If V is the tetrahedron bounded by planes $x=0, y=0, z=0$ and $x+y+z=4$ then express $\int_V f\,dv$ (where f is function of x,y,z) as a treple integral & hence evaluate $\int_V x\,dv$

Sol: Given: $x=0 \rightarrow$ Eq^n of yz-plane.

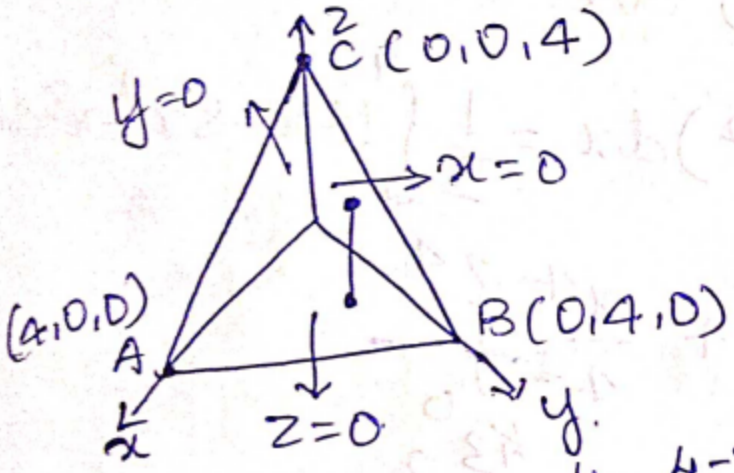
$y=0 \rightarrow$ Eq^n of xz-plane.

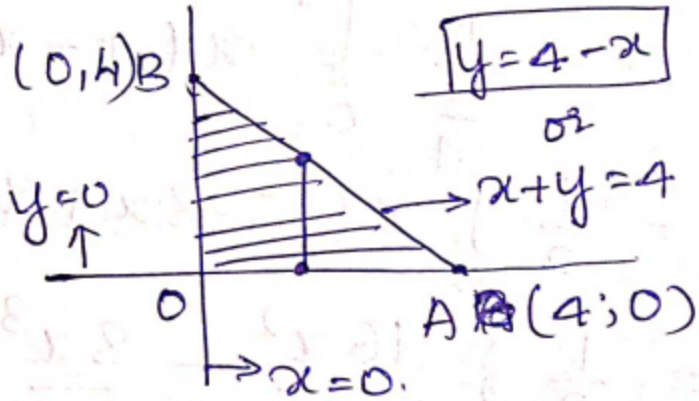
$z=0 \rightarrow$ Eq^n of xy-plane.

$x+y+z=4$ — (1) (Also $z = 4-x-y$).

To find: (i) $\int_V f\,dv$ (ii) $\int_V x\,dv$

Eq^n (1) is a plane that cuts x-axis, y-axis and z-axis at 4





$$\text{(i)}\quad \int_V f\,dv = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} f\,dz\,dy\,dx$$

$$\text{(ii)}\quad \int_V x\,dv = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} x\,dz\,dy\,dx.$$

$$= \int_0^4 \int_0^{4-x} x \left[\int_0^{4-x-y} dz \right] dy\,dx.$$

$$= \int_0^4 \int_0^{4-x} x \left[z \right]_0^{4-x-y} dy\,dx$$

$$= \int_0^4 \int_0^{4-x} x\,[4-x-y]\,dy\,dx$$

$$= \int_0^4 x \int_0^{4-x} [(4-x)-y]\,dy\,dx$$

$$= \int_0^4 x \left[(4-x)y - \frac{y^2}{2}\right]_0^{4-x} dx$$

$$= \int_0^4 x \left[(4-x)^2 - \frac{(4-x)^2}{2}\right] dx.$$

$$= \int_0^4 x\,(4-x)^2 \left[1 - \frac{1}{2}\right] dx$$

$$= \frac{1}{2} \int_0^4 x\,(4-x)^2\,dx.$$

$$= \frac{1}{2} \int_0^4 x(16 - 8x + x^2)\,dx = \frac{1}{2} \int_0^4 16x - 8x^2 + x^3\,dx$$

$$= \frac{1}{2} \left\{ \frac{16x^2}{2} - \frac{8x^3}{3} + \frac{x^4}{4} \right\}_0^4$$

$$= \frac{1}{2} \left\{ 8(16) - \frac{8}{3}(64) + \frac{4^{\cancel{4}3}}{\cancel{4}} \right\}$$

$$= \frac{16}{2} \left\{ 8 - \frac{8}{3}(4) + 4 \right\} = 4 \left\{ 8 - \frac{32}{3} + 4 \right\}$$

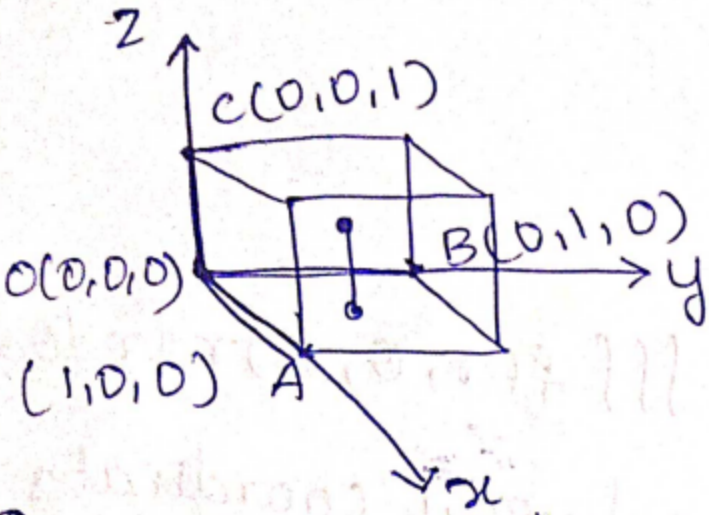
$$= 4(12 - 10.6667) = 4(2.6667)$$

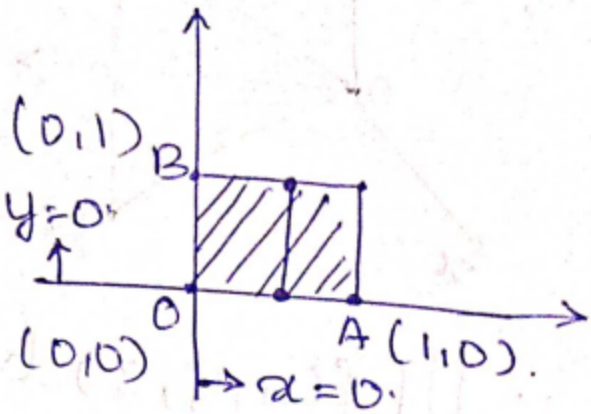
$$= 10.6667 \;//$$

6) Consider the cube of side 1. Then express the integral $\int_V f\,dV$ (where f is a function of x, y, z) as a triple integral and hence evaluate $\int_V (y^2 + z^2)\,dV$

Sol^n: Given : cube of side = 1

To find : (i) $\iiint_V f\,dv$ (ii) $\iiint_V (y^2+z^2)\,dv$





(i) $\iiint_V f\,dv = \int_0^1 \int_0^1 \int_0^1 f\,dz\,dy\,dx$

(ii) $\iiint_V (y^2+z^2)\,dv = \int_0^1 \int_0^1 \int_0^1 (y^2+z^2)\,dz\,dy\,dx$

$= \int_0^1 \int_0^1 \left[y^2 z + \frac{z^3}{3} \right]_0^1 dy\,dx = \int_0^1 \int_0^1 \left[y^2 + \frac{1}{3} \right] dy\,dx$

$= \int_0^1 dx \int_0^1 y^2 + \frac{1}{3}\,dy = [x]_0^1 \left[\frac{y^3}{3} + \frac{y}{3} \right]_0^1$

$= [1-0]\left[\frac{1}{3}+\frac{1}{3}\right] = \frac{2}{3}$ //.

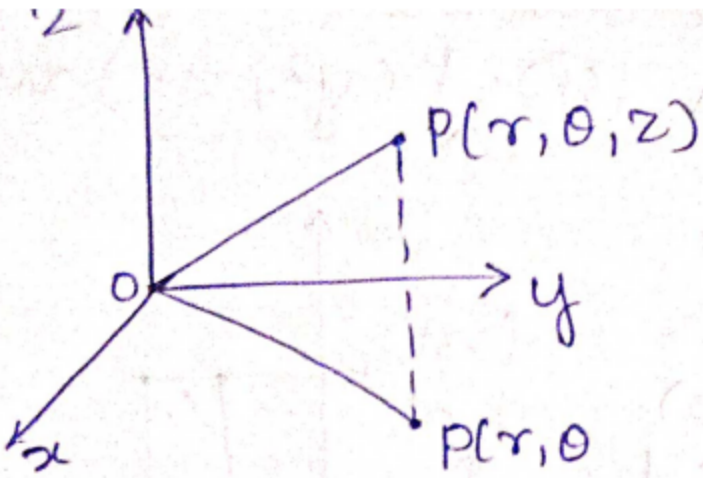
Triple Integrals in Cylendrical co-ordenates :

In cylendrical co-ordenates ,

$x = r\cos\theta$

$y = r\sin\theta$

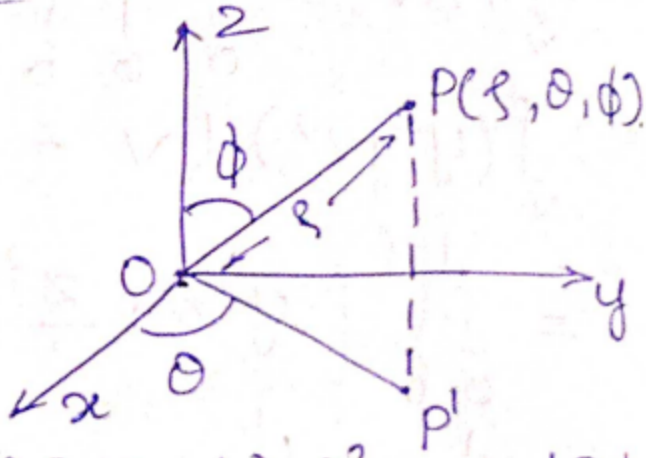
$z = z$



$$\therefore \iiint_V f(x,y,z)\,dV = \iiint f(r,\theta,z)\,r\,dr\,d\theta\,dz$$

Triple Integrals on spherical coordenates

$$x = \rho \operatorname{sen}\phi \cos\theta$$
$$y = \rho \operatorname{sen}\phi \operatorname{sen}\theta$$
$$z = \rho \cos\phi$$



$$\iiint_V f(x,y,z)\,dV = \iiint f(\rho,\theta,\phi)\,\rho^2 \operatorname{sen}\phi\,d\rho\,d\theta\,d\phi$$

where $\rho \geq 0$, $0 \leq \theta \leq 2\pi$,

$0 \leq \phi \leq \pi$