CS 230 : Discrete Computational Structures

Fall Semester, 2023
HOMEWORK ASSIGNMENT #2
Due Date: Tuesday, September 12

Suggested Reading: Rosen Sections 1.4 - 1.6; LLM Chapter 3

For the problems below, explain your answers and show your reasoning. Even if you get a problem wrong, you are more likely to get partial credit if you show your work.

1. [**8 Pts**] For the following problems, let $R(x)$, $C(x)$ and $D(x,y)$ be the statements "x has read the notes", "x is in your class" and "x has discussed problems with y", respectively. *Assume that $D(x,x)$ evaluates to false.*

   Let the domain be all people at ISU. Translate from English to logic.

   (a) Someone in your class has read the notes but has not discussed problems with anyone in your class.
   x, y: student
   $\exists x \forall y((x \neq y) \wedge R(x) \wedge C(x) \wedge C(y) \rightarrow \neg D(x,y))$

   This is my solution because we have a single x for every y, meaning that the y is linked to the x. We must make sure that x and y are not the same student, so we add that at the beginning. Furthermore, we establish that x has read the notes and is in your class and that all the y's are in your class. Lastly, we use an implication to connect them because if the left half of the proposition is not true then we should not be concerned about whether or not they discussed it with someone in our class.

   (b) There are two students in your class who have, between them, discussed problems with everyone in the class.
   x, y, z: student
   $\exists x \exists y \forall z(C(x) \wedge C(y) \wedge C(z) \rightarrow (D(x,z) \vee D(y,z)))$

   I came to this solution because it connects the x and y to the z, which is the more general and less specific quantifier. It says that x and y are individual students in class and z is the rest of the class. It then says that, given the left half of the implication is true, either x or y has discussed the problems with z. Therefore, either x or y has discussed the problems with all of the z's, which is what we want this to translate to.

2. **[15 Pts]** Define predicates and prove the following using the appropriate rules of inference:

   (a) **[5 Pts]** Mary, a student in class, has visited Paris. Everyone who visits Paris goes to the Arc de Triomphe. Therefore, someone in class has visited the Arc de Triomphe.
   Universe: all ISU students

   p: Has visited Paris
   a: Has visited the Arc de Triomphe
   c: Is in class

1. p	Given Premise 1
2. $p \to a$	Given Premise 2
3. c	Given Premise 3
4. a	Modus Ponens (2)
5. c and a $\equiv c \wedge a$	Conjunction (3, 4)
6. $\exists x(c(x) \wedge a(x))$	Conclusion

   Because Mary is in class and has visited Paris, we can use the given fact that everyone who visits Paris goes to the Arc de Triomphe and the modus ponens rule of inference to conclude that Mary has been to the Arc de Triomphe.

(b) **[5 Pts]** There are college towns in the midwest. All college towns are fun places to live. There is a town in the midwest that is a fun to live in.
Universe: all towns

c: Is a college town
f: Is a fun place to live
m: Is in the Midwest

1. $c \land m$	Given Premise 1
2. $c \to f$	Given Premise 2
3. $c \land m \equiv c$	Simplification (1)
4. c and $c \to f \equiv f$	Modus Ponens (2, 3)
5. $c \land m \equiv m$	Simplification (1)
6. m and f $\equiv m \land f$	Conjunction (4, 5)
7. $\exists x(m(x) \land f(x))$	Conclusion from (6)

Using the fact that all college towns are fun places to live, we can use the other given fact that there are college towns in the Midwest to conclude that there are fun places in the Midwest to live. This simple logical leap can be shown through modus ponens and simplification, then finally combining the previous steps with conjunction to get to our conclusion that there exists some town in the midwest that is a fun place to live.

(c) **[5 Pts]** All bears are good swimmers. If you can catch fish, you will not go hungry. If you can't catch fish, you are not a good swimmer. Therefore, no bears go hungry.
Universe: all animals

s: Is a good swimmer
f: Can catch fish
h: Will go hungry

1. s	Given Premise 1
2. $f \to \neg h$	Given Premise 2
3. $\neg f \to \neg s$	Given Premise 3
4. $s \to f$	Contrapositive of (2)
5. s and $s \to f \equiv f$	Modus Ponens (1, 3)
6. f and $f \to \neg h \equiv \neg h$	Modus Ponens (2, 5)
7. $\neg h$	Conclusion from (6)
8. $\forall x(\neg h(x))$	Conclusion

Since all bears are good swimmers, we can use the contrapositive of "if you can't catch fish, you are not a good swimmer" to discover that all good swimmers can catch fish. Then we use the last given to conclude that since all bears are good swimmers and all good swimmers can catch fish, no bears will go hungry (or that all bears will not go hungry).

3. [**5 Pts**] Prove, by giving a counterexample, that the propositions $\exists x(P(x) \wedge Q(x))$ and $\exists x P(x) \wedge \exists x Q(x)$ are not logically equivalent. *Hint*: Let your domain be the set of integers. Define $P(x)$ to be 'x is odd' and $Q(x)$ to be 'x is even'.

   P(x): x is odd
   Q(x): x is even

   The proposition $\exists x(P(x) \wedge Q(x))$ would mean there exists some x where x is even and x is odd. Using common sense, we can see that there is no number that is both even and odd. This proposition is **false**.

   On the contrary, $\exists x P(x) \wedge \exists x Q(x)$ translates to there exists some x that is odd and there exists some x that is even. Because the $\exists$ references different x's in this proposition, it is able to be valid. This proposition is **true**.

   Because the first proposition is false and the second one is true, they cannot be logically equivalent.

4. **[12 Pts]** For this problem the universe is $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$. We will construct predicates using logical operations and quantifiers. Also, assume that you are given the predicates

   $A(n, m, k)$, which means $n + m = k$ and $M(n, m, k)$, which means $n * m = k$.

   We can define new predicates from these such as $Zero(n) = A(n, n, n)$ that will return TRUE if $n = 0$ and FALSE otherwise.

   Since we now have the predicate $Zero(n)$, we can use this to build other predicates. For example, $Greater(m, n)$, which means $m > n$, can be defined as $\exists k\ \neg Zero(k) \wedge A(n, k, m)$.

   Note that we defined $Zero(n)$ only in terms of a previously defined predicate; we did not introduce a new predicate (based on a mathematical relation) in the definition, as in "$Zero(n) = (n = 0)$," and we invoked predicates using variables for each argument, not numbers as in "$Zero(n) = A(n, 0, 0)$." Likewise, we defined $Greater(m, n)$ only in terms of previously defined predicates, a quantifier, and logical relations; we did not introduce a separate predicate based on a mathematical relation, as in "$Greater(m, n) = (m > n)$."

   For each of the parts below, define new predicates for each that uses quantifiers, logical operations, and predicates given above or ones that you have already constructed (in a previous part).

   Note that you cannot use mathematical relations in your definitions, and the arguments for the predicates must be variables, not specific numbers (e.g. an expression like "$A(n, m, m)$" is ok but an expression like "$M(n, 5, 10)$" is not).

   For partial credit, give a short explanation of how you would construct the predicate.

   (a) **[4 pts]** $Equal(m, n)$ that is true if and only if $m = n$

   $\exists k(Zero(k) \wedge A(n, k, m))$
   This predicate works because there only needs to be one value of k that makes it true for it to be valid. The value of k is 0, which works because the Zero(k) must be true for the proposition to be true. Because k is 0, the summation will be n + 0, which is equal to n, meaning the right half of the proposition will only be true if n + 0 = m. This allows us to conlude that this predicate is only true when n = m.

   (b) **[4 pts]** $One(n)$ that is true if and only if $n = 1$

   $\forall k(\neg Zero(n) \wedge M(n, k, k)$
   This predicate starts by ensure that n is not zero, because if it is, it could cause an exception that would make this valid if k is also 0. The next check is that n * k = k. The forall ensures that there are no random cases I am forgetting about, if n is truly 1 then it wont matter what k is, 1 * k should always equal k. Then we put them together with an and to ensure they are both true and the predicate is complete.

(c) **[4 pts]** $Two(n)$ that is true if and only if $n = 2$

$\exists k \exists g((A(k, k, g) \wedge M(n, k, g) \wedge \neg Zero(n))$
This predicate works because we can use background knowledge in algebra to know that k + k = k * 2, so it will only be true if the sum and product are both equal to the same value g. Lastly, as long as n isn't 0 there are no edge cases. Adding "$\neg Zero(n)$" doesn't do anything detrimental to the predicate, as n already can't be zero for n to be two.

(d) **[4 pts Extra Credit]** $Prime(p)$ that is true if and only if p is prime

For more practice, work on the problems from Rosen Sections 1.4 - 1.6 and LLM Chapter 3.