CS 230 : Discrete Computational Structures
Fall Semester, 2023
HOMEWORK ASSIGNMENT #4
Due Date: Thursday, Sept 28

Suggested Reading: Rosen Sections 2.1 - 2.3; Lehman et al. Chapter 4.1, 4.3, 4.4.

For the problems below, explain your answers and show your reasoning.

1. **[6 Pts]** Let A and B be non-empty sets. Prove that if $A \neq B$, then $A \times B \neq B \times A$.

   Let A = 1, 2, 3
   Let B = x. y

   $A \times B$ = (1, x), (2, x), (3, x), (1, y), (2, y), (3, y)
   $B \times A$ = (x, 1), (y, 1), (x, 2), (y, 2), (x, 3), (y, 3)

   Set multiplication is not commutative, unlike regular multiplication. This is true because instead of forming a product, set multiplication forms a set of ordered pairs.

   Set multiplication is similar to nested for loops - it starts with the first value of the first set and iterates through the second set before moving on to the second value and repeating. This characteristic makes it impossible for $A \times B$ to equal $B \times A$ unless A and B are equal, which they were already specified to not be.

2. **[4 Pts]** Prove that $(A \cup B) - C = (A - C) \cup (B - C)$ using iff arguments and logical equivalences.

$x \in (A \cup B) - C$	Starting Point
$iff\ (x \in A \vee x \in B) - C$	Definition of Union
$iff\ (x \in A \vee x \in B) \wedge x \notin C$	Definition of Difference
$iff\ (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$	Distributive Law
$iff\ x \in (A - C) \vee (x \in B \wedge x \notin C)$	Definition of Difference
$iff\ x \in (A - C) \vee x \in (B - C)$	Definition of Difference
$iff\ (A - C) \cup (B - C)$	Definition of Union

For this problem, I first broke the union up into its definition, then did the same for the subtraction. This allowed me to distribute the subtraction into both A and B, enabling me to re build the unions and arrive to our conclusions. This works because we can use the distributive law to subtract from both A and B, then put the union back together.

3. [**8 Pts**] Disprove the statements below.

(a) If $A \cup C \subseteq B \cup C$ then $A \subseteq B$.
Let A = {1, 3, 5}
Let B = {1, 2, 4, 5, 6}
Let C = {3, 6, 9}

$A \cup C = \{1, 3, 5, 6, 9\}$
$B \cup C = \{1, 2, 3, 4, 5, 6, 9\}$

In this example, I have created sets A, B and C. As you can see, the statement $A \cup C \subseteq B \cup C$ is **true**. All values of $A \cup C$ are present in $B \cup C$.

However, not all values of A are in B. A contains the element 3, whereas B does not.

The statement "if $A \cup C \subseteq B \cup C$ then $A \subseteq B$" will always be false if B and C contain a value that A does not. Anytime that this is the case, it is impossible for A to be a subset of B, but $A \cup C \subseteq B \cup C$ will remain true because C is able to make up for the missing value. When C is removed, it can no longer make up for the missing value.

(b) If $A \cap C \subseteq B \cap C$ then $A \subseteq B$.
Let A = {1, 3, 5}
Let B = {2, 3, 4, 6}
Let C = {3, 6, 9}

$A \cap C = \{3\}$
$B \cap C = \{3, 6\}$

In this example, $A \cap C$ is a subset of $B \cap C$. This is very easy to see, making it obvious that the left half of the statement "if $A \cap C \subseteq B \cap C$ then $A \subseteq B$" is **true**.

However, it is also obvious that the right half is **false**. A is not a subset of B. This is because although A and C share the same value that B and C share, there are many values in A and B that are not relevant in a intersection operation but are relevant in a subset statement. Because there can be many values left out in an intersection, $A \cap C \subseteq B \cap C$ is not sufficient evidence for proving $A \subseteq B$.

4. **[8 Pts]** Prove that if $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$ then $A \subseteq B$. *Hint: You can either do a proof by contradiction or a proof by cases.*

Proof by contradiction:

We will assume that $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$, then change the last part to be $A \nsubseteq B$.
This means there is some element in A that is not in B.
In more specific terms, $\exists x \in A \wedge x \notin B$.

Case 1: $x \in C$

This means $x \in A \cap C$ due to the definition of intersection.
This means, by our assumption that $A \cap C \subseteq B \cap C$ is true, $x \in B \cap C$.
This directly contradicts the assumption that $A \nsubseteq B$, so something must be wrong.

Case 2: $x \notin C$

If $x \notin B$, then $x \notin B \wedge x \notin C$, so x cannot be in the union.
If $x \notin B \cup C$, then how can the statement $A \cup C \subseteq B \cup C$ or the statement $A \cap C \subseteq B \cap C$ be true? This directly contradicts the entire problem.

Since both cases caused a contradiction, we can conclude that if both $A \cup C \subseteq B \cup C$ AND $A \cap C \subseteq B \cap C$ are true, **A must be a subset of B.**

5. **[8 Pts]** Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ using subset argument. You *may not* use logical equivalences in your proof. Use general proof techniques like 'proof by contradiction' and 'proof by cases'.

First, lets prove$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$.

We'll start with $(A \cup B) - (A \cap B)$.
This means that $x \in (A \cup B) - (A \cap B)$.
We can convert it into $x \in (A \cup B) \wedge x \notin (A \cap B)$ by definition of difference.
We can also change it to $(x \in A \vee x \in B) \wedge x \notin (A \cap B)$ with the definition of union.

This statement says that x is in A OR B and NOT in A and B, which leaves us with two cases:
$x \in A \wedge \neg B$ OR $x \in B \wedge \neg A$.

This is equivalent to the right half of the statement $(A - B) \cup (B - A)$. Because union states that the value is in one of the two sets. Both statements say it is A and not B OR B and not A.

Since both statements are saying the same thing because they match the definition of union and subtraction, we can come to the conclusion that $(A \cup B) - (A \cap B) \rightarrow (A - B) \cup (B - A)$.

This allows us to conclude that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$.

Next, let's prove $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

We'll start with $(A \cup B) - (A \cap B)$.
We can change it to $x \in (A \cup B) \wedge x \notin (A \cap B)$ by definition of difference.
We can change it to $(x \in A \vee x \in B) \wedge x \notin (A \cap B)$.

We have arrived at the same place as the first part, which would bring us to the same two cases, x is in A and not B OR B and not A.
As we have established, we can conclude subset using this knowledge.
$(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$ (From corollary).

Since $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$ and $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$ have both been proven, we can conclude that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ because it fits the definition of equal.

6. [**4 Pts**] Prove that $f(n) = 5n + 9$ is one-to-one, where the domain and co-domain of f is $\mathcal{Z}^+$. Show that f is not onto.

For this problem, our domain and co-domain are both positive integers ($\mathcal{Z}^+$).

$f(n) = 5n + 9$ is one to one.

One way to prove a function is one-to-one is to put equal variables x1 and x2 into it and see if $f(x1) = f(x2)$ because the definition of one-to-one is $(f(x1) = f(x2)) \rightarrow (x = y)$. If a function is truly one-to-one, these would have the same result, showing that if x1 = x2, $f(x1) = f(x2)$.

$f(x1) = 5(x1) + 9$
$f(x2) = 5(x2) + 9$
$5(x1) + 9 = 5(x2) + 9$
$5(x1) = 5(x2)$
$x1 = x2$

Because two equal variables produced the same result, we can conclude that the function is one-to-one. In other words, the function produces the same unique value for a non-unique input and different unique values for unique inputs. This allows us to conclude that the function is one-to-one because it fits the definition of one-to-one, $(f(x1) = f(x2)) \rightarrow (x = y)$.

$f(n) = 5n + 9$ is not onto.

In order for a function to be onto, there must be some value x where f(x) = y **for ALL** y in our co-domain. Since our domain begins at one and the lowest value of our function (x = 1) is fourteen, we can conclude that there are no values x where $f(x) < 14$. Because there are no values within our domain that can get $f(x) < 14$, the function is **not** onto.

7. [**4 Pts**] Prove that $f(m, n) = m + n + mn$ is onto, where the domain of f is $\mathcal{Z} \times \mathcal{Z}$ and the co-domain of f is $\mathcal{Z}$. Show that f is not one-to-one.

For a function to be onto, we must be able to create every single value in the co-domain using values in the domain. In other words, $\exists x \forall y$ for $f(x) = y$.
Our co-domain is all integers. Therefore, as long as we can make every integer, we can conclude our function is onto.

There is a cheat code to this - since 0 is also an integer, we can make any integer, we'll call it x, by having the other input be 0.

$f(m, n) = m + n + m * n$
$f(0, x) = 0 + x + 0x = x$

Since we can input 0 and make any number, we can conclude the function is onto.

For a function to be one-to-one, there must be exactly one possible input to produce each output. In other words, each unique input must create a unique output. This can be disproven for this function.

Function: $f(m, n) = m + n + mn$

If: m = 1, n = 2 (unique input)
$f(1, 2) = 1 + 2 + 2 * 1 = 5$ (unique output)

If: m = 0, n = 1 (unique input)
$f(2, 1) = 1 + 2 + 1 * 2 = 5$ (NOT unique output)

Because two unique inputs gave us the same output, we can conclude that this function is NOT one-to-one. It is easy to see this because addition and multiplication are both commutative, meaning order does not matter. We can simply swap the values to create a unique input without changing the value of the function. Commutative functions are not one-to-one.

8. **Extra Credit [8 Pts]** Let g be a total function from A to B and f be a total function from B to C.

   (a) If $f \circ g$ is one-to-one, then is g one-to-one? Prove or give a counter-example.

   Proof by contradiction:
   Assume that $f \circ g$ is one-to-one and g is not one-to-one.

   Because g is not one-to-one, that means there exists some value in the set A that does not have a unique output in B.

   If $f \circ g$ is one to one, every value in A has a unique value in C. However, it is impossible for all values of A to have a unique value in C if they do not have a unique value in B. This is a contradiction.

   Since it is impossible for there to be a unique value outputted from A to C without a unique value outputted from A to B, we can conclude that g must be one-to-one.

   (b) If $f \circ g$ is onto, then is g onto? Prove or give a counter-example.
   Proof by contradiction:

   Assume that Assume that $f \circ g$ is onto and g is not onto.

   In order for a function to be onto, there must be at least one input that can be used to get every output. Therefore, there must be some element of A that can be inputted into $f \circ g$ to get every element of C.

   However, g is not onto, so there is a value of B that is not possible to obtain by inputting any values of A into g. Since there is at least one value of B that is not mapped to A, it is impossible for A to fully span C.
   This is a **contradiction**.

   Because of the contradiction, if $f \circ g$ is onto, g must be onto.

For more practice, work on the problems from Sections 2.1 - 2.3; Lehman et al. Chapter 4.1, 4.3, 4.4.