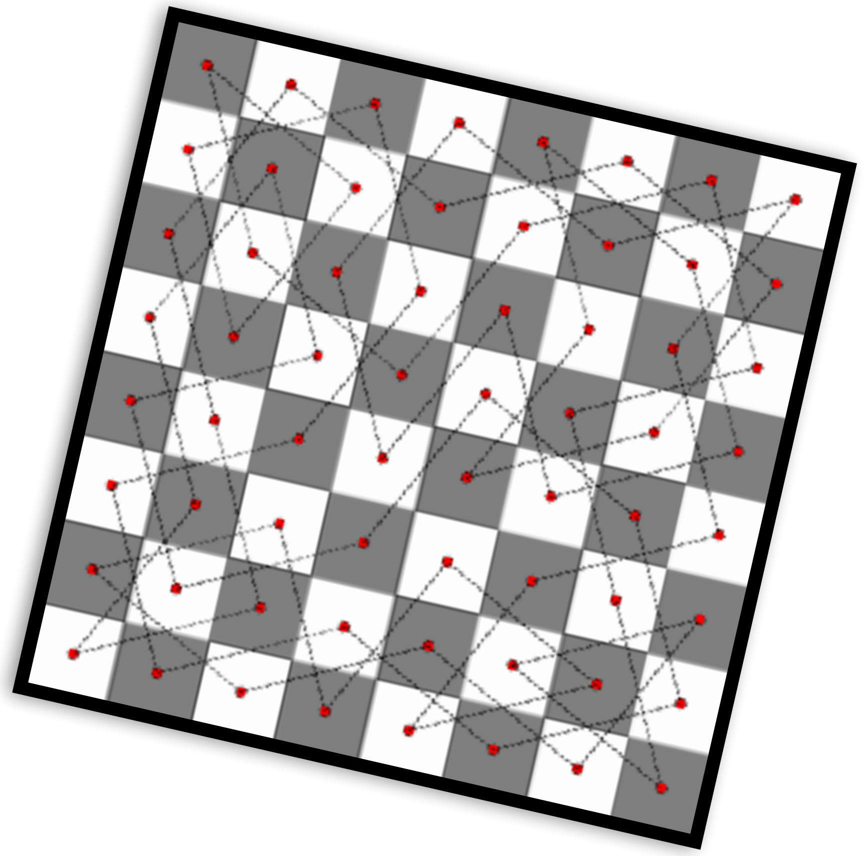
Advanced Dynamic Programming using informed search

LINFO2266

Pierre Schaus



Shortest Path Algorithm: Dijkstra

- Dijkstra solves the single source-shortest path problem.

- Starting from vertex v, it finds all the shortest path toward all the other vertices of a graph.

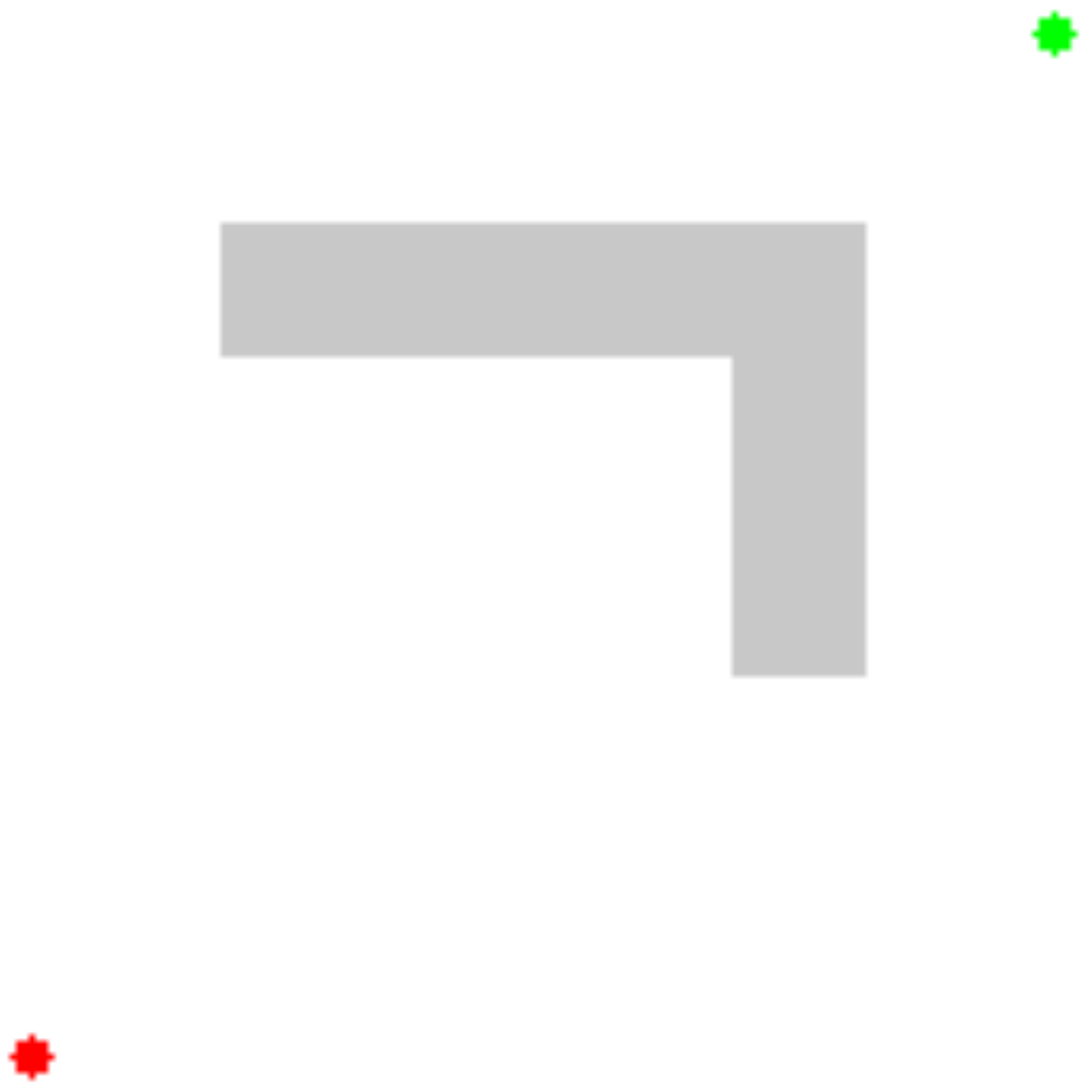
Dijkstra Algorithm (Dijkstra 1959)

```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, and a
      source vertex $s \in V$.
Result: The shortest distance $d[v]$ from $s$ to every vertex $v \in V$.
for each vertex $v \in V$ do
    $d[v] \leftarrow \infty$ ;               // Shortest path distance from $s$ to $v$
    $\pi[v] \leftarrow$ NIL ;                 // Predecessor in the shortest path
end
$d[s] \leftarrow 0$;
$C \leftarrow \emptyset$ ;                                              // Closed Set
$O \leftarrow \emptyset$ ;                          // Open-Set:  Min-priority queue
Insert$(Q, s, d[s])$ ;
while $Q \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $S \leftarrow S \cup \{u\}$;
    for each vertex $v \in Adj[u]$ do
        if $d[v] > d[u] + w(u, v)$ then
            $d[v] \leftarrow d[u] + w(u, v)$;
            $\pi[v] \leftarrow u$;
            InsertOrUpdateKey$(O, v, d[v])$ ;
        end
    end
end
```

$O(E \log V)$ using binary heap for the priority queue

$O(E \log V)$ using binary heap for the priority queue

Dijkstra illustration

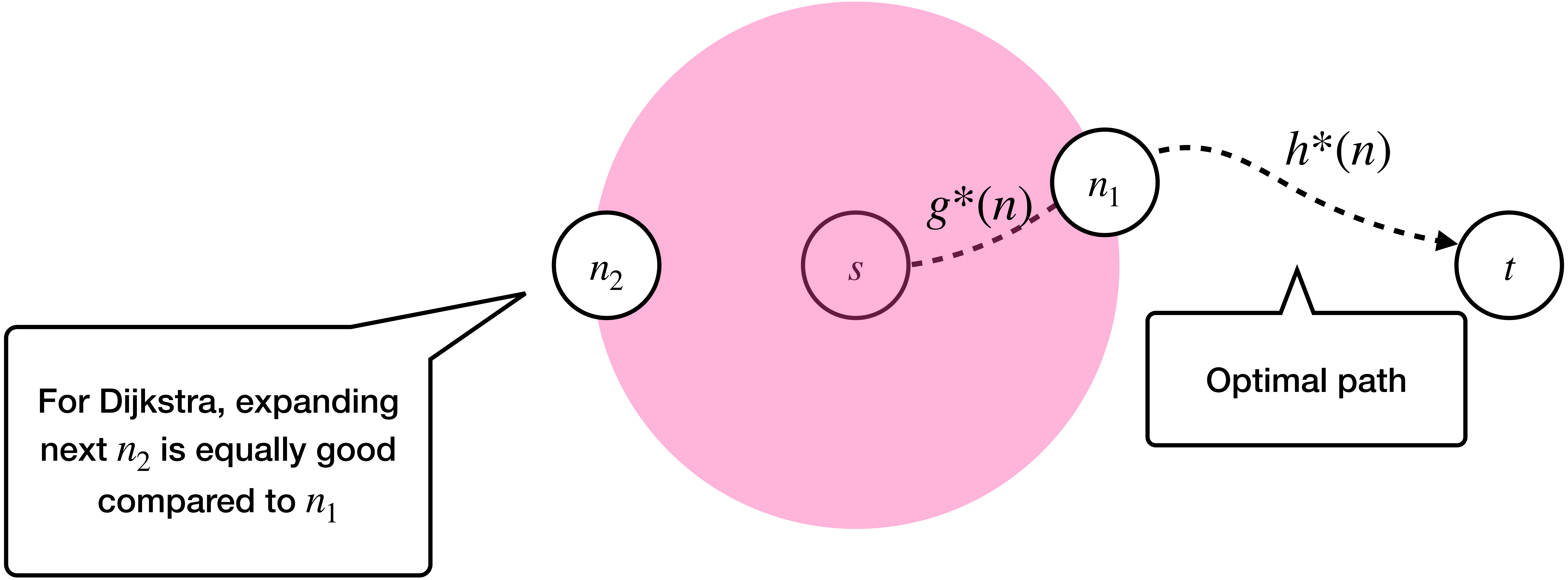


Subh83 wikimedia CC BY 3.0

- The empty circles represent the nodes in the *open set* O, i.e., those that remain to be explored, and
- The filled circles are in the closed set C.

Dijkstra is great but

- It doesn’t look to reach the target, for Dijkstra the target is a regular node of the graph, until you reach it (in this case you can stop Dijkstra earlier).

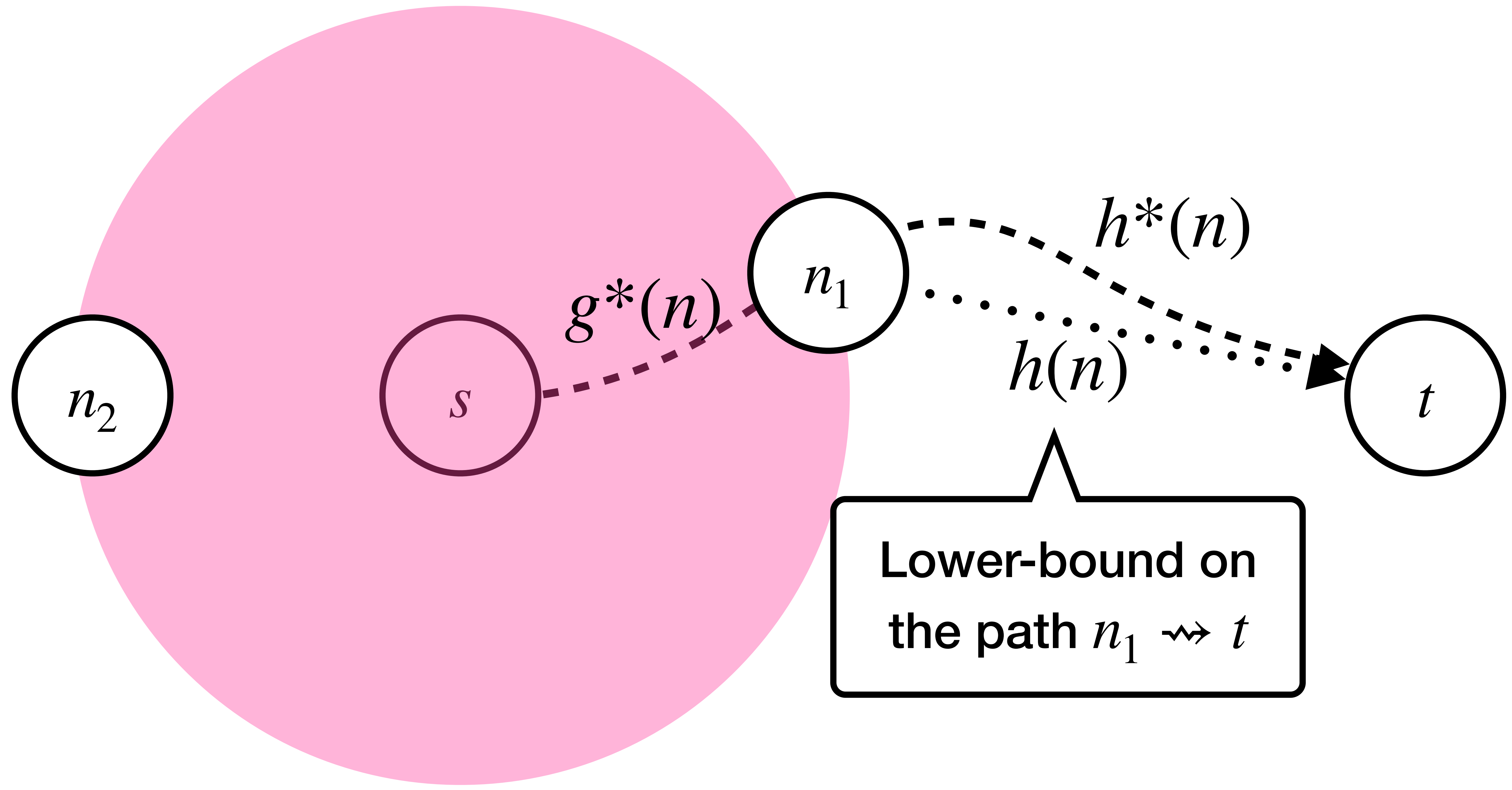


For Dijkstra, expanding next n_2 is equally good compared to n_1

- Dijkstra will thus visit a lot of nodes! We can do better

💡 Use heuristics

- For Euclidian shortest path, you can for instance use the Euclidian distance (straight line distance) to estimate the distance left.



For Dijkstra, expanding next n_2 is equally good compared to n_1

Lower-bound on the path $n_1 \rightsquigarrow t$

- Instead of expanding the node based on $g(n)$ we will expand it based on $g(n) + h(n)$

A* Algorithm (Hart, Nilsson, Raphael 1968)

```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                    // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                    // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                     // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                                // Closed Set
$O \leftarrow \emptyset$ ;        // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                          // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;     // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

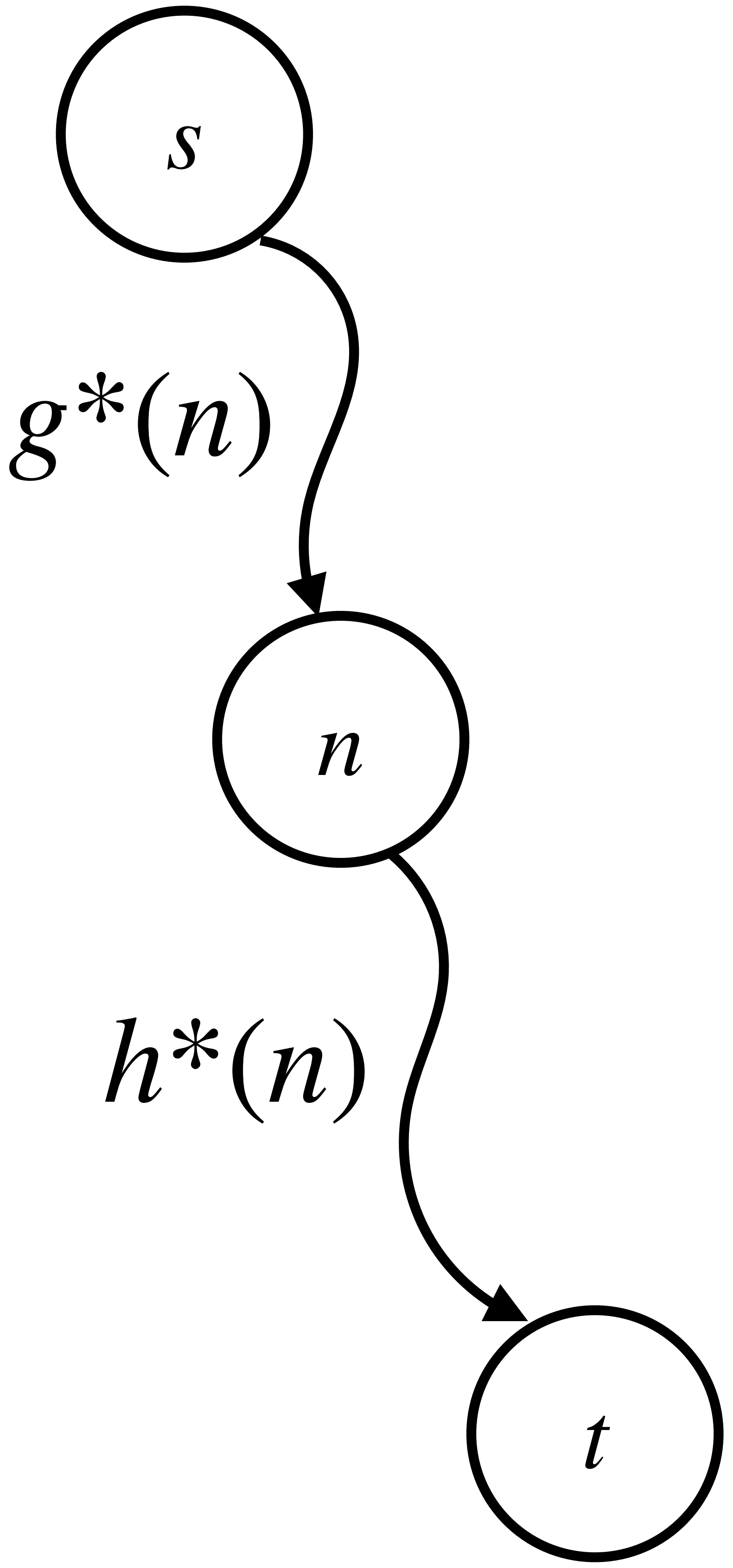
Properties Heuristics h and impact on A*

- A **search algorithm** is **admissible** if it is guaranteed to find an optimal solution
- A **heuristic h** is **admissible** if it never over-estimates the cost to the solution

A* using an admissible heuristic h is admissible

A* with admissible h is admissible (proof by contradiction)

- C^* = optimal cost from s to t, let's assume A* returned a suboptimal one with cost $C > C^*$
- Then there must be some node n on the optimal path that was left unexpanded (i.e. not closed) otherwise we would have returned that solution. Let's denote $g^*(n)$ the cost of the optimal path from s to n, and $h^*(n)$ the cost of the optimal path from n to t. We have:
- $f(n) > C^*$ (otherwise n would have been expanded)
- $f(n) = g(n) + h(n)$ (by definition)
- $f(n) = g^*(n) + h(n)$ (because n is on an optimal path)
- $f(n) \leq g^*(n) + h^*(n)$ (because of admissibility of $h(n) \leq h^*(n)$)
- $f(n) \leq C^*$ (by definition $C^* = g^*(n) + h^*(n)$) contradiction!



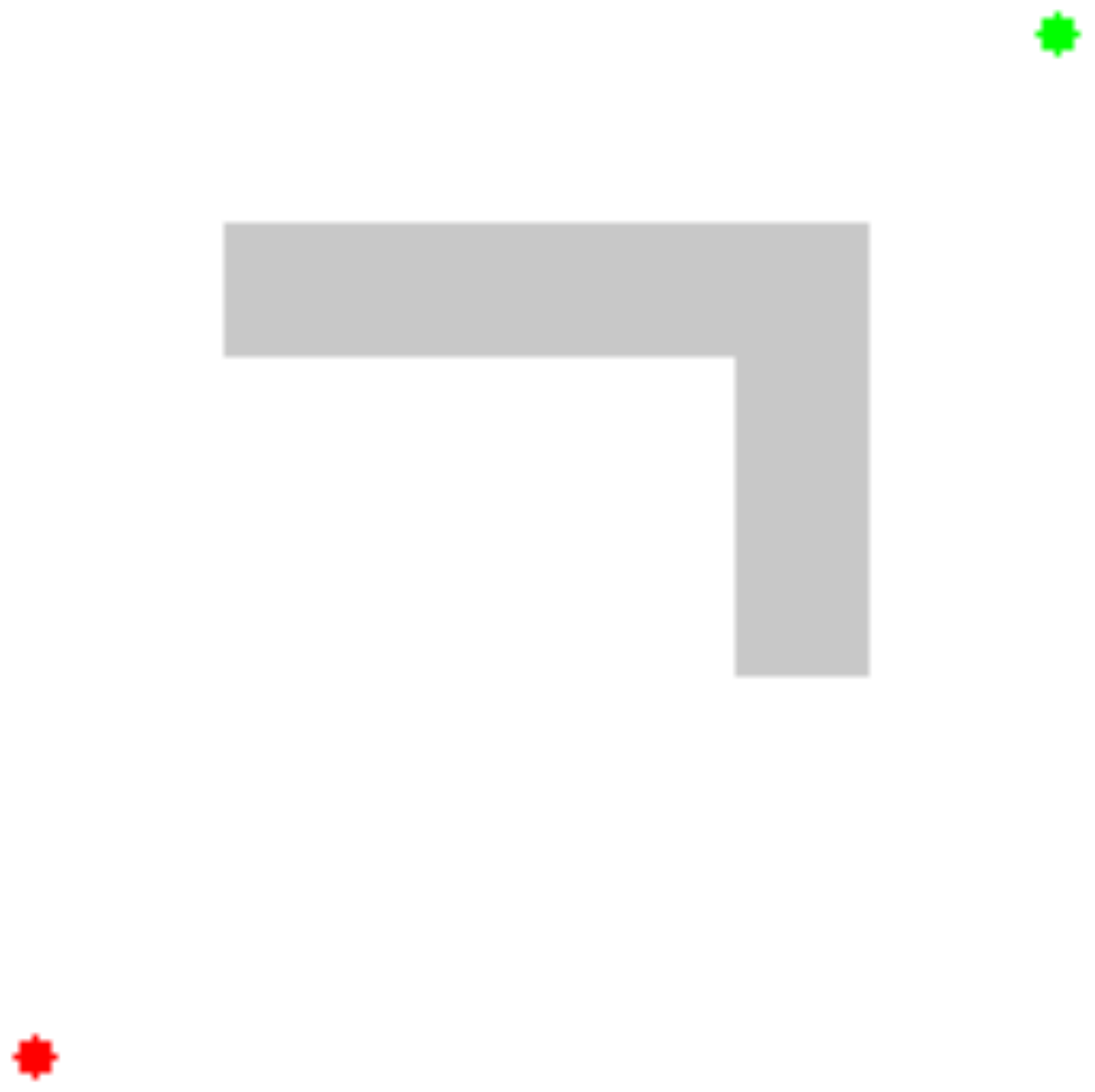
Consistent heuristics

- A heuristic is consistent if for every node n and every successor node n' of n, we have that $h(n) \leq w(n, n') + h(n')$

- With consistent heuristics, the first time we reach a state, it will be on an optimal path, so we never have to re-add it to the queue or to change it's priority.

- In practice, it is quite difficult to design a heuristic that is admissible but not consistent.

A* illustration



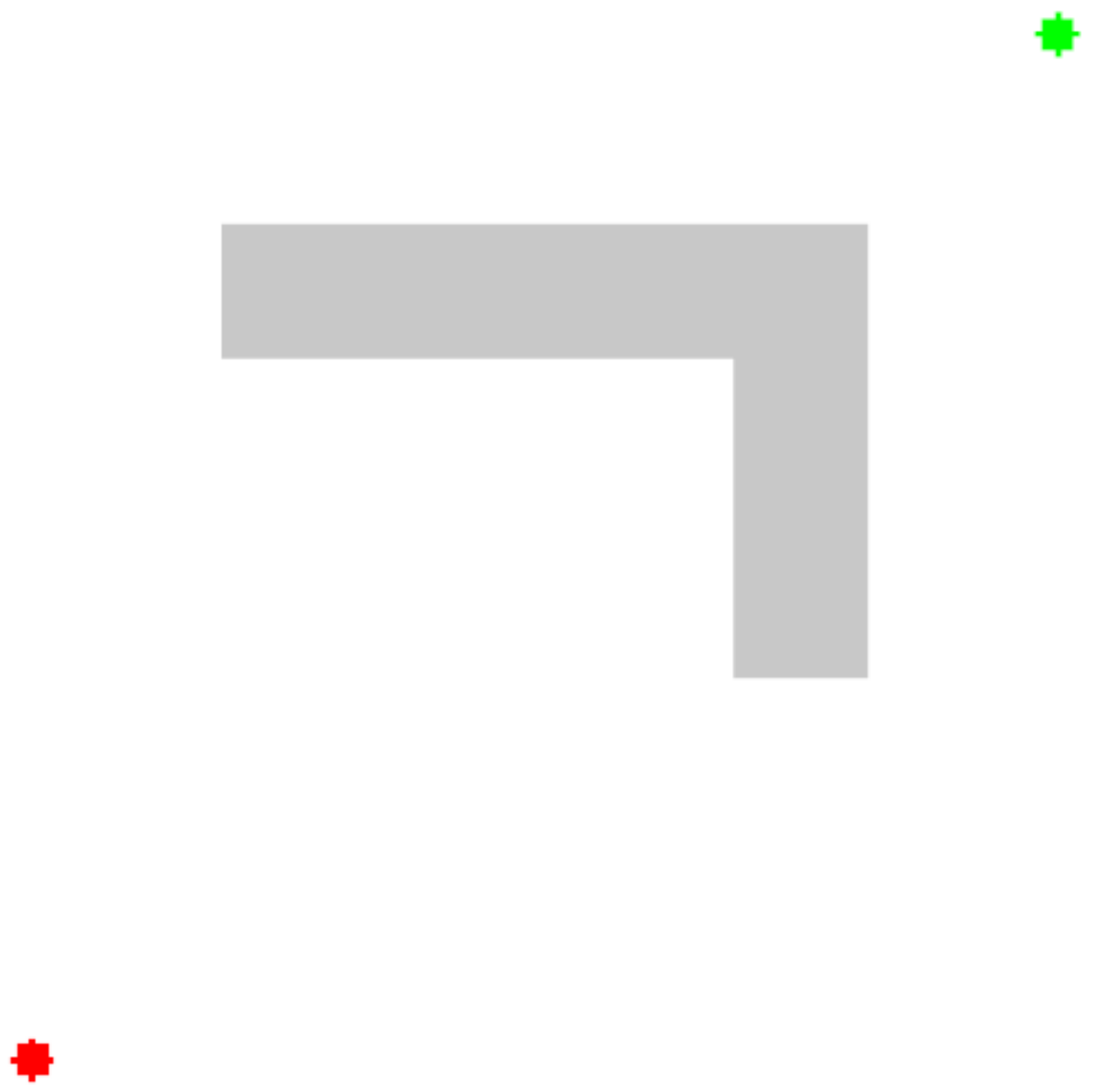
Subh83 wikimedia CC BY 3.0

- The empty circles represent the nodes in the *open set* O, i.e., those that remain to be explored, and
- The filled circles are in the closed set C.
- Color on each closed node indicates the distance from the goal: the greener, the closer.

Weighted A* Algorithm

```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, a heuristic $h(v)$, a weight parameter $w_h \geq 1$ (controls
      heuristic influence).
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                         // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                     // Estimated total cost $(g + w_h \cdot h)$
    $\pi[v] \leftarrow$ NIL ;                     // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow w_h \cdot h(s)$;
$C \leftarrow \emptyset$ ;                                                   // Closed Set
$O \leftarrow \emptyset$ ;       // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t]$ ;                                         // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + w_h \cdot h(v)$ ;              // Weighted score
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return $g[t], \pi$;
```

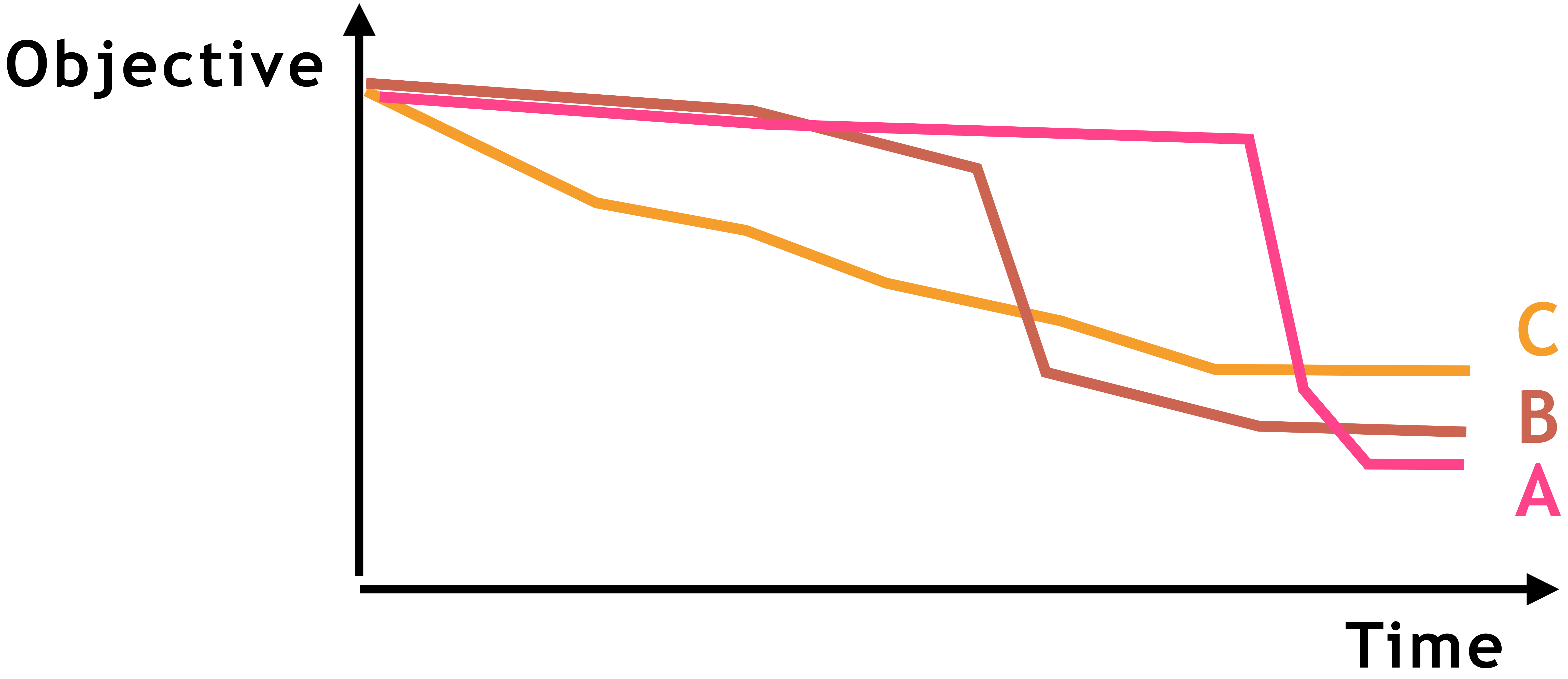
Weighted A* illustration: w = 5



Subh83 wikimedia CC BY 3.0

Reminder: Anytime Algorithm

- An algorithm has good anytime behavior when it is able to find high-quality solutions, even when when the search is stopped before completion.



What algorithm do you prefer ?

Berthold, T. (2013). Measuring the impact of primal heuristics. *Operations Research Letters*, *41*(6), 611-614.

Is it possible to improve the anytime behavior of A* algorithms ?

- First idea: use weighted A*, and decrease gradually the weight w util reaching 1 (w=1 is standard A*)

- Better idea (Hansen & Zoo 2007), bring three changes to (weighted) A*:

1. Use an non-admissible evaluation function $f'(n) = g(n) + h'(n)$, where $h'(n)$ is not admissible (for instance $h'(n) = w \cdot h(n)$ with $w \geq 1$).

2. Continue the search after a solution (reaching node t) is found, but use the cost $g(t)$ as an upper-bound (branch and bound like).

3. Use an admissible $f(n) = g(n) + h(n)$ as lower-bound, and prune it when it is larger than the current upper-bound.

Anytime Weighted A*

```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, a heuristic $h(v)$, a weight parameter $w_h \geq 1$.
Result: The cost of the best solution found, $g_{\text{inc}}$, and the incumbent
        path $\pi_{\text{inc}}$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                          // Cost of best known path $s \rightsquigarrow v$
    $f'[v] \leftarrow \infty$ ;                               // Weighted cost:  $g + w_h \cdot h$
    $\pi[v] \leftarrow$ NIL;                                    // Best known predecessor
$g[s] \leftarrow 0$;
$f'[s] \leftarrow w_h \cdot h(s)$;
$g_{\text{inc}} \leftarrow \infty$ ;                                            // Incumbent cost
$C \leftarrow \emptyset$ ;      // Closed Set (Contains nodes that were expanded)
$O \leftarrow \emptyset$ ;        // Open Set (Min-priority queue ordered by $f'[v]$)
Insert$(O, s, f'[s])$;
while $O \neq \emptyset$ and not user interrupted do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $g[u] + h(u) > g_{inc}$ then
        continue ;                            // Pruning by incumbent cost
    for each vertex $v \in Adj[u]$ do
        if $g[u] + w(u, v) + h(v) \geq g_{inc}$ then
            continue ;                        // Pruning by incumbent cost
        else if $g[u] + w(u, v) < g[v]$ then
            $\pi[v] \leftarrow u$ ;
            $g[v] \leftarrow g[u] + w(u, v)$ ;
            $f'[v] \leftarrow g[v] + w_h \cdot h(v)$ ;
            if $v = t$ then
                $g_{\text{inc}} \leftarrow g[t]$ ;                // Incumbent cost update
            else
                InsertOrUpdateKey$(O, v, f'[v])$;
                $C \leftarrow C \setminus \{v\}$ ;
return $g_{\text{inc}}, \pi$ ;              // $O = \emptyset$ (optimal) or was interrupted
```

Solving Dynamic Programs with A* like algorithms

- Knapsack Problem:

The set of items: I

Variables $x_i \in \{0,1\}$, 1 iff item i selected

Objective: Maximize $\sum_{i \in I} v_i \cdot x_i$

Maximize value of selected items

Constraints: $\sum_{i \in I} w_i \cdot x_i \leq C$

Under capacity constraint

DP formulation of the Knapsack Problem

- Assume $I = \{1 \ldots n\}$
- Notation $O(k, i)$ = optimal objective with capacity k and items $\{1 \ldots j\}$

$$O(k, i) = \begin{cases} 0 & \text{if } i = 0 \\ \max(O(k, j-1), v_j + O(k - w_j, j-1)) & \text{if } w_j \leq k \\ O(k, j-1) & \text{otherwise} \end{cases}$$

The optimal solution is $O(C, n)$

Dynamic Programming = recursive method with caching of the values $O(k, i)$

Knapsack as a minimization problem

- Knapsack Problem:

The set of items: I

Variables $x_i \in \{0,1\}$, 1 iff item i selected

Objective: Minimize $\sum_{i \in I} -v_i \cdot x_i$

Minimize negative value of selected items

Constraints: $\sum_{i \in I} w_i \cdot x_i \leq C$

Under capacity constraint

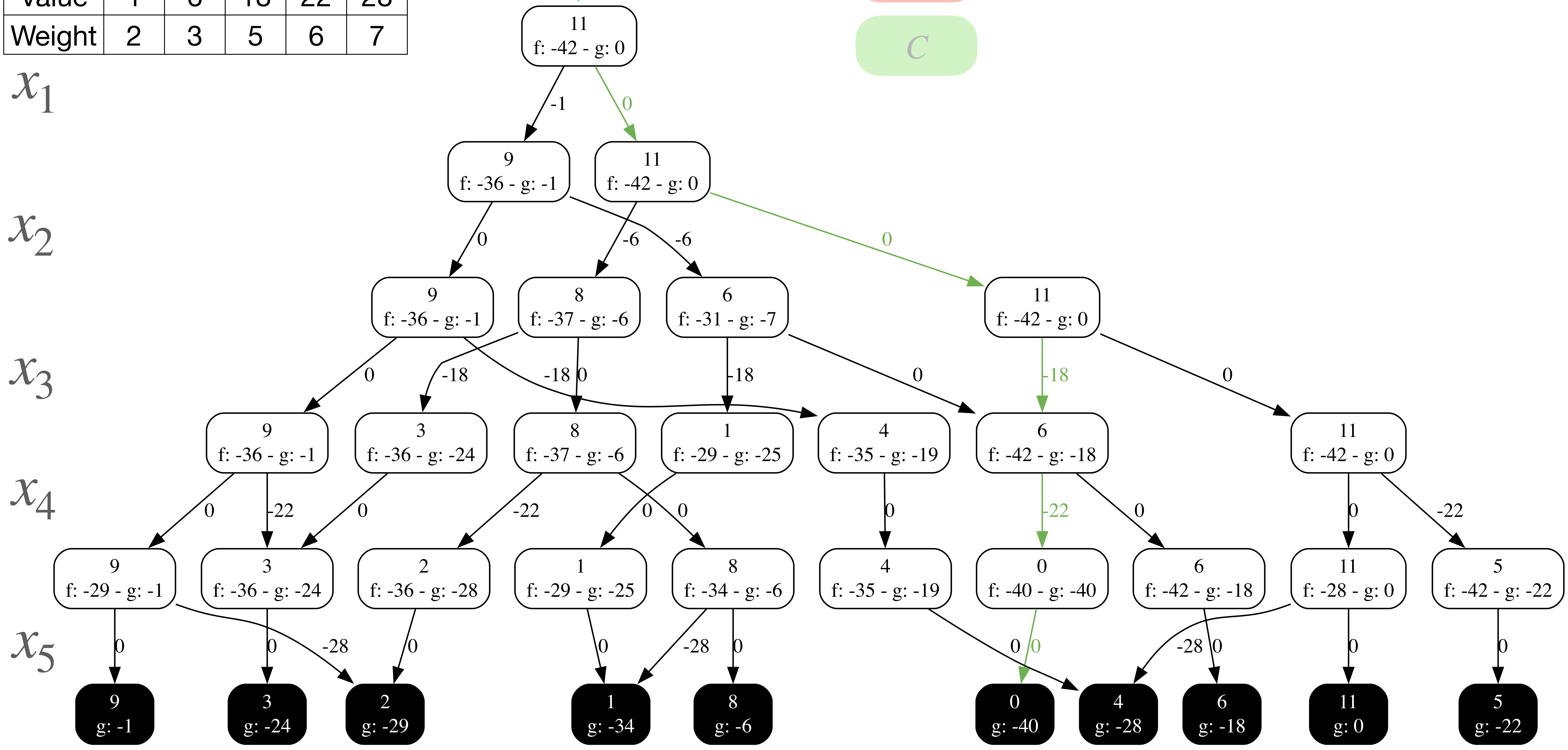
Dynamic Programming Resolution = Shortest Path on DAG

Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7

Initial state = C

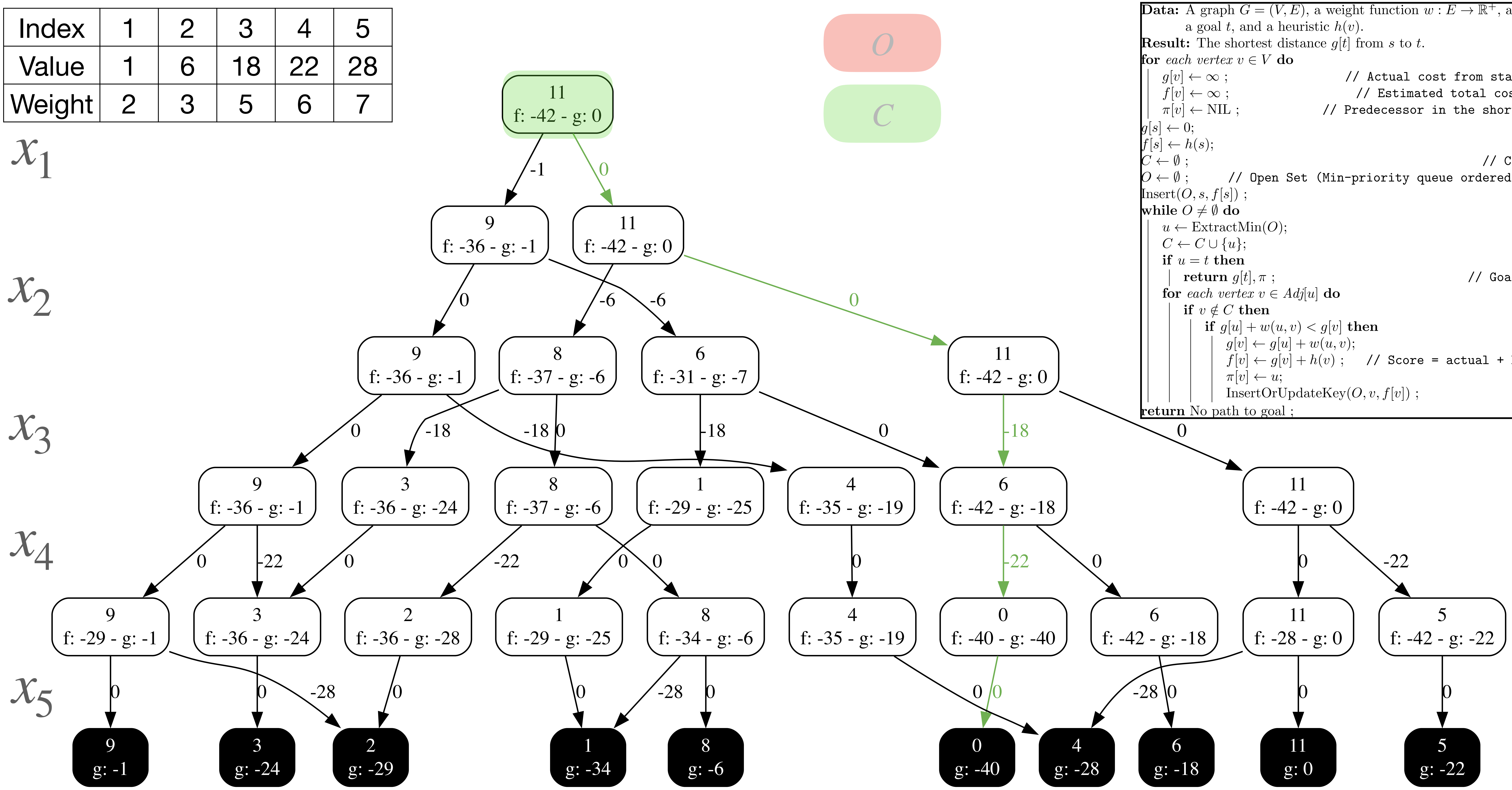
O

C



Resolution with A*, h = linear relaxation

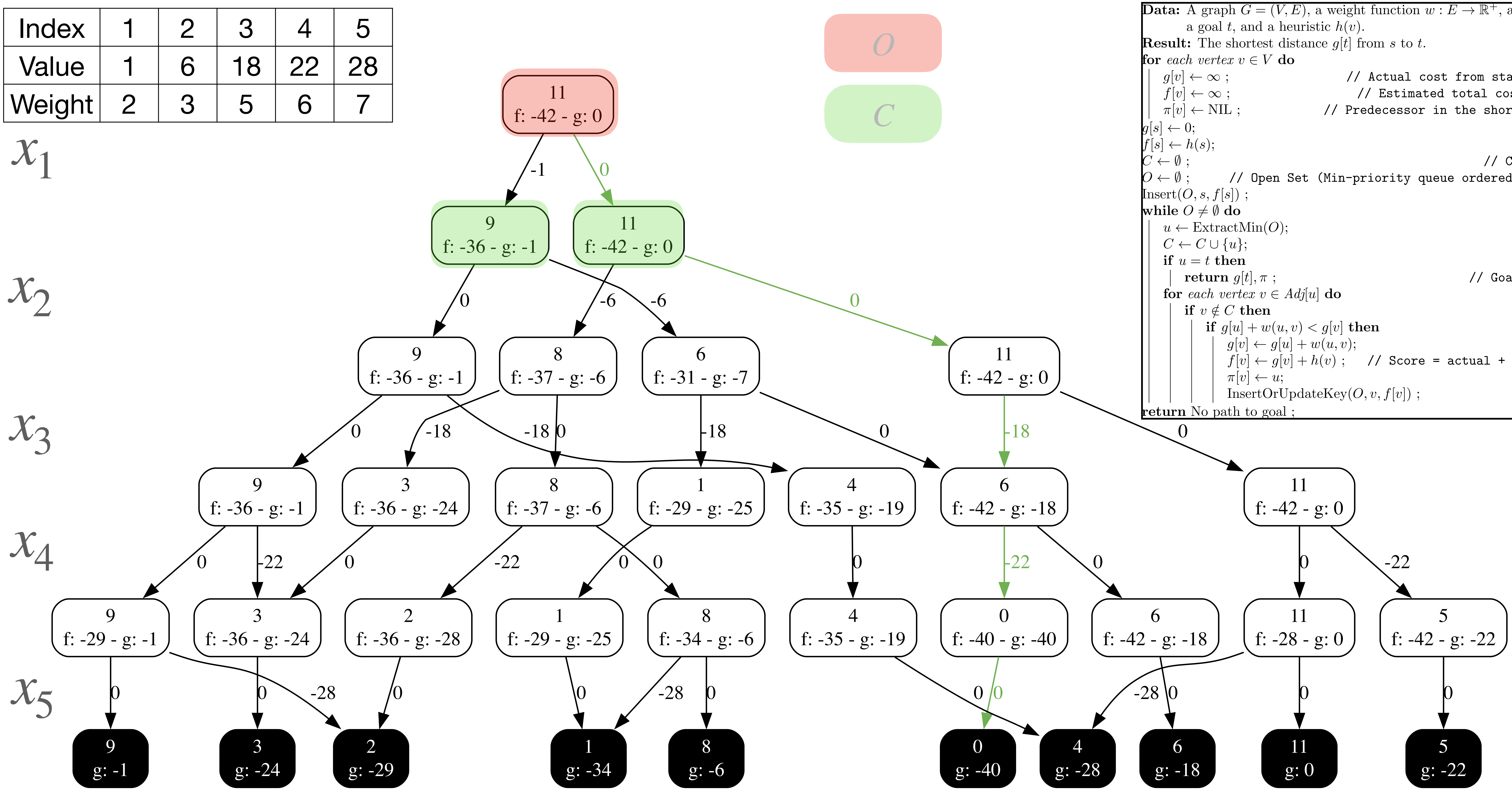
Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7



```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                  // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                  // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                   // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                           // Closed Set
$O \leftarrow \emptyset$ ;    // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                      // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

Resolution with A*, h = linear relaxation

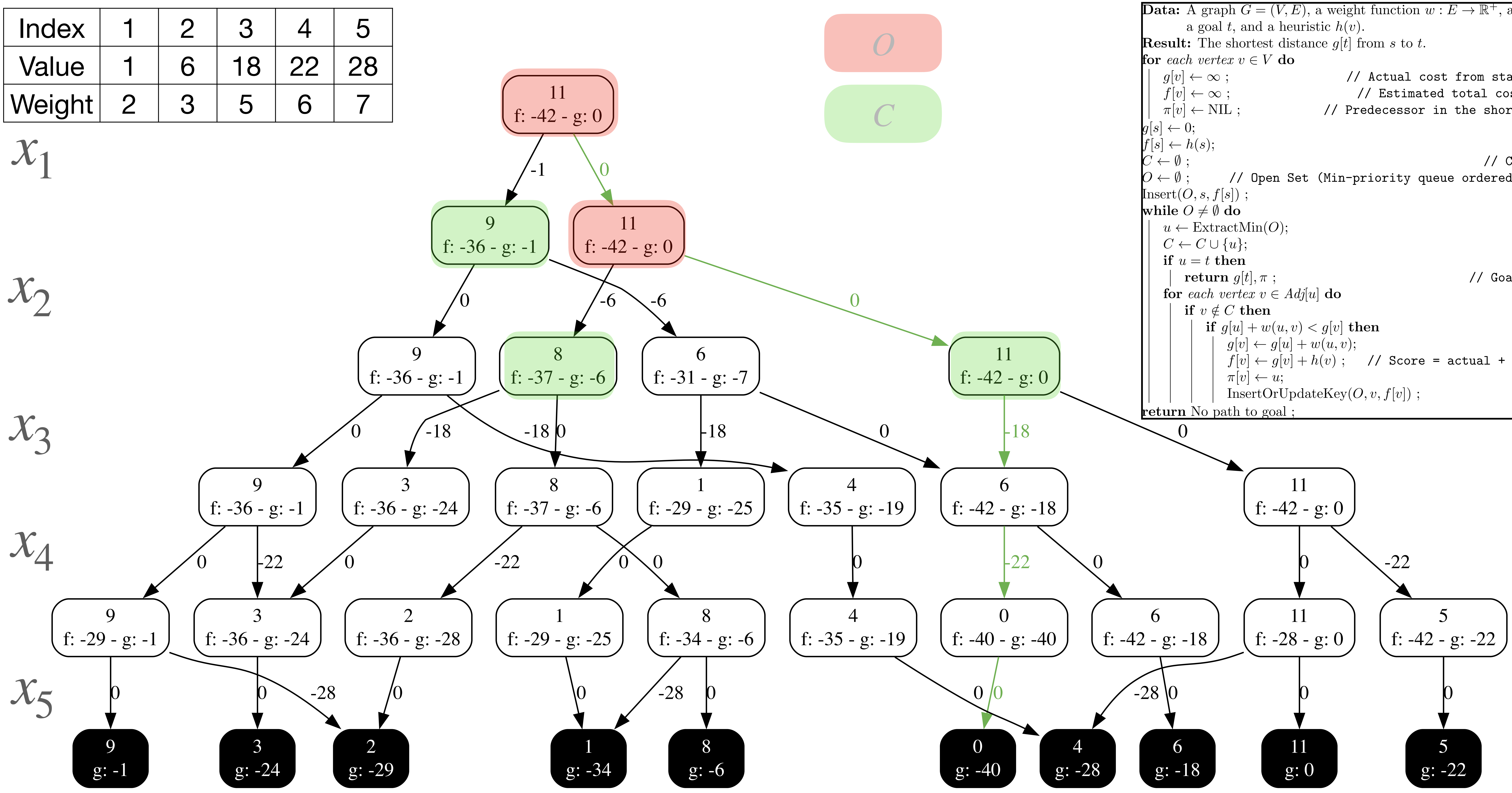
Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7



```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                    // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                     // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                  // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                               // Closed Set
$O \leftarrow \emptyset$ ;      // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                       // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;   // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

Resolution with A*, h = linear relaxation

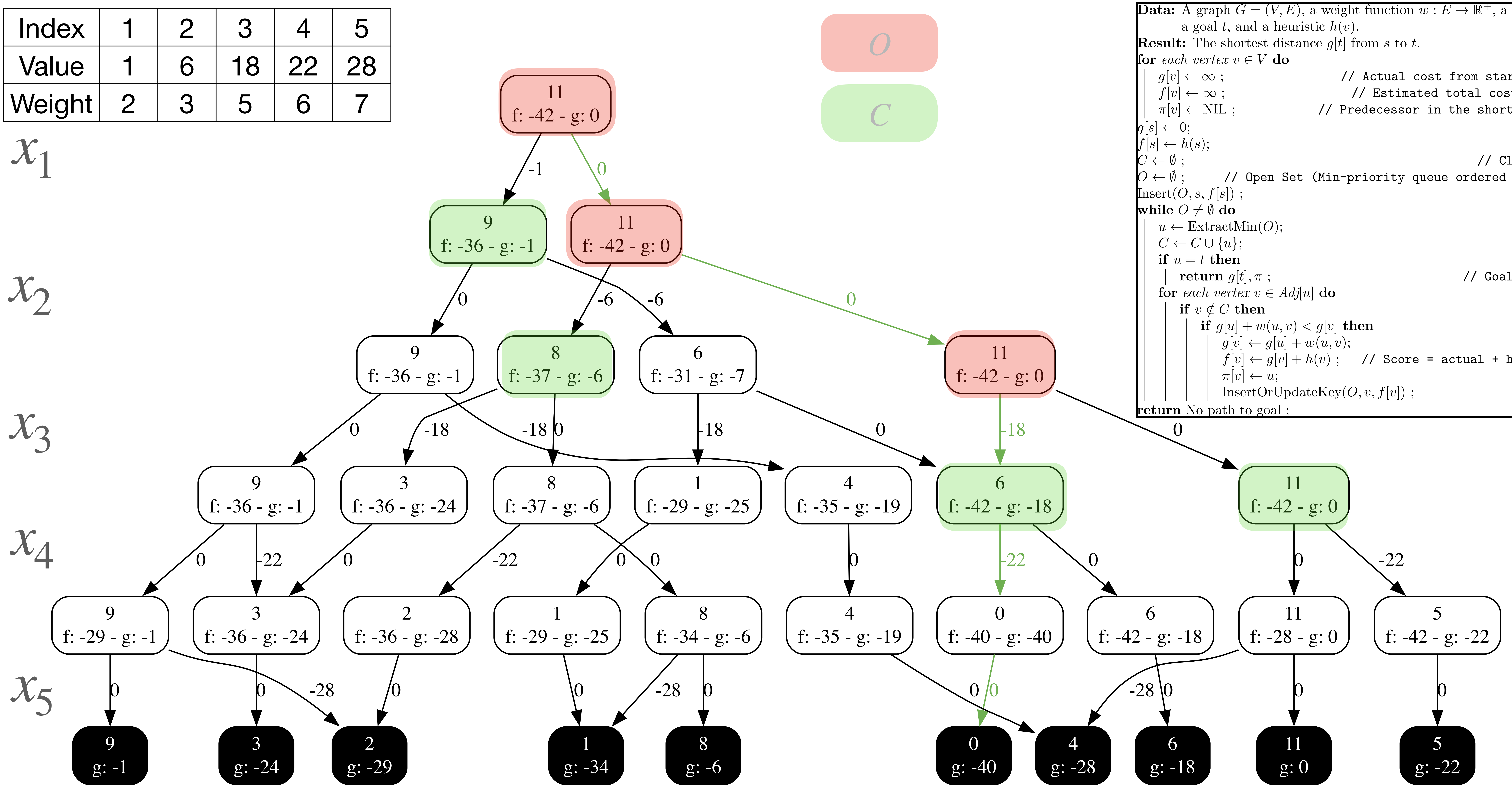
Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7



```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                    // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                     // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                           // Closed Set
$O \leftarrow \emptyset$ ;      // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                   // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

Resolution with A*, h = linear relaxation

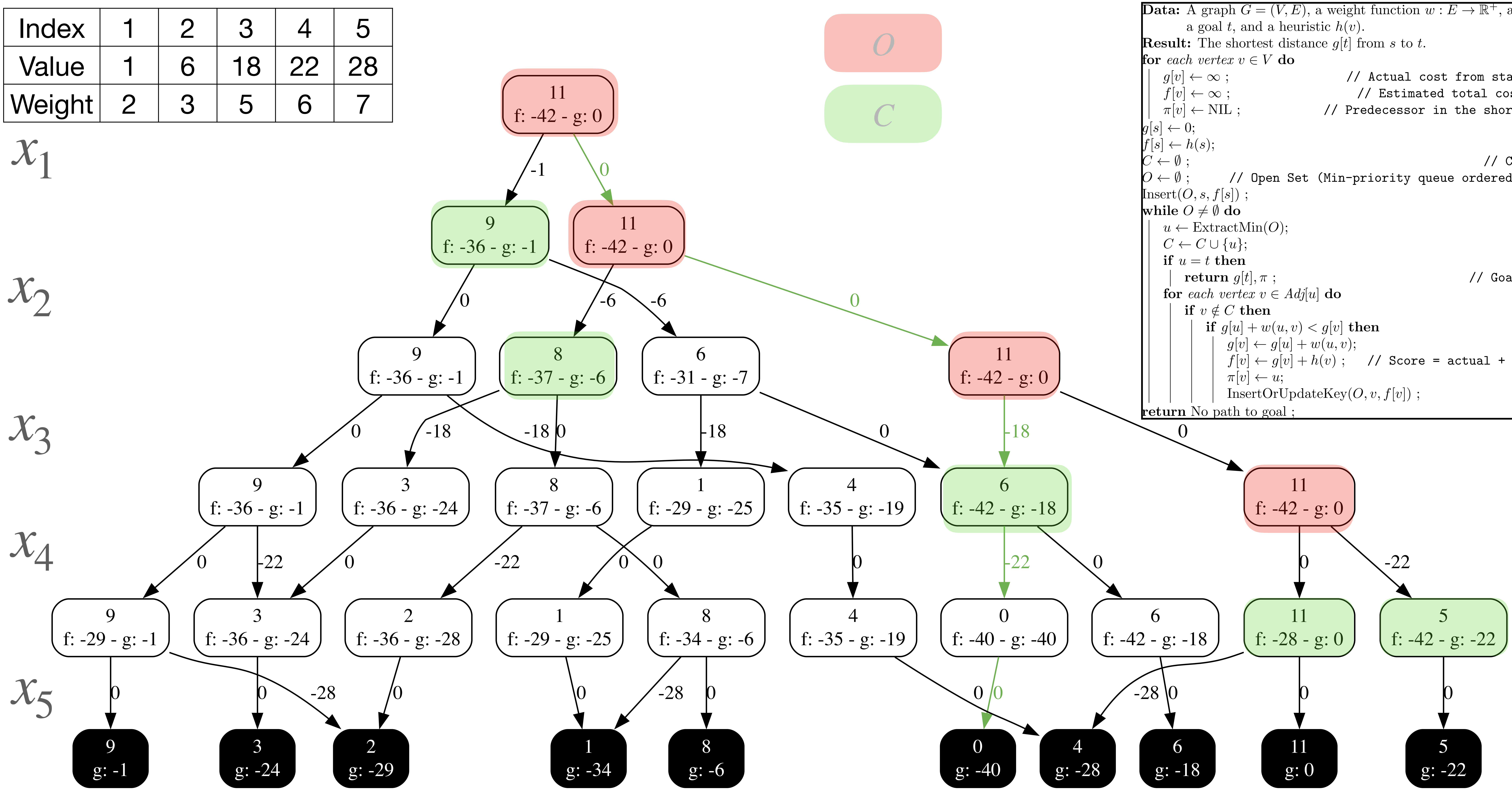
Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7



```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                     // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                      // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;              // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                              // Closed Set
$O \leftarrow \emptyset$ ;      // Open Set (Min-priority queue ordered by f[v])
Insert($O, s, f[s]$) ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin($O$);
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                      // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey($O, v, f[v]$) ;
return No path to goal ;
```

Resolution with A*, h = linear relaxation

Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7

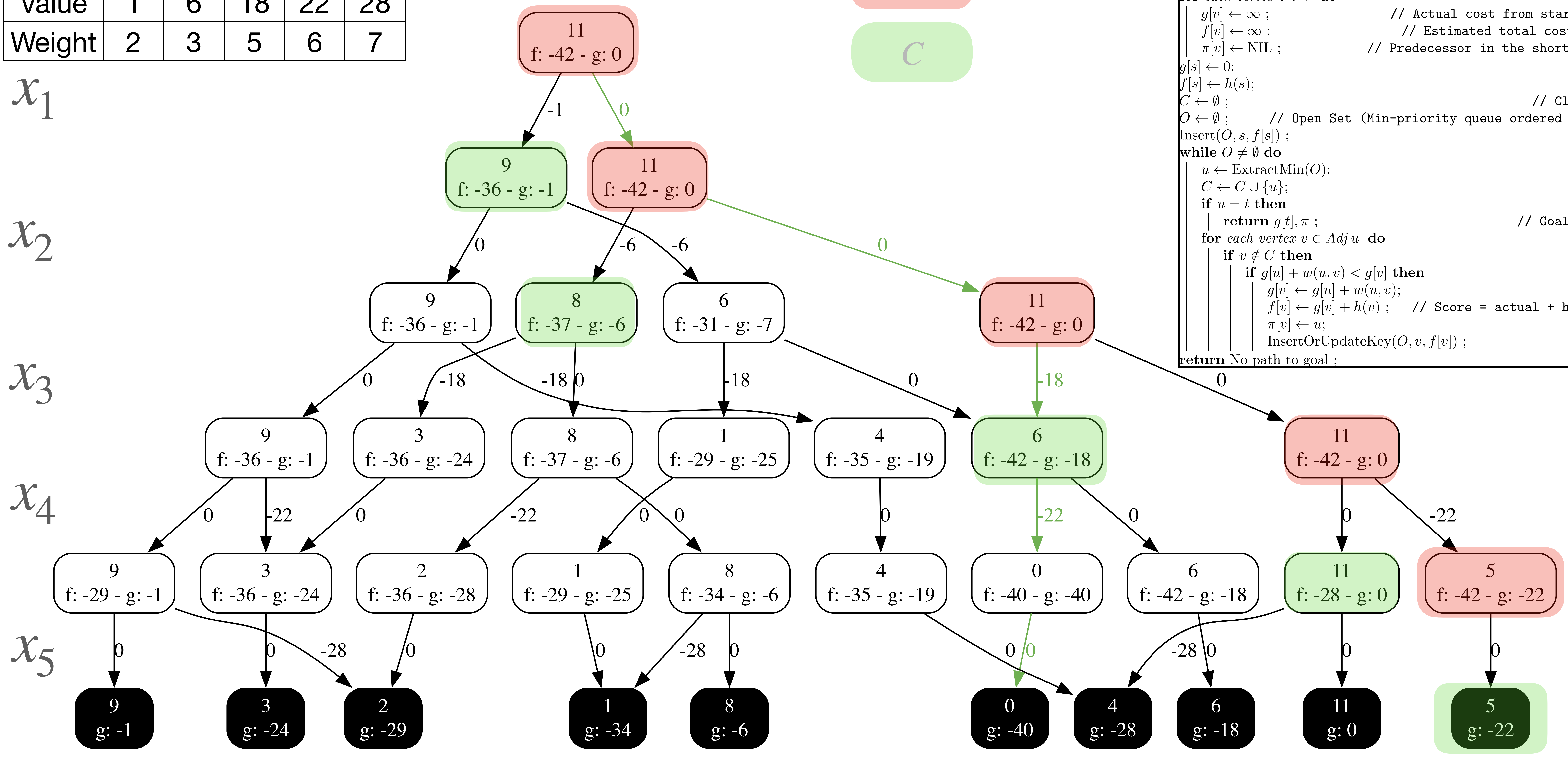


```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                    // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                    // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                     // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                       // Closed Set
$O \leftarrow \emptyset$ ;      // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                  // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7

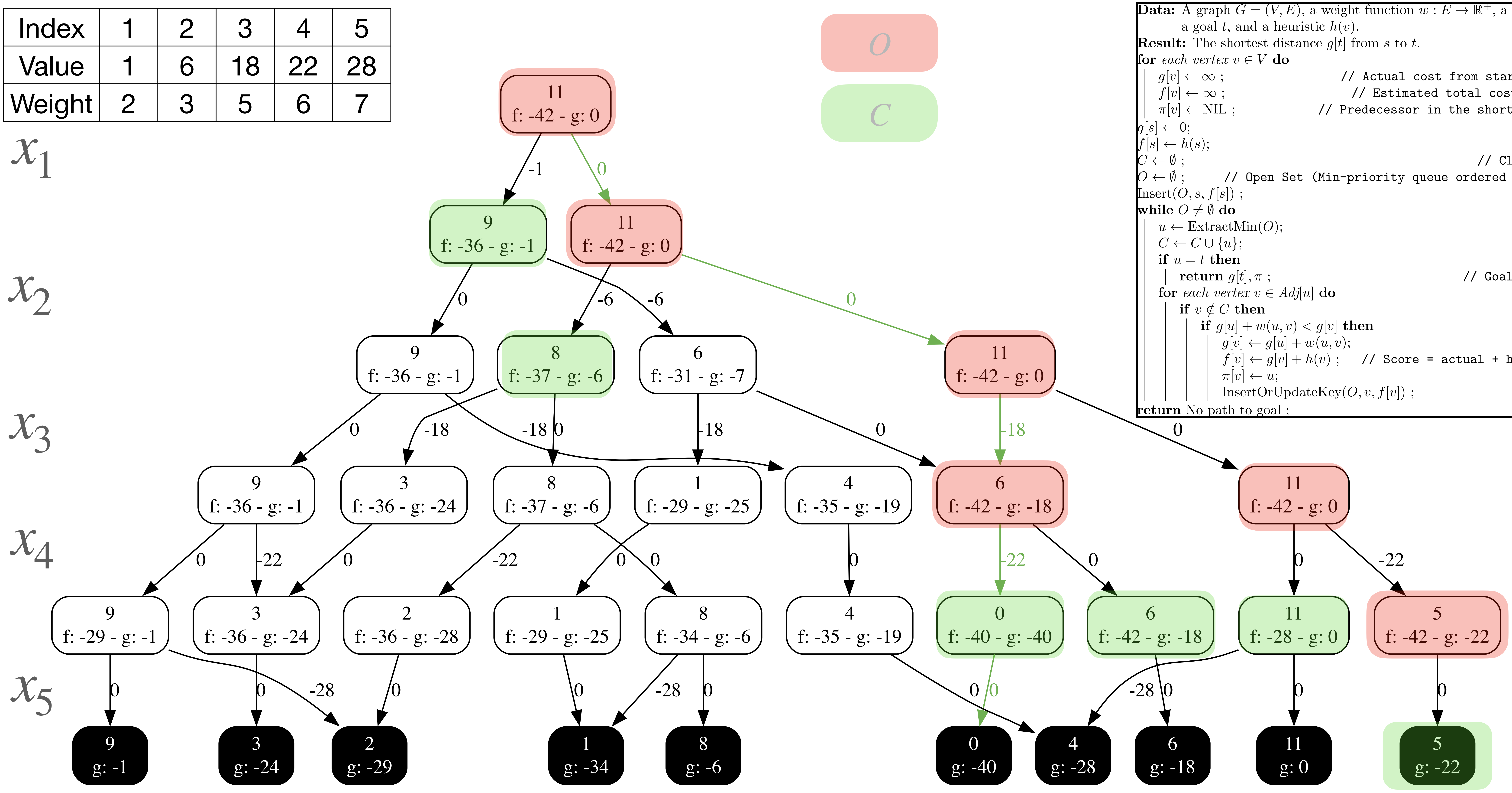
🟥 O

🟩 C



```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                    // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                    // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                     // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                              // Closed Set
$O \leftarrow \emptyset$ ;      // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                     // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7

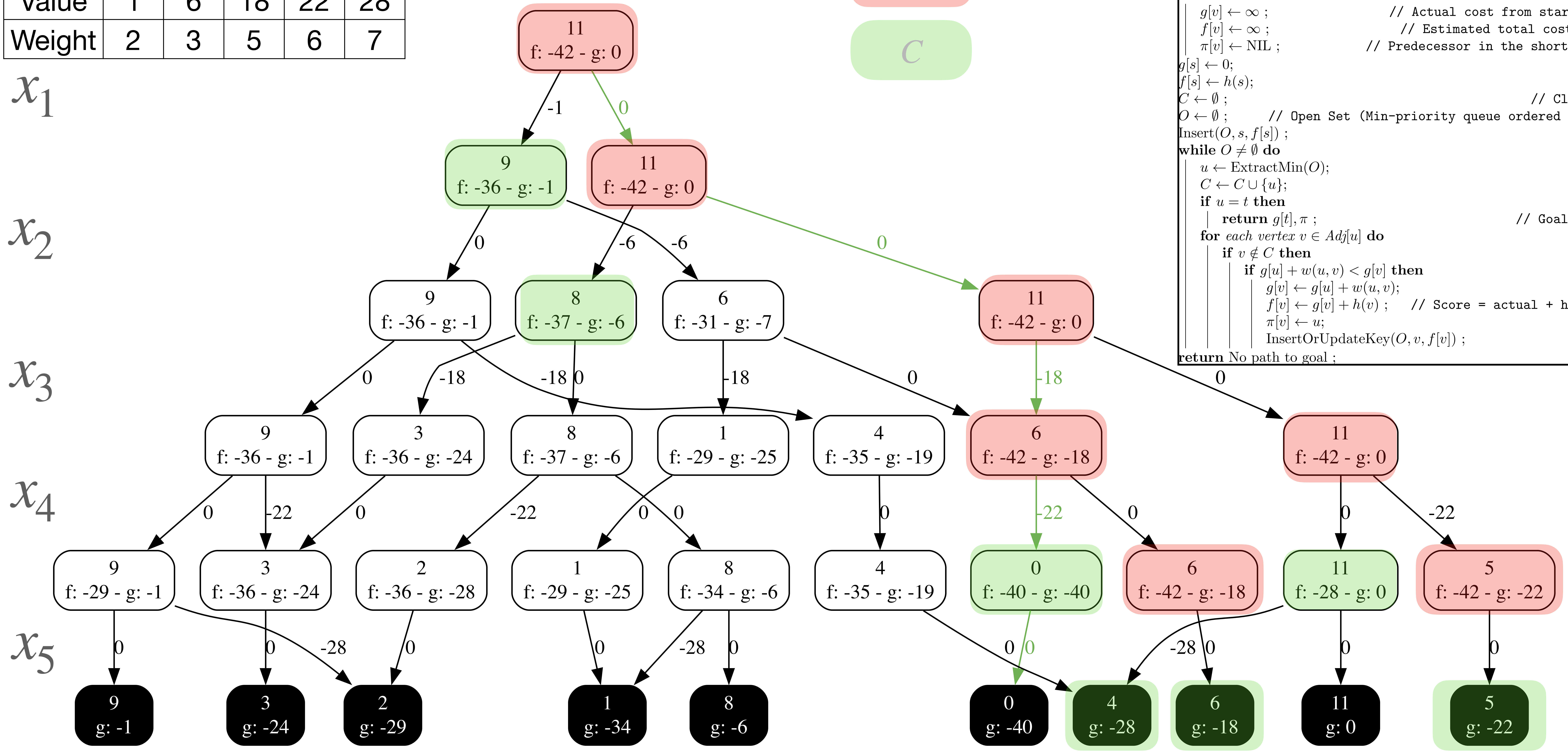


```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                       // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                        // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                  // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                                  // Closed Set
$O \leftarrow \emptyset$ ;      // Open Set (Min-priority queue ordered by f[v])
Insert($O, s, f[s]$) ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin($O$);
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                    // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey($O, v, f[v]$) ;
return No path to goal ;
```

Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7

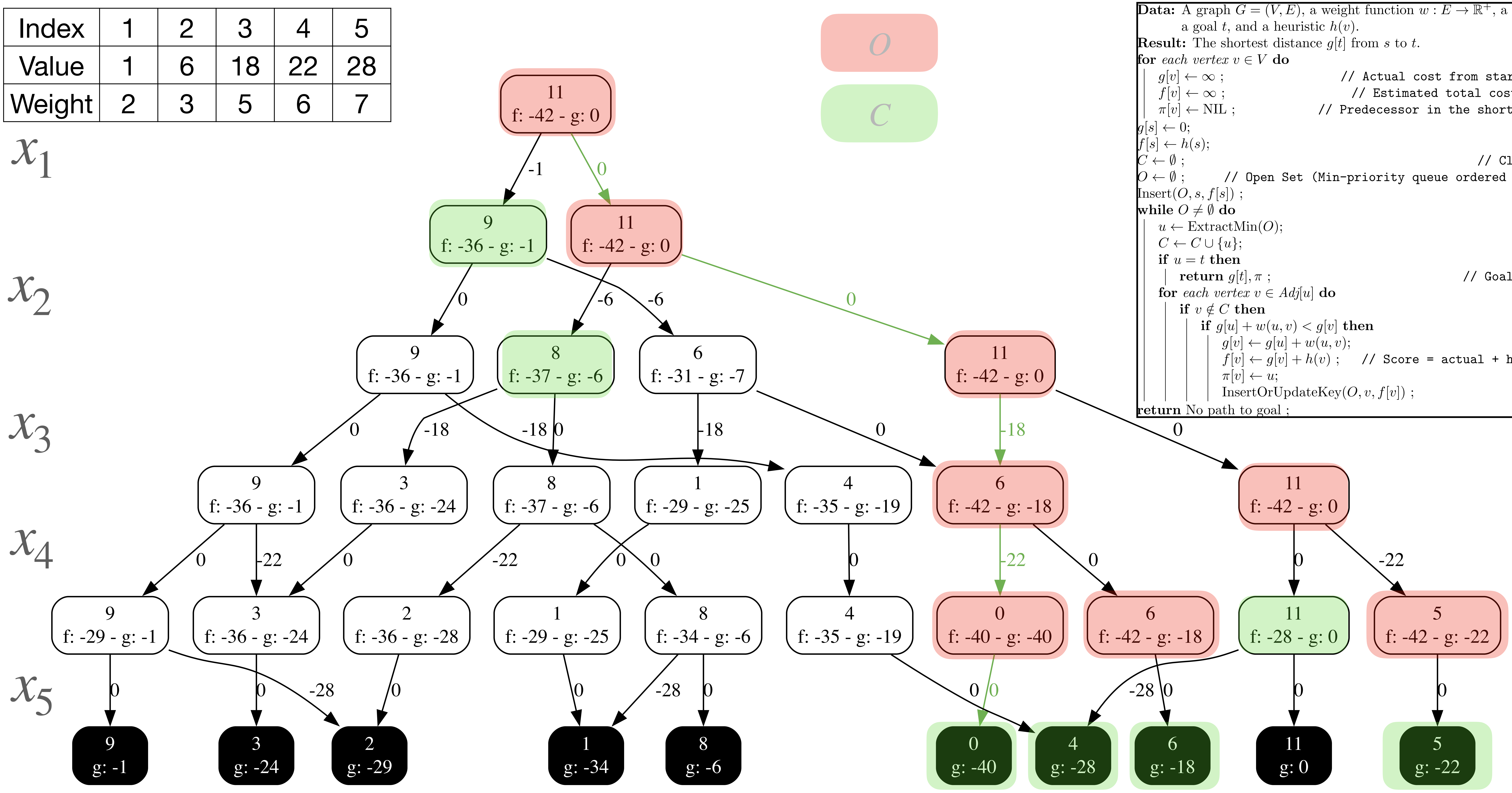
O

C



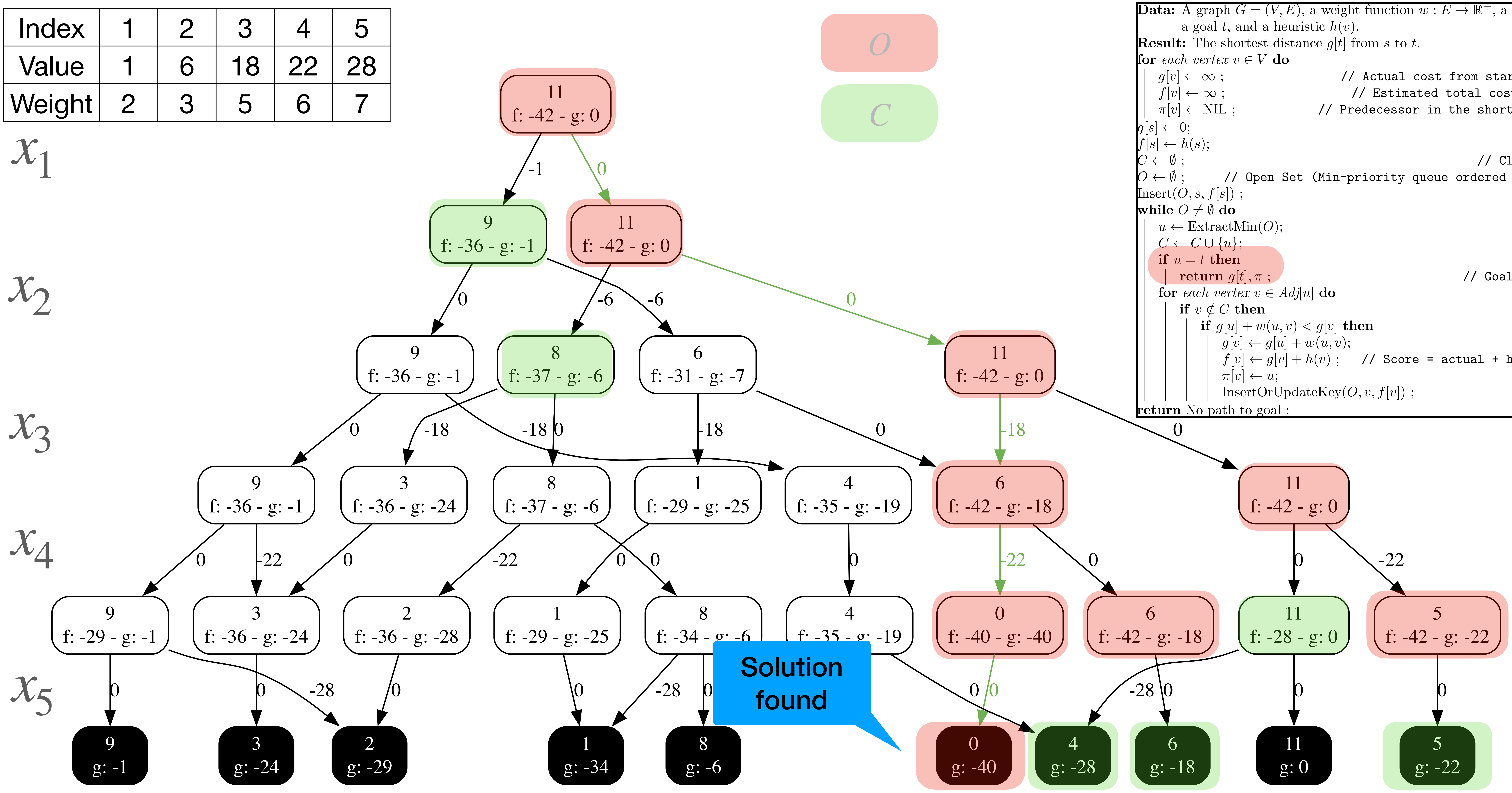
```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                         // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                          // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                      // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                                      // Closed Set
$O \leftarrow \emptyset$ ;      // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                          // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7



```
Data: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                 // Actual cost from start $s$ to $v$
    $f[v] \leftarrow \infty$ ;                 // Estimated total cost $(g + h)$
    $\pi[v] \leftarrow$ NIL ;                  // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                          // Closed Set
$O \leftarrow \emptyset$ ;     // Open Set (Min-priority queue ordered by f[v])
Insert$(O, s, f[s])$ ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                    // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey$(O, v, f[v])$ ;
return No path to goal ;
```

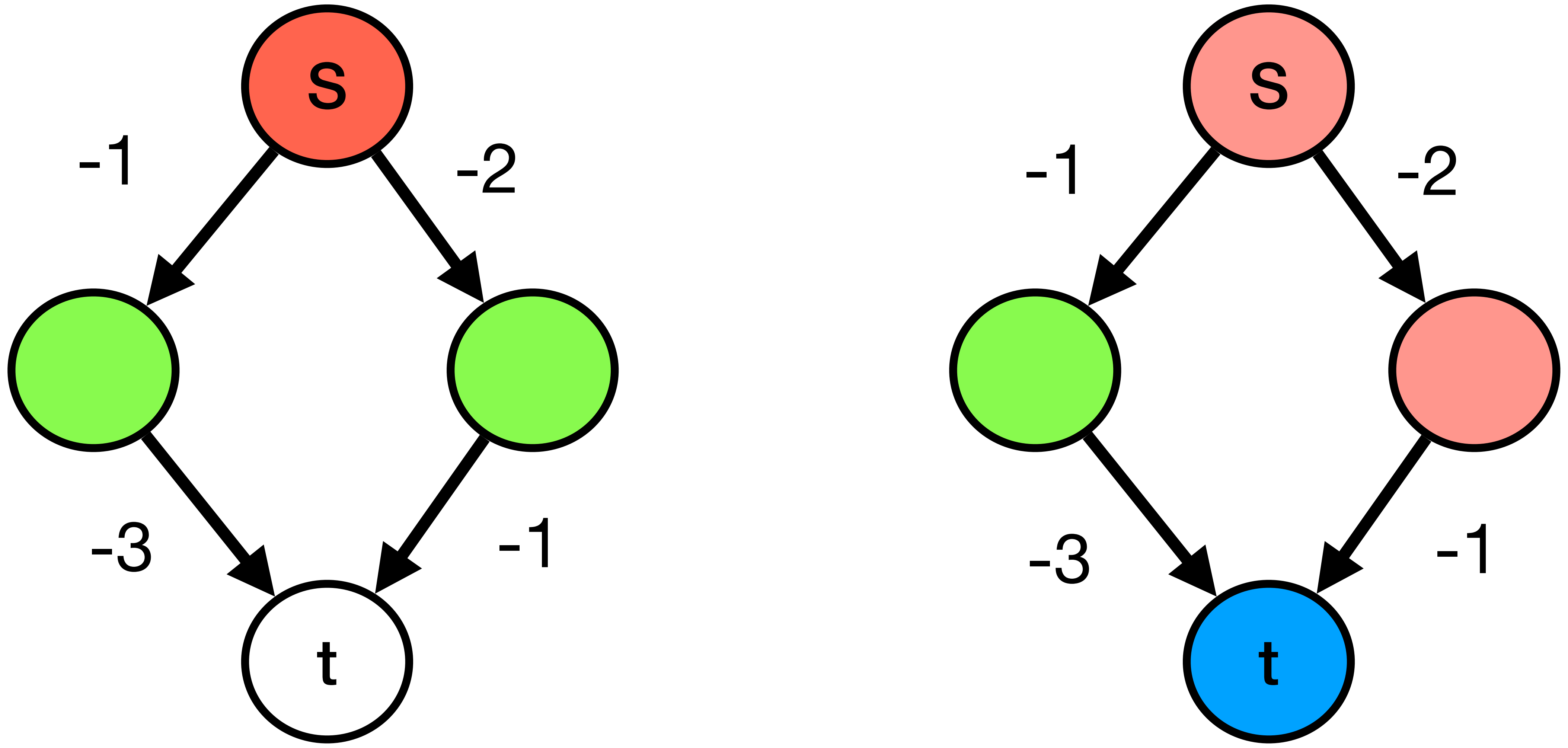
Index	1	2	3	4	5
Value	1	6	18	22	28
Weight	2	3	5	6	7



```
Data: A graph $G = (V, E)$, a weight function $w : E \rightarrow \mathbb{R}^+$, a source $s$,
      a goal $t$, and a heuristic $h(v)$.
Result: The shortest distance $g[t]$ from $s$ to $t$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                    // Actual cost from start s to v
    $f[v] \leftarrow \infty$ ;                    // Estimated total cost (g + h)
    $\pi[v] \leftarrow$ NIL ;                     // Predecessor in the shortest path
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$C \leftarrow \emptyset$ ;                                      // Closed Set
$O \leftarrow \emptyset$ ;    // Open Set (Min-priority queue ordered by f[v])
Insert($O, s, f[s]$) ;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin($O$);
    $C \leftarrow C \cup \{u\}$;
    if $u = t$ then
        return $g[t], \pi$ ;                                // Goal reached
    for each vertex $v \in Adj[u]$ do
        if $v \notin C$ then
            if $g[u] + w(u, v) < g[v]$ then
                $g[v] \leftarrow g[u] + w(u, v)$;
                $f[v] \leftarrow g[v] + h(v)$ ;    // Score = actual + heuristic
                $\pi[v] \leftarrow u$;
                InsertOrUpdateKey($O, v, f[v]$) ;
return No path to goal ;
```

Shortest path problems with negative weights are difficult

- Dijkstra does not work with negative weights, even in a DAG



Stern, R., Kiesel, S., Puzis, R., Felner, A., & Ruml, W. (2014). Max is more than min: Solving maximization problems with heuristic search. In *Proceedings of the International Symposium on Combinatorial Search.*

- Dijkstra algorithm always picks the node with the smallest current distance to process next, it assumes that adding edges to a path can only increase (or keep equal) the total distance. Negative edges violate this assumption because reaching a "farther" node first might allow you to cross a negative edge that reduces the total cost to a "closer" node.

- Thus A* may not work neither! We need to fix it to work with negative weights

Let’s fix A* (we call it Robust A*) to work with negative cycles in a DAG

```
Data: A graph $G = (V, E)$, a weight function $w : E \rightarrow \mathbb{R}^+$, a source $s$,
      a goal $t$, a heuristic $h(v)$, a weight parameter $w_h \geq 1$.
Result: The cost of the best solution found, $g_{\text{inc}}$, and the incumbent
        path $\pi_{\text{inc}}$.
for each vertex $v \in V$ do
    $g[v] \leftarrow \infty$ ;                          // Cost of best known path $s \rightsquigarrow v$
    $f[v] \leftarrow \infty$ ;                          // Estimated cost: $g + h$
    $\pi[v] \leftarrow$ NIL;                             // Best known predecessor
$g[s] \leftarrow 0$;
$f[s] \leftarrow h(s)$;
$g_{\text{inc}} \leftarrow \infty$ ;                                    // Incumbent cost
$C \leftarrow \emptyset$ ;      // Closed Set (Contains nodes that were expanded)
$O \leftarrow \emptyset$ ;       // Open Set (Min-priority queue ordered by $f'[v]$)
Insert$(O, s, f[s])$;
while $O \neq \emptyset$ do
    $u \leftarrow$ ExtractMin$(O)$;
    $C \leftarrow C \cup \{u\}$;
    if $f[u] > g_{inc}$ then
        return $g_{\text{inc}}, \pi$ ;              // Cannot do better than incument
    for each vertex $v \in Adj[u]$ do
        if $g[u] + w(u, v) + h(v) \geq g_{inc}$ then
            continue ;                      // Pruning by incumbent cost
        else if $g[u] + w(u, v) < g[v]$ then
            $\pi[v] \leftarrow u$ ;
            $g[v] \leftarrow g[u] + w(u, v)$ ;
            $f[v] \leftarrow g[v] + h(v)$ ;
            if $v = t$ then
                $g_{\text{inc}} \leftarrow g[t]$ ;                 // Incumbent cost update
            else
                InsertOrUpdateKey$(O, v, f[v])$;
                $C \leftarrow C \setminus \{v\}$ ;
return $g_{\text{inc}}, \pi$ ;                              // $O = \emptyset$ (optimal)
```

A* cannot stop on first solution. It can be stopped only when the set of open-nodes is empty or when the minimum f value is worse than the incumbent solution

Project

- You are given an implementation of A* and a model for the Knapsack
- You have to implement the Anytime Weighted A*
- You have to implement the TSP and solve it using A* and Anytime Weighted A*
  - For that you need to implement a heuristic h for the TSP

KnapsackState

```java
public class KnapsackState extends State {

    int item, capacity;

    public KnapsackState(int index, int capacity) {
        this.item = index;
        this.capacity = capacity;
    }

    @Override
    public int hash() {
        return Objects.hash(item, capacity);
    }

    @Override
    public boolean isEqual(State s) {
        if (s instanceof KnapsackState) {
            KnapsackState state = (KnapsackState) s;
            return item == state.item && capacity == state.capacity;
        }
        return false;
    }
}
```

Model

```java
/**
 * Interface for describing an A* based model
 */
public abstract class Model<S extends State> {

    /**
     * @return true if the state is a base case of the A* model
     */
    public abstract boolean isTerminalState(S state);

    /**
     * @return the value of the base case
     */
    public abstract double getTerminalStateValue(S state);

    /**
     * @return the root state of the A* model
     */
    public abstract S getRootState();

    /**
     * @return the list of transitions from the given state
     */
    public abstract List<Transition<S>> getTransitions(S state);


    public abstract double h(S state);
}
```

Knapsack (Model)

```java
public class Knapsack extends Model<KnapsackState> {

    KnapsackInstance instance;
    KnapsackState root;

    public Knapsack(KnapsackInstance instance) {
        this.instance = instance;
        this.root = new KnapsackState(0, instance.capacity);
    }

    @Override
    public boolean isTerminalState(KnapsackState state) {
        return state.item == instance.n || state.capacity == 0;
    }

    @Override
    public double getTerminalStateValue(KnapsackState state) {
        return 0;
    }

    @Override
    public KnapsackState getRootState() {
        return root;
    }
```

Knapsack (Model)

```
@Override
public List<Transition<KnapsackState>> getTransitions(KnapsackState state) {
    List<Transition<KnapsackState>> transitions = new LinkedList<>();

    // do not take the item
    transitions.add(new Transition<KnapsackState>(
            new KnapsackState(state.item + 1, state.capacity),
            0,
            0
    ));

    // take the item if remaining capacity allows
    if (instance.weight[state.item] <= state.capacity) {
        transitions.add(new Transition<KnapsackState>(
                new KnapsackState(state.item + 1, state.capacity - instance.weight[state.item]),
                1,
                -instance.value[state.item]
        ));
    }

    return transitions;
}
```

Heuristic h for Knapsack = Linear Programming

```java
@Override
public double h(KnapsackState state) {
    double[] ratio = new double[instance.n];
    int capacity = state.capacity;
    for (int i = state.item; i < instance.n; i++) {
        ratio[i] = ((double) instance.value[i] / instance.weight[i]);
    }
    class RatioComparator implements Comparator<Integer> {
        @Override
        public int compare(Integer o1, Integer o2) {
            return Double.compare(ratio[o1], ratio[o2]);
        }
    }
    Integer[] sortedVariables = new Integer[instance.n - state.item];
    for (int i = state.item; i < instance.n; i++) {
        sortedVariables[i - state.item] = i;
    }
    Arrays.sort(sortedVariables, new RatioComparator().reversed());
    int maxProfit = 0;
    Iterator<Integer> itemIterator = Arrays.stream(sortedVariables).iterator();
    while (capacity > 0 && itemIterator.hasNext()) {
        int item = itemIterator.next();
        if (capacity >= instance.weight[item]) {
            maxProfit += instance.value[item];
            capacity -= instance.weight[item];
        } else {
            double itemProfit = ratio[item] * capacity;
            maxProfit += (int) Math.floor(itemProfit);
            capacity = 0;
        }
    }
    return -maxProfit;
}
```

Putting it all together and start the computation

```
KnapsackInstance instance =
   new KnapsackInstance(instance.capacity, instance.value, instance.weight);
Knapsack model = new Knapsack(instance);
Astar<KnapsackState> solver = new Astar<>(model);
Solution solution = solver.getSolution();
for (int decision : solution.getDecisions()) {
    if (decision == 1) {
        checkValue += instance.value[item];
        checkWeight += instance.weight[item];
    }

    item++;
}
```