$A = \frac{3\pi}{2}$ sq. units

Unit - II . 3. Triple Integrals

If the region V is bounded by the surfaces $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$, $z = z_1$ & $z = z_2$ then

$$\iiint_V f(x,y,z)\,dv = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$$

1) Evaluate the Triple Integrals.

i) $\int_0^3\int_0^1\int_0^{\sqrt{1-x^2}} z e^y \,dz\,dx\,dy$.

Sol: $I = \int_0^3\int_0^1 e^y\left[\int_0^{\sqrt{1-x^2}} z\,dz\right]dx\,dy$

$I = \frac{1}{2}\int_0^3\int_0^1 e^y\left[z^2\right]_0^{\sqrt{1-x^2}} dx\,dy$

$I = \frac{1}{2}\int_0^3\int_0^1 e^y(1-x^2)\,dx\,dy$

$I = \frac{1}{2}\int_0^3 e^y\,dy\int_0^1(1-x^2)\,dx$

$= \frac{1}{2}\left[e^y\right]_0^3\left[x - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\left[e^3 - e^0\right]\left[1-\frac{1}{3}\right]$

$I = \frac{1}{2}\left[e^3-1\right]\left[2/3\right] = \frac{1}{3}(e^3-1)$

Note : $\int x e^{-x^2}dx = \frac{e^{-x^2}}{-2}$

Q. $\int_0^1 \int_0^z \int_0^y z e^{-y^2} \, dx \, dy \, dz$

Sol: $I = \int_0^1 \int_0^z z e^{-y^2} \left[\int_0^y dx \right] dy \, dz$.

$I = \int_0^1 \int_0^z z e^{-y^2} [x]_0^y \, dy \, dz$

$I = \int_0^1 \int_0^z z e^{-y^2} [y] \, dy \, dz$

$I = \int_0^1 z \left[\int_0^z y e^{-y^2} dy \right]_0^z dz = \frac{1}{-2} \int_0^1 z \left[e^{-y^2} \right]_0^z dz$

$I = \frac{1}{-2} \int_0^1 z \left[e^{-z^2} - e^{-0} \right] dz = \frac{1}{-2} \int_0^1 z \left[e^{-z^2} - 1 \right] dz$

$I = \frac{1}{-2} \left\{ \int_0^1 z e^{-z^2} dz - \int_0^1 z \, dz \right\}$.

$I = \frac{1}{-2} \left\{ \frac{1}{-2} \left[e^{-z^2} \right]_0^1 - \frac{1}{2} \left[z^2 \right]_0^1 \right\}$.

$I = \frac{1}{4} \left\{ e^{-z^2} + z^2 \right\} = \frac{1}{4} \left\{ (e^{-1} + 1) - (e^0) \right\}$.

$I = \frac{1}{4} \left\{ \frac{1}{e} + 1 - 1 \right\} = \frac{1}{4e}$ //

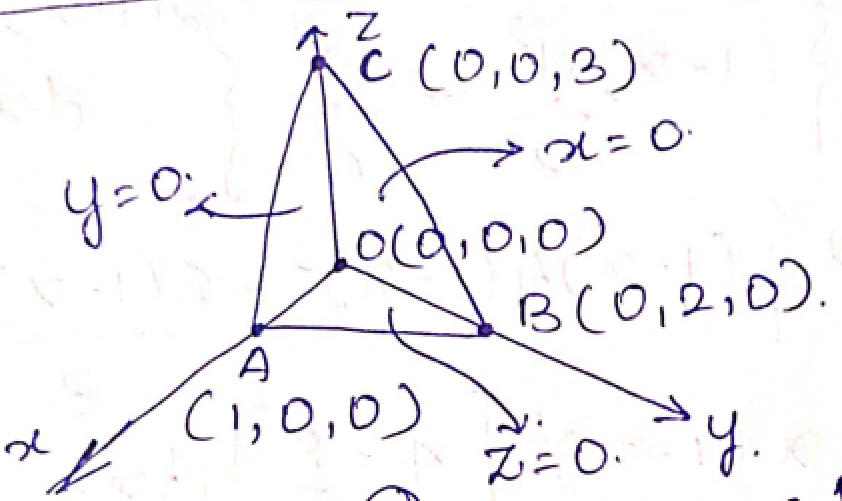
<u>NOTE</u>: Intercept form of a plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

999C [GM] $\iiint_E xy\,dV$, E is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ & $(0,0,3)$

Sol: Eq of planes passing through $(1,0,0)$, $(0,2,0)$ & $(0,0,3)$ is

$$\boxed{\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1} \quad \text{— (1)}$$

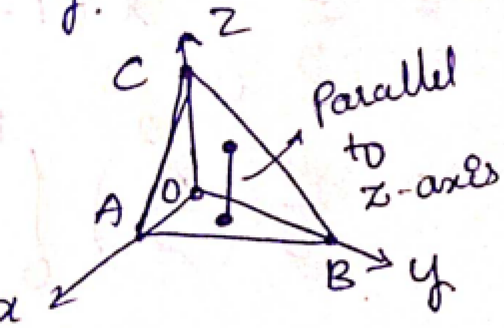


Eq of ABC is eq (1)

Eq of OAB is $z=0$

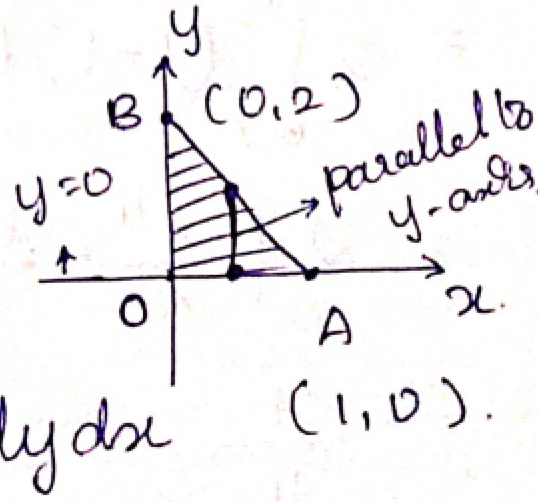
Eq of OAC is $y=0$

Eq of OCB is $x=0$.



From (1), $\frac{z}{3}=1-x-\frac{y}{2}$ $\Rightarrow$ $\boxed{z=3\left(1-x-\frac{y}{2}\right)}$

$$\iiint_E xy\,dV=\iiint xy\,dz\,dy\,dx.$$



$$=\int_0^1\int_0^{2(1-x)}\int_0^{3(1-x-y/2)} xy\,dz\,dy\,dx$$

$$=\int_0^1\int_0^{2(1-x)} xy\left[\int_0^{3(1-x-y/2)} dz\right]dy\,dx$$

$$= \int_0^1 \int_0^{2(1-x)} xy \left[z \right]_0^{3(1-x-y/2)} dy\,dx$$

$$= \int_0^1 \int_0^{2(1-x)} xy \left[3\left(1-x-\frac{y}{2}\right) \right] dy\,dx$$

$$= 3\int_0^1 x \int_0^{2(1-x)} \left[(1-x)y - \frac{y^2}{2} \right] dy\,dx.$$

$$= 3\int_0^1 x \left\{ (1-x)\frac{y^2}{2} - \frac{y^3}{6} \right\}_0^{2(1-x)} dx.$$

$$= \frac{3}{2}\int_0^1 x \left\{ (1-x)4(1-x)^2 - \frac{8(1-x)^3}{3} \right\} dx.$$

$$= \frac{3}{2}\int_0^1 x(1-x)^3 \left[4 - \frac{8}{3} \right] dx.$$

$$= \frac{\cancel{3}}{\cancel{2}}\left(\frac{\cancel{4}^{2}}{\cancel{3}}\right) \int_0^1 x(1-x)^3\,dx. = 2\int_0^1 x(1-x)^3\,dx.$$

$\Rightarrow$ put $1-x=t$ $\quad x=0\,;\ t=1$

$\Rightarrow 1-t=x$ $\quad x=1\,;\ t=0.$

$\Rightarrow dx=-dt$

$$= 2\int_1^0 (1-t)t^3(-dt). = 2\int_0^1 (t^3-t^4)\,dt$$

$$= 2\left[\frac{t^4}{4} - \frac{t^5}{5} \right]_0^1 = 2\left[\frac{1}{4} - \frac{1}{5} \right] = \cancel{2}^{1}\left[\frac{5-4}{\cancel{20}_{10}} \right]$$

$$= \frac{1}{10} = \underline{\underline{0.1}}$$