CS 330 : Discrete Computational Structures

Fall Semester, 2023

ASSIGNMENT #10

Due Date: Thursday, Nov 30

Suggested Reading: Rosen Sections 6.1 - 6.4.

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. **[2 Pts]** How many integers between 1 and 60000, inclusive, are divisible by 3 or 5 or 7?

   Let S = { 1 ... 60000 }.

   $D_3 = \{ \ n \mid 3|n, n \in S \ \}$
   $D_5 = \{ \ n \mid 5|n, n \in S \ \}$
   $D_7 = \{ \ n \mid 7|n, n \in S \ \}$

   $|D_3 \cup D_5 \cup D_7| = |D_3| + |D_5| + |D_7| - |D_3 \cup D_5| - |D_3 \cup D_7| - |D_5 \cup D_7| + |D_3 \cup D_5 \cup D_7|$

   $= \frac{60000}{3} + \frac{60000}{5} + \frac{60000}{7} - \frac{60000}{15} - \frac{60000}{21} - \frac{60000}{35} + \frac{60000}{105}$

   $= 20000 + 12000 + 8571 - 4000 - 2857 - 1714 + 571$

   $= 32571$

2. [**2 Pts**] An ISU Computer Science shirt is sold in 7 colors, 5 sizes, striped or solid, and long sleeve or short sleeve. (a) How many different shirts are being sold? (b) What if the red and gold shirts do not come striped?

   a) Product rule: 7 colors $\cdot$ 5 sizes $\cdot$ 2 patterns $\cdot$ 2 sleeve type
   $= 7 \cdot 5 \cdot 2 \cdot 2 = 35 \cdot 4 = 140$

   b) This combines the product and sum rules:

   Product rule for either side:
   Color is NOT Red/Gold $= 5 \cdot 5 \cdot 2 \cdot 2 = 100$
   Color is Red/Gold $= 2 \cdot 5 \cdot 1 \cdot 2 = 20$

   Sum Rule to combine:
   100+20 = 120 different shirt combinations

3. [**8 Pts**] Let A and B be sets of 6 elements and 8 elements, respectively.
(a) How many different functions possible from A to B? from B to A?

(a) A to B: $8^6 = 262{,}144$

(b) B to A: $6^8 = = 1{,}679{,}616$

(b) How many different relations possible from A to B?

2 points in a relation to the power of the combinations - $2^{6*8} = 2^{48}$

(c) How many of the functions from A to B are one-to-one?

6 Inputs, 8 Outputs = P(6, 8) = 1679616

(d) How many of the functions from B to A are onto?

8 Inputs, 6 Outputs = P(8, 6) = 262144

4. [**4 Pts**] A sack contains 40 movie tickets, 5 for each of 8 different movies. Five friends want to go to a movie. How many tickets would you have to remove from the sack to guarantee that everyone will be able to watch the same movie? What principle did you use? What if everyone wants to go to 'Oppenheimer'? How many tickets would you have to remove from the sack in that case?

   In order to ensure everyone goes to the same movie, you would have to remove 33 tickets. This is because in the worst case scenario, we could remove 32 tickets, 4 from each of the 8 different movies, leaving 8 tickets to different movies. However, if we instead remove 33, we could remove 5 tickets for at least one movie and 4 at most from the rest, leaving 5 tickets all to the same movie.

   This is an example of the pigeonhole principal. In order to ensure at least one box gets 5 tickets, we first have to put 4 in each box.

   If everyone wants to go to Oppenheimer, you must remove all 40 tickets to ensure that you only draw Oppenheimer tickets. This is because in a worst case scenario, you would draw 4 Oppenheimer tickets in 39 draws, leaving only an Oppenheimer ticket with 39 selections, so you must remove all 40 to ensure each friend gets a ticket to Oppenheimer.

5. **[4 Pts]** In how many ways can a photographer arrange 8 people in a row from a family of 12 people, if (a) Mom and Dad are in the photo, (b) Mom and Dad are next to each other in the center of the photo (three people on each side of them).

(a): If mom and dad are in the photo, then there are 8 remaining spots for the other 10 people. We can use the product rule for the three selection tasks:
Mom and Dad = select two people out of two = 1
Other people = select six people from ten = $\frac{10!}{(6!)*(10-6)!} = 210$
Arrangement of the 8 people: $8! = 40320$

1 * 210 * 40320 = 8,467,200

There are 8,467,200 different combinations of people in the picture.

(b): Mom and Dad in the center: We have the same calculation, but we reduce the arrangement to 6! because there are no longer 8 open spots for 8 people, there are 6 spots for 6 people.

Mom and Dad = select two people out of two = 1
Other people = select six people from ten = $\frac{8!}{(6!)*(8-6)!} = 28$
Arrangement of the 6 people outside Mom and Dad: $6! = 720$
Arrangement of mom and dad (the could switch places) = 2

1 * 210 * 720 * 2 = 302,400

There are 302,400 combinations for the picture if Mom and Dad are in the center.

6. [**4 Pts**] A coin is flipped nine times where each flip comes up either head or tails. How many possible outcomes contain at least five heads?

   There has to be at least 5 heads, so we will use the sum rule for the combinations of at least 5.

   C(9, 5) + C(9, 6) + C(9, 7) + C(9, 8) + C(9, 9)
   126 + 84 + 36 + 9 + 1 = 256

   There are 256 possible combination where at least 5 heads are flipped.

7. **[6 Pts]** 12 women and 10 men are on the faculty. How many ways are there to pick a committee of 6 if (a) Ann and Beth will not serve together, (b) at least one woman must be chosen, (c) at least one man and one woman must be chosen.

   (a):
   Total combinations of 6 people chosen from 21 = 230230
   We will multiply this by two to represent the fact that Ann or Beth but not both will be on there.
   Total combinations: 460460

   (b):
   Total combinations of 6 from 22: 296010
   Total combinations of only men: 5005
   Total combinations minus the ones with only men: 296010 - 5005 = 219,005

   (c):
   Total combinations of 6 from 22: 296010
   Total combinations of only men: 5005
   Total combinations of only women: 12376
   Total combinations minus the ones that are only one gender: 296010 - 5005 - 12376 = 278,629

8. [**4 Pts**] Prove, using a combinatorial argument, that $C(m+n,2) = C(m,2) + C(n,2) + mn$, where $m, n \geq 2$.

9. [**6 Pts**] Prove, (a) using a combinatorial argument, and (b) using an algebraic proof, that $P(n,3)C(n-3,k-3) = C(n,k)P(k,3)$.

For more practice, work on problems in Rosen Sections 6.1 - 6.4.