CS 230 : Discrete Computational Structures

Fall Semester, 2023
HOMEWORK ASSIGNMENT #3
Due Date: Tuesday, September 19

Suggested Reading: Rosen Sections 1.7 - 1.8; Lehman et al. Chapter 1

For the problems below, explain your answers and show your reasoning. You may not get full credit unless you show all your work. Use paragraph-style proofs as shown in class.

1. [**6 Pts**] Prove that for all integers p, p is even if and only if $p^3 + 3$ is odd. *Note: You need to do two proofs, one in each direction. You may not use the theorems shown in class regarding odd and even products.*

   Proof by cases:
   (Only cases are as follows because p must be even for this problem, limiting the cases to 2)

   (a) **Case 1: p is even and $p^3 + 3$ is even.**
   Let k = any integer.
   Let p = 2k, an even integer.
   $p^3 = (2k)^3 = 8k^3 = 2(4k^3)$
   We know that any number multiplied by 2 is even, so $2(4k^3)$ must be even. For this case to be true, that means that $2(4k^3)$ + 3, an even number plus an odd number, must be even. Any even number plus an odd number is odd, so this case cannot be true. That means that either p is not even or $p^3 + 3$ is not odd.

   (b) **Case 2: p is even and $p^3 + 3$ is odd.**
   Let k = any integer.
   Let p = 2k, an even integer.
   $p^3 = (2k)^3 = 8k^3 = 2(4k^3)$
   As we established, $2(4k^3)$ is even. An even number plus an odd number is odd, so $2(4k^3)$ + 3 is odd. Because there are no contradictions in this case, we can conclude it is true.

2. [**5 Pts**] Prove that if x is irrational then $3x$ is irrational. What proof technique did you use and why?

   Proof by counter example:
   x is irrational and $3x$ is rational.

   For a number to be rational, it must be able to be written as a over b, with a and b both being real integers and b is not zero.

   $3x = \frac{3}{1} * x$. x is irrational, so we cannot write it as the division of two integers.

   However, we have to be able to write 3x as the division of two integers.

   How can we write $\frac{3x}{1}$ without the x if x is irrational?

   Because x cannot be written as the division of two integers, we cannot write 3x as the division of two integers, therefore we have a contradiction and can conclude that $3x$ is irrational.

3. **[6 Pts]** Let m and n be positive integers. Prove that if $mn > 63$, then $m \geq 8$ or $n \geq 10$. What proof technique did you use and why?

   For this proof, we will use the contrapositive.

   Original statement: $(m * n > 63) \rightarrow (m \geq 8) \wedge (n \geq 10)$

   The contrapositive of this statement: $\neg[(m \geq 8) \wedge (n \geq 10)] \rightarrow \neg[(m * n > 63)]$
   $\equiv (m < 8) \vee (n < 10) \rightarrow m * n \leq 63$

   With $(m < 8) \vee (n < 10)$, the greatest values that m and n can be are 7 and 9 respectively. Multiply these and we get 63, which is ≤ 63.

   Using the contrapositive rule, we can conclude that because the contrapositive is true, the original proposition must be true.
   $(m * n > 63) \rightarrow (m \geq 8) \wedge (n \geq 10)$ is true.

4. **[6 Pts]** Let x and y be rational numbers and let z be an irrational number. Prove that $xy + z$ is irrational. Can you use a direct proof? Why or why not?

First, let's prove that xy is rational. It will simplify the problem.
Any rational number can be written as the division of two integers.
$x = \frac{a}{b}$, where a and b are integers and b $\neq 0$.
$y = \frac{c}{d}$, where c and d are integers and d $\neq 0$.
$xy = \frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$
Since all numbers involved are integers, ac and bd are integers, meaning xy is rational.

Now that we know xy is rational, we can rewrite the equation.
Let p = xy. This is rational(proven above).
Let q = the sum of p and z. This is rational (asssumed from problem, xy + z = rational).
$q = p + z$. This can be rewritten as $q - p = z$.
Since p and q are both rational, let's rewrite them as fractions just like we did with x and y.
$p = \frac{e}{f}$, where a and b are integers and b $\neq 0$.
$q = \frac{g}{h}$, where c and d are integers and d $\neq 0$.
$p - q = \frac{e}{f} - \frac{g}{h} = \frac{eh}{fh} - \frac{gf}{fh} = \frac{eh-gf}{fh}$.
Therefore, z = $\frac{eh-gf}{fh}$.
But wait, e, f, g and h are all integers? How can z be irrational if the fraction only contains integers.
From this, we can conclude that q must be irrational, as q being rational caused a contradiction.
Therefore, xy + z is irrational.

This is a proof by counter example: the assumption that a rational number plus an irrational number was proven false, leading us to the conclusion that the opposite is true.
It is difficult to use a direct proof or a proof by cases because there are too many outcomes and xy + z can contain any numbers. The simplest way to solve this problem was to rewrite it and solve for z as a rational number then get a contradiction.

5. [**6 Pts**] Suppose your student organization has 26 scheduled meetings for the calendar year. Prove that at least three of the meetings fall on the same month. What proof technique do you use?

   Let's start by trying to use direct proof.

   There are 12 months in a year. If we hold two meetings per month, $2m = 2 * 12 = 24$. Because $24 < 26$, even if every month had two meetings, we would not be able to reach the 26 meeting mark without putting at least one more meeting on the calendar.

   $2m + 2 = 26$. **Two months must have 3 meetings** to fulfill the 26 annual meeting requirements.

6. [**6 Pts**] Prove that if $p \geq 5$ or $p \leq -3$ then $(p-1)^2 \geq 16$. What proof technique did you use and why?
Proof by contrapositive:

Original statement $((p \geq 5) \vee (p \leq -3)) \rightarrow ((p-1)^2 \geq 16)$

Contrapositive: $((p-1)^2 < 16) \rightarrow ((p < 5) \wedge (p > -3)$
Rewritten for better clarity: $((p - 1^2) \leq 15) \rightarrow ((p \leq 4) \wedge (p \geq -2))$
If $(p-1)^2$ is less than 15, then p is between -2 and 4.

We can test this by testing each case.

(a) p = 4:
$p - 1 = 3, 3^2 = 9, 9 \leq 15 \Rightarrow$ TRUE

(b) p = 3:
$p - 1 = 2, 2^2 = 4, 4 \leq 15 \Rightarrow$ TRUE

(c) p = 2:
$p - 1 = 1, 1^2 = 1, 1 \leq 15 \Rightarrow$ TRUE

(d) p = 1:
$p - 1 = 0, 0^2 = 0, 0 \leq 15 \Rightarrow$ TRUE

(e) p = 0:
$p - 1 = -1, -1^2 = 1, 1 \leq 15 \Rightarrow$ TRUE

(f) p = -1:
$p - 1 = -2, -2^2 = 4, 4 \leq 15 \Rightarrow$ TRUE

(g) p = -2:
$p - 1 = -3, -3^2 = 9, 9 \leq 15 \Rightarrow$ TRUE

Since each case results in a comparison that is true, we can conclude that the contrapositive is true.
Since the contrapositive is true, **the original statement is true**.

7. [**5 Pts**] Prove that there exist irrational numbers x and y whose sum is rational. Is your proof constructive or non-constructive? Explain.

   There exists some rational number that is the sum of two irrational numbers. To prove this, we will use a constructive proof, meaning we will prove that at least one case exists that causes it to be true.

   The case:
   $(5+\sqrt{2})+(5-\sqrt{2})$
   First, we'll use the corollary from number 4 (integer + irrational = irrational) to prove that both sides of the plus side are in fact irrational. There is no way to express either as the division of integers.
   Next, we can remove the parenthesis and switch the order of the first root two and the second 5. This is possible because addition is commutative.

   Rewritten (but still equal) equation: $5+5+\sqrt{2}-\sqrt{2}$
   Finally, we can add them up. $5+5=10$ and $\sqrt{2}-\sqrt{2}=0$. $10+0=10$.
   Our equation is equal to 10, which is a rational number.

   Therefore, there exists some irrational numbers x and y whose sum is rational.

For more practice, work on the problems from Rosen Sections 1.7 - 1.8; Lehman et al. Chapter 1.