CS 230 : Discrete Computational Structures
Fall Semester, 2023
ASSIGNMENT #6
Due Date: Thursday, October 19

Suggested Reading: Rosen Section 5.1 - 5.2; Lehman et al. Chapter 5.1 - 5.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

For Problems 1-5, prove the statements by mathematical induction. Clearly state your basis step and prove it. What is your inductive hypothesis? Prove the inductive step and show clearly where you used the inductive hypothesis (assumption).

1. **[5 Pts]** $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2))^2$, for all positive integers n.

   P(n) = $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3$

   Prove: P(n) = $(n(n+1)/2))^2$

   Basis: $1^3 = 1$, $(1(1+1)/2)^2 = (1(2)/2)^2 = 1^2 = 1$ 1=1, P(1) is true.
   Since 1 = 1, P(1) is true and our basis step is true.
   Induction Step:
   Inductive Hypothesis: Assume P(k) = $\sum_{n=1}^{k} n^3 = (k(k+1)/2))^2$
   Prove P(k + 1) = $((k+1)(k+2)/2))^2$:
   P(k) = $1^3 + 2^3 + \cdots + k^3$
   P(k + 1) = $1^3 + 2^3 + \cdots + k^3 + (k+1)^3$
   In other words, P(k+1) = P(k) + $(k+1)^3$ and P(k) = $(k(k+1)/2))^2$ (Using **IH**)

   Therefore, P(k+1) = $(k(k+1)/2))^2 + (k+1)^3$
   $= (k(k+1))/2 * (k(k+1)/2 + (k+1)^3$
   $= ((k^2+k) * (k^2+k))/4 + (k+1)(k+1)(k+1)$
   $= (k^4 + k^3 + k^3 + k^2)/4 + (k^2 + k + k + 1)(k+1)$
   $= (k^4 + 2k^3 + k^2)/4 + (k^2 + 2k + 1)(k+1)$
   $= (k(k^3 + 2k^+1)/4 + (k^3 + 2k^2 + k + k^2 + 2k + 1)$
   $= (k^4 + 2k^3 + k^2)/4 + (k^3 + 3k^2 + 3k + 1)$
   $= (k^4 + 2k^3 + k^2)/4 + 4(k^3 + 3k^2 + 3k + 1)/4$
   $= (k^4 + 2k^3 + k^2)/4 + (4k^3 + 12k^2 + 12k + 4)/4$
   $= (k^4 + 6k^3 + 13k^2 + 12k + 4)/4$
   $= (k^4 + 2k^3 + k^2 + 4k^3 + 8k^2 + 4k + 4k^2 + 8k + 4)/4$
   $= ((k^2 + 2k + 1)(k^2 + 4k + 4))/4$
   $= ((k+1)(k+1)(k+2)(k+2))/2^2$
   $= (((k+1)(k+2))/2)^2$

   Because P(k + 1) = $(((k+1)(k+2))/2)^2 = (k(k+1)/2))^2 + (k+1)^3$, we have proved P(k + 1) = $(((k+1)(k+2))/2)^2$ for all positive integers through mathematical induction using our inductive hypothesis.

2. [**5 Pts**] $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$, for all positive integers n.

P(n) = $\sum_{k=1}^{n} k \cdot k! = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$

Prove: P(n) = (n+1)! - 1 for all n $\in \mathbb{Z}^+$

Basis: P(1) = $1 \cdot 1!$ = 1, (1+1)! - 1 = 2! - 1 = 2 - 1 = 1, 1 = 1 so P(1) is true.

Induction Step:
Inductive Hypothesis: P(k) = $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$

Prove: P(k+1) = $((k + 1) + 1)! - 1 = (k + 2)! - 1 = (k + 2)(k + 1)! - 1$

P(k+1) = $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1) \cdot (k + 1)!$
Therefore, P(k+1) = P(k) + $(k + 1) \cdot (k + 1)!$ (via **IH**)
Since P(k) = (k+1)! - 1 (from **IH**), P(k+1) = (k+1)! - 1 + $(k + 2) \cdot (k + 2)!$

P(k+1) = (k+1)! + (k+1)(k+1)! - 1
P(k+1) = (k+1)!(1+k+1) - 1 (Factored out (k+1)!)
P(k+1) = (k+1)!(k+2) - 1
P(k+1) = (k+2)! - 1 (Used def. of factorial

Because P(k+1) = (k+2)! - 1, we have proved P(n) = $(n+1)! - 1$ via mathematical induction.

3. **[5 Pts]** $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$, for all non-negative integers n.

P(n) = $\sum_{k=1}^{n} 2 \cdot (-k)^n = 2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n$

Basis: P(1) = 2 + 2 * -7 = 2 - 14 = -12, $(1 - (-7)^2)/4$ = 1 - 49 /4 = -48/4 = -12
Since -12 = -12, P(1) is true.

Induction Step:
Inductive Hypothesis: P(k) = $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k = (1 - (-7)^{k+1})/4$
Prove: P(k+1) = $(1 - (-7)^{k+2})/4$

P(k+1) = $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^k + 2 \cdot (-7)^{k+1}$
P(k+1) = P(k) + $2 \cdot (-7)^{k+1}$
P(k+1) = $(1 - (-7)^{k+1})/4 + 2 \cdot (-7)^{k+1}$
P(k+1) = $(1 - (-7)^{k+1})/4 + (8 \cdot (-7)^{k+1})/4$
P(k+1) = $(1 - (-7)^{k+1} + 8 \cdot (-7)^{k+1})/4$
P(k+1) = $(1 + 7(-7)^{k+1})/4$
P(k+1) = $(1 + 7(-7)^{k+1})/4$
P(k+1) = $(1 - (-7)^{k+2})/4$

Since P(k+1) = $(1 - (-7)^{k+2})/4$, we have proven P(n) = $(1 - (-7)^{n+1})/4$ for all non negative integers.

4. **[5 Pts]** $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$, for all positive integers n.

P(n) $= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)}$

Prove: P(n) $= \frac{n}{n+1}$, for all n $\in \mathbb{Z}^+$

Basis: P(1) $= \frac{1}{1\cdot 2} = \frac{1}{2}$, $\quad \frac{1}{1+1} = \frac{1}{2}$

Inductive Step:
Inductive Hypothesis: P(k) $= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k\cdot(k+1)} = \frac{k}{k+1}$
Prove: P(k+1) $= \frac{k+1}{k+2}$, for all positive integers.

P(k+1) $= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k\cdot(k+1)} + \frac{1}{(k+1)\cdot(k+2)}$, so P(k+1) = P(k) + $\frac{1}{(k+1)\cdot(k+2)}$.
Therefore, P(k+1) $= \frac{k}{k+1} + \frac{1}{(k+1)\cdot(k+2)}$ (via **IH**)
$= [\frac{k}{k+1} \cdot \frac{k+2}{k+2}] + \frac{1}{(k+1)\cdot(k+2)}$
$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$
$= \frac{k^2+2k}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$
$= \frac{k^2+2k+1}{(k+1)(k+2)}$
$= \frac{k^2+2k+1}{(k+1)(k+2)}$
$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$
P(k+1) $= \frac{(k+1)}{(k+2)}$

Because P(k+1) $= \frac{(k+1)}{(k+2)}$, we have proven P(n) $= \frac{n}{n+1}$ by Mathematical Induction $\forall n \in \mathbb{Z}^+$

5. [**5 Pts**] 6 divides $n^3 - n$, for all positive integers n.

P(n) = is true is $n^3 - n$ is divisible by 6

Basis: P(1) = $1^3 - 1 = 0$, (6)0 = 0, P(1) is true.

Induction Step:
Inductive Hypothesis: P(k) = 6p, where p $\in \mathbb{Z}^+$ (aka divisible by 6)
Prove: P(k+1) = 6p, $p \in \mathbb{Z}^+$
P(k+1) = 6p = $(k+1)^3 - k$ 6p = $(k+1)(k+1)(k+1) - (k+1)$
6p = $(k^2 + 2k + 1)(k+1) - (k+1)$
6p = $k^3 + 3k^2 + k + k^2 + 2k + 1 - (k+1)$
6p = $k^3 + 3k^2 + 3k + 1 - 1$
P(k+1) = $(k^3 - k) + 3k^2 + 3k$
P(k+1) = $(k^3 - k) + 3k^2 + 3k$
Since $(k^3 - k)$ = 6p as established in **IH**, we can sub it in to see:
P(k+1) = 6p + $3k^2 + 3k$
And since 6p, $3k^2, 3k$ are all integers, therefore P(k+1) is also an integer and because of both of those factors, P(k+1) is proven by Mathematical Induction.

6. **[9 Pts]** Let $P(n)$ be the statement that n-cent postage can be formed using just 4-cent and 7-cent stamps. Prove that $P(n)$ is true for all $n \geq 18$, using the steps below.

(a) First, prove $P(n)$ by regular induction. State your basis step and inductive step clearly and prove them.

Basis:
P(18) $= 7 + 7 + 4 = 18$. P(18) is true.

Induction Step:
Inductive Hypothesis: Let k > 18. Assume k cent postage can be formed using 4 and 7 cent coins.
Prove: P(k+1) can be made with 7 and 4 cent coins.
Proof by cases:
Case 1: No 7 cent coins were used to make k
We can replace five 4 cent coins with three 7 cent coins to make k + 1

Case 2: At least one 7 cent coins were used to make k
We can replace one 7 cent coins with two 4 cent coins to make k + 1

Thus, P(k+1) has been proven by Mathematical Induction

(b) Now, prove $P(n)$ by strong induction. Again, state and prove your basis step and inductive step. Your basis step should have multiple cases.

Basis:
P(18) $= 7 + 7 + 4 = 18$. P(18) is true.
P(19) $= 7 + 4 + 4 + 4 = 19$. P(19) is true.
P(20) $= 4 + 4 + 4 + 4 + 4 = 20$. P(20) is true.
P(21) $= 7 + 7 + 7 = 18$. P(21) is true.

Induction Step:
Inductive Hypothesis: Let k > 21. Assume k cent postage can be formed using 4 and 7 cent coins.
There are two cases where we need to step by 1:

No 7 cent coins - replace five 4 cent coins with three 7 cent coins (will always have at least 5, 20 is lowest value and does)
This steps k to k + 1

At least one One 7 cent coin - replace the 7 cent coin with two 4 cent coins
This steps k to k + 1

Since both cases can go from k to k + 1

7. [**6 Pts**] Suppose $P(n)$ is true for an infinite number of positive integers n. Also, suppose that $P(k+1) \rightarrow P(k)$ for all positive integers k. Now, prove that $P(n)$ is true for all positive integers. This is the *reverse induction principle.*

   Basis: P(n) is true for an infinite number of positive integers. Therefore, P(n).

   Induction Step: Since $P(k+1) \rightarrow P(k)$, P(n) implies P(n-1). This is Modus Ponens
   If P(n) implies P(n-1) for all n, then there is some integer > 1 that will eventually imply P(1) via a chain of Modus Ponens.

For more practice, you are encouraged to work on other problems in Rosen Sections 5.1 and 5.2 and in LLM Chapter 5.