CS 230 : Discrete Computational Structures
Fall Semester, 2023
ASSIGNMENT #8
Due Date: Thursday, November 2

Suggested Reading: Rosen Sections 5.3; Lehman et al. Chapter 7

For the problems below, explain your answers and show your reasoning.

1. **[12 Pts]** Let S defined recursively by (1) $4 \in S$ and (2) if $s \in S$ and $t \in S$, then $st \in S$. Let $A = \{4^i \mid i \in \mathbf{Z}^+\}$. Prove that

   (a) **[6 Pts]** $A \subseteq S$ by mathematical induction.

   P(n) = $4^n \in S$
   Basis:
   P(1) = $4^1 = 4 \Rightarrow 4 \in S$, so P(1)

   Induction Step:
   Inductive Hypothesis: Assume P(k): $4^k \in S$
   Prove: P(k + 1): $4^{k+1} \in S$
   Since x = $4^1 \in S$ (By Basis Step) and y = $4^k \in S$ (By **IH**)
   By rule, if x, y ∈ S, then xy ∈ S.
   $4^{k+1} = 4^k \cdot 4^1$ = xy, so 4^{k+1} ∈ S.
   Therefore P(k+1).

   (b) **[6 Pts]** $S \subseteq A$ by structural induction.

   Basis:
   By Inductive Definition of S, 4 ∈ S.
   Since $4^1 = 4$ and $1 \in Z^+$, 4 ∈ A.

   Induction Step
   Consider x, y ∈ S. Assume x, y ∈ A. ← Inductive Hypothesis
   Since x, y ∈ S, xy ∈ S (via Inductive Definintion of S)
   Prove: xy ∈ A

   Since x, y ∈ S, x = 4^a and y = 4^b, a, b $\in Z^+$ because S contains multiples of 4.

   Therefore, xy = $4^a \cdot 4^b = 4^{a+b}$
   Since xy = 4^{a+b} and a+b $\in Z^+$, xy ∈ A by definition of A.
   Therefore, xy ∈ A.

2. [**5 Pts**] Give an inductive definition of the set of palindromes over the alphabet $\{a, b, c\}$. You do not need to prove that your construction is correct. *Note*: a, b, c, aa, cc, aba are all palindromes.

   Let S be the set of palindromes.
   Our set starts with all single characters, as any character by itself is a palindrome, and the empty set, ϵ.

   Basis: Strings a, b, c, and $\epsilon \in$ S

   Induction Step: If string $x \in$ S,
   Then: $axa \in S$
   $bxb \in S$
   $cxc \in S$

3. [**5 Pts**] Define the set $S = \{2^k 3^m \mid k, m \in \mathcal{Z}^+\}$ inductively. You do not need to prove that your construction is correct.

   Basis: $6 \in S \quad (2^1 \cdot 3^1 = 6)$
   Induction Step: If x $\in S$
   Then: 2x $\in$ S
   3x $\in$ S

4. [**8 Pts**] Given the inductive definition of full binary trees (FBTs), define $n(T)$, the number of vertices in tree T, and $\ell(T)$, the number of leaves in tree T, inductively. Then, use structural induction to prove that for all FBTs T, $n(T) = 2\ell(T) - 1$.

   $n(T)$: If the tree is one node, there is one vertex, so $n(T) = 1$.
   Otherwise, there are $1 + n(T_L) + n(T_R)$, one plus the number of vertices in the sub-trees.

   $\ell(T)$: If the tree is one node, there is one leaf, so $\ell(T) = 1$.
   If there is more than one node, $\ell(T) = 1 + \ell(T_L) + \ell(T_R)$.

5. **[15 Pts]** Let $L = \{(a, b) \mid a, b \in \mathcal{Z}, (a - b) \bmod 4 = 0\}$. We want to program a robot that can get to each point $(x, y) \in L$ starting at $(0, 0)$.

   (a) **[5 Pts]** Give an inductive definition of L. This will describe the steps you want the robot to take to get to points in L starting at $(0, 0)$. Let L' be the set obtained by your inductive definition.

   L':
   Basis: (0, 0), 0-0 = 0, 0/4 = 0, so (0, 0) $\in L'$
   Inductive Step: Given (a, b) $\in L'$,
   (a + 4, b) $\in L'$
   (a, b + 4) $\in L'$
   (a + 1, b + 3) $\in L'$
   (a + 3, b + 1) $\in L'$
   (a + 2, b + 2) $\in L'$

   (b) **[5 Pts]** Prove inductively that $L' \subseteq L$, i.e., every point that the robot can get to is in L.

   P(a, b) = (a - b) = 4m, (aka (a - b) is divisible by 4) = (a, b) is a reachable point

   Basis:
   P(0, 0) = 0 - 0 = 0, 0/4 = 0, so P(0, 0).
   Induction Step:
   Assume P(a, b) is a reachable square.
   Prove: (a - b) = 4m
   (a + 4, b): a + 4 - b = a - b + 4 = 4m + 4 = 4(m + 1), so (a + 4, b) $\in L'$
   (a, b + 4): a - b + 4 = 4m + 4 = 4(m + 1), so (a, b + 4) $\in L'$
   (a + 1, b + 3): a + 1 - b + 3 = a - b + 4 = 4m + 4 = 4(m + 1), so (a + 1, b + 3) $\in L'$
   (a + 3, b + 1): a + 3 - b + 1 = a - b + 4 = 4m + 4 = 4(m + 1), so (a + 3, b + 1) $\in L'$
   (a + 2, b + 2): a + 2 - b + 2 = a - b + 4 = 4m + 4 = 4(m + 1), so (a + 2, b + 2) $\in L'$

   (c) **[5 Pts] Extra Credit** Prove that $L \subseteq L'$, i.e., the robot can get to every point in L. To prove this, you need to give the path the robot would take to get to every point in L from $(0, 0)$, following the steps defined by your inductive rules.

For more practice, you are encouraged to work on other problems in Rosen Sections 5.3 and in LLM Chapter 7.