii) $y^2 (2a-x) = x^3$

A]

7M

$$y = \sqrt{\frac{x^3}{2a-x}}$$

$$y = \pm x \sqrt{\frac{x}{2a-x}}$$

(1) Domain :-

$\Rightarrow x \geq 0$ & $2a - x > 0$

$\Rightarrow x \geq 0$ & $x < 2a$

$\Rightarrow 0 \leq x < 2a$

② Intercepts :-

x-int : put $y=0$
$\Rightarrow x=0 \quad \Rightarrow (0,0)$

y-int : put $x=0$
$\Rightarrow y=0 \quad \Rightarrow (0,0)$

③ Symmetry : about x-axis because only y powers are even.

④ Origin : Passes through origin ($\because x=0 \Rightarrow y=0$)
from Eq ① : $2ay^2 - 2xy^2 = x^3$
Equating lowest degree term to zero we get,

$$2ay^2 = 0$$
$$y = 0, 0$$

$\Rightarrow$ origin is cusp.

⑤ Asymptote :-

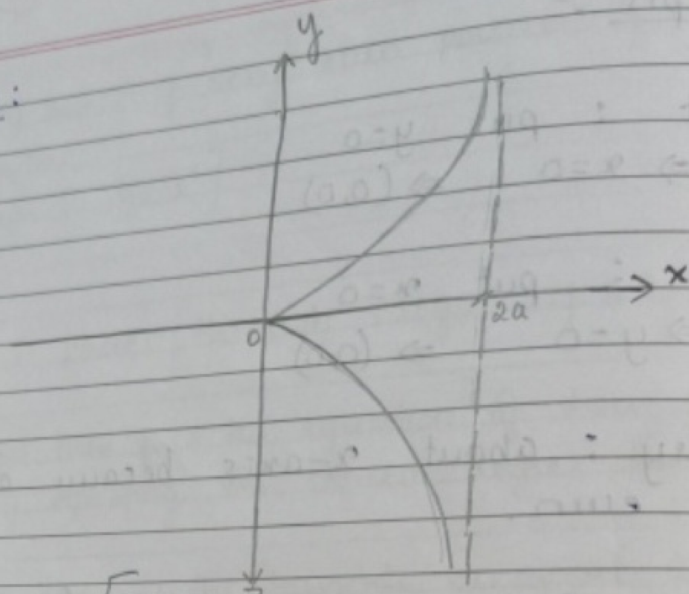
(i) HA :- $\lim_{x \to \infty} y = \lim_{x \to \infty} x\sqrt{\frac{x}{2a-x}}$

$\Rightarrow$ No HA

(ii) VA : $y \to \infty$
$x\sqrt{\frac{x}{2a-x}} \longrightarrow \infty$

$2a - x = 0$
$x = 2a$, $\therefore x = 2a$ is V.A

curve :



[Cissoid]

V] $y^2(a-x) = x^2(a+x)$

$$y^2(a-x) = x^2(a+x) \quad \text{--- (1)}$$

$$y = x\sqrt{\frac{a+x}{a-x}}$$

(1) Domain :-

$a+x \geq 0$ & $a-x > 0$

$x \geq -a$ & $x < a$

$-a \leq x < a$

(2) Intercepts :-

x-int : put $y=0$; $x=0$

y-int : put $x=0$; $y=0$

(3) Symmetry :- About x-axis [$\because$ powers of y is even]

Origin is <u>Node</u>. $y = +x$, $y = -x$

5] <u>Asymptote</u> :-

HA : $\lim\limits_{x \to \infty} y = \lim\limits_{x \to \infty} x\sqrt{\dfrac{a+x}{a-x}}$

$\Rightarrow$ NO H.A

VA :- $y \to \infty$

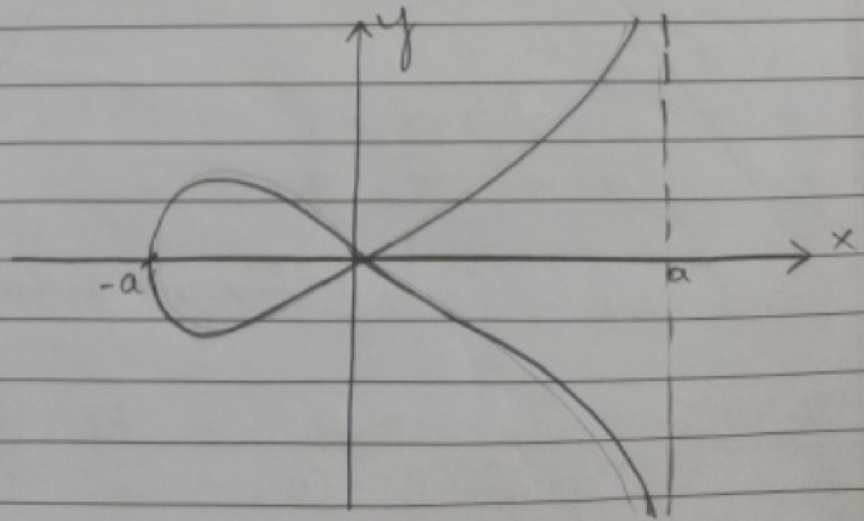
$x\sqrt{\dfrac{a+x}{a-x}} \longrightarrow \infty$

$a - x = 0$

$x = a$

$\therefore$ $x = a$ is VA.

<u>Curve</u> :



24/01/23
Tuesday

1) (i) $y^2(a^2+x^2) = x^2(a^2-x^2)$ —— ①

$\Rightarrow y^2 = x^2 \dfrac{(a^2-x^2)}{(a^2+x^2)}$

$y = \pm x\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ —— ②

root of minus should greater than zero
root of $0 > 0$

① Domain : $a^2 - x^2 \geq 0$

$\Rightarrow x^2 \leq a^2$

$\Rightarrow \sqrt{x^2} \leq \sqrt{a^2}$

$\Rightarrow |x| \leq a$

curve exist withind region.

$\boxed{-a \leq x \leq a}$

② Intercept :-

X-int : put $\boxed{y=0}$ in Eqⁿ ①

$x^2(a^2-x^2) = 0$

$x=0 \,;\; a^2-x^2=0$
$x=0 \,;\; (a-x)(a+x)=0$
$x=0 \,;\; x=a \,;\; x=-a$

points are ; $(0,0)$, $(-a,0)$ & $(a,0)$

Y-int :- put $x=0$ in Eq ②
$y=0 \quad \therefore (0,0)$

③ Symmetry :

About both the axes
(∵ x & y powers are both even)

④ Origin: Passes through Origin

from ①,

$$a^2y^2 + \cancel{x^2y^2} = a^2x^2 - \cancel{x^4}$$

$$\Rightarrow \cancel{a^2}y^2 = \cancel{a^2}x^2$$

$$y^2 = x^2$$

$$\boxed{y = \pm x}$$ (real & distinct)

∴ origin is node.

⑤ Asymptote :

① HA : $\lim_{x\to\infty} y = \lim_{x\to\infty} x\sqrt{\dfrac{a^2-x^2}{a^2+x^2}} = \infty$

NO HA

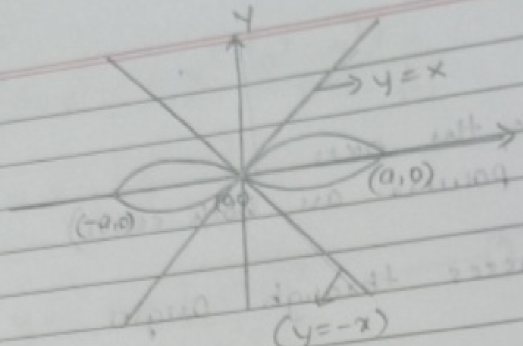
② VA : $y \longrightarrow \infty$

$$x\sqrt{\frac{a^2-x^2}{a^2+x^2}} \longrightarrow \infty \Rightarrow a^2+x^2 \neq 0$$

∴ NO VA

Curve :

[Strophoid]



[lemniscate]

25|01|23
Wednesday

(V) $xy^2 = a^2(a-x)$

$x y^2 = a^2(a-x)$ —— ①

$y^2 = a^2\left(\dfrac{a-x}{x}\right)$

$\boxed{y = \pm a\sqrt{\dfrac{a-x}{x}}}$ —— ②

① Domain : $a-x \geq 0$ & $x > 0$

$a \geq x$ & $x > 0$

$\boxed{0 < x \leq a}$

② Intercepts :-

x-int : Put $y=0$ in Eqⁿ ① $\Rightarrow x = a$

$\therefore$ pt is $(a,0)$

Y-int : Put $x=0$ in Eqⁿ ② $\Rightarrow y = \infty$

③ Symmetry : Abt x-axis ($\because$ only y power is even).

④ Origin : Does not pass through origin
($\because x = 0 \Rightarrow y = \infty$).

⑤ Asymptote : (i) HA : $\lim_{x\to\infty} y = \lim_{x\to\infty} a\sqrt{\frac{a-x}{x}}$

$= \text{not real}$.

$\therefore$ No HA

(ii) VA :- $y \to \infty \Rightarrow a\sqrt{\frac{a-x}{x}} \to \infty$

$\Rightarrow \boxed{x=0}$ is VA

⑥ Special Point :

$$\frac{dy}{dx} = a\left\{\frac{\sqrt{x}\cdot\frac{d}{dx}\left(\sqrt{a-x}\right) - \left(\sqrt{a-x}\right)\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}\right\}$$

$$= \frac{a}{x}\left\{\sqrt{x}\left(\frac{-1}{2\sqrt{a-x}}\right) - \left(\sqrt{a-x}\right)\frac{1}{-2\sqrt{x}}\right\}$$

$$= \frac{-a}{2x}\left\{\frac{\sqrt{x}}{\sqrt{a-x}} + \frac{\sqrt{a-x}}{\sqrt{x}}\right\}$$

$$\frac{dy}{dx} = \frac{-a}{2x}\left\{\frac{\cancel{x}+a-\cancel{x}}{\sqrt{x}\sqrt{a-x}}\right\}$$

versiera — turn
aversiera

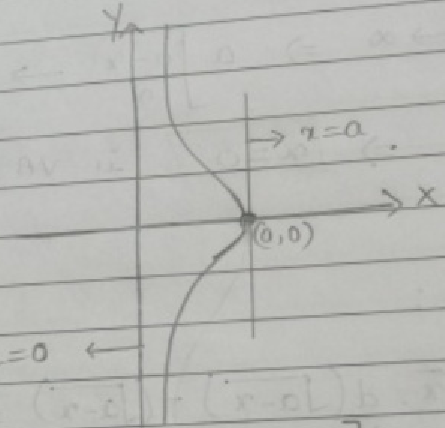
$$\boxed{\frac{dy}{dx} = \frac{-a^2}{2x^{3/2}\sqrt{a-x}}}$$

at $x=a$: $\frac{dy}{dx} = \infty$

$\Rightarrow$ curve has a vertical tangent at $x=a$.

$$\boxed{0 < x \le a}$$

Curve :-



[Witch of Agnesi]

Tracing of Parametric Curves :-

Given :- $x = f(t)$ and
$y = g(t)$

① Symmetry :

(i) About x-axis :- If $x = f(t)$ is even and
$y = g(t)$ is odd.

(ii) About Y-axis :- If $x = f(t)$ is odd &
$y = g(t)$ is even.

(iii) About Opposite quadrants :-

If $x = f(t)$ & $y = g(t)$ are both odd.

(2) Find $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

(3) Table : For x, y, t & $\frac{dy}{dx}$

(4) Origin : Passes thro' origin if $x = 0$ gives $y = 0$

(5) Greatest & least values of x.

Q1] (ii) $x^{2/3} + y^{2/3} = a^{2/3}$

7m ⇒ Parametric Eqn are :-

$x = a\cos^3 t$
$y = a\sin^3 t$

1) Symmetry :

$x = a\cos^3 t$ = even fun.
$y = a\sin^3 t$ = odd fun.

2) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t}$

$$\boxed{\frac{dy}{dx} = -\tan t}$$

3) Table:

t	0	$\pi/2$	π	$3\pi/2$	2π
$x = a\cos^3 t$	a	0	$-a$	0	a
$y = a\sin^3 t$	0	a	0	$-a$	0
$\frac{dy}{dx} = -\tan t$	0	$-\infty$	0	∞	0

4) Does not pass through origin $(\because x = 0 \not\Rightarrow y = 0)$

5) $-a < x < a.$

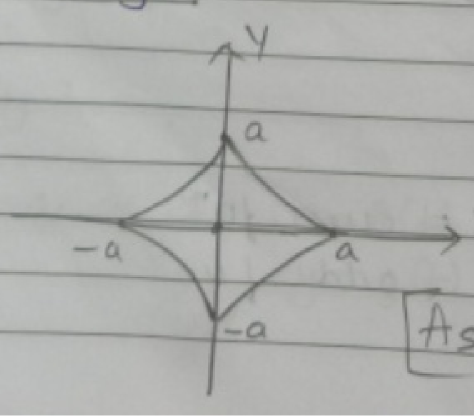
$-a < y < a$

$x\uparrow$	&	$y\uparrow$
$x\downarrow$	&	$y\downarrow$

$\Rightarrow f\uparrow$

$x\uparrow$	&	$y\downarrow$
$x\downarrow$	&	$y\uparrow$

$\Rightarrow f\downarrow$



Astroid

Q1 (vi) $x = a(\theta + \sin\theta)$
$y = a(1 + \cos\theta)$

$\Rightarrow$ ① Symmetry : $x = a(\theta + \sin\theta)$ = odd
$y = a(1+\cos\theta)$ = even.

About Y-axis

② $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(-\sin\theta)}{a(1+\cos\theta)}$$

$$= \frac{-\sin\theta}{1+\cos\theta}$$

$$\frac{dy}{dx} = \frac{-\cancel{2}\sin\theta/2 \ \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

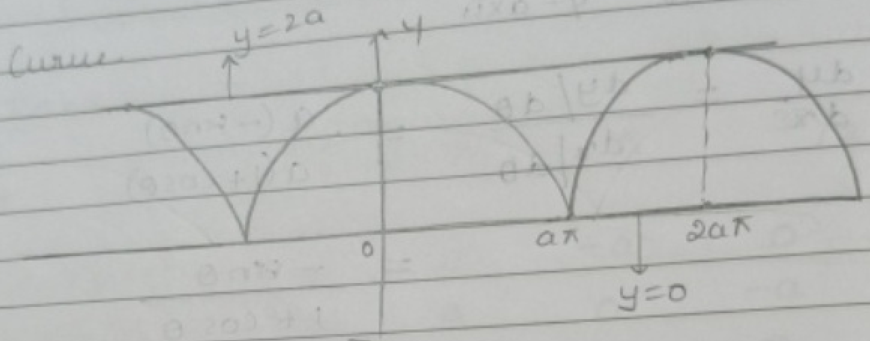
$$\boxed{\frac{dy}{dx} = -\tan\frac{\theta}{2}}$$

③ Table :-

θ	0	$\pi/2$	π	$3\pi/2$	2π
$x = a(\theta + \sin\theta)$	0	$a(\pi/2 + 1)$	$a\pi$	$a(3\pi/2 - 1)$	$a2\pi$
$y = a(1+\cos\theta)$	$2a$	a	0	a	$2a$
$\frac{dy}{dx} = -\tan\left(\frac{\theta}{2}\right)$	0	-1	$-\infty$	1	0

④ Origin : Does not pass thro' origin
($\because x = 0 \nRightarrow y = 0$)

⑤ $0 \leq x \leq 2a\pi$
$0 \leq y \leq 2a$



[cycloid]

Ex: $x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$

⇒ ① Symmetry :- About Y-axis.

$\because x = a(\theta + \sin\theta) = \text{odd}$
$y = a(1 - \cos\theta) = \text{even}$

② $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

③ Table :

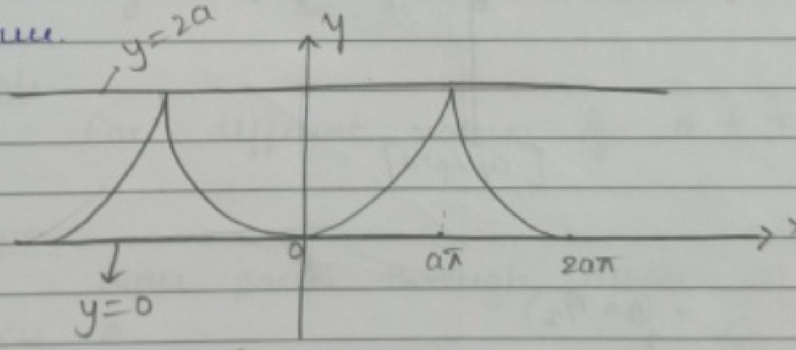
θ	0	$\pi/2$	π	$3\pi/2$	2π
$x = a(\theta + \sin\theta)$	0	$a(\pi/2 + 1)$	$a\pi$	$a(3\pi/2 - 1)$	$2a\pi$
$y = a(1 - \cos\theta)$	0	a	$2a$	a	0
$\frac{dy}{dx} = \tan\left(\frac{\theta}{2}\right)$	0	1	∞	-1	0

④ Origin :- Passes thro' origin.

⑤ $0 \le x \le 2a\pi$

$0 \le y \le 2a$

⑥ Curve.



[cycloid]

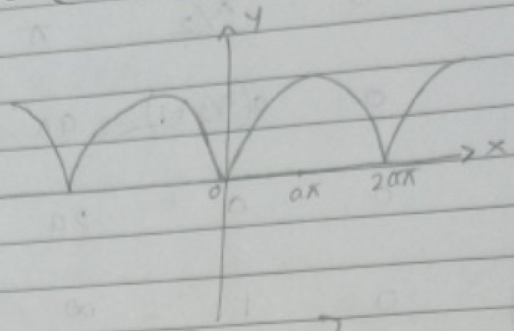
Ex: $x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$
↓
downward open

Ex: $x = a(\theta - \sin\theta)$
$y = a(1 + \cos\theta)$
↓
upward open

1/02/23 Wednesday

(3) $x = a(\theta - \sin\theta)$

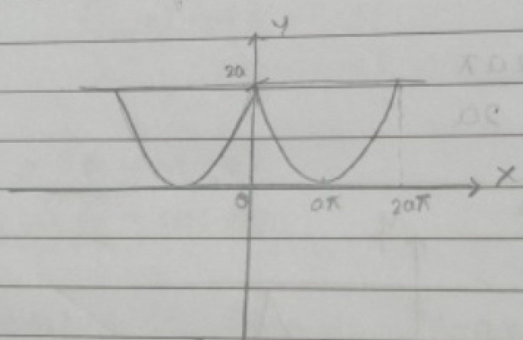
$y = a(1 - \cos\theta)$



[Cycloid]

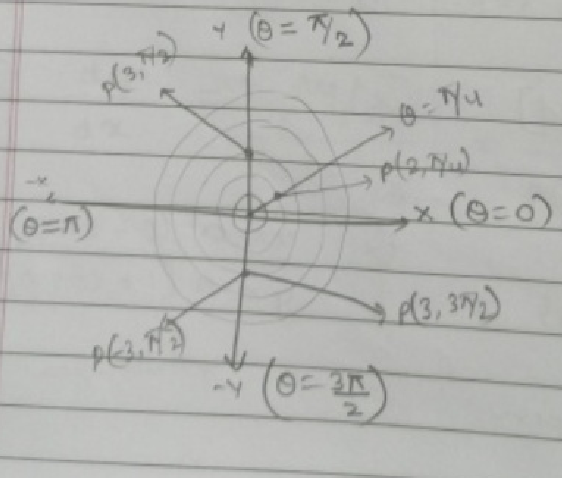
4) $x = a(\theta - \sin\theta)$

$y = a(1 + \cos\theta)$



[Cycloid]

#



Tracing of Polar Curves :-

Given :- $r = f(\theta)$

1. Symmetry :-

(i) About initial line ($\theta = 0$) :-

If equation remains unchanged when θ is changed to $-\theta$.
(i.e. only $\cos\theta$ occurs in the equation).

(ii) About the line $\theta = \pi/2$:

If Equation remains unchanged when θ is changed to $\pi - \theta$
(i.e only $\sin\theta$ occurs in the equation).

2. Table :

For different values of θ & r.

3. Origin :

Curve passes through origin if $r = 0$ for some θ.

4. Greatest value of r.

1) (vii): $r = a(1+\cos\theta)$, $a > 0$.

① Symmetry:

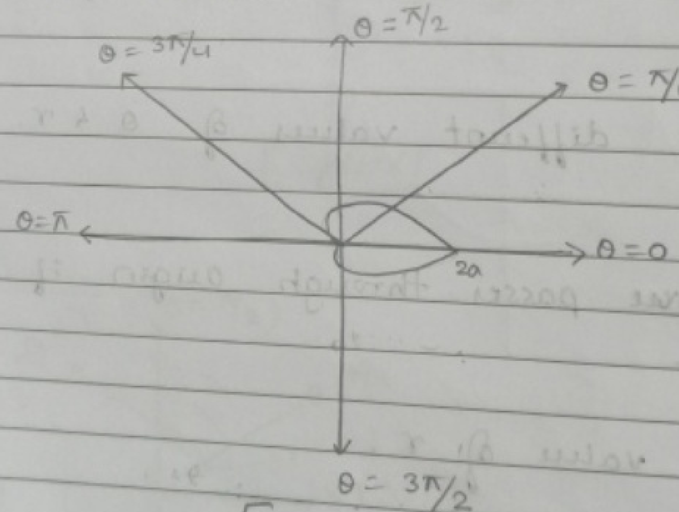
About initial line
($\because$ only $\cos\theta$ occurs in the Equation).

② Table:

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$r = a(1+\cos\theta)$	$2a$	$a(1+1/\sqrt{2})$	a	$a(1-1/\sqrt{2})$	0

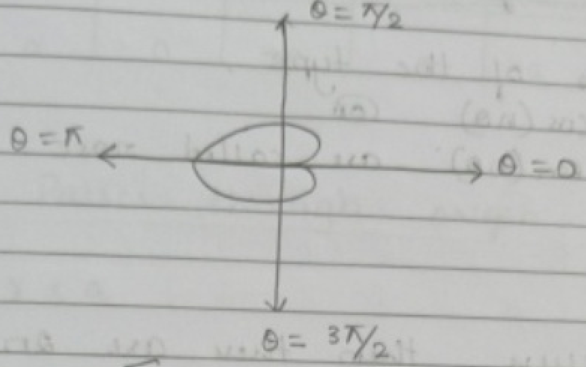
③ Origin: Passes through origin
($\because r=0$ for $\theta=\pi$)

④ $0 \leq r \leq 2a$



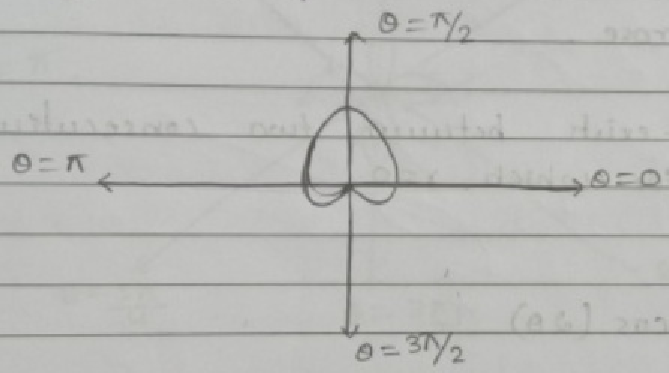
[Cardioid]

Ex: $r = a(1-\cos\theta)$, $a > 0$



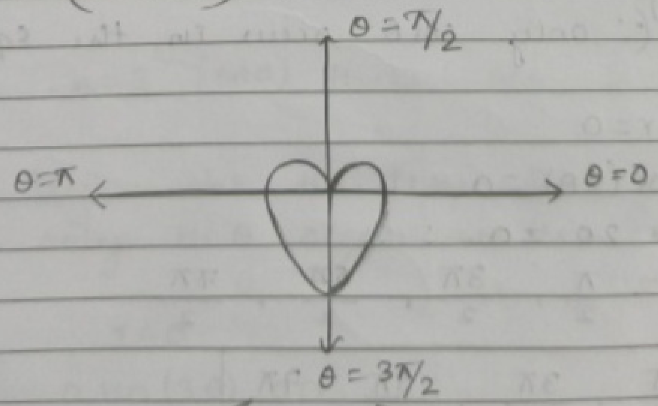
[Cardioid]

Ex: $r = a(1+\sin\theta)$, $a > 0$



[Cardioid]

Ex: $r = a(1-\sin\theta)$, $a > 0$



[Cardioid]

Tracing of Rose :

Equations of the type ,
$r = a\cos(n\theta)$ (or)
$r = a\sin(n\theta)$ are called rose .

NOTE :-

① If n = even , then there are $2n$ loops (or petals) in the rose .

② If n = odd , then there are n loops (or petals) in the rose .

③ A loop exists between two consecutive values of 'θ' for which $r = 0$.

1] (ex) $r = a\cos(2\theta)$

Ans: Since $n = 2$ there are $2(n) = 2(2) = 4$ loops.

① Symmetry : Abt initial line .
($\because$ only $\cos\theta$ occurs in the equation).

② Table : $r = 0$

$\Rightarrow a\cos(2\theta) = 0$

$\Rightarrow \cos 2\theta = 0$

$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

$\Rightarrow \boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

ps (or petals)

or petals)

values

Loops.

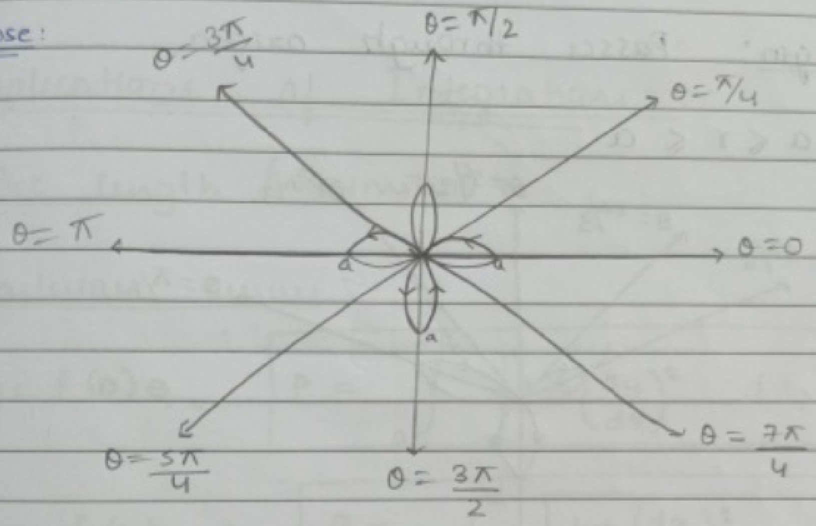
tion).

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$
$r = a\cos(2\theta)$	a	0	$-a$	0	a	0	$-a$	0

③ Origin: Passes through origin.

④ $-a \leq r \leq a$

Rose:



4- Leaved Rose

1) (viii) $r = a\sin(3\theta)$

$\Rightarrow$ Since $n = 3$ (odd) there are 3 loops.

① Symmetry: abt the line $\theta = \pi/2$
(∵ only $\sin\theta$ occur in the Eqⁿ).

② Table: $r = 0$

$a\sin(3\theta) = 0$

$\Rightarrow \sin(3\theta) = 0$

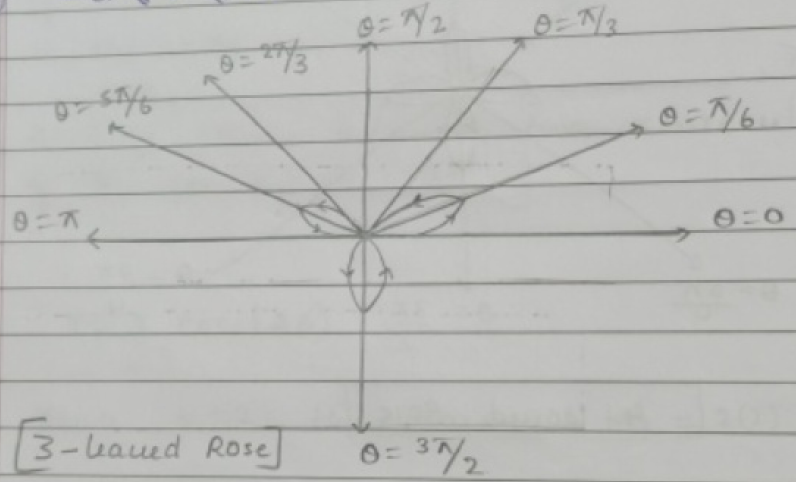
$\Rightarrow \quad 3\theta = 0, \quad \pi, \quad 2\pi, \quad 3\pi$

$\Rightarrow \quad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi.$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$r = a\sin(3\theta)$	0	a	0	$-a$	0	a	0

③ Origin: Passes through origin.

④ $-a \leqslant r \leqslant a$

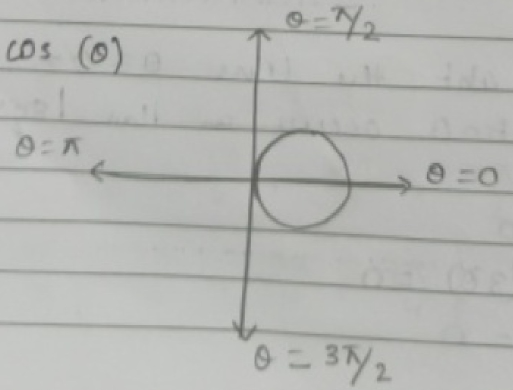


[3-leaved Rose]

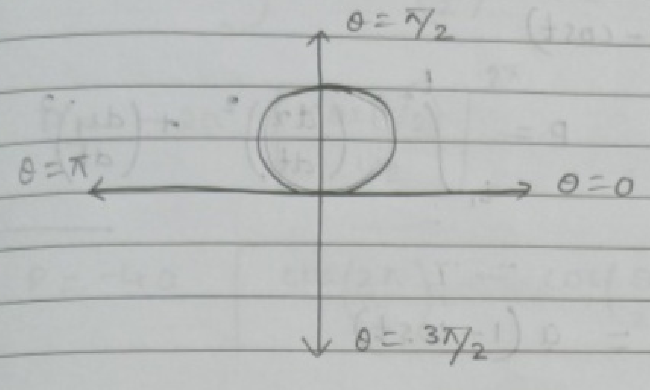
Ex: $r = a\cos(3\theta)$

Ex: $r = a\sin(2\theta)$

1] (X) $r = a\cos(\theta)$



Ex: $r = a\sin\theta$.



Applications of Integrations :-

I. Arc length (Perimeter) :

① Cartesian curves :-

(i) $y = f(x)$: $\boxed{P = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$

(ii) $x = f(y)$: $\boxed{P = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, . dy}$

② Parameter :

$x = f(t)$ & $y = g(t)$

$$\boxed{P = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, . dt}$$

③ Polar : $r = f(\theta)$

4) (i) $x = a(t - \sin t)$

$y = a(1 - \cos t)$

Ans Formula : $P = \int_{t_1}^{t_2} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \, dt$ ——— (1)

Here, $\frac{dx}{dt} = a(1 - \cos t)$

$\frac{dy}{dt} = a \sin t$.

For cycloid, $0 < t < 2\pi$

Eqn reduces to

$$P = \int_0^{2\pi} \sqrt{a^2 (1 - \cos t)^2 + a^2 \sin^2 t} \; dt$$

$$P = a \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} \; . dt$$

$$P = a \int_0^{2\pi} \sqrt{(1 - 2\cos t) + \cos^2 t + \sin^2 t} \; . dt$$

$$P = a \int_0^{2\pi} \sqrt{2 - 2\cos t} \; . dt.$$

$$P = a \int_0^{2\pi} \sqrt{2(1 - \cos t)} \; . dt.$$

$$P = a \int_0^{2\pi} \sqrt{2\left(2\sin^2 \frac{t}{2}\right)} \; . dt$$

$$P = 2a \int_0^{2\pi} \sin\left(\frac{t}{2}\right) \cdot dt$$

$$P = -2a \left[\frac{\cos(t/2)}{1/2} \right]_0^{2\pi}$$

$$P = -4a \left[\cos\left(\frac{2\pi}{2}\right) - \cos\left(\frac{0}{2}\right) \right]$$

$$P = -4a \; [-1-1]$$

$$\boxed{P = +8a \text{ units.}}$$

2/02/23
Thursday

4 (iii) $y^2 = 4ax$ cut off by the line $3y = 8x$

$\Rightarrow$ $y^2 = 4ax$ —— (1)

$3y = 8x$ —— (2)

from (2), $\boxed{y = \left(\frac{8}{3}\right)x}$ —— (3)

Points of x^n :

from (3), $x = \frac{3}{8}y$

put x in (1)

$$y^2 = \cancel{4}a\left(\frac{3}{\cancel{8}_2}\right)y$$

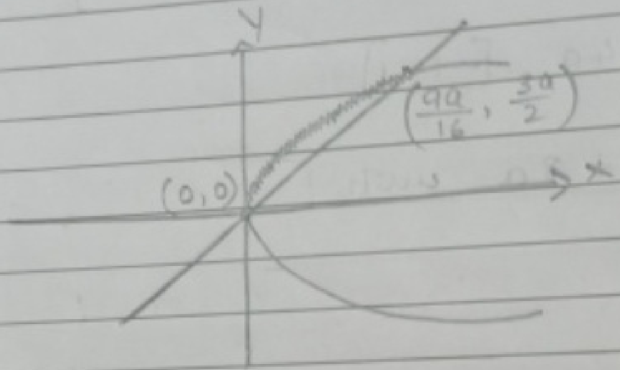
$$y^2 = \left(\frac{3a}{2}\right)y$$

$$y^2 - \frac{3}{2}ay = 0$$

$$y\left(y - \frac{3a}{2}\right) = 0$$

$$\Rightarrow y = 0 \quad ; \quad y = \frac{3a}{2}$$

$$\Rightarrow x = 0 \quad ; \quad x = \frac{9a}{16}$$



from ①, $x = \frac{y^2}{4a}$

$$P = \int_c^d \left[\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right] dy$$

$$\therefore \boxed{\frac{dx}{dy} = \frac{y}{2a}}$$

$$\therefore P = \int_0^{\frac{3a}{2}} \left(\left[\sqrt{1 + \left(\frac{y}{2a}\right)^2}\right]\right) dy$$

$$P = \int_0^{3a/2} \left(\sqrt{1 + \frac{y^2}{(2a)^2}}\right) dy$$

$$P = \int_0^{3a/2} \frac{\sqrt{(2a)^2 + y^2}}{2a} \, dy$$

$$P = \frac{1}{2a} \int_0^{3a/2} \left[\sqrt{(2a)^2 + y^2}\right] dy$$

WKT :-

$$\int \left(\sqrt{a^2 + x^2}\right) dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \log\left[x + \sqrt{a^2+x^2}\right]$$

$$\therefore P = \frac{1}{2a} \left\{ \frac{y\sqrt{(2a)^2 + y^2}}{2} + \frac{(2a)^2}{2} \log\left[y + \sqrt{(2a)^2 + y^2}\right] \right\}_0^{3a/2}$$

$$P = \frac{1}{4a} \left\{ \frac{3a}{2}\sqrt{4a^2 + \frac{9a^2}{4}} + 4a^2 \log\left[\frac{3a}{2} + 4a^2 + \frac{9a^2}{4}\right] - 4a^2 \log(2a) \right\}$$

$$P = \frac{1}{4a} \left\{ \frac{3a}{2}\left(\frac{5a}{2}\right) + 4a^2 \log\left[\frac{3a}{2} + \frac{5a}{2}\right] - 4a^2 \log 2a \right\}$$

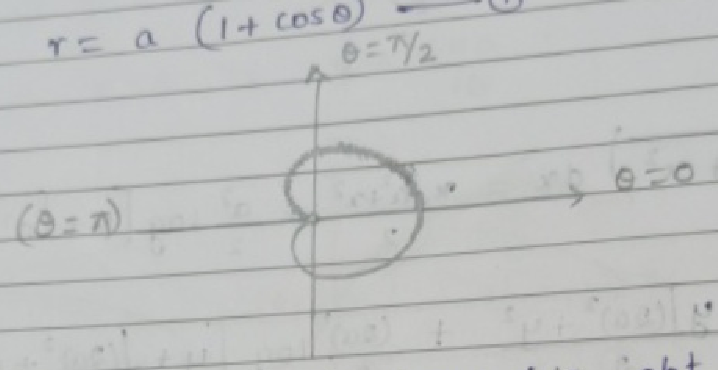
$$P = \frac{1}{4a} \left\{ \frac{15a^2}{4} + 4a^2 \log(4a) - 4a^2 \log 2a \right\}$$

$$P = \frac{a^2}{4a} \left\{ \frac{15}{4} + 4\left(\log 4a - \log 2a\right) \right\}$$

$$P = \frac{a}{4}\left\{\frac{15}{4} + 4\log 2\right\} \text{ units.}$$

Q4] (ii) $r = a(1+\cos\theta)$

$\Rightarrow \quad r = a(1+\cos\theta)$ —— (1)



Given curve is symmetric abt initial line

$\therefore P = 2$ (Perimeter above x-axis)

$$P = 2\int_0^{\pi}\left[\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right] d\theta \quad \text{—— (2)}$$

diff^n Eq^n (1) wrt θ

$$\boxed{\frac{dr}{d\theta} = -a\sin\theta}$$

Eq (2) reduces to

$$P = 2\int_0^{\pi}\left[\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\right] d\theta$$

$$P = 2a\int_0^{\pi}\left(\sqrt{(1+\cos\theta)^2 + \sin^2\theta}\right) d\theta$$

$$P = 2a \int_0^{\pi} \left[\sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\right] d\theta$$

$$P = 2a \int_0^{\pi} \left[\sqrt{2 + 2\cos\theta}\right] d\theta$$

$$P = 2a \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta$$

WKT: $1 + \cos 2A = 2\cos^2 A$

$$P = 2a \int_0^{\pi} \sqrt{2\left(2\cos^2\frac{\theta}{2}\right)}\, d\theta$$

$$P = 4a \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$$

$$P = 4a \left[\frac{\sin(\theta/2)}{(1/2)}\right]_0^{\pi}$$

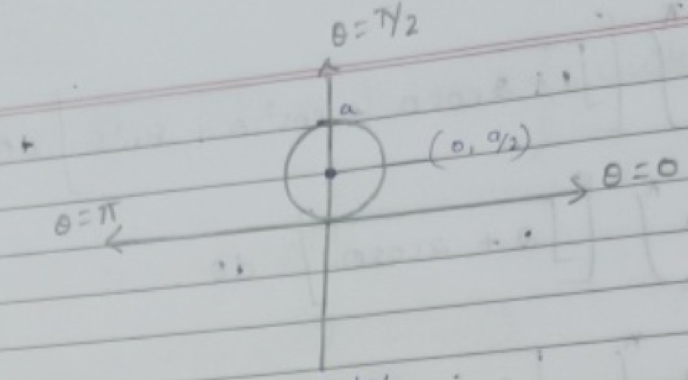
$$P = 8a \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{0}{2}\right)\right]$$

$$P = 8a\,[1 - 0]$$

$$\boxed{P = 8a \text{ units}}$$

(iv) $r = a\sin\theta$

$\Rightarrow r = a\sin\theta$ ——— (1)



diffⁿ ① wrt 'θ'

$$\boxed{\frac{r}{d\theta} = a\cos\theta}$$

$$P = \int_0^{\pi} \left(\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right) d\theta$$

$$P = \int_0^{\pi} \left[\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\right] d\theta$$

$$P = a\int_0^{\pi} \left[\sqrt{\sin^2\theta + \cos^2\theta}\right] d\theta$$

$$P = a\int_0^{\pi} [1]\, d\theta$$

$$P = a\,[\theta]_0^{\pi}$$

$$P = a[\pi - 0]$$

$$\boxed{P = a\pi \text{ units}}$$

AREA :-

1] Cartesian Curve :-

(1) $y = f(x)$ $\quad A = \int_a^b y\,dx$, bounded by x-axis and the lines $x=a$ & $x=b$

(2) $A = \int_c^d x\,dy$, bounded by Y-axis & the lines $y=c$ & $y=d$.

2] Parametric Curve :-

(1) $A = \int_{t_1}^{t_2} y \frac{dx}{dt}\,dt$, bounded by x-axis.

(2) $A = \int_{t_1}^{t_2} x \cdot \frac{dy}{dt}\,dt$. bounded by Y-axis.

3] Polar Curve :-

$$\boxed{A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta}$$

[5] Q Find the area.

(i) of the curve $a^2y^2 = x^3(2a-x)$ $\longrightarrow$ (1)

$\Rightarrow$ Given curve is symmetric about x-axis.

$$a^3y^2 = x^3(2a-x)$$

X-int : Put $y=0$ in Eq. 1

$$\Rightarrow x^3(2a-x) = 0$$

$$\boxed{x=0} \; ; \; \boxed{x=2a}$$

from Eq^n 1,

$$y^2 = \frac{x^3(2a-x)}{a^2}$$

$$y^2 = x(2a-x)\frac{x^2}{a^2}$$

$$\boxed{y = \pm \frac{x}{a}\sqrt{x(2a-x)}}$$

Area = 2 (Area above x-axis)

$$\int \frac{1}{a-x}dx$$

$$\int \frac{1}{a+x}dx$$

$$\text{Area} = 2\int_0^{2a} \boxed{y}\, dx$$

$$\text{Area} = 2\int_0^{2a} \frac{x}{a}\sqrt{x(2a-x)}\; dx$$

put $x = 2a\sin^2\theta$

$$\Rightarrow dx = (2a)\, 2\sin\theta\cos\theta\, d\theta$$

$$\boxed{dx = 4a\sin\theta\cos\theta\, d\theta}$$

$x=0 \quad ; \quad \theta = 0$

$x=2a \quad ; \quad \theta = \frac{\pi}{2}$

$$A = \frac{2}{a}\int_0^{\pi/2} (2a\sin^2\theta)\left[\sqrt{(2a\sin^2\theta)(2a - 2a\sin^2\theta)}\right](4a\sin\theta\cos\theta)\, d\theta$$

$$A = 16a\int_0^{\pi/2} \sin^3\theta\cos\theta\left[\sqrt{4a^2\sin^2\theta\cos^2\theta}\right] d\theta$$

$$A = (16a)(2a)\int_0^{\pi/2} \sin^3\theta\cos\theta\,\sin\theta\cos\theta\, d\theta$$

$$A = 32a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta \cdot d\theta$$

$(p = 4 \;\&\; q = 2)$

$$A = 32a^2\left[\frac{\Gamma\frac{p+1}{2}\;\Gamma\frac{q+1}{2}}{2\,\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}\right]$$

$$A = 32a^2\,\frac{\Gamma\frac{5}{2}\;\Gamma\frac{3}{2}}{2\,\Gamma\frac{8}{2}}$$

$$A = 32a^2\,\frac{\left(\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\right)\left(\frac{1}{2}\cdot\Gamma\frac{1}{2}\right)}{2\cdot\Gamma 4}$$

$$A = \frac{16a^2}{3!}\left(\frac{3}{4}\sqrt{\pi}\right)\left(\frac{1}{2}\sqrt{\pi}\right)$$

$$A = \frac{\cancel{16}a^2}{\cancel{6}}\left(\frac{\cancel{3}}{\cancel{8}}\right)\pi \Rightarrow \boxed{A = a^2\pi \text{ sq. unit}}$$

(ii) Area included between the curve and its asymptote $xy^2 = a^2(a-x)$.

$$xy^2 = a^2(a-x) \quad \text{--- (1)}$$

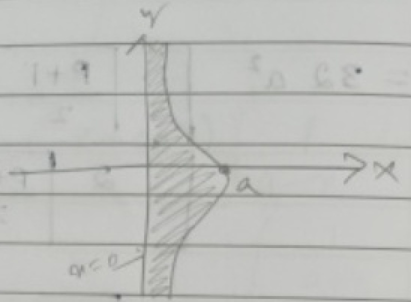
Given curve is symmetric about x-axis.

<u>X-int</u> :- Put $y=0$ in Eq (1)

$$0 = a^2(a-x)$$

$$(a \neq 0) \Rightarrow \boxed{x=a}$$

From (1) $\quad y^2 = a^2\dfrac{(a-x)}{x}$



$$\Rightarrow \boxed{y = a\sqrt{\frac{a-x}{x}}}$$

$\therefore x=0$ is the VA.

Area = 2 (Area above x-axis)

$$A = 2\int_0^a y\,dx$$

$$A = 2\int_0^a a\sqrt{\frac{a-x}{x}}\,dx$$

$$A = 2a\int_0^a \sqrt{\frac{a-x}{x}}\,dx.$$

Put $x = a\sin^2\theta$.

$$\Rightarrow \boxed{dx = 2a \sin\theta \cos\theta \, d\theta}$$

$x = 0 \; ; \; \theta = 0$

$x = a \; ; \; \theta = \frac{\pi}{2}$

$$A = 2a \int_0^{\pi/2} \left[\sqrt{\frac{a - a\sin^2\theta}{a\sin^2\theta}} \right] 2a \sin\theta \cos\theta \, d\theta$$

$$A = 4a^2 \int_0^{\pi/2} \sqrt{\frac{1 - \sin^2\theta}{\sin^2\theta}} \sin\theta \cos\theta \, d\theta$$

$$A = 4a^2 \int_0^{\pi/2} \left(\frac{\cos\theta}{\cancel{\sin\theta}}\right) \cancel{\sin\theta} \cos\theta \, d\theta$$

$$A = 4a^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

Here $p = 0 \; ; \; q = 2$

$$A = \overset{2}{\cancel{4}}a^2 \left[\frac{\Gamma\frac{0+1}{2} \, \Gamma\frac{2+1}{2}}{\cancel{2}\,\Gamma\frac{4}{2}} \right]$$

$$A = 2a^2 \frac{\Gamma\frac{1}{2} \, \Gamma\frac{3}{2}}{\Gamma 2}$$

$$A = \frac{\cancel{2}a^2}{1!} \sqrt{\pi} \, \frac{1}{\cancel{2}} \sqrt{\pi} = a^2\pi \text{ sq. units.}$$

(iii) Loop of the curve $ay^2 = x^2(a-x)$

$\Rightarrow \quad ay^2 = x^2(a-x)$ —— (1)

Given curve is symmetric about x-axis.

X-int :- put $y=0$ in Eqn (1)

$$x^2(a-x) = 0$$

$$\Rightarrow \boxed{x=0} \; ; \; \boxed{x=a}$$

from (1), $y^2 = x^2 \dfrac{(a-x)}{a}$

$$\Rightarrow \boxed{y = x\sqrt{\frac{a-x}{a}}}$$

Area = 2 (Area above x-axis)

$$A = 2\int_0^a y\,dx$$

$$A = 2\int_0^a x\sqrt{\frac{a-x}{a}}\,dx$$

put $x = a\sin^2\theta$

$dx = 2a\sin\theta\cos\theta\,d\theta$

$x=0 \;;\; \theta = 0$

$x=a \;;\; \theta = \frac{\pi}{2}$

$$A = 2\int_0^{\pi/2} a\sin^2\theta\sqrt{\frac{\cancel{a} - \cancel{a}\sin^2\theta}{\cancel{a}}}\,[2a\sin\theta\cos\theta\,d\theta]$$

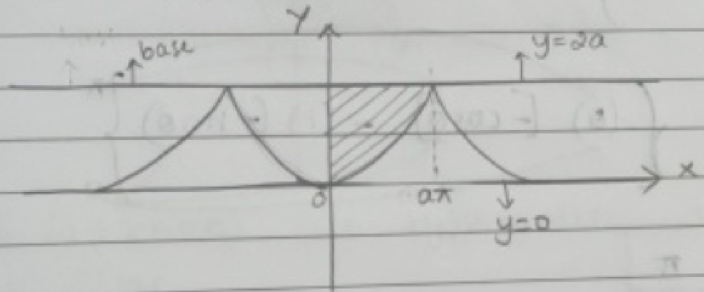
$$A = 4a^2 \int_0^{\pi/2} \sin^3\theta \cos\theta \sqrt{\cos^2\theta} \, d\theta$$

$$A = 4a^2 \int_0^{\pi/2} \sin^3\theta \cos^2\theta \, d\theta$$

$$\boxed{A = \frac{8a^2}{15} \text{ sq. units.}}$$

5/02/22 Monday

(iv) included between cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ and its base.



Given cycloid is symmetric about Y-axis

$\therefore$ Area $= 2$ (Area towards the right of Y-axis)

$$A = 2\int_0^{\pi} x \cdot \frac{dy}{d\theta} \, d\theta$$

$$A = 2\int_0^{\pi} a(\theta + \sin\theta) \frac{d[a(1-\cos\theta)]}{d\theta} \, d\theta$$

$$A = 2a\int_0^{\pi} (\theta + \sin\theta) \, a \sin\theta \, d\theta$$

$d\theta]$

$$A = 2a^2 \int_0^{\pi} (\theta \sin\theta + \sin^2\theta)\, d\theta$$

$$A = 2a^2 \left\{ \int_0^{\pi} \theta \sin\theta \, d\theta + \int_0^{\pi} \sin^2\theta \, d\theta \right\}$$

$$A = 2a^2 \left\{ I_1 + I_2 \right\} \text{ ——— (1)}$$

$$I_1 = \int_0^{\pi} \underset{u}{\theta} \underbrace{\sin\theta}_{v} \, d\theta .$$

WKT :- $\int uv\,dx = uv_1 + u'v_2 + u''v_3 - u'''v_4 + \cdots$

$$I_1 = \left\{ (\theta)\left[-\cos\theta\right] - (1)(-\sin\theta) \right\}_0^{\pi}$$

$$\boxed{I_1 = \pi}$$

$$I_2 = \int_0^{\pi} \sin^2\theta \, d\theta$$

$$I_2 = 2\int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$\left(\because \int_0^{a} f(x)\,dx = 2\int_0^{a/2} f(x)\,dx \text{ if } f(a-x) = f(x) \right)$$

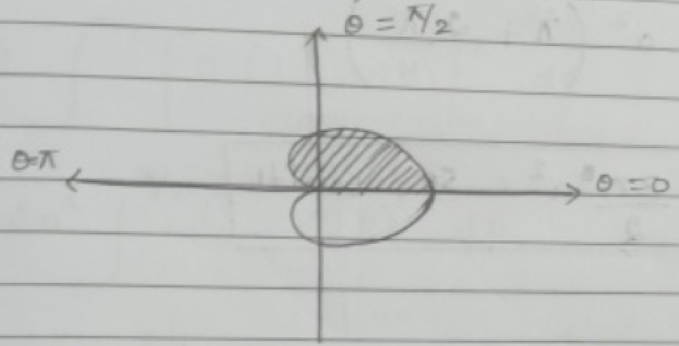
$$I_2 = \frac{2\,\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} = \frac{\pi}{2}$$

Eqn ① reduces to

$$A = 2a^2 \left\{ \pi + \frac{\pi}{2} \right\}$$

$$\boxed{A = 3\pi a^2 \text{ sq. units}}$$

(V) $r = a(1+\cos\theta)$, $a > 0$



$r = a(1+\cos\theta)$ is symmetric about initial line.

Area = 2 (Area above initial line)

$$A = \cancel{2}\left(\frac{1}{\cancel{2}}\int_0^{\pi} r^2\, d\theta\right)$$

$$A = \int_0^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$

$$A = a^2 \int_0^{\pi} (1+\cos\theta)^2\, d\theta$$

$$A = a^2 \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$A = a^2\left(\left[\theta + 2\sin\theta\right]_0^{\pi} + \int_0^{\pi} \cos^2\theta \, d\theta\right)$$

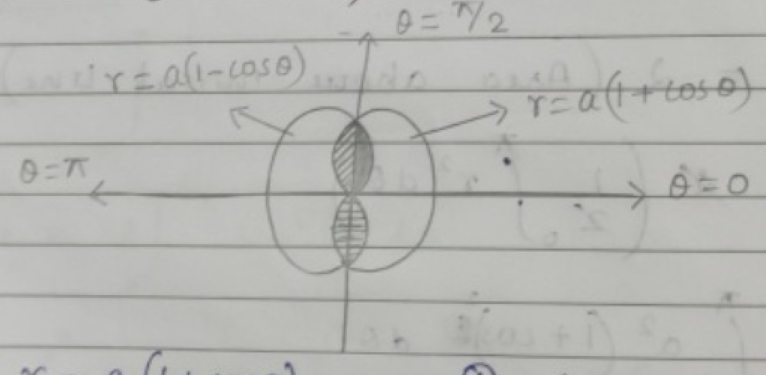
$$A = a^2\left([\pi] + 2\int_0^{\pi/2} \cos^2\theta \, d\theta\right)$$

$$\left(\because \int_0^{a} f(x)\,dx = 2\int_0^{a/2} f(x)\,dx \text{ if } f(a-x) = f(x)\right)$$

$$A = a^2\left(\pi + 2\frac{\pi}{4}\right)$$

$$\boxed{A = \frac{3\pi}{2}a^2 \text{ sq. units}}$$

(vi) Common to cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.



$r = a(1+\cos\theta)$ —— (1)

$r = a(1-\cos\theta)$ —— (2)

From (1) & (2)

$\cancel{a}(1+\cos\theta) = \cancel{a}(1-\cos\theta)$

$1+\cos\theta = 1-\cos\theta$

$$\cos\theta = -\cos\theta$$
$$2\cos\theta = 0$$
$$\cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

Area = 4 (area in 1st quadrant)

$$A = \cancel{4}^{2}\left(\frac{1}{\cancel{2}}\int_0^{\pi/2} r^2\, d\theta\right)$$

$$A = 2\int_0^{\pi/2}\left[a\left(1-\cos\theta\right)\right]^2 d\theta$$

$$A = 2a^2\int_0^{\pi/2}(1-\cos\theta)^2\, d\theta$$

$$A = 2a^2\int_0^{\pi/2}\left[1-2\cos\theta+\cos^2\theta\right] d\theta$$

$$A = 2a^2\left\{\left[\theta - 2\sin\theta\right]_0^{\pi/2} + \int_0^{\pi/2}\cos^2\theta\, d\theta\right\}$$

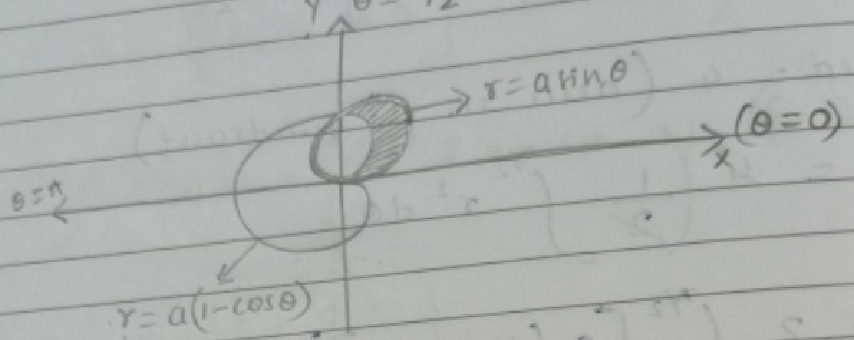
$$A = 2a^2\left\{\left[\frac{\pi}{2}-2\right]+\frac{\pi}{4}\right\}$$

$$A = 2a^2\left\{\frac{3\pi}{4}-2\right\}$$

$$\boxed{A = \left(\frac{3\pi-8}{2}\right)a^2 \text{ sq. units}}$$

7/02/23
Tuesday

(vii) Inside the circle $r_1 = a\sin\theta$ and outside the cardioids $r_2 = a(1-\cos\theta)$.



$$r_1 = a\sin\theta \quad \text{—— (1)}$$
$$r_2 = a(1-\cos\theta) \quad \text{—— (2)}$$

$$\text{Area} = \begin{pmatrix}\text{area of} \\ \text{circle in } 1^{st} \\ \text{quadrant}\end{pmatrix} - \begin{pmatrix}\text{Area of cardioid} \\ \text{in } 1^{st} \\ \text{quadrant}\end{pmatrix}$$

Pt. of intersection of (1) & (2) :

$$a\sin\theta = a(1-\cos\theta)$$
$$\sin\theta = 1-\cos\theta$$
$$\sin\theta + \cos\theta = 1$$

$$\Rightarrow \boxed{\theta = 0} \ ; \ \boxed{\theta = \pi/2}$$

$$A = \left[\int_0^{\pi/2} \frac{1}{2} r_1^2 \, d\theta\right] - \left[\int_0^{\pi/2} \frac{1}{2} r_2^2 \, d\theta\right]$$

$$A = \frac{1}{2}\int_0^{\pi/2} \left\{ r_1^2 - r_2^2 \right\} d\theta$$

$$A = \frac{1}{2}\int_0^{\pi/2} \left\{ a^2 \sin^2\theta - a^2 (1-\cos\theta)^2 \right\} d\theta$$

$$A = \frac{a^2}{2}\int_0^{\pi/2} \left\{ \sin^2\theta - (1 - 2\cos\theta + \cos^2\theta) \right\} d\theta$$

$$A = \frac{a^2}{2}\int_0^{\pi/2} \left\{ \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta \right\} d\theta$$

$$A = \frac{a^2}{2} \left\{ \cancel{\frac{\pi}{4}} \left(-\theta + 2\sin\theta \right)_0^{\pi/2} - \cancel{\frac{\pi}{4}} \right\}$$

$$A = \frac{a^2}{2} \left\{ \left(-\frac{\pi}{2} + 2 \right) \right\}$$

$$A = \frac{a^2}{2} \left\{ 2 - \frac{\pi}{2} \right\}$$

$$\boxed{A = a^2 \left\{ \frac{4-\pi}{4} \right\} \text{ sq. units.}}$$

Ex: Find the area of one loop of the rose, $r = a\sin(2\theta)$

$\Rightarrow$ For 1 loop, $r = 0$

$\Rightarrow a\sin(2\theta) = 0$

$\Rightarrow \sin(2\theta) = 0$

$\Rightarrow 2\theta = 0, \pi$

$\boxed{\theta = 0, \frac{\pi}{2}}$

$$A = \int_{0} \frac{1}{2} r^2 \, d\theta$$

$$A = \frac{1}{2} \int_{0}^{\pi/2} a^2 \sin^2(2\theta)\, d\theta$$

$$A = \frac{a^2}{2} \int_{0}^{\pi/2} \sin^2(2\theta) . \, d\theta$$

put $\boxed{2\theta = t}$

$2\, d\theta = dt$

$\boxed{d\theta = \frac{dt}{2}}$

when, $\theta = 0$; $t = 0$

$\theta = \frac{\pi}{2}$; $t = \pi$

$$A = \frac{a^2}{2} \int_{0}^{\pi} (\sin^2 t) \frac{dt}{2}$$

$$A = \frac{a^2}{4} \int_{0}^{\pi} \sin^2 \underbrace{t}_{(\pi - t)} \, dt$$

$$\left\{ \because \int_{0}^{a} f(x)\, dx = 2 \int_{0}^{a/2} f(x)\, dx \ \text{ if } f(a-x) = f(x) \right\}$$

$$A = \frac{a^2}{4} \cdot 2 \int_{0}^{\pi/2} \sin^2 t \, dt$$

$p = 2$, $q = 0$

$$A = \frac{a^2}{2}\left(\frac{\pi}{4}\right)$$

$$\boxed{A = \frac{a^2\pi}{8} \text{ sq. units.}}$$

III) Surface Area :-

1] Cartesian curves :-

(i) $S = \int_a^b 2\pi y \frac{dP}{dx}\, dx$,

of the solid of revolution abt x-axis, where

$$\frac{dP}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(ii) $S = \int_c^d 2\pi x \frac{dP}{dx}\, dy$, about Y-axis,

$$\frac{dP}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

2] Parametric curve :-

(i) $S = \int_{t_1}^{t_2} 2\pi y \frac{dP}{dt}\, dt$, about x-axis,

$$\frac{dP}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

classmate
Date
Page

(ii) $S = \int_{t_1}^{t_2} 2\pi x \frac{dP}{dt} dt$, about Y-axis,

$$\frac{dP}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(3) Polar Curves :-

(i) $S = \int_{\theta_1}^{\theta_2} 2\pi r \sin\theta \frac{dP}{d\theta} d\theta$, about initial line,

$$\frac{dP}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

(ii) $S = \int_{\theta_1}^{\theta_2} 2\pi (r\cos\theta) \frac{dP}{d\theta} d\theta$, about $\theta = \frac{\pi}{2}$,

$$\frac{dP}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

IV) Volume :-

1] Cartesian Curve :-

(i) $V = \int_a^b \pi y^2 dx$, about X-axis

(ii) $V = \int_c^d \pi x^2 dy$, about Y-axis.

2] Parametric Curves :-

(i) $V = \int_{t_1}^{t_2} \pi y^2 \frac{dx}{dt} \, dt$, about X-axis

(ii) $V = \int_{t_1}^{t_2} \pi x^2 \frac{dy}{dt} \, dt$, about Y-axis

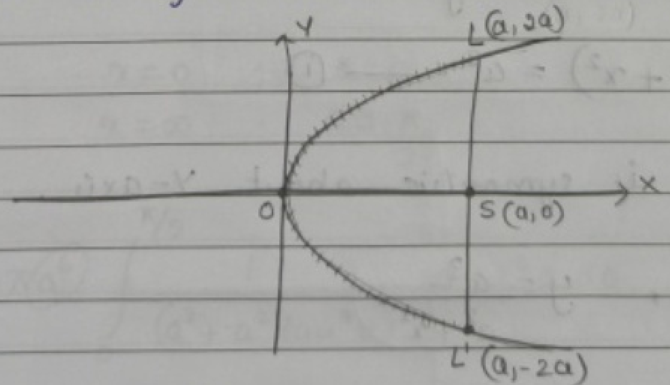
3] Polar Curve :-

(i) $V = \int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \sin\theta \, d\theta$, about initial line.

(ii) $V = \int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \cos\theta \, d\theta$, about $\theta = \frac{\pi}{2}$

Q7] Given : $y^2 = 4ax$ —— (1)

Eqn (1) is symmetric about x-axis.



Volume = 2 (volume abt. x-axis)

$\therefore V = 2 \int_{0}^{2a} \pi x^2 \, dy$

$$V = 2\pi \int_0^{2a} \left(\frac{y^2}{4a}\right)^2 dy \quad \left[\text{from } ①, x = \frac{y^2}{4a}\right]$$

$$V = 2\pi \int_0^{2a} \frac{y^4}{16a^2}\, dy$$

$$V = \frac{\pi}{8a^2} \int_0^{2a} y^4\, dy$$

$$V = \frac{\pi}{8a^2} \left[\frac{y^5}{5}\right]_0^{2a}$$

$$V = \frac{\pi}{40a^2}\, 32a^5$$

$$\boxed{V = \frac{4}{5}\pi a^3 \text{ cubic units}}$$

8/02/23 Wednesday

Q] Find the volume generated by the revolution of the curve $y(a^2+x^2) = a^3$ about its asymptote

$\Rightarrow$ $y(a^2+x^2) = a^3$ —— ①

Eq^n ① is symmetric about Y-axis.

from ①, $y = \dfrac{a^3}{a^2+x^2}$

HA :- $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \dfrac{a^3}{a^2+x^2}$

$\lim\limits_{x\to\infty} y = 0$

$\therefore$ $y=0$ is HA

$\Rightarrow$ X-axis is HA

Since curve is symmetric about Y-axis we revolve part of the curve towards the right of Y-axis to the get the solid

$$\therefore \text{ Volume } = 2\int_0^\infty \pi y^2\, dx$$

$$V = 2\pi \int_0^\infty \left(\frac{a^3}{a^2+x^2}\right)^2 dx \quad \text{(refer Eqn (2))}$$

$$V = 2\pi a^6 \int_0^\infty \frac{1}{(a^2+x^2)^2}\, dx$$

$$\left\{ \int \frac{1}{a^2-x^2} \quad x = a\sin\theta \right\}$$

put $x = a\tan\theta$

$\Rightarrow$ $\boxed{dx = a\sec^2\theta\, d\theta}$

when, $x=0$; $\theta=0$
$x=\infty$; $\theta=\frac{\pi}{2}$

$$V = 2\pi a^6 \int_0^{\pi/2} \frac{1}{(a^2+a^2\tan^2\theta)^2}\, a\sec^2\theta\, d\theta$$

$$V = 2\pi a^7 \int_0^{\pi/2} \frac{\cancel{\sec^2}\theta}{a^4\sec^{\cancel{4}}\theta}\, d\theta$$

$$V = 2\pi a^3 \int_0^{\pi/2} \frac{1}{\sec^2\theta}\, d\theta$$

$$V = 2\pi a^3 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$V = \cancel{2}\pi a^3 \left(\frac{\pi}{\cancel{4}\,2}\right)$$

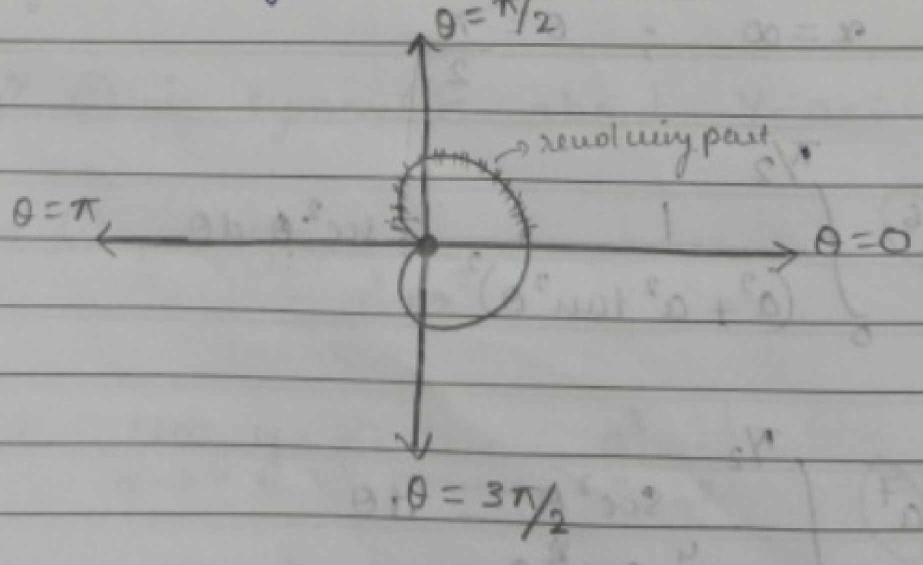
$$\boxed{V = \frac{\pi^2 a^3}{2} \text{ cubic units.}}$$

Q 6] Find the surface area and volume of the solid formed by revolving the cardioid, $r = a(1+\cos\theta)$ about the initial line.

$\Rightarrow$ Given: $r = a(1+\cos\theta)$ —— ①

To find: Surface area (S), volume (V) } abt initial line.

Eqn ① is symmetric about initial line.



Surface Area :-

$$S = \int_0^{\pi} 2\pi (r\sin\theta) \frac{dP}{d\theta} d\theta$$

$$S = 2\pi \int_0^{\pi} r\sin\theta \frac{dP}{d\theta} d\theta \quad \text{——— (2)}$$

where, $\frac{dP}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$$\frac{dP}{d\theta} = \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}$$

$$\frac{dP}{d\theta} = a\sqrt{(1+\cos\theta)^2 + \sin^2\theta}$$

$$\frac{dP}{d\theta} = a\sqrt{1+2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$\frac{dP}{d\theta} = a\sqrt{2+2\cos\theta}$$

Eq^n (2) reduces to,

$$S = 2\pi \int_0^{\pi} a(1+\cos\theta)\sin\theta \left(a\sqrt{2+2\cos\theta}\right) d\theta$$

$$S = 2\pi a^2 \int_0^{\pi} (1+\cos\theta)\sin\theta \sqrt{2}(1+\cos\theta)^{1/2} d\theta$$

$$S = (2\sqrt{2})\pi a^2 \int_0^{\pi} (1+\cos\theta)^{3/2} \sin\theta \, d\theta$$

WKT:

$$\int [F(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$S = 2\sqrt{2}\pi a^2 \left[\frac{-(1+\cos\theta)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)}\right]_0^{\pi}$$

$$S = -2\sqrt{2}\pi a^2 \frac{2}{5}\left[(1+\cos\theta)^{5/2}\right]_0^{\pi}$$

$$S = \frac{-4\sqrt{2}\pi a^2}{5} \cdot \left[0^{5/2} - 2^{5/2}\right].$$

$$S = +\frac{4}{5} 2^{1/2} 2^{5/2} \pi a^2$$

$$\boxed{S = \frac{32}{5}\pi a^2 \text{ sq. units.}}$$

Volume :-

$$V = \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta\, d\theta$$

$$V = \frac{2}{3}\pi \int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta\, d\theta$$

$$V = \frac{2}{3}\pi a^3 \int_0^{\pi} (1+\cos\theta)^3 \sin\theta\, d\theta$$

WKT :-

$$\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$V = \frac{2}{3}\pi a^3 \left[-\frac{(1+\cos\theta)^4}{4} \right]_0^\pi$$

$$V = \frac{-\cancel{2}\pi a^3}{3\,\cancel{(4)}_2} \left[0^4 - 2^4 \right]$$

$$V = \frac{\pi a^3}{6}(2^4)$$

$$\boxed{V = \frac{8}{3}\pi a^3 \text{ cubic units}}$$

Q 8] Find the surface area and the volume of the spindle shaped solid generated by revolution of the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

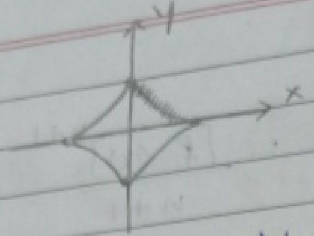
$\Rightarrow \quad x^{2/3} + y^{2/3} = a^{2/3}$ —— (1)

To find :- Surface area (S), Volume (V) } abt x-axis.

Parametric Eq^n of (1) are :-

$x = a\cos^3 t$

$y = a\sin^3 t$



Eqⁿ (1) is symmetric abt x-axis.

Surface area (S) :-

$$S = 2\int_0^{\pi/2} 2\pi y \frac{dP}{dt}\, dt$$

$$S = 4\pi\int_0^{\pi/2} y \frac{dP}{dt}\, dt \quad \text{—— (2)}$$

$$\frac{dP}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{dP}{dt} = \sqrt{(-3a\cos^2 t \sin t)^2 + (3a\sin^2 t \cos t)^2}$$

$$\frac{dP}{dt} = \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}$$

$$\frac{dP}{dt} = \sqrt{9a^2\cos^2 t \sin^2 t\, (\underbrace{\cos^2 t + \sin^2 t}_{1})}$$

$$\boxed{\frac{dP}{dt} = 3a\cos t \sin t}$$

Eqⁿ (2) reduces to,

$$S = 4\pi \int_0^{\pi/2} (a \sin^3 t)(3a \cos t \sin t)\, dt$$

$$S = 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t \, dt$$

$$S = 12\pi a^2 \left[\frac{\sin^5 t}{5}\right]_0^{\pi/2}$$

$$S = \frac{12\pi a^2}{5} \left[\sin^5 \frac{\pi}{2} - \sin^5 0\right]$$

$$\boxed{S = \frac{12\pi a^2}{5} \text{ sq. units}}$$

Volume (V) :-

$$V = 2\int_0^{\pi/2} \pi y^2 \frac{dx}{dt}$$

$$V = 2\pi \int_0^{\pi/2} (a \sin^3 t)^2 \frac{d}{dt} (a \cos^3 t)\, dt$$

$$V = 2\pi a^3 \int_0^{\pi/2} \sin^6 t \left[-3 \cos^2 t \sin t\right] dt$$

$$V = -6\pi a^3 \int_0^{\pi/2} \sin^7 t \cos^2 t \, dt$$

$$V = -\cancel{6}^{3}\pi a^3 \int \frac{\Gamma\frac{8}{2}\ \Gamma\frac{3}{2}}{\cancel{2}\ \Gamma\frac{11}{2}}$$

$$V = -3\pi a^3 \frac{\Gamma 4 \; \Gamma 3/2}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma 3/2}$$

$$V = \frac{-\cancel{3}\pi a^3 \,(3!)\,(16)}{9.7.5.\cancel{3}}$$

$$V = \frac{-\cancel{6}^{2}\pi a^3 \,(16)}{\cancel{9}_{3}.7.5}$$

$$V = \frac{-32\pi a^3}{105}$$

$$\boxed{V = \frac{-32\pi a^3}{105} \text{ (Numerically)}}$$

$\int x\,dx = \frac{x^2}{2}$

$\int f(x)\,dx$

x	0	1	2	3	4
$f(x)$	150	240	276	1794	3517

$$\int e^{x^2}\,dx = , \quad \frac{d}{dx}(f(x)) = e^{x^2}$$

Numerical Integration :-

1) Trapezoidal Rule :-

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right)\right]$$

where $h = \frac{b-a}{n}$, h = width

n = number of sub intervals

2] Simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule :-

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\Big]$$

Where, $h = \frac{b-a}{n}$

n = number of sub-intervals //

9/02/23 Thursday

NOTE :- For simpson's rule
n = multiple of 2.

12] $\int_0^{1/2} \sin t\,dt$, $n = 8$

7M

Here, $a = 0$, $b = 1/2$, $f(t) = \sin t$, $n = 8$

WKT, $h = \frac{b-a}{n} = \frac{1/2 - 0}{8} = \frac{1}{16}$

t	0	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16 = 1/2
$f(t) = \sin t$	0	0.0625	0.12467	0.1864	0.2474	0.30743	0.36627	0.4236	0.4794
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Trapezoidal Rule :-

$$\int_0^{1/2} \sin t\,dt = \frac{h}{2}\Big[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\Big]$$

$$I = \frac{1}{32}\left[(0+0.4794) + 2\left(0.0625 + 0.1247 + 0.1864 + 0.2474 + 0.3074 + 0.3663 + 0.4237\right)\right]$$

$$I = \frac{1}{32}\left[(0.4794) + 2(1.7184)\right]$$

$$= \frac{1}{32}\left[0.4794 + 3.4368\right]$$

$$= \frac{1}{32}\left[3.9162\right]$$

$$\boxed{I = 0.1224}$$

Q 10] $\int_0^1 \frac{dx}{1+x^2}$, $n=6$

$\Rightarrow$ To find : (i) $\int_0^1 \frac{1}{1+x^2}\,dx$

(ii) value of π

$\Rightarrow$ Given: $a=0$, $b=1$, $f(x) = \frac{1}{1+x^2}$, $n=6$

WKT = $h = b-a/n = 1-0/6 = 1/6$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$f(x) = \frac{1}{1+x^2}$	1	0.9729	0.9	0.8	0.6923	0.5902	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$1 \div (1 + (1 \div 6)^2)$

Simpsons $1/3^{rd}$ rule :-

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4+y_6)+4(y_1+y_3+y_5)\right]$$

$$\cancel{= \frac{1}{48}\left[(1+2)+2(1.1+1.44)+4(1.023+1.25+1.694)\right]}$$

$$\cancel{= \frac{1}{48}\left[(3)+2(2.55)+4(3.971)\right]}$$

$$\cancel{= \frac{1}{48}}$$

$$= \frac{1}{18}\left[(1+0.5)+2(0.9+0.6923)+4(0.9729+0.8+0.5902)\right]$$

$$= \frac{1}{18}\left[(1.5)+2(1.5923)+4(2.3631)\right]$$

$$= \frac{1}{18}\left[(1.5)+3.1846+9.4524\right]$$

$$\boxed{I = 0.7854} \text{ —— } (1)$$

Also, $I = \int_0^1 \frac{1}{1+x^2}\,dx$

$$I = \left[\tan^{-1}x\right]_0^1$$

$$I = \tan^{-1}1 - \tan^{-1}0$$

$$I = \frac{\pi}{4} - 0$$

$$\boxed{I = \frac{\pi}{4}} \text{ —— } (2)$$

From (1) & (2)

$\frac{\pi}{4} = 0.7854 \Rightarrow \pi = 0.7854 \times 4$

$$\boxed{\pi = 3.1416}$$

classmate

Date

Page

Q11] $\int_0^1 e^{-x^2}\,dx$, $n=10$

$\Rightarrow$ $a=0$, $b=1$, $f(x)=e^{-x^2}$, $n=10$

$h=\dfrac{b-a}{n}=\dfrac{1-0}{10}=\dfrac{1}{10}=0.1$

$\boxed{e(-(\boxed{0.1}\wedge 2))}$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)=e^{-x^2}$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6976	0.6126	0.5272	0.4449	0.3679
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Simpson's rule :-

$$\int_0^1 e^{-x^2}dx=\frac{h}{3}\left[(y_0+y_{10})+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$=\frac{0.1}{3}\left[(1+0.3679)+2(0.9608+0.8521+0.6976+0.5272)+4(0.99+0.9139+0.7788+0.6126+0.4449)\right]$$

I = 0.7468

$=\dfrac{0.1}{3}$

0/02/23
Friday

Q 13]

M
To find :- Area

Formula :- $A = \int_a^b y\,dx$

From table, $a = 0$, $b = 80$, $n = 8$, $h = \frac{b-a}{n} = \frac{80-0}{8} = 10$

Simpson's rule :-

$$\int_0^{80} y\,dx = \frac{h}{3}\left[(y_0 + y_8) + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)\right]$$

$$A = \frac{10}{3}\left[(0+3) + 2(7+12+14) + 4(4+9+15+8)\right]$$
$$(3 + 66 + 144)$$
$$10/3 \times 213$$

$A = 710$ sq. units //

Q 14] **To find :-** Volume.
Formula: $V = \int_a^b \pi y^2\,dx$

Given:-

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415
y^2	1	0.9793	0.9195	0.8261	0.7081
	y_0	y_1	y_2	y_3	y_4

$b = 1$, $h = 0.25$, $n = 4$

classmate

Date
Page

Simpson's rule :-

$$\int_0^1 \pi y^2 dx = \pi \frac{h}{3}\left[(y_0+y_4) + 2(y_2) + 4(y_1+y_3)\right]$$

$$V = \pi\left(\frac{0.25}{3}\right)\left[(1+0.7081) + 2(0.9195) + 4(0.9793 + 0.8261)\right]$$

$V = 2.8192$ cubic units

Q16] Given: $3y = x^3$ —— ①

Points : $(0,0)$ to $(1, 1/3)$
n = no. of sub intervals $= 8$

Arc length (Perimeter) :-

$$P = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{—— ②}$$

from ①, $y = \frac{x^3}{3}$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3x^2}{3} = x^2}$$

Eqⁿ ② reduces to

$$P = \int_a^b \sqrt{1+(x^2)^2}\, dx$$

$$\boxed{P = \int_0^1 \sqrt{1+x^4}\, dx}$$

Here, $a = 0$, $b = 1$, $f(x) = \sqrt{1+x^4}$, $n = 8$.

$$h = \frac{b-a}{n} = \frac{1-0}{8} = \frac{1}{8}$$

x	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8 = 1
$f(x) = \sqrt{1+x^4}$	1.00	1.0012	1.0019	(1.0097)' 1.0199	1.0308	1.0736	1.1473	1.2992	1.414
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Simpson's Rule :-

$$P = \frac{h}{3}\left[(y_0+y_8) + 2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{1}{24}\left[(1+1.414) + 2(1.0020 + 1.0308 + 1.1473) + 4(1.0002 + 1.0199 + 1.0736 + 1.2592)\right]$$

doubt

$$\boxed{P = 1.0894 \text{ units}}$$

Q16] To find :- time to travel 60 ft.

7M

WKT :- $V = \frac{ds}{dt}$

$$dt = \frac{1}{V} ds$$

Integrating we got,

$$\int dt = \int_0^{60} \frac{1}{V} ds$$

$$\Rightarrow \boxed{t = \int_0^{60} \frac{1}{V} ds}$$

S	0	10	20	30	40	50	60
V	47	58	64	65	61	52	38
$\frac{1}{V}$	0.0212	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's rule:

$$\int_0^{60} \frac{1}{V}\, ds = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$t = \frac{10}{3}\left[(0.0212 + 0.0263) + 2(0.0156 + 0.0164) + 4(0.0172 + 0.0154 + 0.0192)\right]$$

$$\boxed{t = 1.0627 \text{ sec.}}$$

Q 17] To find : Work done :

Formula :- $W = \int_a^b f(x)\, dx$

Given:

x	0	3	6	9	12	15	18
f(x)	9.8	9.1	8.5	8.0	7.7	7.5	7.4
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

From table, $a=0$, $b=18$, $h=3$, $n=6$

$$\int_0^{18} f(x)\, dx = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= 148 \text{ J}$$

13/02/23 Monday

Note :-

	circle	centre	radius
1]	$r = a\cos\theta$	$\left(\frac{a}{2}, 0\right)$	$\left(\frac{a}{2}\right)$
2]	$r = a\sin\theta$	$\left(0, \frac{a}{2}\right)$	$\left(\frac{a}{2}\right)$
3]	$r = a$	$(0, 0)$	a