CS 230 : Discrete Computational Structures
Fall Semester, 2023
ASSIGNMENT #9
Due Date: Thursday, Nov 16

Suggested Reading: Rosen Section 2.5; LLM Chapter 7.1

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[12 Pts]** Show that the following sets are countably infinite, by defining a bijection between $\mathbb{N}$ (or $\mathbb{Z}^+$) and that set. You do not need to prove that your function is bijective.

   (a) **[6 Pts]** the set of integers divisible by 6
   Let A = the set of integers divisble by 6.
   f(x) = 6x f: $\mathbb{N} \to A$
   A = { 6k | k $\in \mathbb{N}$ }

   (b) **[6 Pts]** $A \times \mathbb{Z}^+$ where $A = \{0, 1, 2, 3\}$

   This cartesian product creates a the following enumerated sets:
   {(0, 0), (1, 0), (2, 0), (3, 0) ... }
   {(0, 1), (1, 1), (2, 1), (3, 1) ... }
   {(0, 2), (1, 2), (2, 2), (3, 2) ... }
   {(0, 3), (1, 3), (2, 3), (3, 3) ... }

   This leaves us with four countably infinite sets, as $\mathbb{Z}^+$ is countably infinite, so it's product with a single element would be countably infinite.
   This is a finite amount of infinite sets, therefore proving $\{0, 1, 2, 3\} \times \mathbb{Z}^+$ is countably infinite.

2. **[7 Pts]** Prove that the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable, by using a diagonalization argument.

Assume for the purpose of contradiction that the set of functions F from $\mathbb{N}$ to $\mathbb{N}$ is countable. Therefore, it is possible to enumerate them, e.g. $f_1, f_2, f_3, ... = F$.
In order to use diagonalization, we need to prove that something is different element to element by at least one digit.
In order to do this, we will create the function p(n) $= f_i(n) + 1$.
This makes p(n) one digit different than every function, creating a new function that wasn't in the list. This is contradiction because F contains all functions from $\mathbb{N}$ to $\mathbb{N}$, but we know p(n) $\notin$ F, therefore proving that the set of functions F from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

3. [**12 Pts**] Determine whether the following sets are countable or uncountable. Prove your answer. To prove countable, describe your enumeration precisely, using dovetailing. There is no need to define a bijection.

   (a) [**6 Pts**] the set of real numbers with decimal representation consisting of all 9's (9.99 and 99.999... are such numbers).

   We can group the numbers in the set by how long they are, excluding the decimal point. This would group 9, .9 together, 9.9, 99, .99 together, and so on. This creates a infinite amount of finite sets, proving this set is countably infinite.

   (b) [**6 Pts**] the set of real numbers with decimal representation consisting of 8's and 9's.

   For the purpose of contradiction, let's assume that the set of real numbers with decimal representation consisting of 8's and 9's is countable.
   This would mean it can be enumerated as 8, 9, 9.8, 8.9, 8.8, 9.9, etc.

   Now let's take any element of the set, so x $\in$ A. For each digit in x, flip it. In other words, iterate through the digits of x, and if the digit is a 9, change it to an 8, but if the digit is an 8 change it to a 9. This would mean our new value is different from x. If we do this for every digit in the set, it would create values not in the set, creating a contradiction and therefore proving that the set of real numbers with decimal representation consisting of 8's and 9's is uncountably infinite.

4. **[9 Pts]** Give an example of two uncountable sets A and B (along with a justification) such that $A \cap B$ is (a) finite (b) countably infinite (c) uncountably infinite

(a) finite

Uncountable sets:

A is the set of all reals from [-1,1).

B is the set of all reals from (1, 3].

Both of these sets are unaccountably infinite because there are infinitely many unique decimals between the values. This is proven with diagonalization. However, there are no values that are in both A and B, meaning $A \cap B = \epsilon$, or the empty set, which contains no values. This is a finite set, showing that A and B are uncountably infinite and $A \cap B$ is finite.

(b) countably infinite

Uncountable sets:

A is the set of all reals from ($[-1, 1) \cup \mathbb{N}$).

B is the set of all reals from ($(1, 3] \cup \mathbb{N}$).

A and B are both uncountable - the sets [-1, 1] and [1, 3] are both uncountably infinite, as they contain an uncountable amount of unique decimals between the values. However, $(A \cap B)$ is countable, as the only common values are $\mathbb{N}$, which is countable. Therefore A and B are uncountably infite but the intersection between them is countably infinite.

(c) uncountably infinite

Uncountable sets:

A is the set of all reals from [0,1).

B is the set of all reals from [0,2).

Since [0,1) is contained within [0,2), we know that $A \subseteq B$.

Since A is uncountably infinite and $A \subseteq B$, then B is uncountably infinite because it is larger than an uncountably infinite set.

Since $A \subseteq B$, then $A \cap B = A$, as the intersection will only contain values from A. Since we already know A is uncountably infinite and $A \cap B = A$, then $A \cap B$ is uncountably infinte.