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CS 230 : Discrete Computational Structures

Fall Semester, 2023
HOMEWORK ASSIGNMENT #1
Due Date: Tuesday, September 5

Suggested Reading: Rosen Sections 1.1 - 1.3; LLM Sections 1.1, 3.1 - 3.4

These are the problems that you need to hand in for grading. Always explain your answers and show your reasoning.

1. **[6 Pts]** Translate the following English sentences into logic. First, define your basic propositions and use logical operations to connect them.
(a) If n is prime, then $n = 2$ or n is odd.

   p: n is a prime number.
   q: $n = 2$.
   r: n is odd.
   $p \rightarrow (p \vee r)$

   (b) The differentiability of f is sufficient for f to be continuous and smooth.

   p: f is differentiable.
   q: f is continuous.
   r: f is smooth.
   $p \leftrightarrow (p \wedge r)$

   (c) Being a good programmer and having good communication skills are necessary to get a good job.

   p: Steve is a good programmer.
   q: Steve is a good communicator.
   r: Steve will get a good job.
   $(p \wedge q) \rightarrow r$

2. **[6 Pts]** Determine whether $((p \to q) \land (q \to r)) \leftrightarrow (p \to r)$ is a tautology using truth tables.

p	q	r	$p \to q$	$(p \to q) \land (q \to r)$	$q \to r$	$((p \to q) \land (q \to r)) \leftrightarrow (p \to r)$	$p \to r$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	F	F	F	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	T	F
F	T	F	F	F	T	F	T
F	F	T	T	F	F	T	F
F	F	F	T	T	T	T	T

Since there are two possible p, q, and r combinations that result in the compound proposition being false (T, F, T) and (F, T, F), we can conclude that the compound proposition is not a tautology.

3. **[6 Pts]** Prove that $(p \to (q \to r))$ and $(q \to (p \to r))$ are logically equivalent by deduction using a series of logical equivalences studied in class (truth tables will not be allowed).

1. $(p \to (q \to r))$	Given
2. $(p \to (\neg q \vee r))$	Implication Rule 1
3. $\neg p \vee (\neg q \vee r)$	Implication Rule 1
4. $\neg p \vee \neg q \vee r$	Associative Law
5. $\neg q \vee \neg p \vee r$	Commutative Law
6. $\neg q \vee (\neg p \vee r)$	Associative Law
7. $\neg q \vee (p \to r)$	Implication Rule 1
8. $q \to (p \to r)$	Implication Rule 1
9. $p \to (q \to r) \equiv q \to (p \to r)$	Conclusion

Since I have used a series of logical operations to get from the starting point, $p \to (q \to r)$, to our ending point, $q \to (p \to r)$, we can conclude that the two compound propositions are logically equivalent.

4. **[10 Pts]** Use logical reasoning to solve the following puzzle:
A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook. The cook and the gardener are either both telling the truth or they are both lying. Either the gardener or the handyman is telling the truth, but not both. If the handyman is lying then so is the cook. Determine which witnesses are lying and which are telling the truth. Define four basic propositions and use these to describe the four statements above. Then use deductive reasoning, not truth tables, to derive the truth of each of those basic propositions.

Propositions:
b: The butler is telling the truth.
c: The cook is telling the truth.
g: The gardener is telling the truth.
h: The handyman is telling the truth.

Statements:
1. If the butler is telling the truth then so is the cook $\equiv b \rightarrow c$
2. The cook and the gardener are either both telling the truth or they are both lying $\equiv c \leftrightarrow g$
3. Either the gardener or the handyman is telling the truth, but not both $\equiv g \oplus h$
4. If the handyman is lying then so is the cook $\equiv \neg h \rightarrow \neg c$

Assumption: the butler is telling the truth.
Using statement 1 ($b \rightarrow c$), the cook is telling the truth since the cook will tell the truth if the butler does.
Using statement 2 ($c \leftrightarrow g$), the gardener is telling the truth because c and g must have the same value.
Using statement 3 ($g \oplus h$), the handyman is lying since the gardener is telling the truth.
Using statement 4 ($\neg h \rightarrow \neg c$), the cook must be lying since the cook will lie if the handyman does.
This gives us a contradiction. Statements 1 and 4 say opposite things. This means our assumption is false. **The butler must be lying.**

Assumption: the cook is lying.
Using statement 2 ($c \leftrightarrow g$), the gardener must be lying because c and g will always have the same value.
Using statement 3 ($g \oplus h$), the handyman is telling the truth because h and g will always have opposite values.

Since our second assumption didn't cause any contradictions, we can conclude it is true.

This means that the **handyman** is **telling the truth** and the **gardener**, **butler** and **cook** are **lying**.

5. **[6 Pts]** A proposition is said to be in CNF (conjunctive normal form) if it is a conjunction (and) of one or more clauses, where each clause is a disjunction (or) of basic propositions or their negations. For example, $(p \vee \neg q) \wedge (\neg p \vee \neg q \vee r) \wedge (q \vee r)$ is in CNF. Now argue that any compound proposition is logically equivalent to some proposition in CNF and show your construction. *Hint: Look at the argument on DNF in the notes and come up with some sort of algorithm. Start by constructing a truth table.*

   Any compound proposition can be written in CNF by creating a truth table and using it to create a separate CNF proposition. As long as they have the same truth table, they are logically equivalent, just like any other compound proposition. To do this, first find the false values of the truth table and take note of them. Connect each value with an or. Do that for each row with a false value, then put them in parenthesis and connect with an and. Any variable that is true can be left and any that is false should get negated. Put it all together and you should have a result that lines up with the truth table. This is essentially the opposite creating a DNF proposition.

   To prove this, I have created a random truth table. Function F does not exist and I made up the values. However, I have created a CNF proposition that lines up with the truth table.

p	q	r	F(p, q, r)
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

   $(\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee \neg q \vee r)$

   As you can see, this is a compound proposition in CNF that has the same truth table as the random truth table above, proving that any compound proposition is able to be written in CNF.

6. **[6 Pts]** Prove that $\{\vee, \oplus, \text{TRUE}\}$ is functionally complete, i.e.,any propositional formula is equivalent to one whose only connectives are $\vee$, $\oplus$, and the constant TRUE. Prove using a series of logical equivalences. You may assume any logical equivalences we studied in class and the fact that any formula is equivalent to some formula in DNF. *Note: If you make the statement that a set of operators is functionally complete, and use this in your proof, then you need to justify your statement.*

   To prove something is functionally complete, we need to make $\{\vee, \neg\}$ as that set is functionally complete (explanation later in answer). For this problem, we only need to make NOT because OR is given.

   First, $\neg$.
   We can make $\neg p$ using $(p \oplus TRUE)$.

   This is valid because when p is false, the expression $p \oplus TRUE$ will be true because the XOR operator is true when exactly one operand is true. Conversely, when p is true the expression will be false because the XOR operator is false when the operands have the same truth value.

   We have a functionally complete set with $\{\vee, \neg\}$. It is a known fact that disjunctive normal form is a functionally complete set, however, the AND operator is actually not necessary. This is because we can use De'Morgan's law to make the AND operator with negation and the OR operator.

1. $p \wedge q$	Starting point
2. $\neg\neg(p \wedge q)$	Double Negation Law - adding two does nothing
3. $\neg(\neg p \vee \neg q)$	De Morgans Law
4. $p \wedge q \equiv \neg(\neg p \vee \neg q)$	Conclusion

   Because we have made AND from only NOT and OR, we have shown that the set $\{\vee, \oplus, \text{TRUE}\}$ is functionally complete because we have made the functionally complete set $\{\vee, \wedge, \neg\}$ with only the operators given.

For more practice, you are encouraged to work on the problems given at the end of Rosen, Sections 1.1 - 1.3.