CS 230 : Discrete Computational Structures

Fall Semester, 2023

HOMEWORK ASSIGNMENT #5

Due Date: Thursday, October 12

Suggested Reading: Rosen Sections 9.1 and 9.5; Lehman et al. Chapter 10.5, 10.6 and 10.10

For the problems below, explain your answers and show your reasoning.

1. **[10 Pts]** For each of these relations decide whether it is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive. Justify your answers. R_1 and R_2 are over the set of real numbers.

   (a) $(x, y) \in R_1$ if and only if $xy = 4$

   R_1 is neither reflexive nor anti reflexive. Both can be proved by counterexample.

   R_1 is not reflexive. The definition of reflexive is $\forall a \in A, (a, a) \in R$. The counter example is that the pairs (1, 4) and (4, 1) are present yet (4, 4) and (1, 1) are not as they are not equal to four when multiplied.

   R_1 is not anti reflexive because the definition of anti reflexive is $\forall a \in A, (a, a) \notin R$ and the pair (2, 2) is present in the relation.

   R_1 is symmetric because multiplication is commutative, meaning a * b = b * a. Because of this, x * y = y * x, meaning (x, y) and (y, x) produce the same result in the relation. This allows us to conclude it is symmetrical, as x and y are interchangeable while producing the same result.

   This fits the definition of symmetric.
   $\forall a \in A, \forall b \in A, (a, b) \in R \wedge (b, a) \in R$

   R_1 is not transitive. This is easy to disprove with a counterexample. The definition of transitive is:
   $\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$
   Let a = 1, b = 4 and c = 1.
   (a, b) = (1, 4), 1 * 4 = 4.
   (b, c) = (4, 1), 4 * 1 = 4.
   (a, c) = (1, 1), 1 * 1 = 1 $\neq$ 4.
   Since (a, c) is not a member of R_1, it cannot be transitive because it directly conflicts the definition of transitive.

(b) $(x, y) \in R_2$ if and only if $x \geq 2y$

R_2 is anti reflexive. This is because the definition of anti reflexive is $\forall a \in A, (a, a) \notin R$, and anything multiplied by two will be greater than itself, invalidating the inequality.
For example, (2, 2) cannot be a member of R because $2 \geq 4$ cannot be true.

R_2 is neither symmetric nor anti symmetric.

R_2 cannot be symmetric because the definition of symmetric is
$\forall a \in A, \forall b \in A, (a, b) \in R \wedge (b, a) \in R$
and x must be at least two times greater than y for the inequality to be true. If x is not, 2y will be greater. For example, (10, 5) is in the relation but (5, 10) is not because $10 \geq 2 * 5$ is valid but $5 \geq 10 * 2$ is not valid.

R_2 cannot be anti symmetric because the definition of anti symmetric is
$\forall a \in A, \forall b \in A, [(a, b) \in R \wedge (b, a) \in R \rightarrow a = b]$ and as mentioned earlier, (a,b) and (b,a) cannot both be in the relation. To fulfill the definition of anti symmetric, (a,b) and (b,a) must both be present in the relation.

R_2 is not transitive because the definition of transitive is
$\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$
and this is not true for negative numbers, which are present in the set $\mathbb{R}$.
Let a = -4, b = -2 and c = -1.
$-4 \geq 2(-2)$ is true.
$-2 \geq 2(-1)$ is true.
$-4 \geq 2(-1)$ is NOT true.
Since R_2 does not fit the definition of transitive for all values, R_2 is not transitive.

2. **[5 Pts]** Consider relation R_3 defined below on the set of positive real numbers. Prove that R_3 is a partial order, *i.e.*, it is reflexive, anti-symmetric and transitive.

$$(x, y) \in R_3 \text{ if and only if } x/y \in \mathbb{Z}$$

The easiest one to prove is reflexive, which means that for all a, (a, a) is a member of R_3. This is easy to see because a/a will always be 1, regardless of what a is, and 1 is a member of $\mathbb{Z}$.
More formally, the definition of reflexive is $\forall a \in A, (a, a) \in R$.
If x = a and y = a, then for all a, $x/y = 1$, and $1 \in \mathbb{Z}$.

Next, let's prove that R_3 is anti symmetric. We know R_3 is NOT symmetric because division is not commutative, meaning x/y $\neq$ y/x. However, the definition of anti symmetric is
$\forall a \in A, \forall b \in A, [(a, b) \in R \wedge (b, a) \in R \rightarrow a = b]$
and R_3 fits this definition because the domain is positive **integers**, meaning non whole numbers are not valid.
Proof by cases:

(a) Case 1: b > a
(b, a) can be present in R_3 because b/a could possibly create an integer.
However, (a, b) cannot be present because if a > b, then a/b would be less than 1, meaning it is a fraction and not an integer, meaning it is not present in $\mathbb{Z}$.

(b) Case 2: a > b
Just like the last case, (a, b) can be in R_3 because a/b could possibly create an integer, but (b, a) cannot be present because it would be less than 1 and not an integer.

(c) Case 3: a = b
This case fulfills our definition!

Since the only case that would be valid is a = b, we can conclude that R_3 is anti symmetric because it fits the definition of anti symmetric.

R_3 is transitive because the definition of transitive is
$\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$
Let a/b = p, b/c = q and a/c = r where p, q, r are all integers and b, c are not zero.
b = cq (by algebra), a/b = a/cq (by substitution), a/cq = p, so a/c = pq.
Since p and q are both integers and we know that the product of two integers is an integer, a/c is an integer and would be in $\mathbb{Z}$, meaning R_3 fits the definition and we can conclude it is transitive.

3. [**7 Pts**] Consider relation R_4 defined below on the set of real numbers. Prove that R_4 is an equivalence relation. What is the equivalence class of 2? of π? Describe all the equivalence classes using set-builder notation.

$$(x, y) \in R_4 \text{ if and only if } a = b + 3n \text{ for some integer } n.$$

R_4 is reflexive because 0 is an integer, 3 * 0 = 0 and any value plus itself is equal to itself. This means (a, a) will be true for all a and n = 0 because a = a + 0 is valid. Since (a, a) is present for all a and the definition of reflexive is $\forall a \in A, (a, a) \in R$, we can conclude that R_4 is reflexive.

R_4 is symmetrical because $\mathbb{Z}$ contains negative numbers, so (a, b) is present for n = k and (b, a) is present for n = -k. The definition of symmetrical is $\forall a \in A, \forall b \in A, (a, b) \in R \wedge (b, a) \in R$, and this is true for R_4. Since it fits the definition of symmetrical, we can conclude that R_4 is symmetrical.

R_4 is transitive because the definition of transitive is
$\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$
Let's say that for (a, b), n is i and for (b, c) n is j, where i, j are integers. The difference for a and b would be 3i and the difference between b and c would be 3j. Therefore the difference between a and c is 3i + 3j, or 3(i + j). Since i + j is an integer because the sum of two integers is always an integer, i + j is an integer and would make the function valid. This means that $\forall a, b, c, \exists i, j$ to satisfy the equation. Because there exists some value to get from a to c, (a,c) will be in R if (a,b) and (b,c) are both in R, fulfilling the definition of transitivity. Since R_4 fits the definition, we can conclude R_4 is transitive.

Equivalence class for 2: $[2] = \{2, 5, 8, 11...2 + 3n_k\}$

Equivalence class for pi: $[\pi] = \{\pi, \pi + 3, \pi + 6, \pi + 9...\pi + 3n_k\}$

All equivalence classes will follow the pattern $\{k + 3n | n \in \mathbb{Z}\}$, where k is the number that the equivalence class is based off.

The equivalence classes are uncountably infinite because there are an infinite number of decimal places that can have an integer added to them.
The values within the classes are countably infinite because we are only adding integers, meaning it is any number k + each number in $\mathbb{Z}$, which is countably infinite.

4. **[6 Pts]** Let R_5 be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ where $((a, b), (c, d)) \in R_5$ if and only if $a/c = b/d$.

(a) Prove that R_5 is an equivalence relation.
R_5 is reflexive because f(a, b) = f(a, b) evaluates to a/a = b/b, which is true and therefore would be in the relation, fulfilling the definition of reflexive, $\forall a \in A, (a, a) \in R$.

R_5 is symmetrical because the numerators and denominators are interchangeable, so (a, b) would be the same as (b, a). This fulfills the definition of $\forall a \in A, \forall b \in A, (a, b) \in R \wedge (b, a) \in R$.

R_5 is transitive because f(a, b) and f(b, c) imply f(a, c) because a/c = c/d is valid. This fits the definition of transitive, $\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$.

Because R_5 is reflexive, symmetrical and transitive, R_5 is an equivalence relation.

(b) Define a function f such that $f(a, b) = f(c, d)$ if and only if $((a, b), (c, d)) \in R_5$.
f(x, y) = x * n = y

(c) Define the equivalence class containing $(1, 1)$.
The equivalence class containing (1, 1) is either any 1/c = 1/d, where c, d $\neq$ 0 or where c/1 = d/1 or any scalar upon these, meaning this equivalence class is any k * a/c or b/d.

(d) Describe the equivalence classes using set-builder notation.
$\forall c, d\{(n * 1/c = 1/d) | n \in \mathbb{R}\}$

5. **[12 Pts]** Prove that these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations. Describe the equivalence classes.

   (a) $R_6 = \{(f, g) \mid f(0) = g(0)\}$

   R_6 is reflexive because f(0) = f(0), so (a, a) would be in the relation for all A, fitting the definition of reflexive, $\forall a \in A, (a, a) \in R$.

   R_6 is symmetric because the relation checks if they are equal, and equality is commutative. f(0) = g(0) is the same as g(0) = f(0), so (f, g) is the same as (g, f), meaning they both appear in the set but do not need to be equal, so long as they are equal when 0 is inputted. This means R_6 fits the definition of symmetric, $\forall a \in A, \forall b \in A, (a, b) \in R \wedge (b, a) \in R$.

   R_6 is transitive because equality is transitive, if f(0) = g(0) and g(0) = h(0), then f(0) = h(0). This is a well known concept in math that works because f(0) = g(0) = h(0) and we can just take out the g(0). Because of the transitivity of equals, R_6 fits the definition of transitive, $\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$, we can conclude R_6 is transitive.

   Since R_6 is reflexive, symmetric and transitive, R_6 is an equivalence function.

   The equivalence classes for R_6 are uncountably infinite, as any real or imaginary number can be multiplied by 0 to make 0, so f(x) = px + r and g(x) = qx + r for all p, q, r.

   The elements within the class are also uncountably infinite because the functions can produce any real or imaginary value, so long as f(0) = g(0).

(b) $R_7 = \{(f, g) \mid \exists C \in \mathbb{Z}, \forall x \in \mathbb{Z}, f(x) - g(x) = C\}$

R_7 is reflexive because 0 is an integer and f(0) - f(0) = 0 for all f, fulfilling the definition of reflexive, $\forall a \in A, (a, a) \in R$.

R_7 is symmetric because $\mathbb{Z}$ includes negative integers, and if f(x) - g(x) = C then g(x) - f(x) = -C, which is also an integer and thus would be in the relation. Therefore (f, g) and (g, f) would both be in R_7, fulfilling the definition of symmetric, $\forall a \in A, \forall b \in A, (a, b) \in R \wedge (b, a) \in R$.

R_7 is transitive because the sum of two integers is an integer and subtraction is just addition with a negative number, so the difference of two integers would also be an integer. Therefore any (f, h) would be in the relation if both (f, g) and (g, h) are in the relation because f(x) - h(x) = (f(x) - g(x)) + (g(x) - h(x)), which would produce an integer. This fulfills the definition of transitive, $\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$, which means we can conclude R_7 is transitive.

R_7 is an equivalence relation because it is reflexive, symmetric and transitive.

The equivalence classes for R_7 would be uncountably infinite because as proved in a previous homework, the sum or difference of two irrational numbers can still produce an integer, meaning f(x) and g(x) could contain any real or imaginary number. This can be proved with the imaginary number i, which is the square root of -1. We can make an integer by taking any k and performing (k + i) - i, which equals k (an integer). This means there are an uncountably infinite amount of equivalence classes.

The values in the equivalence classes are countably infinite because they have to be in $\mathbb{Z}$, as $C \in \mathbb{Z}$, as specified in the problem. This means that there would be at most $\mathbb{Z}$ values in each class, which is countably infinite.