CS 230 : Discrete Computational Structures
Fall Semester, 2023
ASSIGNMENT #7
Due Date: Thursday, October 26

Suggested Reading: Rosen Sections 5.2 - 5.3; Lehman et al. Chapters 5, 6.1 - 6.3

For the problems below, explain your answers and show your reasoning.

1. **[8 Pts]** Prove that $f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ where f_i are the Fibonacci numbers.

P(n) $= f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n}$
Prove: P(n) $= = f_{2n-1} - 1$ for all n $\in F$ (Fibonnaci numbers)

Fibonnaci Sequence: 1, 2, 3, 5, 8, 13, 21, 34, 55 ... (For reference)

Basis:
P(1) = 1 = 1,   2(1) - 1 = 1   P(1) is true

Induction Step:
Induction Hypothesis: P(k) $= f_{2k-1} - 1$
Prove: P(k+1) $= f_{2(k+1)-1} - 1 = f_{2k+2-1} - 1 = f_{2k+1} - 1$

P(k) $= f_0 - f_1 + f_2 - \cdots - f_{2k-1} + f_{2k} = f_{2k-1} - 1$
P(k+1) $= f_0 - f_1 + f_2 - \cdots - f_{2k-1} + f_{2k} - f_{2k+1} + f_{2(k+1)}$
P(k+1) $= f_0 - f_1 + f_2 - \cdots - f_{2k-1} + f_{2k} - f_{2k+1} + f_{2k+2}$
P(k+1) = P(k) - $f_{2k+1} + f_{2k+2}$ (via **IH**)
P(k+1) $= f_{2k-1} - 1 - f_{2k+1} + f_{2k+2}$ (via **IH**)
P(k+1) $= f_{2k-1} - 1 - f_{2k+1} + f_{2k} + f_{2k+1}$ by Def. of Fibonacci Number
P(k+1) $= f_{2k-1} - 1 + f_{2k}$
P(k+1) $= f_{2k-1} - 1 + f_{2k}$
P(k+1) $= f_{2k-1} + f_{2k} - 1$
P(k+1) $= f_{2k+1} - 1$ by Def. of Fibonacci Number

Since we have proven P(k+1) $= f_{2k+1} - 1$, we have proven P(n) $= f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ for all n $\in F$ (Fibonnaci numbers) using Mathematical Induction.

2. **[8 Pts]** Consider the following state machine with six states, labeled 0, 1, 2, 3, 4 and 5. The start state is 0. The transitions are $0 \to 1$, $0 \to 2$, $1 \to 3$, $2 \to 4$, $3 \to 5$, $4 \to 5$, and $5 \to 0$.

Prove that if we take n steps in the state machine we will end up in state 0 if and only if n is divisible by 4. Argue that to prove the statement above by induction, we first have to *strengthen the induction hypothesis.* State the strengthened hypothesis and prove it.

The reason regular induction will not work is because there are two possible cases for P(0) and because we need to prove that it is not , Since there are two cases we cannot prove it based on only one. P(0) can lead to 1 or 2, which have separate cases in the Induction Step from the two possible states of the strengthened hypothesis.

State Machine Definition:

States: {0, 1, 2, 3, 4, 5, 6}
Start State: 0

Transitions Relation: $\{0 \to 1, 0 \to 2, 1 \to 3, 2 \to 4, 3 \to 5, 4 \to 5, 5 \to 0\}$

P(n) = After n steps, the state machine will be at state k
Prove: P(n) is at 0 if and only if there exists some m where n = 4m (n is divisible by 4).

Basis:
P(0) = 0 steps, so the SM is at 0, 0 | 4 is true, so P(0).

P(1) = 1 step, $0 \to 1$ or $0 \to 2$, so SM is at 1 or 2 and NOT at 0, so ¬ P(1).
1 | 4 is false, so ¬P(1).

P(2) = 2 steps, $1 \to 3$ or $2 \to 4$, so the SM is at 3 or 4 and NOT at 0, so ¬ P(2).
2 | 4 is false, so ¬P(2).

P(3) = 3 steps, $3 \to 5$ or $4 \to 5$, so the SM is at 5 and NOT at 0, so ¬ P(3).
3 | 4 is false, so ¬P(3).

P(4) = 4 steps, $5 \to 0$ so the SM is at 0, so P(4).
4 | 4 is true, so 4 is divisible by 4, so P(4).

Induction Step:
Inductive Hypothesis: P(k), which means k steps, and SM is at 0 iff $k \mid 4$ for all $k > 4$.

Prove: P(k+1), which means k + 1 steps, prove SM is at 0 iff $k+1 \mid 4$ for all $k > 4$.

Since we know that $k+1 > 4$ and that every 4 steps we return to the start state (via **IH**), we can take (k+1) - 4m, where m is the greatest integer where k+1 is still greater than 4m. This allows us to reduce (k+1) to 1, 2, 3, or 4 and gives us four clear cases that were established in the basis.

Case 1: (k+1) - 4m = 1
1 step, $0 \rightarrow 1$ or $0 \rightarrow 2$, so SM is at 1 or 2 and NOT at 0, so $\neg$ P(k+1).
$1 \mid 4$ is false, so $\neg$P(k+1).

Case 2: (k+1) - 4m = 2
2 steps, $1 \rightarrow 3$ or $2 \rightarrow 4$, so the SM is at 3 or 4 and NOT at 0, so $\neg$ P(k+1).
$2 \mid 4$ is false, so $\neg$P(k+1).

Case 3: (k+1) - 4m = 3
3 steps, $3 \rightarrow 5$ or $4 \rightarrow 5$, so the SM is at 5 and NOT at 0, so $\neg$ P(k+1).
$3 \mid 4$ is false, so $\neg$P(k+1).

Case 4: (k+1) - 4m = 4
4 steps, $5 \rightarrow 0$ so the SM is at 0, so P(k+1).
$4 \mid 4$ is true, so 4 is divisible by 4, so P(k+1).

Since P(k+1) would only be true if (k+1) - 4m = 4 and (k+1) - 4m = 4 means that $(k+1) \mid 4$ is true, we have proven that the SM will be at 0 iff $k+1 \mid 4$ for all $k > 4$.

3. [**8 Pts**] Let $P(n)$ be the statement that n can be written as a sum of distinct powers of 2. For example, $5 = 2^0 + 2^2$ and $10 = 2^1 + 2^3$. Note that $10 = 2^0 + 2^0 + 2^3$ is not a sum of *distinct* powers of 2. Prove that $P(n)$ is true for all positive integers n by strong induction.

P(n) = $2^k + 2^j + \cdots + 2^i$ (AKA n can be written by sum of distinct powers of two)

Prove: P(n) for all n $\in Z^+$

Basis:
First we need to prove P(1).
P(1) = 2^0 = 1

Induction Step:
Induction Hypothesis: P(k) can be constructed with distinct powers of two
Prove: P(k+1) can be constructed by distinct powers of two

Case 1: k + 1 is a power of two
If k+1 is a power of two, then k+1 = 2^j for some j, so P(k+1).

Case 2: k + 1 is not a power of two
If k + 1 is not a power of two, that means k + 1 = 2^i + j, where 2^i is the greatest power of two that satisfies k + 1 > 2^i.
k + 1 - 2^i = j. This means that j cannot be larger than k, since the lowest i can be is 0 and 2^i = 1, so k = j. Since P(n) is true for $1 \leq n \leq k$ via the **IH** and j cannot be greater than k, we know that P(j) can be represented with distinct powers of two.

The only caveat is that P(j) may contain a duplicate power of two. However, this is impossible because k+1 must be less than 2^{i+1}, as 2^i is the greatest power of two that satisfies k + 1 > 2^i. This makes it impossible to contain a duplicate because unless k is a power of two, P(j) would not contain that value as $j \leq k$, and if k is a power of two we can add a 2^0 to it to create P(k+1) with distinct values.

Because P(k+1) can be true for either case, we have proven P(k+1) can be constructed with distinct powers of two and we have proven P(k+1) for all k $\in Z$.

4. [**8 Pts**] A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (+1,+3), (+1,-1), (-4,0) and (0,+4). Define a state machine for this problem. Then, define a Preserved Invariant and prove that the robot will never get to (1,1).

State Machine Definition:

States: $\{(a, b) \mid a, b \in Z\}$
Start State: (0,0)

Transitions Relation:

$$\{(a, b) \rightarrow (a + 1, b + 3),$$
$$(a, b) \rightarrow (a + 1, b - 1),$$
$$(a, b) \rightarrow (a + 4, b),$$
$$(a, b) \rightarrow (a, b + 4) \mid (a, b) \in Z\}$$

Preserved Invariant: (a, b) is in the set if (a + b) | 4

Basis:
(0,0) is in the set because 0+0 = 0 | 4

Induction Step:
Assume (a_1, b_1) is a possible square to get to
Inductive Hypothesis: (a+b) = 4m where m is any integer. (This means a + b is divisible by 4)
Prove that (a_2, b_2) is a possible square to reach if $a_2 + b_2$ — 4

(a_1, b_1) can only become (a_2, b_2) through the relation, meaning the only possible changes are (a + 1, b + 3), (a + 1, b - 1), (a + 4, b), and (a, b + 4).

Case 1: (a + 1, b + 3)
a + b + 1 + 3 = 4m + 1 + 3 = 4m + 4 = 4(m + 1) (subbed (a+b) for 4m from (**IH**)
Divisible by 4
Case 2: (a + 1, b - 1) (subbed (a+b) for 4m from (**IH**)
a + b + 1 - 1 = a + b = 4m
Divisible by 4
Case 3: (a + 4, b)
a + 4 + b = 4m + 4 = 4m + 4 = 4(m + 1) (subbed (a+b) for 4m from (**IH**)
Divisible by 4
Case 4: (a, b + 4)
a + b + 4 = 4m + 4 = 4m + 4 = 4(m + 1) (subbed (a+b) for 4m from (**IH**)
Divisible by 4

Since all cases show that reachable squares are divisible by four, we have proven that (a, b) is reachable if (a+b) | 4 via Mathematical Induction.
Because (1,1) does not add to be divisible by 4, it is not a reachable square based on the preserved invariant.

5. [**8 Pts**] Show that if a predicate $P(n)$ can be proven true for all positive integers n by strong induction, then it can be proven true also by regular induction, once you strengthen the inductive hypothesis. In other words, Strong Induction isn't really stronger than Regular Induction. *Hint: Given any statement $P(n)$, define a new (stronger) statement $Q(n)$ so that proving $P(n)$ by strong induction is similar to proving $Q(n)$ by regular induction.*

Strong induction uses every predicate before P(k+1) to prove P(k+1), aka
$P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$.
If P(n) holds true for P(1) through P(k+1), we can redefine P(n) into Q(n), where Q is a predicate that is more specific and holds true for P(1) and $P(k) \rightarrow P(k+1)$.

Now that we know Q(n) is true for Q(1) and $Q(k) \rightarrow Q(k+1)$, we would be able to use induction to assume Q(k) and prove Q(k+1), proving strong and regular induction are the same things but regular requires a stronger inductive hypothesis.

For more practice, you are encouraged to work on other problems in Rosen Sections 5.2 and 5.3 and in LLM Chapter 5, 6.1 - 6.3.