

Standardizing Representation for Equality with a Population Seat Index

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ABSTRACT

The *proportion of seats to population* (PSP) has long been used to quantify the contributions (weights) of individuals or votes in representation equality studies. We show that the PSP has a bias in estimating the true contribution; thus, *proportionality schemes* (*proportional representation*) are insufficient for ensuring equality. To address this issue, we introduce a *standard function* $f(p)$ for the number of seats for a population p and a *population seat index* (PSI) $\frac{f^{-1}(s)}{p}$ to replace the PSP for assigning s seats to population p , where f^{-1} is the inverse of f . In contrast to the PSP, the PSI has no bias. We use it to derive an apportioning scheme with absolute or relative individual equality. If $f(p) \propto p^\gamma$ for a constant γ , this scheme distributes seats proportionally to the γ -th powers of the populations. Real-world observations indicate that $f(p) \propto p^\gamma$ with a constant $\gamma < 1$, showing that proportionality schemes represent individuals in less populous groups *less* than individuals in more populous groups, while the proposed subproportionality scheme guarantees equality.

Keywords: Equal representation, “one person, one vote”, apportionment, proportional representation, degressive proportionality, subproportionality, standardized representation, population seat index

1 INTRODUCTION

Since the late eighteenth century, there have been increased movements toward equality (Piketty, 2022). *Political equality* is one of the most important types of equality. As a core element of political equality, *proportionality in representation* (PR), or *proportional representation*, is well-known. As indicated by the slogan “one person, one vote”, PR reflects subgroups of a population (or an electorate or votes; in the following work, we use “population” for ease of writing.) *proportionally* in a legislature or an elected body. PR is considered the ideal system for ensuring the equality of individuals since it considers the contribution of *all* people and votes (see, e.g., Lijphart, 1998). It has been adopted worldwide in modern comparative politics, apportionment, and elections (see, e.g., Allen and Taagepera, 2017; Benoit, 2000; Lijphart, 1998, 2012; Pukelsheim, 2017; Puyenbroeck, 2008; Samuels and Snyder, 2001; Taagepera and Grofman, 2003; US House of Representatives, 2022).

To study the degree of representation inequality between two groups, PR schemes use an indicator, namely, the *proportion of seats to population* (PSP), to estimate the contributions (weights) of individuals within a group. For example, in apportioning seats in a legislative body to different subgroups, the PSP of subgroup i is defined as $\text{PSP}_i = \frac{s_i}{p_i}$, where s_i and p_i denote the number of distributed seats and the population of subgroup i , respectively. PR approaches require that $\text{PSP}_i = c$, or equivalently, that $s_i = cp_i$ for all i for some constant $c > 0$. An apportionment with different PSPs between subgroups is referred to as a *malapportionment* and is considered to violate representation equality among individuals (see, e.g., Auerbach, 1964; Eckman, 2021; Frederick, 2008; Huntington, 1942; Samuels and Snyder, 2001).

Unfortunately, as we will see later, the PSP has a bias in estimating the true contribution; hence, PR is insufficient for ensuring equality. This study was motivated by a paradox of real-world PR schemes. For each subgroup i , the number of seats s_i is said to be “proportional” to the population p_i ; however, *no proportionality can be observed between the total number of seats $S = \sum s_i$ and the overall population $P = \sum p_i$* in the real world, as discussed in Subsection 2.1. We observed that this *inconsistency* occurs because the PSP (i.e., $\frac{S}{P} = \frac{\sum s_i}{\sum p_i}$) is *not* constant in real-world scenarios. We explain why this inconsistency is critical by posing the following question.

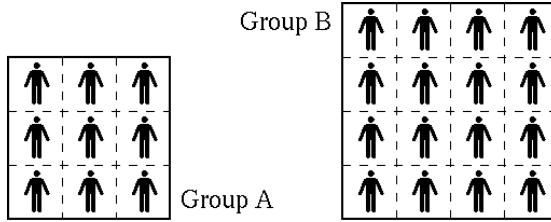


Figure 1. Illustration of the bias in the proportion for evaluating the contribution of an individual. Identical individuals in different groups are evaluated differently unless the outcome is proportional to the group size.

Question 1 Suppose that a group of p people contributed equally to an outcome s . What is the equal contribution of an individual within that group?

A simple answer is the proportion $\frac{s}{p}$. It depends on the group size. We use an example to study this bias on equality, as shown in Figure 1. Assume that $s = \sqrt{p}$, a value proportional to the *perimeter* of the square consisting of p people, for all p . Then, $\frac{s}{p} = \frac{1}{\sqrt{p}}$ estimates that the contribution of an individual in a larger group is *less* than that of an individual in a smaller group, even if the two individuals are completely identical. Moreover, if $s > 0$ is *independent* of p , an intuitive definition of the contribution of an individual is *zero*, as no one can contribute to the outcome. However, $\frac{s}{p} \neq 0$. This population-dependent bias exists unless $s \propto p$ (e.g., s is the *area* of the square in Figure 1). Summarizing the above observation, we propose the following theorem.

Theorem 1 The proportion of outcome to population has a population-dependent bias in estimating the contribution of an individual within a group unless the outcome is proportional to the population.

This theorem implies that, in general, the PSP, and hence PR schemes, cannot correctly estimate the contribution of an individual. A common belief underlying the PR theory is that, as long as the PSP between two subgroups is equal for a single apportionment, there is no (relative) inequality. However, this assumption is not true. Consider the example shown in Figure 1. Assume that $s^* = \sqrt{p}$ is a *standard* (e.g., the average) number of seats for a population p . Suppose that there are 5 seats to apportion. The PR scheme assigns $\frac{9}{25} \times 5 = 1.8$ and $\frac{16}{25} \times 5 = 3.2$ seats to Groups A and B, respectively (with the PSP 0.2). These numbers are rounded to the nearest whole numbers, namely, 2 and 3 seats, respectively. With this apportionment, the PR scheme claims that people in *Group B* are *underrepresented*, since $\text{PSP}_B = \frac{3}{16} < \frac{2}{9} = \text{PSP}_A$.

However, this argument neglects the population-dependent bias of the PSP. In fact, since 2 seats is the standard number of seats for 4 people according to the given assumption ($2 = \sqrt{4}$), assigning 2 seats to Group A gives the 9 people in Group A the same weight as 4 people, with respect to the standard. Thus, the *true* (effective) weight of an individual in Group A is $\frac{4}{9}$. Analogously, assigning 3 seats to Group B gives the 16 people in Group B the same weight as $3^2 = 9$ people. Therefore, the true weight of an individual in Group B is $\frac{9}{16}$. Since $\frac{4}{9} < \frac{9}{16}$, the people underrepresented are those within Group A, not Group B. This shows that the bias of the PR scheme is critical.

This example demonstrates that the PSP, and thus the PR scheme, is insufficient for ensuring equality. The PR approach advocates that assigning 1.8 and 3.2 seats to Groups A and B, respectively, is equal. However, there is a gap in the weights as large as $(1.8^2/9) : (3.2^2/16) = 9 : 16$, with respect to a standard $s^* = \sqrt{p}$. To ensure equality among individuals, an indicator and apportioning scheme should be developed to counteract this population-dependent bias with an *appropriate* standard. This concept is analogous to the design of the body mass index (BMI). The BMI is defined as $\frac{\text{weight}}{\text{height}^2}$ as opposed to $\frac{\text{weight}}{\text{height}}$, because empirical studies have shown that the standard (i.e., *average*) weight of an adult is approximately proportional to the square of their height (Eknayan, 2007). The PSP and PR approach are limited because they implicitly assume proportionality between the optimal (or average) number of representatives (seats) and the population, which is unfortunately not observed in real-world scenarios, as discussed in Section 2.

In the following work, we propose a *standardized representation* theory with an unbiased *population seat index* (PSI) by linking the number of representatives (seats) with an apportioning scheme (Section 3).

Moreover, we show that, in the real world, the PR approach *overestimates* (*underestimates*) the weights of individuals in less (more) populous groups, thus resulting in *underrepresentation* (*overrepresentation*) for the people within those groups. In contrast, the proposed scheme, which is a special case of the degressive proportionality, guarantees equality (Section 4).

2 LITERATURE REVIEW

Determining the optimal number of representatives (seats) for a population is sometimes called the most susceptible political problem (Madison, 1788). This problem persisted for more than 130 years since the late eighteenth century and was divided into two subproblems in the 1920s (Chafee, 1929). The first subproblem is determining the optimal size of a legislative body, i.e., the *total* number of seats. The other subproblem is optimally apportioning a *fixed* number of seats to different subgroups. These two subproblems were originally linked by the Framers of the U.S. Constitution (C. A. Kromkowski and J. A. Kromkowski, 1991; Madison, 1789; see also the 1790–1830 data in Figure 3) but were unlinked by Title 2 of the U.S. Code (C. A. Kromkowski and J. A. Kromkowski, 1991). However, as discussed in Section 1, these two problems are indeed *dependent* (Theorem 1). In this section, we review the literature on both problems and relink them in the next section.

2.1 Standard number of representatives (seats) for a population

There are many studies on the standard number $s^* = f(p)$ of representatives (seats) for a population p . Note that $s^* = f(p)$ assumes the equality of individuals since it depends only on the population and not on any individual properties. Table 1 shows a summary of the key results. We observe that all of these models can be formulated or approximated with different values of γ in Formula 1, except for the logarithmic model developed by Magdon-Ismail and Xia, 2018, which has a polynomial voting cost (omitted in Table 1).

$$s^* = f(p) \propto p^\gamma \text{ for a constant } \gamma, 0 \leq \gamma \leq 1. \quad (1)$$

Table 1. Key studies on the standard number of representatives (see Formula 1 for the meaning of γ).

Scheme	γ	Reference	Data/Model/Remark
Empirical (Regression)	≈ 0.37	Stigler, 1976	37 democratic countries
	≈ 0.41	Auriol and Gary-Bobo, 2012	111 countries
	≈ 0.39	Zhao and Peng, 2020	192 countries
	≈ 0.37	Figure 2 in this article	U.S. Congress between 1790 and 1920
Theoretical		Sorted by γ	
Fixed-size	0	U.S. Senate	100 (two seats per state)
		U.S. House after the 1920s	435
Cubic root	1/3	Taagepera, 1972	Social mobilization-based model
Sublinear	$\frac{1}{3} \leq \gamma \leq \frac{5}{9}$ (e.g.) 0.4	Zhao and Peng, 2020 (Same as above)	Social network-based model First model matching real-world data
Square root	1/2	Penrose, 1946	Voting power-based model
		Auriol and Gary-Bobo, 2012	Mechanism design
		Godefroy and Klein, 2018	Mechanism design
		Gamberi et al., 2021	Complex network-based model
		Margaritondo, 2021	Revision of Taagepera, 1972
		Blonder, 2021	Derived from Madison, 1789
Proportional	1	U.S. House before 1830	Approximately (see Figure 3)
		Magdon-Ismail and Xia, 2018	Voting model (fixed voting cost)
		Revel, Lin, and Halpern, 2022	Voting model

For clarity, we refer to the numbers obtained by empirical studies as the *average* numbers since they were determined through regression. The other values, i.e., the numbers obtained by theoretical studies, are referred to as the *optimal* numbers since they were determined through optimization models. The real-world data showed $\gamma \approx 0.4$ (Auriol and Gary-Bobo, 2012; Stigler, 1976; Zhao and Peng, 2020;

Figure 2 of this study). This phenomenon is surprising, suggesting *the existence of a standard number of representatives and that this number depends largely on the population and little on other factors*, such as location, race, culture, religion, economics, or political system (Stigler, 1976; Taagepera, 1972; Zhao and Peng, 2020). Moreover, voting game- and mechanism design-based theoretical models (Auriol and Gary-Bobo, 2012; Godefroy and Klein, 2018; Magdon-Ismail and Xia, 2018; Penrose, 1946; Revel, Lin, and Halpern, 2022) are limited in explaining this phenomenon, as they have different values of γ and lack theoretical connections to social representations.

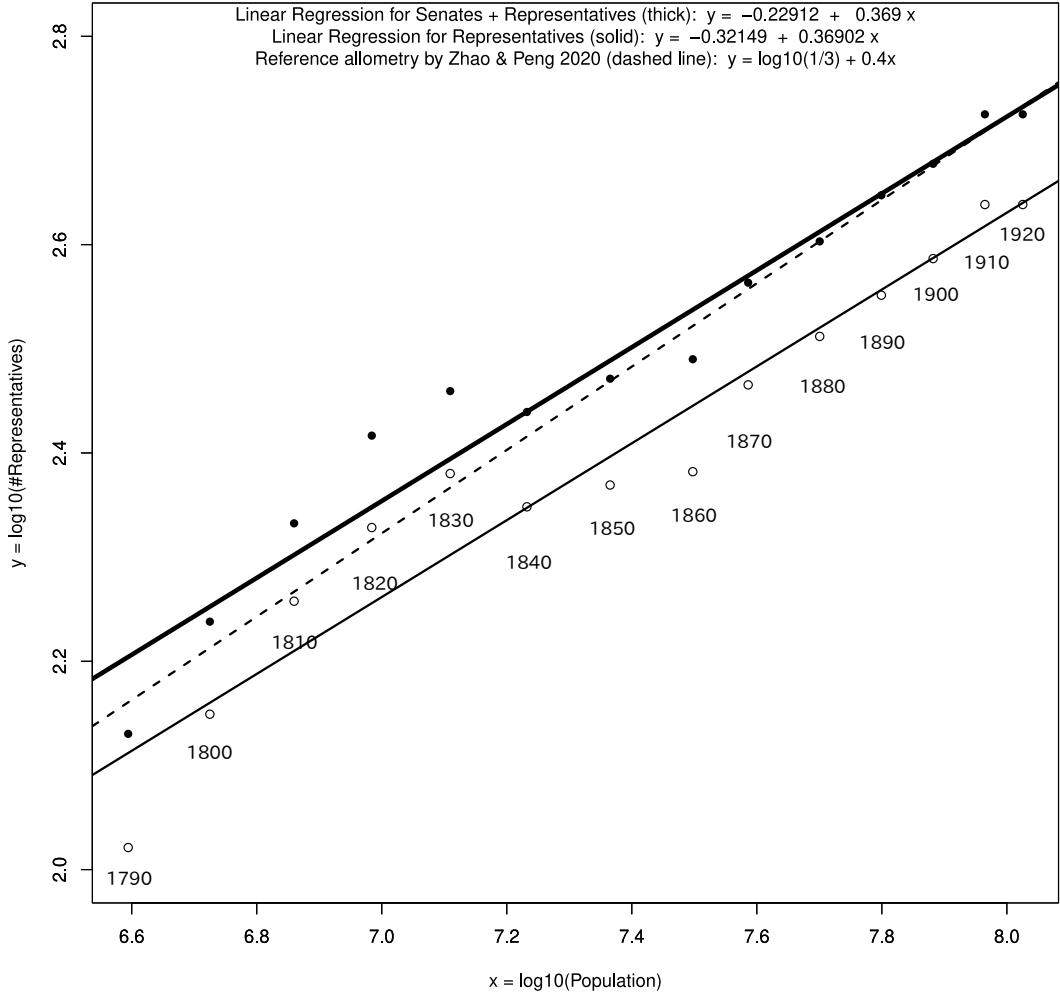


Figure 2. Population of the U.S. (x-axis) and seats of the U.S. Congress (y-axis) between 1790 and 1920 in log-scale. Black dots denote the total size of the Congress (Senate + House), whereas white dots denote the size of the House. The regression result (thick line) shows that $y \propto x^{0.37}$, with a p-value of $1.8e-09$ and an adjusted $R^2 = 0.95$. For reference, we also plot a standard formula $y = \frac{1}{3}x^{0.4}$ (Zhao and Peng, 2020, dashed line). Data source: US Census Bureau, 2022 for population data and Office of the Historian, 2022 for seat data.

In particular, we remark that the formula $f(p) = \frac{1}{3} \times p^{0.4}$ proposed by Zhao and Peng, 2020 is the first model derived from a theoretical analysis that matches the value of γ observed in current real-world data. Surprisingly, this γ value also matches the size of the U.S. Congress (Senate + House of Representatives) between 1790 and 1920 with high accuracy (Figure 2). The existence of this social network analysis-derived formula supports that the (optimal) number of representatives may be largely determined by social

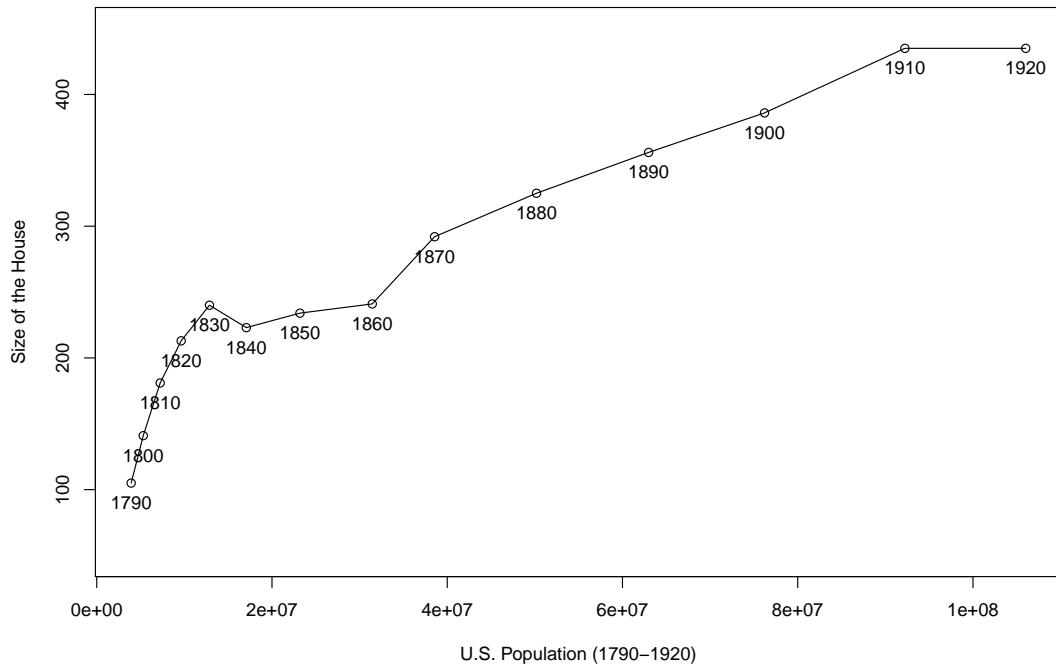


Figure 3. Population of the U.S. and the size of the House between 1790 and 1920 (after 1929, the size was fixed to 435). Data sources: US Census Bureau, [2022](#) and Office of the Historian, [2022](#).

connections, which is the basic idea underlying Taagepera, [1972](#) (see also Margaritondo, [2021](#)), Gamberi et al., [2021](#), and Zhao and Peng, [2020](#). We note that $\gamma < 1$ suggests that human society is efficient in aggregating public voices in a superlinear manner.

Formula 1 with $\gamma < 1$ is called *subproportional*. James Madison (1751–1836) was the first to note a subproportionality between the number of representatives and the population. He proposed an amendment to the U.S. Constitution (Madison, [1789](#)) that can be considered a square root scheme (Blonder, [2021](#)). The most interesting idea of his proposal is that it is *subproportional* in the *long-term* but *proportional* in the *short-term* (Blonder, [2021](#); Zhao, Tanimoto, and Lyu, [2022](#)). This piecewise subproportionality is considered important, as it is easier to be accepted and implemented than the straightforward nonlinear formula shown in (1).

The idea proposed by Madison still affects the House today, as shown by changes in the size of the House. The U.S. Constitution states that the seats in the House must be reapportioned to the states according to their populations every ten years (Article I, Section 2, Clause 3). It seems that this clause had an implicit understanding that the size of the House should increase *proportionally* to the total population (C. A. Kromkowski and J. A. Kromkowski, [1991](#); Madison, [1789](#)), which can be approximately observed between 1790 and 1830 (Figure 3). However, the size of the House between 1840 and 1920 increased by substantially *less* than the size of the population (see Figure 3). Therefore, the size of the House followed Madison's piecewise subproportionality proposal.

This piecewise subproportionality scheme went to an extreme in the 1920s, when the size of the House was fixed to 435. The reason behind this decision was that Congress failed to reapportion seats since they did not have an optimal method for *rounding fractional seats to whole numbers* (Chafee, [1929](#)). As a result, the House failed to perform a reapportionment according to the 1920 census, thereby violating the Constitution. Finally, after several failed attempts, due to the pressure of the approaching 1930 census, Congress “hastily passed reapportionment legislation” (C. A. Kromkowski and J. A. Kromkowski, [1991](#), 134) and permanently fixed the size of the House for *ease of operation*.

The discord between the total number of representatives and apportionments has led to several issues (C. A. Kromkowski and J. A. Kromkowski, [1991](#), [1992](#)). We note that this discord introduces an

inconsistency in the philosophy of the Constitution, namely, that fixing the size of the House suggests that the number of representatives is *independent* of the population, whereas the Constitution states that the number of representatives should be proportional to the population. Many people have suggested increasing the size of the House (Bowen, 2021; Frederick, 2008, 2009; C. A. Kromkowski and J. A. Kromkowski, 1991, 1992; Leib and Webster, 1998; Lijphart, 1998). Our study supports these voices.

2.2 Seat apportionment schemes

There are three schemes for apportioning a (fixed) number of seats to different subgroups:

1. *Fixed Apportionment* (FA): This scheme assigns a fixed number of seats to each subgroup. An FA scheme is adopted by the U.S. Senate (two seats per state). Single and fixed-member district electoral systems are also examples of this type of scheme.
2. *Proportional Apportionment* (PA): This scheme assigns seats proportionally according to the populations of subgroups. The House and many legislatures worldwide use this type of scheme.
3. *Degressive Proportionality* (DP): This is a new type of apportionment scheme that has been adopted by the European Parliament (European Council, 2018). Let us discuss it in the following.

As one of the most important steps toward individual equality, PA and PR schemes began to dominate the literature in the late eighteenth century (US House of Representatives, 2022). Nevertheless, they have been challenged by DP schemes recently. According to the official definition in European Council, 2018 (see also Cegiełka, Łyko, and Rudek, 2019; Grimmett et al., 2017), DP is defined as follows. For any two subgroups A and B with populations p_A and p_B , respectively, the numbers of seats $s_A > 0$ and $s_B > 0$ before rounding to whole numbers should satisfy the following constraints:

$$\left(\frac{p_A}{s_A} - \frac{p_B}{s_B} \right) (p_A - p_B) > 0 \quad \text{if } p_A \neq p_B, \quad (2)$$

$$(s_A - s_B)(p_A - p_B) > 0 \quad \text{if } p_A \neq p_B, \quad (3)$$

$$s_A = s_B \quad \text{otherwise } (p_A = p_B). \quad (4)$$

For example, Formula 1 is a type of DP when $0 < \gamma < 1$. Note that proportionality requires that $\frac{p_A}{s_A} = \frac{p_B}{s_B}$, thereby satisfying (3) and (4) but never satisfying (2). Hence, despite its name, DP is *not* a type of proportionality approach. Instead, we propose to use “subproportionality” to replace DP.

The nonproportionality of DP schemes has received criticism for “unequal” representations of individuals. The European Parliament has explained (see Grimmett et al., 2017) that DP *compromises* between *individual* equality (per capita) and the equality of *state* (per state). This “compromise” can also be observed in Canada, Germany, and the EU Council (Allen and Taagepera, 2017). In fact, the U.S. also utilizes this compromise with its *two* chambers: The Senate follows a per-state principle, whereas the House follows a per-capita principle. By adopting these two principles simultaneously, the U.S. Congress implements a type of DP scheme. Therefore, we used the total number in our regression study (Figure 2). Nevertheless, later we will show that there is actually *no compromise*, and the DP method is better than PA/PR approaches in terms of *individual* equality.

Finally, we remark on the minor but extensively studied issue of an “optimal” method for *rounding* fractional numbers of seats to whole numbers. The only consensus is that there is no method that is optimal in terms of all aspects. We refer readers to Balinski and Young, 2001; Chafee, 1929; Huntington, 1942; C. A. Kromkowski and J. A. Kromkowski, 1992; Squire and Hamm, 2005 for discussions, and Benoit, 2000; Kalandrakis and Rueda, 2021; Puyenbroeck, 2008; Samuels and Snyder, 2001; Taagepera and Grofman, 2003 for measuring disproportionality.

3 STANDARDIZING REPRESENTATION WITH AN UNBIASED INEQUALITY INDICATOR

Thus far, we have described how the PSP and PR scheme are limited. In this section, we propose an indicator to quantify the weight of an individual in a subgroup without the bias of the PSP. Then, we use this indicator to derive an apportionment scheme with unbiased individual equality.

We assume that a function $f = f(p)$ for the standard number of representatives (seats) for population p is available. As discussed in Section 1, such a standard function is necessary to ensure equality among

individuals. It can be determined according to the average number obtained by an empirical study, the optimal number obtained by a theoretical study, or a model adopted by policy makers (e.g., the formula adopted by the European Parliament). See Subsection 2.1 for examples.

We first assume that f is invertible. This assumption is true for all existing models (Table 1) except for the $\gamma = 0$ case. Let f^{-1} denote the inverse function of f . Suppose that s seats are assigned to population p . As discussed in Section 1, for estimating the weight of an individual, we should use the ratio $\frac{p^*}{p}$ of the *effective* population p^* to the *real* population p , where the effective population is the standard population that deserves s seats, i.e., $p^* = f^{-1}(s)$. Therefore, we propose the next *population seat index* (PSI) as an indicator for estimating the contribution (weight) of an individual.

$$w(s, p) = \frac{f^{-1}(s)}{p}. \quad (5)$$

Note that (5) calculates a *scalar*. This value is equal to 1 if $s = f(p)$ is the standard number of seats for population p . Thus, we can estimate the *absolute inequality* of an assignment with respect to the standard value. If the value calculated by Formula 5 is greater than 1, the number of seats s is greater than the standard number, i.e., overrepresentation; however, if the value is less than 1, the number of seats s is less than the standard number, i.e., underrepresentation. This fact is independent of the population p . Therefore, the proposed PSI indicator has no population-dependent bias.

We illustrate this concept with the example in Section 1, where the standard is $f(p) = \sqrt{p}$ (see Figure 1). Suppose that we assign 2 seats to Group A (9 people) and 3 seats to Group B (16 people). According to Formula 5, the (unbiased) weight of an individual in Group A is $w(2, 9) = \frac{2^2}{9} = \frac{4}{9}$, which is less than the weight $w(3, 16) = \frac{9}{16}$ of Group B. Therefore, people in Group A are *less* represented than people in Group B, in contrast to the (biased) analysis according to the PSP. In fact, we can determine an apportionment with *relative equality* by solving $w(s_A, p_A) = w(s_B, p_B) \Leftrightarrow \frac{s_A^2}{p_A} = \frac{s_B^2}{p_B}$ for $p_A = 9$, $p_B = 16$, and $s_B = 5 - s_A$. A simple calculation shows that $s_A = 15/7$ and $s_B = 20/7$ (with weight $25/49$). Rounding these values to the nearest whole numbers, we obtain 2 and 3 seats, respectively. This apportionment gives less representation to Group A ($15/7 > 2$) and more representation to Group B ($20/7 < 3$), matching the above analysis. Note that the absolute equality condition, i.e., $w(s, p) = 1$ for all groups, occurs if and only if the total number of seats is $3 + 4 = 7$.

In general, we can use the PSI to determine an apportionment with absolute or relative individual equality. Assume that there are k groups with populations p_1, p_2, \dots, p_k . Given a total number S of seats, the apportionment problem with (unbiased) *relative equal weight* w^* can be formulated as determining the number s_i of seats assigned to group i , where $i = 1, 2, \dots, k$, as follows:

$$w^* = \frac{f^{-1}(s_1)}{p_1} = \frac{f^{-1}(s_2)}{p_2} = \dots = \frac{f^{-1}(s_k)}{p_k}, \quad (6)$$

$$s_1 + s_2 + \dots + s_k = S. \quad (7)$$

According to (6), we have

$$s_i = f(w^* p_i) \text{ for } i = 1, 2, \dots, k. \quad (8)$$

Then, according to (7), the weight w^* can be calculated by solving:

$$\sum_{i=1}^k f(w^* p_i) = S. \quad (9)$$

Once w^* is found, the apportionment of seats can be determined with Formula 8. We note that this scheme is the same as the traditional PR scheme when the standard function $f(p) \propto p$, i.e., the total number of seats is proportional to the population.

The solution of Formula 9 depends on f . If $f(p) = cp^\gamma$ for some $c > 0$ and $\gamma \neq 0$, which is a special case of Formula 1 that has been observed worldwide, the proposed indicator PSI is

$$w(s, p) = \frac{s^{1/\gamma}}{c^{1/\gamma} p}. \quad (10)$$

The constant $\frac{1}{c^{1/\gamma}}$ can be removed if we are interested in only *relative* equality. This situation is sometimes convenient since not all theoretical models for determining the optimal number of seats have a simple estimation on c . With this simplification, the $\gamma = 1$ case degenerates to the PSP, i.e., $\frac{s}{p}$. For a general $c > 0$ and $\gamma \neq 0$, we can derive an apportionment scheme with relative individual equality as follows. A simple calculation with (9) shows that the weight is

$$w^* = \left(\frac{S}{c \sum_{i=1}^k p_i^\gamma} \right)^{1/\gamma}. \quad (11)$$

Therefore, according to (8), the number of seats can be calculated as

$$s_i = \frac{p_i^\gamma}{\sum_{j=1}^k p_j^\gamma} \times S \text{ for } i = 1, 2, \dots, k. \quad (12)$$

Thus, the proposed scheme distributes seats proportionally to the γ -th powers of the populations. For absolute equality, i.e., weight $w^* = 1$, the total number of seats must be equal to $S = c \sum_{j=1}^k p_j^\gamma$.

Finally, we consider a function f with no inverse function. We consider only $f = c$ for a constant $c > 0$, which is adopted by the U.S. Senate. Let $f_\varepsilon(p) = c + \varepsilon p$, where $\varepsilon > 0$ is a small number. We define the weight $w(s, p)$ for f according to the limit of $w_\varepsilon(s, p)$ when $\varepsilon \rightarrow 0$:

$$w_\varepsilon(s, p) = \frac{s - c}{\varepsilon p} \rightarrow \begin{cases} 0, & s = c, \varepsilon \rightarrow 0, \\ +\infty, & s > c, \varepsilon \rightarrow 0, \\ -\infty, & s < c, \varepsilon \rightarrow 0. \end{cases} \quad (13)$$

The weight is $w(s, p) = 0$ if $s = c$. This result matches our intuition, as no one contributes to the number of seats. Otherwise, if $s > c$ (respectively, $s < c$), the weight is $+\infty$ ($-\infty$), which is reasonable. In either case, the result is independent of the population. The only equal apportionment is $s_i = c$ for all i (thus, $S = kc$). Therefore, the apportionment of the U.S. Senate is consistent with respect to individual equality, with each individual having a constant weight of zero.

4 IMPLICATIONS

We discuss the implications of the proposed theory for existing studies. First, we empirically compare the PSI with the PSP using data from G20 countries (except for the EU). For the standard function, we used $f(p) = \frac{1}{3}p^{0.4}$ (Zhao and Peng, 2020), as this function shows the average size of the congress in the world through regression. Figure 4 shows the results, where “Effective weight” denotes the PSI calculated by Formula 5. We note several interesting findings in the figure.

Next, we consider the theoretical implications for existing apportioning schemes. Suppose that the standard number of seats for a population p is given by the function $f(p) = c_1 p^{\gamma_1}$ for constants $c_1 > 0$ and γ_1 . We use “Absolute Eq.” and “Relative Eq.” to show the conditions for absolute equality (i.e., an unbiased weight of 1) and relative equality (i.e., the same weight for two different groups). For the latter, we use p_1 and p_2 to denote the populations of the two groups.

Table 2 summarizes the results for an FA scheme. Note that for a fixed-member district electoral system, regardless of how the function $f(p)$ is chosen, relative equality is always possible if the populations are equal. Thus, our theory is compatible with equal redistricting (see Auerbach, 1964). We note that absolute equality can be achieved under certain conditions.

Table 2. Implications for the fixed apportionment (FA) scheme (assuming it assigns c_2 seats to each group).

Standard function $f(p)$	PSI ($= w(c_2, p)$)	Absolute Eq.	Relative Eq.
c_1 (by $c_1 + \varepsilon p, \varepsilon \rightarrow 0$)	0 if $c_1 = c_2, \infty$ otherwise	$c_1 = c_2$	Always
$c_1 p^{\gamma_1}$, where $\gamma_1 \neq 0$	$\frac{c_2^{1/\gamma_1}}{c_1^{1/\gamma_1} p} \propto \frac{1}{p}$	$p = \left(\frac{c_2}{c_1}\right)^{1/\gamma_1}$	$p_1 = p_2$

Table 3 summarizes the results of the subproportional apportionment (SA) scheme (i.e., the DP scheme), assuming that $c_2 p^{\gamma_2}$ seats are assigned to population p for some constants $c_2 > 0$ and γ_2 , with

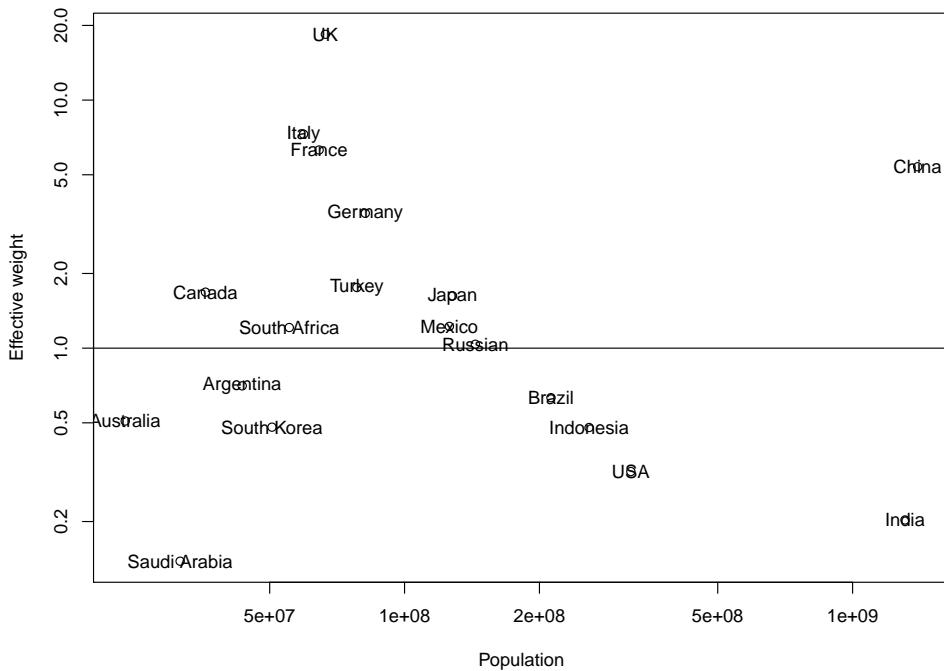
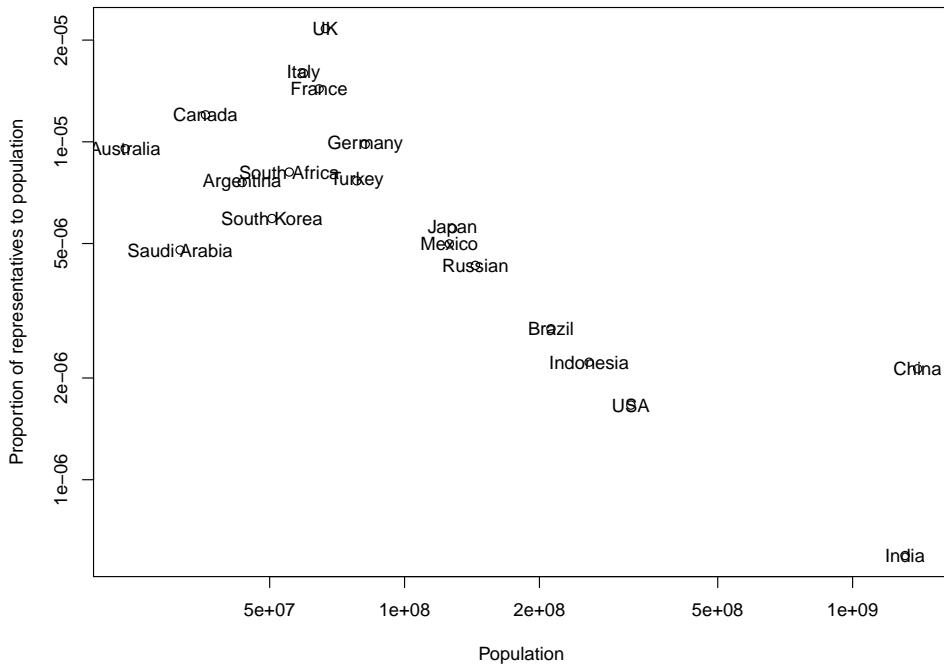


Figure 4. Comparing G20 countries (except for the EU) according to the indicators PSP (upper) and PSI (lower), where the total size of the congress is used for a bicameral country. For the PSI, $f(p) = \frac{1}{3}p^{0.4}$ (Zhao and Peng, 2020) is used as the standard function, as it shows the global average. Note that the PSP provides no measure of the appropriateness, while the proposed PSI metric does: PSI = 1 indicates absolute compatibility with the global standard (average); PSI > 1 (respectively, PSI < 1) indicates that the size of the congress is more (less) than the global average. Differences in the order due to different indicators can be confirmed, e.g., Canada and Germany, Saudi Arabia and the USA, etc. Additionally, Australia and Germany have almost the same PSP value but considerably different PSI values. Data source: IPU, 2021.

$0 < \gamma_2 < 1$. Relative equality can be achieved by selecting $\gamma_2 = \gamma_1$ or by redistricting groups to equal populations. For the European Parliament, since equal populations are not an option, $\gamma_2 = \gamma_1$ must be used, in contrast to existing “compromise” proposals which try to keep $\gamma_2 = 1$. We note that absolute equality can be achieved under certain conditions.

Table 3. Implications for the subproportional apportionment (SA) scheme (i.e., the DP scheme) with $\gamma_2 > 0$.

Standard function $f(p)$	PSI ($= w(c_2 p^{\gamma_2}, p)$)	Absolute Eq.	Relative Eq.
c_1 (by $c_1 + \varepsilon p, \varepsilon \rightarrow 0$)	0 if $p = (\frac{c_1}{c_2})^{1/\gamma_2}$, else ∞	$p = (\frac{c_1}{c_2})^{1/\gamma_2}$	$\frac{c_2 p_1^{\gamma_2} - c_1}{p_1} = \frac{c_2 p_2^{\gamma_2} - c_1}{p_2}$
$c_1 p^{\gamma_1}, \gamma_1 \neq 0$	$\left(\frac{c_2}{c_1}\right)^{1/\gamma_1} p^{\gamma_2/\gamma_1 - 1}$	$\gamma_1 = \gamma_2, c_1 = c_2$ or $\gamma_1 \neq \gamma_2, p = \left(\frac{c_1}{c_2}\right)^{\frac{1}{\gamma_2 - \gamma_1}}$	$\gamma_1 = \gamma_2$ or $p_1 = p_2$

For the proportional apportionment (PA) scheme, the results can be obtained by simply substituting $\gamma_2 = 1$ in Table 3. However, since $f(p) \propto p^{\gamma_1}$ for some $0 < \gamma_1 < 1$ in the real world, the only option for individual equality with the PA (i.e., the PR) scheme is choosing $p_1 = p_2$, i.e., equal redistricting. This means that the PA/PR scheme can be completely replaced by the SA/DP scheme, since equal redistricting is also supported by the SA/DP scheme.

We remark that the PA scheme adopted by the House and various legislatures worldwide is not truly proportional. A general idea of such an approach is to first determine a population size d , which should ideally be $d = \frac{\sum_j p_j}{S}$ for a total of S seats, then, assign $\frac{p_i}{d}$ seats (before rounding to whole numbers) to group i with population p_i for all i . However, in general, d is not a constant; thus, $\frac{p_i}{d} = \frac{S p_i}{\sum_j p_j}$ is not proportional to p_i . As previously discussed, this approach has a population-dependent bias and thus cannot ensure representation equality among individuals. To achieve true equality, we should either resize the total number of seats in proportion with the total population, as the House did in 1790–1830, or adopt a subproportional apportionment scheme.

5 CONCLUSION

In the previous century, the literature has noted the existence of a standard (average or optimal) number of representatives (seats) for a population, and that follows some subproportionality scheme with respect to the size of the population. Based on these previous works, this article pointed out a bias inherent in the PSP metric and thus in the PR scheme in estimating the contribution (weight) of an individual. To address this issue, it introduced a standard function $f(p)$ for the number of seats for a population p and a nonproportional indicator PSI.

It is shown that the proposed indicator does not have the bias inherent to the PSP. By using it as the indicator to develop an apportionment scheme, a standardized representation theory is proposed with absolute or relative individual equality. In particular, if $f(p) \propto p^\gamma$ for some constant γ , the proposed scheme distributes seats proportionally to the γ -th power of the populations. Because $0 \leq \gamma < 1$ in the real world, it is concluded that the PR scheme represents people in smaller groups *less* than people in larger groups, whereas the proposed subproportionality scheme, which is a type of degressive proportionality approach, guarantees equality.

In the future, rounding methods should be revised since existing methods were designed for PR schemes only. Empirical studies are also required to investigate the issue in detail, e.g., the implication to the apportionment of the EU Parliament. We also suggest that related sectors reconsider existing apportionment schemes to better ensure individual equality.

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