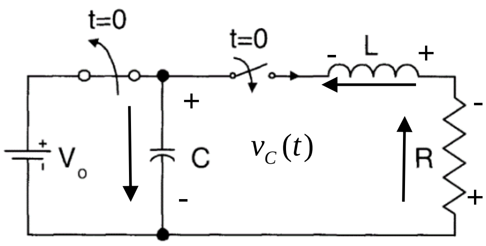
1081 Circuit Theory HW9 Solution

Ch 7 Transient Analysis Sec 7.3 Second-Order Circuits

Sec 7.3.1 Circuits Excited by Initial Conditions

Problem 1 RLC Circuit excited by initial conditions

Consider the circuit shown below, where the DC source voltage is given by V_0=5V. R=1Ω, L=1H, C=1F. Assuming zero initial condition for the current through the inductor. Derive the expression for the voltage across the capacitor and the current through the inductor. Is this circuit overdamped? Critically damped? Underdamped?



Ans: Label the voltages and currents across R, L, C with passive sign convention. For this circuit (RLC connected in series), it is more convenient to solve the capacitor voltage $\boldsymbol{v_C(t)}$ first, then find the inductor current.

(1) t<0, find the two initial conditions $v_C(0_+)$, $\dfrac{dv_C(0_+)}{dt}$.

First find $v_C(0_+)$, $i_L(0_+)$, then find $\dfrac{dv_C(0_+)}{dt}$ from $i_L(0_+)$.

$v_C(0_+) = v_C(0_-) = v_S(0) = 5(\text{V})$

$i_L(0_+) = i_L(0_-) = 0(\text{A}) \quad \because i_L(t) = i_C(t) = C\dfrac{dv_C(t)}{dt} \quad \therefore \dfrac{dv_C(0_+)}{dt} = \dfrac{1}{C} i_L(0_+) = 0.$

(2) t>0, find $v_C(t) = v_p(t) + v_h(t)$

(a) Find $v_p(t)$. For t>0 the voltage source is disconnected from the RLC circuit, so $v_p(t) = 0$.

(b) Find $v_h(t) = v_C(t)$. Derive the differential equation (ODE) from KVL.

$$v_C(t) + v_L(t) + v_R(t) = 0$$

$$v_C(t) + L\frac{di(t)}{dt} + Ri(t) = 0 \qquad i(t) = C\frac{dv_C(t)}{dt}$$

$$v_C(t) + L\frac{d}{dt}\left(C\frac{dv_C(t)}{dt}\right) + R\left(C\frac{dv_C(t)}{dt}\right) = 0$$

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$ Second-Order Homogeneous ordinary differential equation (ODE)

$v_C(t) = Ae^{st}$ The homogeneous solution (guessed, must be exponential function)

$s^2LCAe^{st} + sRCAe^{st} + Ae^{st} = 0$ subsitute guessed solution into homogeneous ODE

$Ae^{st}(s^2LC + sRC + 1) = 0$

$s^2LC + sRC + 1 = 0$ characterisic equation is obtained after canceling out common term Ae^{st}

$Z(s) = R + sL + \dfrac{1}{sC} = 0$ the same equation can be obtained by setting the impedance to zero

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$ solutions (roots) of characteristics equation

$$d = \frac{R^2}{4L^2} - \frac{1}{LC}$$ discriminant

$$\boxed{v_C(t) = \begin{cases} A_1 e^{s_1 t} + A_2 e^{s_2 t} & \text{, if } d > 0 \\ A_1 e^{st} + A_2 t e^{st} & \text{, if } d = 0 \\ e^{-\sigma t}(A_1 \cos \omega t + A_2 \sin \omega t) & \text{, if } d < 0 \end{cases}}$$

$$d = \frac{R^2}{4L^2} - \frac{1}{LC} = \frac{1^2}{4(1)^2} - \frac{1}{(1)(1)} = -\frac{3}{4}$$

$\because$ d $<$ 0

$\therefore$ This is an underdamped ciucuit.

$$\sigma = \frac{R}{2L} = \frac{1}{2(1)} = \frac{1}{2}$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\frac{1}{(1)(1)} - \frac{1^2}{4(1)^2}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$v_C(t) = v_h(t) = e^{-\sigma t}(\mathrm{A}_1 \cos \omega t + \mathrm{A}_2 \sin \omega t).$ $\mathrm{A}_1, \mathrm{A}_2$ can be found from two initial conditions

(c) Find total response $v_C(t) = v_p(t) + v_h(t) = v_h(t)$.

Solve coefficients of the homogeneous solution A_1, A_2 from two initial conditions satisfied by the complete response $v_C(0), \frac{d}{dt} v_C(t)|_{t=0}$.

The two initial conditions have been found in (1) , $v_C(0) = 5,\ \frac{dv_C(t)}{dt}|_{t=0} = 0.$

$$v_C(t) = v_p(t) + v_h(t) = v_h(t) = e^{-\sigma t}(\mathrm{A}_1 \cos \omega t + A_2 \sin \omega t).$$

$$v_C(0) = 5 = e^{-\sigma 0}(\mathrm{A}_1 \cos \omega 0 + \mathrm{A}_2 \sin \omega 0) = 1(\mathrm{A}_1(1) + \mathrm{A}_2(0)) = A_1$$

$$\frac{dv_C(t)}{dt}|_{t=0} = 0 = -\sigma e^{-\sigma 0}(A_1 \cos \omega 0 + A_2 \sin \omega 0) \qquad \frac{d}{dt}(f(t)g(t)) = (\frac{d}{dt} f(t))g(t) + f(t)(\frac{d}{dt} g(t))$$

$$+ e^{-\sigma 0}(-\omega A_1 \sin \omega 0 + \omega A_2 \cos \omega 0)$$

$$= -\sigma(A_1) + 1(\omega A_2)$$

$$-\sigma A_1 + \omega A_2 = 0 \qquad \sigma = 1/2, \quad \omega = \sqrt{3}/2$$

$$A_2 = \frac{\sigma A_1}{\omega} = \frac{5\sigma}{\omega} = \frac{5(1/2)}{(\sqrt{3}/2)} = \frac{5}{\sqrt{3}}$$

The final solution (total response) is :

$$v_C(t) = e^{-\sigma t}(\mathrm{A}_1 \cos \omega t + A_2 \sin \omega t)$$

$$= e^{-\frac{1}{2}t}(5\cos\frac{\sqrt{3}}{2}t + \frac{5}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)$$

$$i_L(t) = i_C(t) = C\frac{dv_C(t)}{dt} = (1)\frac{d}{dt}[e^{-\frac{1}{2}t}(5\cos\frac{\sqrt{3}}{2}t + \frac{5}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)]$$

$$= (-\frac{1}{2})e^{-\frac{1}{2}t}(5\cos\frac{\sqrt{3}}{2}t + \frac{5}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t) + e^{-\frac{1}{2}t}(\frac{-5\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + \frac{5}{2}\cos\frac{\sqrt{3}}{2}t)$$

$$= e^{-\frac{1}{2}t}(-\frac{5}{2}\cos\frac{\sqrt{3}}{2}t + \frac{5}{2}\cos\frac{\sqrt{3}}{2}t - \frac{5}{2\sqrt{3}}\sin\frac{\sqrt{3}}{2}t - \frac{5\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t)$$

$$= e^{-0.5t}(-\frac{5\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t - \frac{15\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t)$$

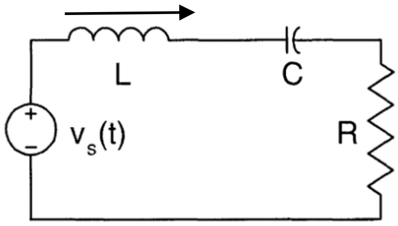
$$= e^{-0.5t}(-\frac{20\sqrt{3}}{6}\sin\frac{\sqrt{3}}{2}t)$$

$$= e^{-0.5t}(-\frac{10\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t)$$

Sec 7.3.2 Circuit Excited by Sources

Problem 2 RLC Circuit excited by sources

Assuming zero initial condition for the circuit below $v_C(0)=0$, $i_L(0)=0$. For t>0, the RLC circuit is connected to a voltage source as shown below. The AC source voltage is given by $V_s(t)=12\cos(t)$V. R=2Ω, L=1H, C=1F. Derive the expression for the current through the inductor. Is this circuit overdamped? Critically damped? Underdamped?



Ans: For this circuit (RLC connected in series), it is more convenient to solve the capacitor voltage $\boldsymbol{v_C(t)}$ first, then find the inductor current.

(1) t<0, find the two initial conditions $v_C(0_+)$, $\dfrac{dv_C(0_+)}{dt}$.

$$v_C(0_+)=v_C(0\_)=0(\text{V})$$

$$i_L(0_+)=i_L(0\_)=0(\text{A}) \quad \because i_L(t)=i_C(t)=C\frac{dv_C(t)}{dt} \ \therefore \frac{dv_C(0_+)}{dt}=\frac{1}{C}i_L(0_+)=0.$$

(2) t>0, find $v_C(t)=v_p(t)+v_h(t)$

(a) Find $v_p(t)$. Use phasor/impedance method, find phasor $\hat{V}_C$, then find $v_p(t)$.

$$\hat{V}_C = \hat{V}_S\left(\frac{Z_C}{Z_R+Z_L+Z_C}\right)=12\left(\frac{-j}{2+j-j}\right)=-6j$$

$$v_p(t)=6\sin t.$$

(b) Find $v_h(t)$. Find the homogeneous ODE from the nonhomogeneous differential equation (ODE).

$$v_C(t)+v_L(t)+v_R(t)=v_S(t) \quad KVL$$

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=v_S(t) \quad \text{Second-Order nonhomogeneous ODE}$$

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=0 \quad \text{Second-Order homogeneous ODE}$$

$$s^2LC+sRC+1=0 \qquad \text{characterisic equation}$$

$$s^2+2s+1=0 \qquad s=\frac{-2\pm\sqrt{2^2-4}}{2}=-1,-1 \quad \text{Critically Damped}$$

$$v_h(t)=A_1e^{-t}+A_2te^{-t}$$

(c) Find $v_C(t)=v_p(t)+v_h(t)$. Find A_1, A_2 from two initial conditions $v_C(0)=0, v'_C(t)=0$.

$$v_C(0)=v_p(0)+v_h(0)=6\sin(0)+A_1e^{-0}+A_2(0)e^{-0}=0+A_1+0=0\rightarrow A_1=0.$$

$$v'_C(0)=6\cos(0)+(-1)A_1e^{-0}+A_2(e^{-0}+0(-1)e^{-0})=6+(-1)(0)(1)+A_2(1+0)=0\rightarrow A_2=-6.$$

$$v_C(t)=v_p(t)+v_h(t)=6\sin t+A_1e^{-t}+A_2te^{-t}=6\sin t-6te^{-t}.$$

$$i_L(t)=i_C(t)=C\frac{dv_C(t)}{dt}=(1)\frac{d}{dt}(6\sin t-6te^{-t})=6\cos t-6e^{-t}+6te^{-t}.$$

Sec 7.4 Transfer Function

Problem 3 Transfer Function

Find the transfer functions $H_R(s)$, $H_L(s)$, $H_C(s)$ for the RLC circuit in problem 2 if (a) The output signal is the resistor voltage $v_R(t)$. (b) The output signal is the inductor voltage $v_L(t)$. (c) The output signal is the capacitor voltage $v_C(t)$. Find the poles and zeros for each transfer function.

Ans: The transfer functions are the voltage divider ratios.

$$H_R(j\omega)=\frac{\hat{v}_R}{\hat{v}_s}=\frac{R}{j\omega L+\dfrac{1}{j\omega C}+R}=\frac{2}{j\omega+\dfrac{1}{j\omega}+2}$$

$$\hat{H}_R(s)=\frac{2}{s+\dfrac{1}{s}+2}=\frac{2s}{s^2+2s+1}=\frac{2s}{(s+1)(s+1)}$$

has one zero $\mathrm{s}=0$

two poles $\mathrm{s}=-1,-1$

$$H_L(j\omega)=\frac{\hat{v}_L}{\hat{v}_s}=\frac{j\omega L}{j\omega L+\dfrac{1}{j\omega C}+R}=\frac{j\omega}{j\omega+\dfrac{1}{j\omega}+2}$$

$$\hat{H}_L(s)=\frac{s}{s+\dfrac{1}{s}+2}=\frac{s^2}{s^2+2s+1}=\frac{s^2}{(s+1)(s+1)}$$

has two zeros $\mathrm{s}=0,0$

two poles $\mathrm{s}=-1,-1$

$$H_C(j\omega)=\frac{\hat{v}_C}{\hat{v}_s}=\frac{\dfrac{1}{j\omega C}}{j\omega L+\dfrac{1}{j\omega C}+R}=\frac{\dfrac{1}{j\omega}}{j\omega+\dfrac{1}{j\omega}+2}$$

$$\hat{H}_C(s)=\frac{\dfrac{1}{s}}{s+\dfrac{1}{s}+2}=\frac{1}{s^2+2s+1}=\frac{1}{(s+1)(s+1)}$$

has no zeros

two poles $\mathrm{s}=-1,-1$