27/07/23 Monday

UNIT - III ...

Ordinary Differential Equation of First Order.

Linear Differential Equation (LDE) :-

An Equation of the type $\boxed{\frac{dy}{dx} + Py = Q}$, where P & Q are functions of x is called LDE.

Solution of LDE :-

① Integrating factor = $IF = e^{\int P dx}$

② Sol^n is :-

$$y(IF) = \int Q(IF)\,dx + C$$

NOTE: ① $e^{\log_e f(x)} = f(x)$

② $\frac{dx}{dy} + Px = Q$ is a LDE, where P & Q are functions of y.

Q 1] Solve:

① $x^2y' + 2xy = \cos^2 x$

6-7M

$\Rightarrow x^2\frac{dy}{dx} + (2x)y = \cos^2 x$

$\left(\frac{dy}{dx} + Py = Q\right)$

$\div$ by x^2

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{\cos^2 x}{x^2} \quad \text{—— ①}$$

Eq ① is a LDE with

$\boxed{P = \frac{2}{x}}$ $\boxed{Q = \frac{\cos^2 x}{x^2}}$

$$IF = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{\int \frac{2}{x} dx} = e^{2\int \frac{1}{x} dx} = e^{2\log x}$$

$$= e^{\log x^2}$$

$$\Rightarrow \boxed{IF = x^2}$$

sol^n is: $$y(IF) = \int Q(IF)\, dx + C$$

$$y x^2 = \int \frac{\cos^2 x}{x^2} (x^2)\, dx + C$$

$$y x^2 = \int \cos^2 x \, dx + C$$

$$y x^2 = \int \left(\frac{1 + \cos 2x}{2}\right) dx + C$$

$$\boxed{y x^2 = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C} \rightarrow \text{General sol}^n$$

(ii) $x y' \cancel{=} y + x^2 \sin x, \quad y(\pi) = 0$

$x y' - y = x^2 \sin x$

$$\Rightarrow x \frac{dy}{dx} \cancel{+} - y = x^2 \sin x$$

$\div$ by x

$$\frac{dy}{dx} + \left(\frac{-1}{x}\right) y = x \sin x \quad \text{—— (1)}$$

Eq^n (1) is a LDE with $\boxed{P = -1/x}$ & $\boxed{Q = x \sin x}$

$$IF = e^{\int P dx}$$

$$IF = e^{\int -1/x \, dx}$$

$$IF = e^{-\int 1/x \, dx}$$

$$IF = e^{-\log x}$$

$$e = e^{\log x^{-1}}$$

$$\boxed{IF = x^{-1} = \frac{1}{x}}$$

Sol^n is: $y(IF) = \int Q(IF)\, dx + C$

$y\left(\frac{1}{x}\right) = \int (\cancel{x} \sin x)\left(\frac{1}{\cancel{x}}\right) dx + C$

$y\left(\frac{1}{x}\right) = \int \sin x \, dx + C$

$\boxed{y\left(\frac{1}{x}\right) = -\cos x + C}$ —— ② (Particular

put $x = \pi$ & $y = 0$ in Eq^n ②

$0 = -\cos\pi + c$

$0 = 1 + c \Rightarrow \boxed{c = -1}$

put $c = -1$ in eq ②

$y\left(\frac{1}{x}\right) = -\cos x + 1$

$\boxed{y = -x\cos x - x}$

(iii) $e^{-y} \sec^2 y\, dy = dx + x\, dy.$

$\Rightarrow \cancel{\frac{dy}{dx} + Py = Q} \rightarrow \boxed{\frac{dx}{dy} + Px = Q}$ ✓

$\Rightarrow e^{-y} \sec^2 y\, dy - x\, dy = dx.$

$\Rightarrow (e^{-y} \sec^2 y - x)\, dy = dx.$

$\Rightarrow e^{-y} \sec^2 y - x = \frac{dx}{dy}$

$e^{-y} \sec^2 y = \frac{dx}{dy} + x$

$\boxed{\frac{dx}{dy} + 1(x) = e^{-y} \sec^2 y}$ —— ①

Eq^n (1) is a LDE with $\boxed{P=1}$ & $\boxed{Q=e^{-y}\sec^2 y}$

$IF = e^{\int P dy} = e^{\int 1 dy} = e^y$, $\boxed{IF = e^y}$

sol^n is: $x(IF) = \int Q(IF)\,dy + c$

$$x e^y = \int e^{-y}\sec^2 y\,(e^y)\,dy + c$$

$$xe^y = \int \sec^2 y\,dy + c$$

$$\boxed{xe^y = \tan y + c} \rightarrow \text{General sol}^n$$

Bernoulli's LDE :-

An Eq^n of the type $\boxed{\frac{dy}{dx} + Py = Qy^n}$

where n = constant & P & Q are fun^n of x.

Q2 Solve :

(iii) $y - \cos x \frac{dy}{dx} = y^2(1-\sin x)\cos x$, $y=2$ when $x=0$

$\Rightarrow$ $y - \cos x \frac{dy}{dx} = y^2(1-\sin x)\cos x$

$(-\cos x)\frac{dy}{dx} + y = (1-\sin x)(\cos x)\,y^2$

$\frac{dy}{dx} + (-\sec x)\,y = -(1-\sin x)\,y^2$

which is a Bernoulli's LDE $\boxed{\frac{dy}{dx} + Py = Qy^n}$

$\div$ by y^2

$\frac{1}{y^2}\frac{dy}{dx} + (-\sec x)\frac{1}{y} = -(1-\sin x)$

put $\frac{1}{y} = t$

$$\frac{-1}{y^2}\,dy = dt$$

$$\frac{-dt}{dx} + (-\sec x)t = -(1-\sin x)$$

$$\Rightarrow \frac{dt}{dx} + (\sec x)t = (1-\sin x) \quad \text{--- (1)}$$

28/02/2 Tuesday

Eqⁿ (1) is a LDE with $\boxed{P = \sec x}$ & $\boxed{Q = (1-\sin x)}$

$\cancel{IF = \int Q(IF)\,d}$

$$IF = e^{\int P dx} = e^{\int \sec x\,dx} = e^{\log_e(\sec x + \tan x)}$$

$$\boxed{IF = \sec x + \tan x}$$

solⁿ : $t\,(IF) = \int Q\,\cancel{Q}(IF)\,dx + C$

$$\frac{1}{y}(\sec x + \tan x) = \int (1-\sin x)(\sec x + \tan x)\,dx + C$$

$$\frac{1}{y}(\sec x + \tan x) = \int (1-\sin x)\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)dx + C$$

$$= \int (1-\sin x)\left(\frac{1+\sin x}{\cos x}\right)dx + C$$

$$= \int \frac{1-\sin^2 x}{\cos^2 x}\,dx + C$$

$$= \int \frac{\cos^2 x}{\cos^2 x}\,dx + C$$

$$\frac{1}{y}(\sec x + \tan x) = \sin x\,\cancel{dx} + C \quad \text{--- (2)}$$

put $x = 0$ & $y = 2$ in Eq (2)

$$\boxed{\frac{1}{2}(1) = c}$$

Put c in Eq ②

$$\boxed{\frac{1}{y}(\sec x + \tan x) = \sin x + \frac{1}{2}}$$

28/02/23 Tuesday

(i) $\left(xy^2 - e^{1/x^3}\right) dx - x^2 y\, dy = 0$

$\Rightarrow \left(xy^2 - e^{x^{-3}}\right) dx = x^2 y\, dy$

$= xy^2 - e^{x^{-3}} = x^2 y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{xy^2 - e^{x^{-3}}}{x^2 y}$

$\frac{dy}{dx} = \frac{y}{x} - \frac{e^{x^{-3}}}{x^2} \cdot \frac{1}{y}$

$\boxed{\frac{dy}{dx} + Py = Q^n}$

$$\boxed{\frac{dy}{dx} + \left(\frac{-1}{x}\right) y = \frac{-e^{x^{-3}}}{x^2} \cdot y^{-1}} \quad \text{—— ①}$$

Eqⁿ ① is Bernoulli's LDE

multiply Eq ① by 'y'

$y\frac{dy}{dx} + \left(\frac{-1}{x}\right) y^2 = \frac{-e^{x^{-3}}}{x^2}$

put $y^2 = t$

$\Rightarrow 2y\,dy = dt$

$y\,dy = \frac{dt}{2}$

$\frac{1}{2}\frac{dt}{dx} + \left(\frac{-1}{x}\right) t = \frac{-e^{x^{-3}}}{x^2}$

$$\frac{dt}{dx} + \left(\frac{-2}{x}\right)t = \frac{-2e^{x^{-3}}}{x^2} \quad \text{——— (2)}$$

Eqⁿ (2) is a LDE with

$$P = \frac{-2}{x} \quad \& \quad Q = \frac{-2e^{x^{-3}}}{x^2}$$

$$IF = e^{\int P dx} = e^{\int -2/x \, dx} = e^{-2\int \frac{1}{x} dx} = e^{-2\log x}$$

$$IF = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

solⁿ.

$$\cancel{t}\,t(IF) = \int Q(IF)\,dx + C$$

$$y^2\left(\frac{1}{x^2}\right) = \int \frac{-2e^{x^{-3}}}{x^2}\left(\frac{1}{x^2}\right)dx + C$$

$$\frac{y^2}{x^2} = -2\int \frac{e^{x^{-3}}}{x^4}\,dx + C$$

$$\frac{y^2}{x^2} = -2\int e^{x^{-3}}\,x^{-4}\,dx + C$$

$$\boxed{\frac{y^2}{x^2} = -2\left(\frac{e^{x^{-3}}}{-3}\right) + C}$$

$$\left(\int e^{f(x)} f'(x)\,dx = e^{f(x)}\right)$$

(ii) $$\frac{dy}{dx} + \frac{y\ln y}{x} = \frac{y(\ln y)^2}{x^2}$$

solⁿ.

$\Rightarrow$ ÷ by $y(\ln y)^2$

$$\frac{1}{y(\ln y)^2}\left(\frac{dy}{dx}\right) + \frac{1}{x}\left(\frac{1}{\ln y}\right) = \frac{1}{x^2}$$

$$\frac{dy}{dx} + Py = Q \quad (x)$$

put $\frac{1}{\ln y} = t$

$$(\ln y)^{-1} = t \qquad \boxed{x^n = n x^{n-1}}$$

$$(-1)(\ln y)^{-2} \cdot \frac{1}{y}\, dy = dt$$

$$\boxed{\frac{1}{y(\ln y)^2}\, dy = -dt}$$

$$\frac{-dt}{dx} + \left(\frac{1}{x}\right) t = \frac{1}{x^2}$$

$$\Rightarrow \boxed{\frac{dt}{dx} + \left(\frac{-1}{x}\right) t = \frac{-1}{x^2}} \text{ —— } (2)$$

Eqn (2) is LDE with $P = \frac{-1}{x}$ & $Q = \frac{-1}{x^2}$

$$IF = e^{\int P\,dx} = e^{\int -1/x\, dx} = e^{-\log x} = e^{\log x^{-1}}$$

$$IF = e^{+\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$$\boxed{IF = \frac{1}{x}}$$

Soln is:

$$t(IF) = \int Q(IF)\, dx + C$$

$$\frac{1}{\ln y}\left(\frac{1}{x}\right) = \int \frac{-1}{x^2}\left(\frac{1}{x}\right) dx + C$$

$$\frac{1}{x \ln y} = -\int \frac{1}{x^3}\, dx + C$$

$$\frac{1}{x \ln y} = -\int x^{-3}\, dx + C$$

$$\frac{1}{x \ln y} = -\frac{x^{-2}}{-2} + C$$

Exact Differential Equation :-

the condition for a D.E $Mdx + Ndy = 0$ to be exact $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ (∂ = doe → Partial differentia

$N = x + y$

$$\frac{\partial N}{\partial y} = 0 + 1 \qquad \frac{\partial N}{\partial x} = 1 + 0$$

Q→ $M = x^2 y^4$

$$\frac{\partial M}{\partial x} = 2x\,y^4 \qquad \frac{\partial M}{\partial y} = x^2(4y^3)$$

Q→ $M = e^{xy}$

$$\frac{\partial M}{\partial x} = e^{xy} \cdot \frac{\partial (xy)}{\partial x}$$

$$= e^{xy} \cdot y$$

Q→ $M = e^{\frac{x}{y}}$

$$\frac{\partial M}{\partial x} = e^{x/y} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= \frac{1}{y} e^{x/y}$$

$$\frac{\partial N}{\partial y} = e^{x/y} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$= e^{x/y} \frac{-x}{y^2}$$

Solution of Exact Differential Equation :-

If $Mdx + Ndy = 0$ is exact then its solⁿ is

$$\boxed{\underset{y\text{- constant}}{\int M\,dx} + \underset{\text{terms free from } x}{\int N\,dy} = C}$$

to be

differentiation

Q3 (i) $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)\,dy = 0$

$\Rightarrow$ Here, $M = 2xy\cos x^2 - 2xy + 1$; $N = \sin x^2 - x^2$

$\boxed{\dfrac{\partial M}{\partial y} = 2x\cos x^2 - 2x}$; $\boxed{\dfrac{\partial N}{\partial x} = \cos(x^2)[2x] - 2x}$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = $ Eqⁿ (1) is exact.

Solⁿ is: $\underset{y\text{-constant}}{\int (2xy\cos x^2 - 2xy + 1)\,dx} + \underset{\text{Term free from } x}{\int (\sin x^2 - x^2)\,dy} = C$

$\int 2xy\cos x^2\,dx$

$y\int 2x\cos(x^2)\,dx$

$y\sin(x^2)$

$y\sin(x^2) - x^2y + x + 0 = C$

$\Rightarrow \boxed{y\sin(x^2) - x^2y + x = C}$

1/03/23 Wednesday

(ii) $\left(1+e^{\frac{x}{y}}\right)dx + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\,dy = 0$ —— (1)

$\Rightarrow M = 1+e^{\frac{x}{y}}$; $N = \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}$

$\frac{\partial M}{\partial y} = 0 + e^{\frac{x}{y}}\cdot\left(\frac{-x}{y^2}\right)$; $\boxed{\frac{\partial N}{\partial x} = \left(1-\frac{x}{y}\right)e^{x/y}\cdot\left(\frac{1}{y}\right) + e^{x/y}\left(0-\frac{1}{y}\right)}$

$\boxed{\frac{\partial M}{\partial y} = \frac{-x\,e^{x/y}}{y^2}}$; $\boxed{\frac{\partial N}{\partial x} = \cancel{\frac{1}{y}e^{x/y}} - \frac{x e^{x/y}}{y^2} - \cancel{\frac{1}{y}e^{x/y}} = -\frac{x}{y^2}e^{x/y}}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$

$\Rightarrow$ Eq^n (1) is exact

Sol^n: $\underbrace{\int\left(1+e^{x/y}\right)dx}_{y\text{-constant}} + \underbrace{\int\cancel{\left(1-\frac{x}{y}\right)e^{x/y}}^{\,0}\,dy}_{\text{Term free from } x} = C$

$x + ye^{x/y} = C$

Equations Reducible to exact :-

Some times a differential Equation which is not exact can be made exact by multiplying it a suitable factor called integrating factor (IF).

Cases: If $Mdx + Ndy = 0$
is not exact but
(1) is homogenous then $IF = \frac{1}{Mx+Ny}$

(2) is of the form $y f(xy)\,dx + x g(xy)\,dy = 0$ then.
$$IF = \frac{1}{Mx - Ny}$$

(3) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ then $IF = e^{\int f(x)dx}$

(4) $\frac{1}{-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$ then $IF = e^{\int g(y)dy}$

Q4 (1) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

7M $\Rightarrow (x^2y - 2xy^2)\,dx + (-x^3 + 3x^2y)\,dy = 0$ ——— (1)

Here, $\boxed{M = x^2y - 2xy^2}$; $\boxed{N = -x^3 + 3x^2y}$

$\boxed{\frac{\partial M}{\partial y} = x^2 - 4xy}$; $\boxed{\frac{\partial N}{\partial x} = -3x^2 + 6xy}$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ Eq (1) is not exact but homogenous

$$IF = \frac{1}{Mx + Ny} = \frac{1}{\cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2} = \frac{1}{x^2y^2}$$

(Eq (1) $\times$ IF) $\rightarrow$ divide each term with x^2y^2

$$\left(\frac{x^2y}{x^2y^2} - \frac{2xy^2}{x^2y^2}\right)dx + \left(\frac{-x^3}{x^2y^2} + \frac{3x^2y}{x^2y^2}\right)dy = 0$$

$$\Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{-x}{y^2} + \frac{3}{y}\right)dy = 0 \quad \text{——— (2)}$$

Eq ② is exact, solⁿ is

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\left(\frac{-x}{y^2}+\frac{3}{y}\right)dy = C$$

(y-constant) (Terms free from x)

$$\left(\frac{x}{y}-2\log x\right)+3\log y = C$$

(ii) $(x^2y^2+xy+1)y\,dx+(x^2y^2-xy+1)x\,dy=0$

$\Rightarrow (x^2y^3+xy^2+y)dx+(x^3y^2-x^2y+x)dy=0$ — ②

Here $\boxed{M = x^2y^3+xy^2+y}$; $\boxed{N = x^3y^2-x^2y+x}$

$\boxed{\frac{\partial M}{\partial y}=3x^2y^2+2xy+1}$; $\boxed{\frac{\partial N}{\partial x}=3x^2y^2-2xy+1}$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \Rightarrow$ Eqⁿ ① is not exact but is of the type $y\,f(xy)\,dx+x\,g(xy)\,dy=0$

Multiply M×(x) & N×(y)

$$IF=\frac{1}{Mx-Ny}=\frac{1}{\cancel{x^3y^3}+x^2y^2+yx-\cancel{x^3y^3}+x^2y^2+xy}=\frac{1}{2x^2y^2}$$

$\boxed{Eq^n ② \times IF} = \frac{1}{2}\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\frac{1}{2}\left(x-\frac{1}{y}+\frac{1}{xy^2}\right)dy=$

Eqⁿ ③ is exact, solⁿ is,

$$\int\frac{1}{2}\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\int\frac{1}{2}\left(\cancel{x}-\frac{1}{y}+\cancel{\frac{1}{xy^2}}\right)dy = C$$

y-constant (Terms free from x)

$$\boxed{\frac{1}{2}\left(xy + \log x - \frac{1}{yx}\right) - \frac{1}{2}\log y = c}$$

(v) $y\,dx - x\,dy + \ln x\,dx = 0$

$\Rightarrow (y + \ln x)\,dx + (-x)\,dy = 0$ —— (1)

Here, $M = y + \ln x$; $N = -x$.

$$\boxed{\frac{\partial M}{\partial y} = 1 + 0 = 1} \; ; \; \boxed{\frac{\partial N}{\partial x} = -1}$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Eqn (1) is not exact but

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-x}\left(1-(-1)\right) = \frac{-2}{x} = f(x)$$

$$IF = e^{\int f(x)dx} = e^{\int -2/x\,dx} = e^{\int -2/x\,dx} = e^{-2\log x} = e^{\log x^{-2}} = e^{-2} = \frac{1}{x^2}$$

$$[Eq(1) \times IF] = \left(\frac{y}{x^2} + \frac{\ln x}{x^2}\right)dx + \left(\frac{-1}{x}\right)dy = 0 \quad \text{—— (2)}$$

Eqn (2) is exact, sol^n is

$$\underbrace{\int \left(\frac{y}{x^2} + \frac{\ln x}{x^2}\right)dx}_{(y\text{-constant})} + \underbrace{\int \cancel{\left(\frac{-1}{x}\right)}^{0} dy}_{\text{Terms free from } x} = C$$

$$\boxed{\frac{-y}{x} + \frac{-1}{x}\left[\ln x+1\right] = C}$$

$$\int \frac{\ln x}{x^2}\,dx = \int \frac{\ln x}{x}\cdot\frac{1}{x}\cdot dx$$

put $\ln x = t \Rightarrow x = e^t \rightarrow e^{-t} = \frac{1}{x}$

$$\Rightarrow \frac{1}{x}\,dx = dt$$

$$\int \frac{t}{e^t}\,dt = \int \overset{v}{e^{-t}}\,\overset{u}{t}\,.\,dt$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$=(t)\left(-e^{-t}\right) - (1)\left(e^{-t}\right)$$

$$=(\ln x)\left(\frac{-1}{x}\right) - \left(\frac{1}{x}\right) = \frac{-1}{x}\left[\ln x+1\right]$$

$$\boxed{\frac{-y}{x} - \frac{1}{x}\left[\ln x+1\right] = C}$$

[Eq

(vi) $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$

$\Rightarrow$ Here $\boxed{M = xy^3+y}$; $\boxed{N = 2(x^2y^2+x+y^4)}$

$\boxed{\frac{\partial M}{\partial y} = 3y^2x+1}$; $\boxed{\frac{\partial N}{\partial x} = 2(2xy^2+1) = 4xy^2+2}$

[IF

M

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Eq}^n \ (1) \text{ is not } \text{~~Equal~~ Exact.}$$

$$\frac{1}{-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-M}\left(3xy^2+1-4xy^2-2\right) = \frac{(-xy^2-1)}{-(xy^3+y)}$$

$$= \frac{-1}{-y}\frac{(xy^2+1)}{(xy^2+1)} = \frac{1}{y} = g(y)$$

$$IF = e^{\int g(y)\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

$$[IF \times \text{Eq}^n (1)] = (xy^4+y^2)\,dx + 2(x^2y^3+xy+y^5)\,dy = 0 \quad \text{--- } (2)$$

Eq^n (2) is Exact, sol^n is

$$\underbrace{\int (xy^4+y^2)\,dx}_{y\text{-constant}} + \underbrace{2\int (\cancel{x^2y^3} + \cancel{xy} + y^5)\,dy}_{\text{Term free from } x} = C$$

$$\boxed{\left(\frac{x^2}{2}y^4 + xy^2\right) + 2\left[\frac{y^6}{6}\right] = C}$$

(iii) $(x^2+y^2+1)\,dx - 2y\,dy = 0$

$$\left.\begin{aligned}
&\Rightarrow \int xe^x\,dx = e^x(x-1)\\
&\Rightarrow \int x^2e^x\,dx = e^x(x^2-2x+2)\\
&\Rightarrow \int xe^{-x}\,dx = -e^{-x}(x+1)\\
&\Rightarrow \int x^2e^{-x}\,dx = -e^{-x}(x^2+2x+2)
\end{aligned}\right\} \underline{\text{Formula}}$$

$\Rightarrow M = x^2y^2+1$; $N = -2y$

$\frac{\partial M}{\partial y} = 2y$; $\frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Eq^n 1 is not exact, but

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-2y}(2y-0) = -1 = f(x)$

$IF = e^{\int f(x)dx} = e^{\int -1 dx} = e^{-x}$

$(Eq^n\ 1 \times IF) = (x^2e^{-x} + y^2e^{-x} + e^{-x})dx - 2ye^{-x}dy = 0 \hookrightarrow 2$

Eq^n 2 is exact, sol^n is :

$\underset{y\text{-constant}}{\int}(x^2e^{-x} + y^2e^{-x} + e^{-x})dx - \underset{\text{Terms free from } x}{\int 2ye^{-x}dy} = C$

$\boxed{-e^{-x}(x^2+2x+2) - e^{-x}y^2 - e^{-x} = C}$

(iv) $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

$\Rightarrow M = y^4+2y$; $N = xy^3+2y^4-\ldots$

$\boxed{\frac{\partial M}{\partial y} = 4y^3+2}$;

[Eq^n 1

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Eq}^n \ (1) \text{ is not Exact, but}$$

$$\frac{1}{-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-(y^4+2y)}\left(4y^3+2-y^3+4\right)$$

$$= \frac{3y^3+6}{-y(y^3+2)} = \frac{3(y^3+2)}{-y(y^3+2)} = \frac{-3}{y} = g(y)$$

$$IF = e^{\int g(y)dy} = e^{\int -3/y \, dy} = e^{-3\log y} = e^{\log y^{-3}}$$

$$\boxed{IF = \frac{1}{y^3}}$$

$$\left[\text{Eq}^n \ (1) \times IF\right] = \left(y + \frac{2}{y^2}\right) dx + \underbrace{\left(x + 2y - \frac{4x}{y^3}\right) dy = 0}_{(2)}$$

Eq^n (2) is exact, sol^n is

$$\underbrace{\int \left(y + \frac{2}{y^2}\right) dx}_{y\text{-constant}} + \underbrace{\int \left(\cancel{x}^{\,0} + 2y - \cancel{\frac{4x}{y^3}}^{\,0}\right) dy}_{\text{Terms free from } x} = 0$$

$$\boxed{\left(y + \frac{2}{y^2}\right) x + y^2 = C}$$

2/03/23 Thursday

Numerical Method of solving ordinary Differential Equations.

Runge - Kutta Method (RK Method) :-

1] Runge - Kutta Method

Given :- $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$

Calculate : $K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = hf\left(x_0 + h, y_0 + K_3\right)$

$$\therefore \boxed{K = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}}$$

$$\Rightarrow \boxed{y_1 = y_0 + K}$$

Q7] (i) $\frac{dy}{dx} = x^2 + y^2$; $y(1) = 1.5$ compute y in 2 steps.

(7M)

[Eg

$\Rightarrow$ Given: $f(x, y) = x^2 + y^2$; $x_0 = 1$; $y_0 =$

To find: $y(1.2) = ?$ in two steps

$\Rightarrow \boxed{h = 0.1}$

$y(1)$

$y(1.2$

To find: $y(1.1)$

$k_1 = hf(x_0, y_0) = 0.1 f(1, 1.5)$

$= 0.1[1^2 + 1.5^2]$

$$\boxed{K_1 = 0.3250}$$

$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.1 f\left(1 + \frac{0.1}{2},\ 1.5 + \frac{0.325}{2}\right)$$

$$= 0.1 f\left(1.05,\ 1.6625\right)$$

$$= 0.1 \left[(1.05)^2 + (1.6625)^2\right]$$

$$\boxed{K_2 = 0.3866}$$

$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 f\left(1.05,\ 1.5 + \frac{0.3866}{2}\right)$$

$$= 0.1 f\left(1.05,\ 1.6933\right)$$

$$= 0.1 \left[(1.05)^2 + (1.6933)^2\right]$$

$$\boxed{K_3 = 0.3970}$$

$$K_4 = hf\left(x_0 + h,\ y_0 + k_3\right)$$

$$= 0.1 f\left(1 + 0.1,\ 1.5 + 0.3970\right)$$

$$= 0.1 f\left(1.1,\ 1.8970\right)$$

$$= 0.1\left(1.1^2 + 1.8970^2\right)$$

$$\boxed{K_4 = 0.4809}$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.3250 + 2(0.3866) + 2(0.3970) + 0.4809}{6}$$

$$\boxed{K = 0.3955}$$

$y_1 = y_0 + k$
$= 1.5 + 0.3955$
$\boxed{y_1 = 1.8955}$

$\Rightarrow$ <u>To find :-</u> $y(1.2)$

Now $x_1 = 1.1$ & $y_1 = 1.8955$.

$k_1 = hf(x_1, y_1) = 0.1 f(1.1, 1.8955)$
$= 0.1(1.1^2 + 1.8955^2)$
$\boxed{k_1 = 0.4803}$

$k_2 = hf\left(x_p + \frac{h}{2}, y_p + \frac{k_1}{2}\right)$
$= 0.1 f\left(1.1 + \frac{0.1}{2}, 1.8955 + \frac{0.4803}{2}\right)$
$= 0.1 f(1.15, 2.1357)$
$= 0.1[1.15^2 + 2.1357^2]$
$\boxed{k_2 = 0.5884}$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$
$= 0.1 f\left(1.1 + \frac{0.1}{2}, 1.8955 + \frac{0.5884}{2}\right)$
$= 0.1 f(1.15^2 + 2.1897^2)$
$\boxed{k_3 = 0.6117}$

$k_4 = hf(x_1 + h, y_1 + k_3)$
$= 0.1 f(1.1 + 0.1, 1.8955 + 0.6117)$

$= 0.1\,(1.2^2 + 2.5072^2)$

$\boxed{k_4 = 0.7726}$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.4803 + 2(0.5884) + 2(0.6117) + 0.7726}{6}$$

$\boxed{k = 0.6089}$

$y_2 = y_1 + k$

$= 1.8955 + 0.6089$

$\boxed{y_2 = 2.5044}$

(iii) $y' = x^2 - y$; $y(0) = 1$, find $y(0.1)$ & $y(0.2)$ and compare with exact value.

$\Rightarrow$ Given :- $f(x,y) = x^2 - y$; $x_0 = 0$; $y_0 = 1$

$h = 0.1$

To find :- $y(0.1)$ & $y(0.2)$

$\Rightarrow \boxed{h = 0.1}$

$y(0) = 1$

$y(0.1) = ?$

$y(0.2) = ?$

$\rightarrow$ To find : $y(0.1)$

$x_0 = 0$, $y_0 = 1$

$k_1 = hf(x_0, y_0) = 0.1 f(0, 1) = \cancel{0.1(0^2 + 1^2)}$

$= 0.1(0^2 - 1) = \cancel{0.1}$

$k_1 = -0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\, f\left(0 + \frac{0.1}{2},\ 1 - \frac{0.1}{2}\right)$$

$= 0.1f\left(0.05, 1-0.05\right)$
$= 0.1\left(0.05^2 - 0.95\right)$
$k_2 = -0.0948$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$
$= 0.1f\left(0.05, 1 - \frac{0.0948}{2}\right)$
$= 0.1f\left(0.05, 0.9526\right)$
$= 0.1\left(0.05^2 - 0.9526\right)$

$k_3 = -0.0950$

$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$
$= 0.1f\left(0+0.1, 1-0.0950\right)$
$= 0.1f\left(0.1, 0.9050\right)$
$= 0.1\left(0.1^2 - 0.9050\right)$
$k_4 = -0.0895$

$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$
$= \frac{-0.1 - 2(0.0948) - (0.0950)2 - 0.0895}{6}$
$= -0.0949$

$y_1 = y_0 + k$
$= 1 - 0.0949$
$\boxed{y_1 = 0.9052}$

$\rightarrow$ To find :- $y(1.2)$

$x_1 = 0.1$, $y_1 = 0.9052$, $h = 0.1$.

$k_1 = hf(x, y_1) =$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$=$

$=$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$=$

$=$

$=$

.0895

$k_4 = hf(x_1 + h, y_1 + k_3)$

$=$

$=$

$=$

$=$

$k = \frac{k_1 + 2k_2 + 2k_3 + 2k_4}{6}$

$=$

=

$y_2 = y_1 + k$

=

$y_2 = 0.8216.$

Q8 :-

6/03/23 Monday II

Euler's Method :-

Given :- $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$ with step size = h.

An approximate solution at x_n is

$$\boxed{y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})}$$

5 (ii) with step size 0.1 to estimate $y(0.5)$, given

$\boxed{y' = y + xy}$; $\boxed{y(0) = 1}$

⇒ Given :- $x_0 = 0$, $y_0 = 1$,

$f(x,y) = y + xy$, $h = 0.1$

To find :- $y(0.5)$ with step size 0.1

⇒ $x_1 = 0.1$, $y_1 = ?$

$x_2 = 0.2$; $y_2 = ?$

$x_3 = 0.3$; $y_3 = ?$

$x_4 = 0.4$; $y_4 = ?$

$x_5 = 0.5$, $y_5 = ?$

Euler's formula :-

$$\boxed{y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})}$$

$$m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{dy}{dx} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{h}$$

$$\Rightarrow \frac{dy}{dx} = f(x,y)$$

$$\Rightarrow f(x,y) = \frac{y_2 - y_1}{h} \Rightarrow y_2 - y_1 = h f(x,y)$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

$$\Rightarrow y_1 = y_0 + hf(x_0, y_0) = 1 + (0.1) f(0,1)$$

$$\boxed{y_1 = 1 + 0.1[1+(0)(1)] = 1.1}$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = 1.1 + (0.1) f(0.1, 1.1)$$

$$y_2 = 1.1 + (0.1)[1.1 + (0.1)(1.1)]$$

$$\boxed{y_2 = 1.2210}$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$y_3 = (1.221) + (0.1) f(0.2, 1.221)$$

$$y_3 = (1.221) + (0.1)[(1.221) + (0.2)(1.221)]$$

$$\boxed{y_3 = 1.3675}$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$y_4 = (1.3675) + (0.1) f(0.3, 1.3675)$$

$$y_4 = (1.3675) + (0.1)[(1.3675) + (0.3)(1.3675)]$$

$$\boxed{y_4 = 1.5453}$$

$$y_5 = y_4 + hf(x_4, y_4)$$

$$y_5 = (1.5453) + (0.1) f(0.4, 1.5453)$$

$$y_5 = (1.5453) + (0.1)((1.5453) + (0.4)(1.5453))$$

$$\boxed{y_5 = 1.7616}$$

$$\boxed{\therefore y(0.5) = 1.7616}$$

(i) step size 0.2 to estimate $y(1)$, given $y' = 1 - xy$; $y(0) = 0$

$\Rightarrow$

$x_1 = 0.2$	$y_1 = 0.2$
$x_2 = 0.4$	$y_2 = 0.392$
$x_3 = 0.6$	$y_3 = 0.5606$
$x_4 = 0.8$	$y_4 = 0.6934$
$x_5 = 1$	$y_5 = 0.7824$

III) Euler's Modified Method :-

Given :- $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$

Predictor (P) :-

$$y_n^{(P)} = y_{n-1} + h f(x_{n-1}, y_{n-1})$$

corrector : (C) :

$$y_n^{(C)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(P)}) \right.$$

Q (6)(11) step size 0.1 to find $y(0.2)$ given $\frac{dy}{dx} = y - \frac{2x}{y}$;

(7M) $y(0) = 1$;

$\Rightarrow$ Given :- $x_0 = 0$; $y_0 = 1$; $h = 0.1$

$f(x, y) = y - \frac{2x}{y}$

To find : $y(0.2)$ with step size 0.1

$\boxed{x_1 = 0.1}$, $\boxed{y_1 = ?}$

$\boxed{x_2 = 0.2}$, $\boxed{y_2 = ?}$

Euler's Modified formula :

for $x_1 = 0.1$:-

Predictor :- (P) :-

$y_1^{(p)} = y_0 + h f(x_0, y_0)$

$y_1^{(p)} = 1 + (0.1) \boxed{f(0,1)}$

$y_1^{(p)} = 1 + (0.1) \left[1 - \frac{2(0)}{1}\right]$

$\boxed{y_1^{(p)} = 1.1}$

corrector :

$y_1^{(c)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(p)})\right]$

$y_1^{(c_1)} = 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.1)\right]$

$y_1^{(c_1)} = 1 + (0.05)\left[\left(1 - \frac{2(0)}{1}\right) + \left(1.1 - \frac{2(0.1)}{1.1}\right)\right]$

$\boxed{y_1^{(c_1)} = 1.0959}$

$y_1^{(c_2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(c_1)})\right]$

$y_1^{(c_2)} = 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.0959)\right]$

$y_1^{(c_2)} = 1 + (0.05)\left[\left(1 - \frac{2(0)}{1}\right) + \left(1.0959 - \frac{2(0.1)}{1.0959}\right)\right]$

$\boxed{y_1^{(c_2)} = 1.0957}$

$\therefore \boxed{x_1 = 0.1} \Rightarrow \boxed{y_1 = 1.0957}$

For $x_2 = 0.2$:-

Predictor :-

$y_2^{(P)} = y_1 + hf(x_1, y_1)$

$y_2^{(P)} = (1.0957) + (0.1) f(0.1, 1.0957)$

$y_2^{(P)} = (1.0957) + (0.1) \left[1.0957 - \frac{2(0.1)}{(1.0957)} \right]$

$y_2^{(P)} = 1.1870$

corrector :-

$y_2^{(C_1)} = y_1 + \frac{h}{2} \left[f(x_1, y_1) + f(x_2, y_2^{(P)}) \right]$

$y_2^{(C)} = (1.0957) + \frac{0.1}{2} \left[f(0.1, 1.0957) + f(0.2, 1.1870) \right]$

$\boxed{y_2^{(C_1)} = 1.1839}$

$\boxed{y_2^{(C_2)} = 1.1837}$

$\therefore x_2 = 0.2 \quad , \quad y_2 = 1.1837$