

Are Front-running HFTs Harmful?

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Abstract

This paper is concerned with how a front-running high-frequency trader (HFT) influences the large trader: whether and under what conditions the latter is harmed or benefited. We study, in the extended Kyle's model, the interactions between a large informed trader and an HFT who can predict the former's incoming order in some extent. Equilibria under various specific situations are discussed. We conclude that HFT always front-runs and the large trader could be favored in the following circumstances: (1) there is sufficient noise trading with HFT's liquidity-consuming trading; (2) the noise trading with HFT's liquidity-consuming trading is inadequate but HFT's signal is vague enough. Besides, we explore the influences of market noise and signal noise on investors' behavior and profits. We find surprisingly that (1) increasing the noise in HFT's signal might decrease the large trader's profit; (2) when there are few market noises, although HFT nearly does nothing, the large trader is still hurt; (3) in any case, HFT will not front-run more than half of the large trader's order.

Keywords: High-frequency trading; Front-running; Large informed trader

1 Introduction

Front-runner predicts large traders' incoming orders, trades in the same direction in front of them, and then supplies liquidity back when they arrive, hoping to profit from the market impacts caused by large traders. For example, if a front-runner receives the signal that a large trader is going to buy, she will buy in advance and then sell back to the large trader.

In today's market, most front-runners are high-frequency traders (HFTs), who process information faster and send orders with lower latency, compared to normal-speed traders. HFTs' speed advantage enables them to apply and possibly make profits from the front-running strategy.

Front-runners' influence on large traders' actions and profits is an important topic in both predatory trading and high-frequency trading. Conventional wisdom has it that front-runners increase large traders' transaction costs by taking away liquidity that might otherwise have gone to the large traders. However, some studies do believe that under certain conditions, front-runners could benefit large traders by providing liquidity when the large traders come. Empirical work supports both views, a unified conclusion has not been reached.

In this paper, we consider the interactions between an informed large trader and an HFT who can predict, to a certain extent, the large trader's future order, in the extended celebrated Kyle's model. We are interested in whether or not the HFT will conduct the front-running strategy and how the large trader is affected. A comprehensive analysis has been made under various circumstances. We prove that it is optimal for HFT to front-run and we outline situations where HFT harms or benefits the large trader. Specifically, the front-running HFT is favorable to the large trader when (1) there is sufficient noise trading with HFT's liquidity-consuming trading, no matter her signal is accurate or not; (2) the noise trading with HFT's liquidity-consuming trading is inadequate but her signal is vague enough.

We also explore the influences of market noise and signal accuracy on investors' behavior and profits. For HFT, she trades more aggressively when the market is noisier and the signal is more accurate. For the

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large trader, her trading intensity and profit always increase with the size of market noise. Surprisingly, we find that, when the HFT's signal is inaccurate to a certain degree, the large trader can be guaranteed not to be harmed, but her profit tends to decrease as the signal gets noisier, which seems counterintuitive with the common sense that the large trader gets better if her trading intentions are deeper hidden. Hence, for a conservative large trader, if she just aims to protect herself from being harmed by potential HFTs, it is always safer to enter a market where the order anticipation is more inaccurate. However, if she aims to make a profit, it may be not optimal to enter such a market.

The limit results of the above influences are also investigated. We have drawn two interesting conclusions: (1) when there are few market noises, HFT's trading intensity tends to zero, that is she nearly does nothing, but the large trader is still harmed; (2) when the signal is completely accurate and the size of market noise tends to infinity, HFT reaches the maximum front-running volume, which is only half of the large trader's incoming order.

Besides the case where HFT predicts the large trader's order, we also study the case where HFT can predict the aggregate order flow, i.e., the large trader's order and its accompanying noise traders' orders. From a practical point of view, this kind of HFTs has more advanced technology to estimate the trend of the whole market, rather than only the major market trend. In this case, we find that HFT still front-runs and always benefits the large trader.

2 Related Literature

The front-running problem has attracted wide attention since the last century. For example, Harris (1997) [1] introduces that, "Large impatient traders often significantly impact prices when they trade. If other traders know their intentions, they may front-run them."

With the development of high-frequency trading technology, more and more front-runners are HFTs, who receive and process information faster. But how they make profits and affect other market participants remains questionable. In the empirical work Kirilenko et al. (2017) [2], the authors examine the role of market participants before and during Flash Crash by using transaction-level data for the E-mini. They find that before liquidity demanders arrive, HFTs aggressively remove the last few contracts at the best bid and ask, then provide liquidity at a new price level, offsetting their position, which implies that HFTs behave like front-runners at a high frequency. Other supportive work includes Khan and Lu (2013) [3], where the authors find significantly positive short sales in the days leading up to large insider's sales. Manahov (2016) [4] shows that HFTs trade in the same direction as the strategic informed traders, however keeping ahead of them. Hirschev (2021) [5] presents evidence that HFTs identify patterns from trading history to predict future selling or buying pressure and trade ahead of other investors' order flow.

Front-running strategy exploits other investors' need to trade, thus belonging to the topic of predatory trading. Brunnermeier and Pedersen (2005) [6] is a pioneering work to study predatory trading. In their model, when time begins from t_0 but the distressed large trader starts selling at $t_1 > t_0$, predatory traders are actually front-runners who sell before t_1 and buy back after t_1 . They conclude that front-runners bring inferior prices to the large trader.

In [6], the price impact is permanent. When the temporary impact is also considered, Schoenebor and Schied (2009) [7] and Carmona and Yang (2011) [8] mention that under particular market conditions, e.g. low permanent impact and high temporary impact, it is optimal for predatory traders to first buy when the large trader is selling and sell when the price bounces back, the large trader is thus benefited by the liquidity provided by them. Bessembinder et al. (2016) [9] presents a model where the temporary impact is transient. The authors find that when the market is quite resilient and the permanent impact is not so large, front-runners could benefit the large trader. A resilient market can be regarded as active and liquid, which is consistent with our discovery that HFT may be good for the large trader in a market with abundant noise tradings.

Besides front-running the large or informed traders, Bernhardt and Taub (2008) [10] models an investor who is not only aware of the true value of the asset but also the current and future noise tradings. Her optimal strategy includes front-running the future noise tradings.

When it comes to high-frequency trading, Li (2013) [11] models front-running HFTs who predict the sum of informed and noise orders. The author concludes that informed trader is harmed and trades less aggressively in the presence of HFTs. Brogaard et al. (2014) [12] studies whether HFT increases the execution costs of institutional investors, based on the data of LSE from 2007 to 2011. The authors find that HFTs are more active after the exchange speed is improved. However, the costs of institutional investors remain unchanged. Hens et al. (2018) [13] investigates front-running HFTs in a limit order model and shows that they extract rents from investors who need to trade large size quickly. Yang and

Zhu (2020) [14] models back-runners in an extended two-period Kyle's model, where back-runners use past order flow information to predict the informed traders' future path and trade along with them. For a more thorough review of high-frequency trading literature, readers could refer to Menkveld (2016) [15].

3 The Model and Equilibrium

We start by introducing the following two-stage market model for front-running, which is an extension of the classic Kyle's model.

Assets and Participants. In this market, a risky asset is traded whose true value or ex-post liquidation value, v , is normally distributed as

$$v \sim N(p_0, \sigma_v^2).$$

There is also a risk-free asset with zero interest rate, which provides inter-temporal value accumulations only.

There are four types of market participants: (1) *dealers*, who observe the aggregate order flow and are assumed to be competitive and risk-neutral. The Bertrand competition forces them to make zero expected profit and hence set the transaction price of the risky asset as the expectation of v conditional on their information; (2) a normal-speed large *informed trader* (IT, for short), who privately knows v ; (3) a strategic *High-Frequency Trader* (HFT, for short), who is capable to get a signal about IT's future trading and might act as a front-runner; (4) *noise traders*, who trade randomly. We suppose that both IT and HFT are risk-neutral and seek to maximize their expected P&L.

Trading Structure and Prices. For a two-stage model, we consider three time points ¹, $t = 0, 1, 2$. At $t = 0$, IT sends a market order of quantity $i = i(v)$, based on her private knowledge of v . However, for some reasons, e.g., the submission delay, as in [11], [16], the order is not executed until $t = 2$.

Very soon after IT sends her order, HFT recognizes IT's trading intention and gets a noisy signal about the informed order i :

$$\hat{i} = i + z,$$

where the noise z is independent of i and follows $N(0, \sigma_z^2)$. Hirshey (2021) [5] finds evidence supporting that HFTs can recognize non-HFTs' persistent informed order flow in real time. We will not get to the bottom of how HFT predicts the order flow specifically, which is another challenging work that lies beyond the scope of this paper.

The noise in HFT's signal may come from: (1) market regulations about information disclosure, which makes it difficult for HFT to filter useful news about large trader's tradings; (2) the limitation of HFT's technology, which brings the prediction error. The standard deviation of z represents the accuracy of HFT's signal. The smaller the σ_z , the higher the accuracy. When $\sigma_z = 0$, the information hidden in IT's order is perfectly detected by HFT.

At $t = 1$, HFT builds up a position on the risky asset by sending a market order of quantity $x = x(\hat{i})$. We assume that HFT's market orders will always be fulfilled at once. This risky position will be offset when HFT sends a market order of size $-x$ at $t = 2$. If x is in the same direction of \hat{i} , HFT employs the front-running strategy: takes liquidity in front of IT and when IT's order arrives, trades against it to supply liquidity.

To be closer to reality, we suppose that both time-1 and time-2 trades are accompanied by noise tradings. We denote the aggregate noise order flow in each period by u_1 and u_2 , where

$$u_1 \sim N(0, \sigma_1^2), \quad u_2 \sim N(0, \sigma_2^2),$$

are independent of each other and any other random variables.

To sum up, the total order flow y_1 and y_2 executed at $t = 1$ and $t = 2$ respectively are

$$y_1 = x + u_1 \quad \text{and} \quad y_2 = i + u_2 - x.$$

Hence the transaction prices of the risky asset at each time point will be

$$p_1 = \mathbb{E}(v|y_1) \quad \text{and} \quad p_2 = \mathbb{E}(v|y_1, y_2).$$

¹Note that we mark these time stamps just for convenience. It is not required that the time lengths of $[0, 1]$ and $[1, 2]$ are equal.

In the following, we assume $\sigma_2 > 0$ as in Kyle's model and use

$$\theta_z = \frac{\sigma_z^2}{\sigma_2^2} \quad \text{and} \quad \theta_1 = \frac{\sigma_1^2}{\sigma_2^2}$$

to characterize the signal noise and time-1 market noise normalized by σ_2^2 , which are in line with the parameters studied in [14].

Equilibrium. The main purpose of the current paper is to study HFT's trading strategy and how IT is affected by HFT in equilibrium, and to compare the results with those in one-period Kyle's model when there are only IT and dealers. We first give the definition of equilibrium in our model.

Definition 1. *The equilibrium is defined as a collection of strategies of the dealers, IT and HFT: $\{p_1, p_2, i, x\}$, such that the following market-efficiency condition and two optimization conditions are satisfied.*

1. Given IT's strategy i and HFT's strategy x , dealers set price according to the weak-efficiency rule:

$$\begin{aligned} p_1 &= \mathbb{E}(v|y_1), \\ p_2 &= \mathbb{E}(v|y_1, y_2). \end{aligned}$$

2. Given HFT's strategy x and dealers' pricing rule p_1, p_2 , IT's strategy i^* maximizes her expected profit over all measurable strategies $i = i(v)$:

$$i^* = \arg \max_{i=i(v)} \pi^{IT}(i),$$

where $\pi^{IT}(i) = \mathbb{E}((v - p_2)i|v)$.

3. Given IT's strategy i and dealers' pricing rule p_1, p_2 , HFT's strategy x^* maximizes her expected profit over all measurable strategies $x = x(\hat{i})$:

$$x^* = \arg \max_{x=x(\hat{i})} \pi^{HFT}(x),$$

where $\pi^{HFT}(x) = \mathbb{E}((p_2 - p_1)x|\hat{i})$.

Within the Normal-distribution framework, it is natural to conjecture a linear structure of the equilibrium, i.e., in equilibrium, the strategies of IT and HFT as well as liquidation prices are linear functions:

$$p_1 = p_0 + \lambda_1 y_1,$$

$$p_2 = p_0 + \mu_1 y_1 + \mu_2 y_2,$$

$$i = \alpha(v - p_0),$$

$$x = \beta \hat{i}.$$

This is consistent with the conjectures and results in Kyle (1985) [17], Bernhardt and Miao (2004) [18], Bernhardt and Taud (2008) [10] and Yang and Zhu (2020) [14].

We will prove the existence and uniqueness of a linear equilibrium under different circumstances in Section 4. The properties of the optimal strategies will also be verified.

Please note that, till now, we assume HFT predicts the informed order i rather than the aggregate order $i + u_2$. In other words, here we focus on those potential front-running HFTs who are skilled at estimating some major market trends but may not know well the whole market. These HFTs' behavior is consistent with the back-runners' in Yang and Zhu (2020) [14]. Empirically, both kinds of HFTs may exist, we will discuss the second case in Section 6, as a supplement to Li (2018) [11].

4 Main Results

In this section, we investigate market participants', especially HFT's and IT's behavior in equilibrium. It is usually believed that front-runner increases large traders' transaction costs by trading in the same direction in front of them, see [1], [6], [11]. But there is another line of research suggests that, both theoretically and empirically, see, e.g., [9], the front-running behavior might on the contrary decrease the

impact of large traders' order under some circumstances, since it also provides liquidity back. Therefore, under the formulated model, we are concerned about the following questions: (1) whether or not the HFT would do front-running; (2) whether or not the HFT's action would harm the large trader IT; (3) under what circumstances would the HFT's action harm or benefit IT.

In the following, the existence and uniqueness of a linear equilibrium will be given after cautiously analyzing the participants' strategies in different situations, according to the size of time-1 noise trading and the accuracy of HFT's signal. Then we discuss how the HFT's strategy affects IT, especially when will the latter be benefited or harmed. Through comparative static analysis, we explore the influences of θ_1 and θ_z on investors' actions and the results in limit situations.

For the sake of simplicity, in the following discussions, we assume $p_0 = 0$, for it does not bring out any essential changes in results.

Since there is no general equilibrium for the two cases when there is noise trading at time 1 or not, we discuss the two cases separately: (1) $\theta_1 > 0$, that is $\sigma_1 > 0$, which can be characterized as a case that the market is active in both periods, noise tradings might come from high-speed traders as well as normal-speed traders; (2) $\theta_1 = 0$, that is $\sigma_1 = 0$, which can be viewed as a case that the noise traders are of slower speed relative to HFT, and thus the HFT's time-1 trading is accompanied by few noises.

4.1 Equilibrium in the case $\theta_1 > 0$

In this case, noise tradings occur in both periods. We study the strategies of main participants: dealers, IT and HFT respectively, given the others' strategies.

Dealers' quotes. As stated in the former section, risk-neutral and competitive dealers set the transaction prices as the expectation of v , conditioned on the order flow information.

At $t = 1$, when HFT builds up the position x , the price should be

$$p_1 = \mathbb{E}(v|y_1) = \lambda_1 y_1,$$

where, given by the linear conjecture and projection theorem,

$$\lambda_1 = \frac{\sigma_v}{\sigma_{y_1}} \rho_{(v,y_1)} = \frac{\alpha\beta\sigma_v^2}{\beta^2(\alpha^2\sigma_v^2 + \sigma_z^2) + \sigma_1^2}. \quad (1)$$

At $t = 2$, when IT's order i and HFT's offsetting order $-x$ are being executed, the price should be

$$p_2 = \mathbb{E}(v|y_1, y_2) = \mu_1 y_1 + \mu_2 y_2,$$

where, still by the projection theorem,

$$\mu_1 = \frac{\sigma_v}{\sigma_{y_1}} \frac{\rho_{(v,y_1)} - \rho_{(y_1,y_2)}\rho_{(v,y_2)}}{1 - \rho_{(y_1,y_2)}^2} = \frac{\alpha\sigma_v^2(\beta^2\sigma_z^2 + \beta\sigma_2^2)}{\alpha^2\sigma_v^2(\sigma_1^2(1-\beta)^2 + \sigma_2^2\beta^2 + \sigma_z^2\beta^2) + \beta^2\sigma_z^2(\sigma_1^2 + \sigma_2^2) + \sigma_1^2\sigma_2^2}, \quad (2)$$

$$\mu_2 = \frac{\sigma_v}{\sigma_{y_2}} \frac{\rho_{(v,y_2)} - \rho_{(v,y_1)}\rho_{(y_1,y_2)}}{1 - \rho_{(y_1,y_2)}^2} = \frac{\alpha\sigma_v^2(\beta^2\sigma_z^2 + (1-\beta)\sigma_1^2)}{\alpha^2\sigma_v^2(\sigma_1^2(1-\beta)^2 + \sigma_2^2\beta^2 + \sigma_z^2\beta^2) + \beta^2\sigma_z^2(\sigma_1^2 + \sigma_2^2) + \sigma_1^2\sigma_2^2}. \quad (3)$$

The above impact coefficients, λ_1, μ_1, μ_2 , also represent the informativeness of the corresponding order flow. And we will see in latter sections that some heuristic results could be figured out in special cases and in comparative static analysis.

Here we point out that, in the special case when HFT decided to do nothing, that is, when $x = 0$, dealers would confront a market as in Kyle's one-period model, except that there is an additional auction before the IT's order i is completed. However, no information is leaked out in this prior auction. So we find that $p_1 = p_0 = \mathbb{E}[v]$ while

$$p_2 = \lambda_2 y_2, \lambda_2 = \frac{\alpha\sigma_v^2}{\alpha^2\sigma_v^2 + \sigma_2^2},$$

which is the same as the liquidating price in Kyle's one-period model.

Another point to note is, since HFT partly detects the intention of IT and trades against/ along the informed order i , i has indirect impacts on the prices. In fact, under linear structures, the total impact of i on p_2 turns out to be a combination of different-time impacts: $(\mu_1\beta + \mu_2(1-\beta))i$. Let λ_2 denote this total impact, then

$$\begin{aligned} \lambda_2 &= \mu_1\beta + \mu_2(1-\beta) \\ &= \frac{\alpha\sigma_v^2(\sigma_1^2(1-\beta)^2 + \sigma_2^2\beta^2 + \sigma_z^2\beta^2)}{\alpha^2\sigma_v^2(\sigma_1^2(1-\beta)^2 + \sigma_2^2\beta^2 + \sigma_z^2\beta^2) + \beta^2\sigma_z^2(\sigma_1^2 + \sigma_2^2) + \sigma_1^2\sigma_2^2}. \end{aligned} \quad (4)$$

The part $\mu_1\beta$ comes from HFT's time-1 trading x and the part $\mu_2(1-\beta)$ comes from time-2 trading $i-x$.

IT's strategy. IT decides the order i to maximize her expected P&L based on her information. Given dealers' quotes and HFT's strategy, IT's expected P&L is:

$$\begin{aligned}\pi^{\text{IT}} &= \mathbb{E}(i(v - p_2)|v) \\ &= i(v - \lambda_2 i),\end{aligned}$$

which can be uniquely maximized at $i^* = \frac{1}{2\lambda_2}v$, under the second order condition (SOC) $\lambda_2 > 0$. The optimal α is

$$\alpha^* = \frac{1}{2\lambda_2}. \quad (5)$$

HFT's strategy. Given dealers' quotes and IT's strategy, HFT's expected P&L is:

$$\begin{aligned}\pi^{\text{HFT}} &= \mathbb{E}(x(p_2 - p_1)|\hat{i}) \\ &= x \left((\mu_1 - \mu_2 - \lambda_1)x + \mu_2 \mathbb{E}(i|\hat{i}) \right) \\ &= -(\mu_2 + \lambda_1 - \mu_1)x^2 + \mu_2 \frac{\alpha^2 \sigma_v^2}{\alpha^2 \sigma_v^2 + \sigma_z^2} \hat{i}x,\end{aligned}$$

which can be uniquely maximized at $x^* = \frac{\mu_2}{2(\mu_2 + \lambda_1 - \mu_1)} \frac{\alpha^2 \sigma_v^2}{\alpha^2 \sigma_v^2 + \sigma_z^2} \hat{i}$, under the SOC $\mu_2 + \lambda_1 - \mu_1 > 0$. The optimal β is

$$\beta^* = \frac{\mu_2}{2(\mu_2 + \lambda_1 - \mu_1)} \frac{\alpha^2 \sigma_v^2}{\alpha^2 \sigma_v^2 + \sigma_z^2}. \quad (6)$$

Equilibrium. We have analyzed the optimal behavior of different participants and now we give the existence and uniqueness of a linear equilibrium.

Note that according to Equation (4), λ_2 is a function of α and β :

$$\lambda_2 = \lambda_2(\alpha, \beta) = \frac{B\alpha}{B\alpha^2 + C}, \quad (7)$$

where $B = B(\beta) = \sigma_v^2 (\sigma_1^2(1-\beta)^2 + \sigma_2^2\beta^2 + \sigma_z^2\beta^2)$, $C = C(\beta) = \beta^2 \sigma_z^2 (\sigma_1^2 + \sigma_2^2) + \sigma_1^2\sigma_2^2$. Substituting it into Equation (5), we find that α^* could be written as the function of β^* :

$$\alpha^* = \sqrt{\frac{C(\beta^*)}{B(\beta^*)}}. \quad (8)$$

Substituting (8) into (1),(2),(3),(6), we get the equation of β^* which should be satisfied in equilibrium.

Theorem 1. Given $\sigma_1, \sigma_2 > 0, \sigma_z \geq 0$, $(\theta_1 = \frac{\sigma_1^2}{\sigma_2^2} > 0, \theta_z = \frac{\sigma_z^2}{\sigma_2^2} \geq 0,)$ there exists a unique equilibrium $\{p_1, p_2, i^*, x^*\}$, where HFT follows the strategy $x^* = \beta^* \hat{i}$ and $\beta^* \in (0, 1)$ solves the equation

$$\begin{aligned}0 &= \beta^6 (4\theta_1\theta_z^2 + \theta_1\theta_z^3 + 2\theta_1^2\theta_z^2 + 2\theta_z^2 + \theta_z^3) + \beta^5 (4\theta_1\theta_z + 4\theta_1\theta_z^2 + 2\theta_1\theta_z^3 + 8\theta_1^2\theta_z + 4\theta_1^2\theta_z^2 + 4\theta_1^3\theta_z) \\ &\quad + \beta^4 (2\theta_1\theta_z + \theta_1\theta_z^2 - 11\theta_1^2\theta_z - 8\theta_1^2\theta_z^2 - 13\theta_1^3\theta_z) + \beta^3 (2\theta_1^2 + 2\theta_1^3 + 8\theta_1^2\theta_z + 4\theta_1^2\theta_z^2 + 16\theta_1^3\theta_z) \\ &\quad - \beta^2 (\theta_1^2\theta_z + 5\theta_1^3 + 9\theta_1^2\theta_z) + \beta (4\theta_1^3 + 2\theta_1^2\theta_z) - \theta_1^3;\end{aligned} \quad (9)$$

IT follows the strategy $i^* = \alpha^* v$ and

$$\alpha^* = \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\theta_1 + \beta^{*2}\theta_z(\theta_1 + 1)}{\theta_1(1 - \beta^*)^2 + \beta^{*2}(1 + \theta_z)}}; \quad (10)$$

the liquidation price at time 1 is

$$p_1 = \lambda_1^*(x^* + u_1),$$

where

$$\lambda_1^* = \frac{\sigma_v}{2\sigma_2} \frac{2\beta^* \sqrt{(\beta^{*2}\theta_z(\theta_1 + 1) + \theta_1)(\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1))}}{\beta^{*2}(\beta^{*2}\theta_z(\theta_1 + 1) + \theta_1) + (\beta^{*2}\theta_z + \theta_1)(\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1))};$$

the liquidation price at time 2 is

$$p_2 = \mu_1^*(x^* + u_1) + \mu_2^*(i^* + u_2 - x^*),$$

where

$$\begin{aligned}\mu_1^* &= \frac{\sigma_v}{2\sigma_2} \frac{\beta^{*2}\theta_z + \beta^*}{\sqrt{(\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1))(\beta^{*2}\theta_z(\theta_1 + 1) + \theta_1)}}, \\ \mu_2^* &= \frac{\sigma_v}{2\sigma_2} \frac{\beta^{*2}\theta_z + (1 - \beta^*)\theta_1}{\sqrt{(\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1))(\beta^{*2}\theta_z(\theta_1 + 1) + \theta_1)}};\end{aligned}$$

the impact coefficient of the informed order i^* is

$$\lambda_2^* = \mu_1^*\beta^* + \mu_2^*(1 - \beta^*) = \frac{\sigma_v}{2\sigma_2} \sqrt{\frac{\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1)}{\theta_1 + \beta^{*2}\theta_z(\theta_1 + 1)}}.$$

Now we give the maximal expected profit of investors and price discovery variables in equilibrium, representing them in the form of β^* , to prepare for subsequent analysis.

Corollary 1. *The maximal expected profits of IT and HFT are*

$$\begin{aligned}\mathbb{E}(\pi^{IT}) &= \frac{\sigma_v\sigma_2}{2} \sqrt{\frac{\theta_1 + \beta^{*2}\theta_z(\theta_1 + 1)}{\theta_1(1 - \beta^*)^2 + (1 + \theta_z)\beta^{*2}}}, \\ \mathbb{E}(\pi^{HFT}) &= \frac{\sigma_v\sigma_2}{2} \frac{\beta^*\theta_1((1 - \beta^*)\theta_1 + \beta^{*2}\theta_z)}{\sqrt{(\beta^{*2}\theta_z(\theta_1 + 1) + \theta_1)(\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1))}} \\ &\quad \frac{\beta^{*2}\theta_z((1 - 2\beta^*) - \beta^*\theta_z) + (1 - \beta^*)\theta_1(1 - \beta^*(1 - 2\beta^*)\theta_z)}{(1 - \beta^*)^2\theta_1^2 + \beta^{*4}\theta_z(\theta_z + 2) + 2\beta^{*2}\theta_1(1 + (\beta^{*2} - \beta^* + 1)\theta_z)}.\end{aligned}$$

The ex-ante and ex-post price discovery variables are

$$\begin{aligned}\mathbb{E}(v - p_1)^2 &= \sigma_v^2 \frac{(\beta^{*2}\theta_z + \theta_1)(\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1))}{\beta^{*2}(\beta^{*2}\theta_z(\theta_1 + 1) + \theta_1) + (\beta^{*2}\theta_z + \theta_1)(\theta_1(1 - \beta^*)^2 + \beta^{*2}(\theta_z + 1))}, \\ \mathbb{E}(v - p_2)^2 &= \frac{\sigma_v^2}{2}.\end{aligned}$$

4.1.1 How Front-running HFT affects IT

From Theorem 1, we see that in equilibrium $\beta^* \in (0, 1)$, which means that when HFT predicts the future order of IT, she front-runs. So how IT is affected in the presence of the front-running HFT? Compared to the classic Kyle's model, where in equilibrium the expected profit of IT is $\frac{\sigma_v\sigma_2}{2}$, when HFT exists, IT's expected profit is $\frac{\sigma_v^2}{4\sigma_2^2}$, so if $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$, it turns out to get larger.

Recalling Equation (7) and (8), we find that λ_2^* is the maximum of $\lambda_2(\alpha; \beta^*)$, attained at α^* . So in the next proposition, we specifically analyze how HFT's transactions affect informed order i 's total impact $\lambda_2(\alpha; \beta)$, and then influence λ_2^* and IT's profit.

Proposition 1. *Given $\theta_1 > 0, \theta_z \geq 0$ and IT's action $\alpha > 0$, for any $\beta \in (0, 1)$,*

$$\lambda_2(\alpha; \beta) = \mu_1(\alpha; \beta)\beta + \mu_2(\alpha; \beta)(1 - \beta),$$

where the first part increases with β and the second part decreases with β .

Hence, HFT's time-1 trading increases IT's transaction cost, while, her offsetting trading at time 2 decreases it. All in all, front-running HFT is a double-edged sword for IT. However, we can confirm that under certain conditions, HFT is advantageous to IT.

Proposition 2. *In equilibrium, we have*

$$\mu_2^* < \frac{\sigma_v}{2\sigma_2},$$

which makes it possible that $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$, i.e. IT benefits from HFT.

In fact, in Kyle's model, where there are only IT and dealers, the informativeness of order flow $y = i + u_2$ is $\frac{\sigma_v}{2\sigma_2}$. While, in our model, with the front-running HFT, information about v has been exposed through her time-1 trading x , moreover, her time-2 reverse trading $-x$ further reduces the information contained in the order flow y_2 . Thus, the price is less sensitive to the news in y_2 ($y_2 - \mathbb{E}(y_2|y_1)$), which makes $\mu_2^* < \frac{\sigma_v}{2\sigma_2}$.

The above two propositions tell us that HFT's front-running behavior has two-sided effects on IT's profit and might be helpful. A natural question is that when does it happen? We answer this in the following theorem.

Theorem 2. *In the following cases, $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$, i.e., IT makes more profits in the case with a front-running HFT:*

1. $\theta_1 \geq \frac{2\sqrt{3}-3}{3}, \theta_z \geq 0$.
2. $0 < \theta_1 < \frac{2\sqrt{3}-3}{3}, \theta_z > \bar{\theta}_z = \frac{-(\theta_1+5)+2\sqrt{4\theta_1^2+10\theta_1+5}}{-5\theta_1}$.
3. $\theta_z \geq 0, \theta_1 > \bar{\theta}_1 = \frac{-(5\theta_z+3)+2\sqrt{5\theta_z^2+8\theta_z+3}}{-5\theta_z^2+2\theta_z+3}$.

In case 1, when the market noise θ_1 is large enough, HFT cannot harm IT even if her signal is perfectly accurate. In case 2, if noise trading at time 1 is not active enough, IT could be benefited when HFT's signal is relatively vague. What's more, larger the θ_1 , smaller the $\bar{\theta}_z$, that is, as the size of noise trading gets larger, less signal noise is needed to disturb HFT. In case 3, for any accuracy of signal, the IT could be benefited when the market hides her well. In this case, larger the θ_z , smaller the $\bar{\theta}_1$, when the size of signal noise grows, less market noise is needed to cover the information disclosed by HFT's time-1 trading.

Similarly, we could consider situations where HFT harms IT.

Theorem 3. *In the following cases, $\lambda_2^* > \frac{\sigma_v}{2\sigma_2}$, i.e., IT makes less profits in the case with a front-running HFT:*

1. $0 < \theta_1 < \frac{2\sqrt{3}-3}{3}, 0 \leq \theta_z < \bar{\theta}_z$.
2. $\theta_z \geq 0, 0 < \theta_1 < \bar{\theta}_1$.

4.1.2 Influences of noises on traders' behavior

In equilibrium, the optimal intensities of investors, α^* and β^* , both depend on the relative size of noises: θ_1 and θ_z . Now we investigate how α^* and β^* change with them.

Proposition 3. *The optimal intensity of HFT, β^* , increases with θ_1 and decreases with θ_z .*

A larger θ_1 means more noise tradings at time 1, which enables HFT to employ more of her priority information and bear less impact. A larger θ_z means a less accurate signal, which makes HFT less confident about IT's future trading, and hence reduces her trading intensity.

The impact of θ s on IT's behavior is more complicated. However, we can still prove the following results.

Proposition 4. *1. When $\theta_1 \geq \frac{1}{2}$, α^* increases with θ_1 and decreases with θ_z .*

2. *When $0 < \theta_1 < \frac{1}{2}$,*
 - (a) *given $\theta_z \geq 0$, α^* increases with θ_1 when $\theta_1 \geq \tilde{\theta}_1 = \frac{\sqrt{(3\theta_z+1)^2+8}-(3\theta_z+1)}{4}$;*
 - (b) *given $\theta_1 \geq 0.0089$, α^* increases with θ_z when $0 \leq \theta_z < \tilde{\theta}_z = \frac{1-\theta_1-2\theta_1^2}{3\theta_1}$ and decreases with it when $\theta_z \geq \tilde{\theta}_z$.*

For the monotonicity concerned in Proposition 4, we have done a lot of numerical experiments, and the results can be regarded as a supplement to Proposition 4. In fact, we find in general that (see Figure 2, Figure 4 and Figure 6)

1. α^* increases with θ_1 ;
2. given $\theta_1 \geq \frac{1}{2}$, α^* decreases with θ_z , α^* is maximized at $\theta_z = 0$;

3. given $0 < \theta_1 < \frac{1}{2}$, α^* increases with θ_z when $0 \leq \theta_z < \tilde{\theta}_z$ and decreases with it when $\theta_z \geq \tilde{\theta}_z$.

We will illustrate these in Section 5.

Since $\mathbb{E}(\pi^{IT}) = \frac{\sigma_v^2}{2}\alpha^*$, when IT trades more aggressively, her expected profit is higher. Thus, combining Proposition 4 and Theorem 2, we surprisingly find that when $\theta_1 \geq \frac{1}{2}$, IT is benefited by HFT, however, α^* , and consequently, IT's expected profit decrease with the size of signal noise. When the market provides enough noise shelter for IT to protect herself, adding any noise to HFT's signal is unfavorable to IT. This also answers a controversial question: is it harmful to other traders if HFT has too precise information? From our observation, it depends on the size of market noise, if the noise trading is active enough, it is not necessary to increase the noise in HFT's signal.

4.1.3 The limit results

Proposition 5. Given $\theta_z \geq 0$, when $\theta_1 \rightarrow \infty$, β^* converges to the root of the following equation:

$$4\theta_z\beta^5 - 13\theta_z\beta^4 + (2 + 16\theta_z)\beta^3 - (5 + 9\theta_z)\beta^2 + (4 + 2\theta_z)\beta - 1 = 0.$$

The optimal intensity of IT

$$\alpha^* \rightarrow \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\beta^{*2}\theta_z + 1}{(1 - \beta^*)^2}} > \frac{\sigma_2}{\sigma_v}.$$

Limits of other variables could be calculated by just substituting β^* into Theorem 1 and Corollary 1.

It verifies that given $\theta_z \geq 0$, there exists a θ_1 large enough, such that $\alpha^* > \frac{\sigma_2}{\sigma_v}$, i.e. IT trades and profits more when there is a front-running HFT.

Proposition 6. Given $\theta_z \geq 0$, when $\theta_1 \rightarrow 0$,

$$\begin{aligned} \beta^* &\rightarrow 0, \alpha^* \rightarrow \frac{\sigma_2}{\sigma_v} \sqrt{\frac{y^2\theta_z^{2x} + \theta_z}{y^2\theta_z^{2x} + \theta_z + 1}} < \frac{\sigma_2}{\sigma_v}, \\ \mathbb{E}(\pi^{IT}) &\rightarrow \frac{\sigma_2\sigma_v}{2} \sqrt{\frac{y^2\theta_z^{2x} + \theta_z}{y^2\theta_z^{2x} + \theta_z + 1}}, \mathbb{E}(\pi^{HFT}) \rightarrow 0, \\ \lambda_1^*, \mu_1^* &\rightarrow \infty, \mu_2^* \rightarrow \frac{\sigma_v}{2\sigma_2} \sqrt{\frac{y^2\theta_z^{2x} + \theta_z}{y^2\theta_z^{2x} + \theta_z + 1}}, \lambda_2^* \rightarrow \frac{\sigma_v}{2\sigma_2} \sqrt{\frac{y^2\theta_z^{2x} + \theta_z + 1}{y^2\theta_z^{2x} + \theta_z}}, \\ \mathbb{E}(v - p_1)^2 &\rightarrow \sigma_v^2 \frac{y^2\theta_z^{2x} + \theta_z + 1}{y^2\theta_z^{2x} + \theta_z + 2}, \end{aligned}$$

where $x = 0.3245$, $y = 1.3845$.

Given $\theta_z \geq 0$, when θ_1 is small enough, $\alpha^* < \frac{\sigma_2}{\sigma_v}$, i.e., IT trades less and is harmed by HFT. Interestingly, in the case when nearly all noise traders are normal-speed ($\theta_1 \rightarrow 0$), although HFT almost does nothing ($\beta^* \rightarrow 0$), IT is still worse off.

Proposition 7. Given $\theta_1 > 0$, when $\theta_z \rightarrow \infty$,

$$\begin{aligned} \beta^* &\rightarrow 0, \alpha^* \rightarrow \frac{\sigma_2}{\sigma_v}, \mathbb{E}(\pi^{IT}) \rightarrow \frac{\sigma_2\sigma_v}{2}, \mathbb{E}(\pi^{HFT}) \rightarrow 0, \\ \lambda_1^* &\rightarrow 0, \mu_1^* \rightarrow 0, \mu_2^* \rightarrow \frac{\sigma_v}{2\sigma_2}, \lambda_2^* \rightarrow \frac{\sigma_v}{2\sigma_2}, \mathbb{E}(v - p_1)^2 \rightarrow \sigma_v^2. \end{aligned}$$

In other words, when HFT gets a signal which is too noisy, the market converges to the one without HFT. When $\theta_z \rightarrow 0$, it is easy to prove that investors and market's behavior converges to the case $\theta_z = 0$, which will be fully discussed in the next subsection.

4.1.4 A special case: $\theta_z = 0$

To have more intuition about how IT is affected by the front-running HFT, we display the results of a special case that $\theta_1 > 0, \theta_z = 0$, where noise tradings occur in both periods and HFT receives a perfectly accurate signal. From the calculations above Theorem 1, we have

$$\begin{aligned} p_2 &= \mathbb{E}(v|y_1, y_2) = \mu_1 y_1 + \mu_2 y_2, \\ \mu_1 &= \frac{\alpha\theta_v\beta}{\alpha^2\theta_v(\theta_1(1 - \beta)^2 + \beta^2) + \theta_1}, \mu_2 = \frac{\alpha\theta_v\theta_1(1 - \beta)}{\alpha^2\theta_v(\theta_1(1 - \beta)^2 + \beta^2) + \theta_1}, \end{aligned} \tag{11}$$

where $\theta_v = \frac{\sigma_v^2}{\sigma_2^2}$. The impact coefficient for informed order, λ_2 , is:

$$\lambda_2 = \mu_1\beta + \mu_2(1 - \beta) = \frac{\alpha\theta_v(\theta_1(1 - \beta)^2 + \beta^2)}{\alpha^2\theta_v(\theta_1(1 - \beta)^2 + \beta^2) + \theta_1}. \quad (12)$$

Based on (11) and (12), we could conduct further analysis.

Given $\alpha > 0$, $\forall \beta \in (0, 1)$,

$$\begin{aligned} \frac{\partial(\mu_1(\alpha; \beta)\beta)}{\partial\beta} &= \frac{2\alpha\theta_v(\alpha^2\theta_v\theta_1\beta(1 - \beta) + \beta\theta_1)}{(\alpha^2\theta_v(\theta_1(1 - \beta)^2 + \beta^2) + \theta_1)^2} > 0, \\ \frac{\partial(\mu_2(\alpha; \beta)(1 - \beta))}{\partial\beta} &= \frac{-2\alpha\theta_v\theta_1(\alpha^2\theta_v\beta(1 - \beta) + (1 - \beta)\theta_1)}{(\alpha^2\theta_v(\theta_1(1 - \beta)^2 + \beta^2) + \theta_1)^2} < 0, \end{aligned}$$

which implies that front-running HFT both increases and decreases the impact of informed order through moving up and down $\lambda_2(\alpha; \beta)$.

In equilibrium,

$$\begin{aligned} \mu_1^* &= \frac{\sigma_v}{2\sigma_2} \frac{\beta^*}{\sqrt{\theta_1(\theta_1(1 - \beta^*)^2 + \beta^{*2})}}, \mu_2^* = \frac{\sigma_v}{2\sigma_2} \frac{\theta_1(1 - \beta^*)}{\sqrt{\theta_1(\theta_1(1 - \beta^*)^2 + \beta^{*2})}}, \\ \lambda_2^* &= \frac{\sigma_v}{2\sigma_2} \sqrt{\frac{\beta^{*2} + \theta_1(1 - \beta^*)^2}{\theta_1}}. \end{aligned}$$

μ_1^* could be larger than $\frac{\sigma_v}{2\sigma_2}$, the impact coefficient in the classic Kyle's model. But $\mu_2^* < \frac{\sigma_v}{2\sigma_2}$. Since λ_2^* is the linear combination of these two, it is possible that $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$ and IT is favored by HFT. This happens when

$$0 < \beta^* < \frac{2\theta_1}{\theta_1 + 1}. \quad (13)$$

In fact, we could solve β^* explicitly through Equation (9). When $\theta_z = 0$, it becomes:

$$2(\theta_1 + 1)\beta^3 - 5\theta_1\beta^2 + 4\theta_1\beta - \theta_1 = 0,$$

denote $y = \frac{\beta}{1 - \beta}$,

$$2y^3 + \theta_1y - \theta_1 = 0,$$

which we can solve analytically:

$$y = \left(\frac{\theta_1}{4}\right)^{\frac{1}{3}} \left(\left(\sqrt{1 + \frac{2}{27}\theta_1} + 1\right)^{\frac{1}{3}} - \left(\sqrt{1 + \frac{2}{27}\theta_1} - 1\right)^{\frac{1}{3}} \right), \beta^* = \frac{y}{y + 1}.$$

So (13) holds if and only if $\theta_1 > \frac{2\sqrt{3}-3}{3}$.

As for how HFT's optimal intensity β^* changes with θ_1 ,

$$\begin{aligned} \frac{\partial\beta^*}{\partial\theta_1} &> \left(\frac{\theta_1}{4}\right)^{\frac{1}{3}} \frac{(1 - \beta^*)^2}{3} \left(\left(\sqrt{1 + \frac{2}{27}\theta_1} + 1\right)^{\frac{1}{3}} - \left(\sqrt{1 + \frac{2}{27}\theta_1} - 1\right)^{\frac{1}{3}} \right. \\ &\quad \left. - \left(\left(\sqrt{1 + \frac{2}{27}\theta_1} + 1\right)^{-\frac{2}{3}} - \left(\sqrt{1 + \frac{2}{27}\theta_1} - 1\right)^{-\frac{2}{3}}\right) \right) \\ &\geq 0. \end{aligned}$$

For IT, we could also calculate $\frac{\partial\alpha^*(\theta_1)}{\partial\theta_1}$,

$$\alpha^* = \frac{\sigma_2}{\sigma_v} \frac{1}{1 - \beta^*} \sqrt{\frac{\theta_1}{\theta_1 + y^2}}, \quad (14)$$

where both $\frac{1}{1 - \beta^*}$ and $\sqrt{\frac{\theta_1}{\theta_1 + y^2}}$ increase strictly with θ_1 , so $\frac{\partial\alpha^*(\theta_1)}{\partial\theta_1} > 0$. More noise tradings during period 1 enable IT to trade more intensely.

For limit results, we have

$$\begin{aligned} \lim_{\theta_1 \rightarrow 0} \beta^* &= 0, \lim_{\theta_1 \rightarrow 0} \alpha^* = 0; \\ \lim_{\theta_1 \rightarrow \infty} \beta^* &= \frac{1}{2}, \lim_{\theta_1 \rightarrow \infty} \alpha^* = 2\frac{\sigma_2}{\sigma_v}. \end{aligned}$$

When HFT gets precise information and there are few noise tradings in period 1, both HFT and IT do nearly nothing. Overall parameters $\theta_1 > 0, \theta_z \geq 0$,

$$\beta^*(\theta_1, \theta_z) \leq \beta^*(\theta_1 = \infty, \theta_z = 0) = \frac{1}{2},$$

$$\alpha^*(\theta_1, \theta_z) \leq \alpha^*(\theta_1 = \infty, \theta_z = 0) = 2 \frac{\sigma_v^2}{\sigma_v}.$$

Therefore, in any case, HFT will not front-run more than half of the informed order. IT's intensity α^* and expected profit are at most twice of those without HFT.

4.2 Equilibrium in the case $\theta_1 = 0$

Now we analyze the equilibrium in the case with normal-speed noise traders. In this case, $y_1 = x, y_2 = i + u_2 - x$.

Dealers' quotes. Given $i = \alpha v, x = \beta \hat{i}$, dealers quote linearly,

$$\begin{aligned} p_1 &= \mathbb{E}(v|y_1) = \lambda_1 y_1, \\ p_2 &= \mathbb{E}(v|y_1, y_2) = \mu_1 y_1 + \mu_2 y_2. \end{aligned}$$

When $x \neq 0$, i.e., $\beta \neq 0$,

$$\lambda_1 = \frac{\alpha \theta_v}{\beta(\alpha^2 \theta_v + \theta_z)}, \mu_1 = \frac{\alpha \theta_v (\theta_z + 1/\beta)}{\alpha^2 \theta_v (\theta_z + 1) + \theta_z}, \mu_2 = \frac{\alpha \theta_v \theta_z}{\alpha^2 \theta_v (\theta_z + 1) + \theta_z}.$$

When $x = 0$, i.e., $\beta = 0$,

$$\lambda_1 = \mu_1 = \infty, \mu_2 = \frac{\alpha \theta_v}{\alpha^2 \theta_v + 1}.$$

IT's strategy. Given $x = \beta \hat{i}$ and dealers' linear quotes, IT's expected P&L is

$$\pi^{IT} = i(v - \lambda_2 i),$$

where $\lambda_2 = \mu_1 \beta + \mu_2 (1 - \beta)$. It can be uniquely maximized at $i^* = \frac{1}{2\lambda_2} v$ with SOC $\lambda_2 > 0$. Then we have

$$\alpha^* = \frac{1}{2\lambda_2}.$$

HFT's strategy. Given $i = \alpha v$ and dealers' linear quotes, HFT's expected P&L is

$$\pi^{HFT} = -(\mu_2 + \lambda_1 - \mu_1)x^2 + \mu_2 \frac{\alpha^2 \theta_v}{\alpha^2 \theta_v + \theta_z} \hat{i} x, \quad (15)$$

which can be uniquely maximized at $x^* = \frac{\mu_2}{2(\mu_2 + \lambda_1 - \mu_1)} \frac{\alpha^2 \theta_v}{\alpha^2 \theta_v + \theta_z} \hat{i}$ with SOC $\mu_2 + \lambda_1 - \mu_1 > 0$. Then we have

$$\beta^* = \frac{\mu_2}{2(\mu_2 + \lambda_1 - \mu_1)} \frac{\alpha^2 \theta_v}{\alpha^2 \theta_v + \theta_z}.$$

Equilibrium. In summary, we have the following equations in equilibrium.

$$\begin{aligned} \alpha &= \frac{1}{2\lambda_2}, \lambda_2 > 0. \\ \beta &= \frac{\mu_2}{2(\mu_2 + \lambda_1 - \mu_1)} \frac{\alpha^2 \theta_v}{\alpha^2 \theta_v + \theta_z}, \mu_2 + \lambda_1 - \mu_1 > 0. \\ \beta &> 0, \lambda_1 = \frac{\alpha \theta_v}{\beta(\alpha^2 \theta_v + \theta_z)}, \lambda_2 = \frac{\alpha \theta_v \theta_z}{\alpha^2 \theta_v (\theta_z + 1) + \theta_z}, \\ \mu_1 &= \frac{\alpha \theta_v (\theta_z + 1/\beta)}{\alpha^2 \theta_v (\theta_z + 1) + \theta_z}, \mu_2 = \frac{\alpha \theta_v \theta_z}{\alpha^2 \theta_v (\theta_z + 1) + \theta_z}, \\ \beta &= 0, \lambda_1 = \mu_1 = \infty, \mu_2 = \lambda_2 = \frac{\alpha \theta_v}{\alpha^2 \theta_v + 1}. \end{aligned}$$

Unfortunately, the above equations have no solution. So there is no linear equilibrium when $\theta_1 = 0$. However, we can investigate the partial equilibrium between IT and dealers, which is a collection of

strategies $\{p_1, p_2, i^*\}$, satisfying the market-efficiency and IT's optimization conditions, given HFT's strategy $x = \hat{\beta}i$.

When $\beta \in (0, 1)$, i.e., HFT does front-run, $i^* = \alpha^*v$ and

$$\alpha^* = \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\theta_z}{\theta_z + 1}} < \frac{\sigma_2}{\sigma_v}, \quad (16)$$

IT is hurt by HFT's front-running behavior. Substituting (16) into (15), we find that HFT's expected profit is zero. Due to the absence of time-1 noise trading, HFT's front-running causes so large a price impact that harms IT without benefiting herself. When $\beta = 0$, undoubtedly, $\alpha^* = \frac{\sigma_2}{\sigma_v}$, which is in line with the classic result.

5 Numerical Results

In this section, we focus on the case $\theta_1 > 0$, since the case $\theta_1 = 0$ has been fully discussed in Section 4.2. The sixth-order polynomial equation (9) hardly has analytical solutions, so we study the equilibrium through numerical analysis. In order to illustrate front-running HFT's impacts on IT and market quality, we use orange lines to represent the corresponding results in Kyle's one-period model, i.e., the situation without HFT.

From results of Theorem 1 and Corollary 1, given σ_2 and σ_v , we investigate:

$$\frac{\alpha^*}{\sigma_2/\sigma_v}, \beta^*, \frac{\mathbb{E}(\pi^{\text{IT}})}{\sigma_v\sigma_2/2}, \frac{\mathbb{E}(\pi^{\text{HFT}})}{\sigma_v\sigma_2/2}, \frac{\lambda_1^*}{\sigma_v/2\sigma_2}, \frac{\mu_1^*}{\sigma_v/2\sigma_2}, \frac{\mu_2^*}{\sigma_v/2\sigma_2}, \frac{\mathbb{E}(v - p_1)^2}{\sigma_v^2}, \quad (17)$$

which are values only decided by θ_1, θ_z , to see how these two parameters affect investors' actions, profits and market quality. For readability, the x-axis in the figures shows $\sqrt{\theta_1}$ and $\sqrt{\theta_z}$, rather than θ_1 and θ_z .

Note that the ex-post pricing error $\mathbb{E}(v - p_2)^2$ is not included in (17), since it is always $\frac{\sigma_v^2}{2}$, which is the same as that in the situation without HFT. In fact, front-running HFT makes the price discovered earlier through p_1 , but the ex-post pricing error is not reduced. HFT's time-1 trading x injects information into the market before the IT arrives, thus the price discovery is advanced. However, HFT does not produce any extra information: she transfers information into the market through x and reduces its exposure through $-x$. Whatever the IT's intensity α^* is, these two effects offset each other and the ex-ante price discovery is identical.

5.1 Comparative statics with respect to θ_1

Now we investigate how investors and market react when θ_1 changes. Let $\theta_z = 0.04$, i.e., $\sigma_z = 0.2\sigma_2$;

$$\begin{aligned} \theta_1 &\in [10^{-6}, 10^{-4}], \text{ i.e., } \frac{\sigma_1}{\sigma_2} \in [10^{-3}, 10^{-2}]; \\ \theta_1 &\in [10^{-4}, 10^{-2}], \text{ i.e., } \frac{\sigma_1}{\sigma_2} \in [10^{-2}, 10^{-1}]; \\ \theta_1 &\in [10^{-2}, 25], \text{ i.e., } \frac{\sigma_1}{\sigma_2} \in [10^{-1}, 5]. \end{aligned}$$

Since the general shapes of equilibrium are similar for different θ_z , we only display results for $\theta_z = 0.04$.

For HFT, β^* and her profit increase with θ_1 , as shown in Figure 1. HFT is favored by more noise traders since u_1 plays the role of a shield, enabling HFT to employ the signal \hat{i} more intensely and obtain more profits.

For IT, α^* and her profit also increase with θ_1 , as shown in Figure 2. On the one hand, although a larger θ_1 brings a larger β^* , which increases the impact of HFT's time-1 trading, this impact is also decreased by more noise tradings. On the other hand, a larger β^* means that HFT provides more liquidity back at time 2 and shares more of IT's impact. On average, IT gets more profits from the greater size of time-1 noise trading. When θ_1 is large enough ($\theta_1 \geq 0.15$), the advantages of a large θ_1 outweigh the disadvantages, thus IT is benefited by HFT's front-running.

For other market variables, results are similar when θ_1 changes in different regions. So we only display the results when $\theta_1 \in [10^{-2}, 25]$ in Figure 3.

In equilibrium, the intensities of order flow y_1 's impact, λ_1^* and μ_1^* decrease with θ_1 , both because when y_1 contains more noise, its informativeness becomes lower. Correspondingly, the intensities of order flow y_2 's impact, μ_2^* increases with θ_1 : when y_1 is noisier, the news component in y_2 , $y_2 - \mathbb{E}(y_2|y_1)$,

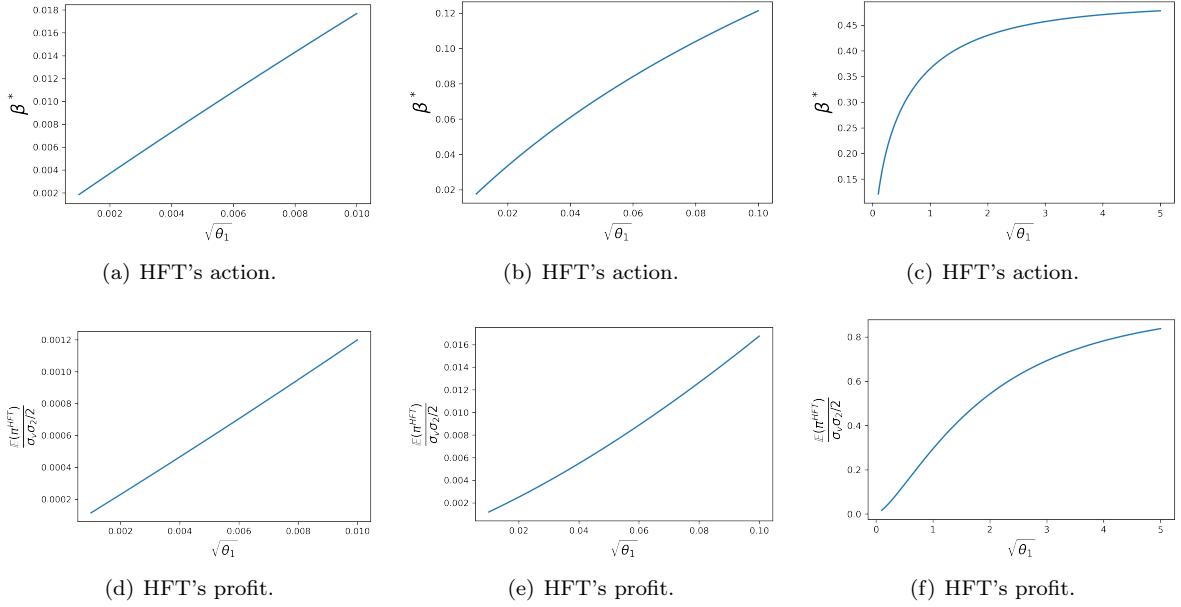


Figure 1: How HFTs' action and profits change with θ_1 .

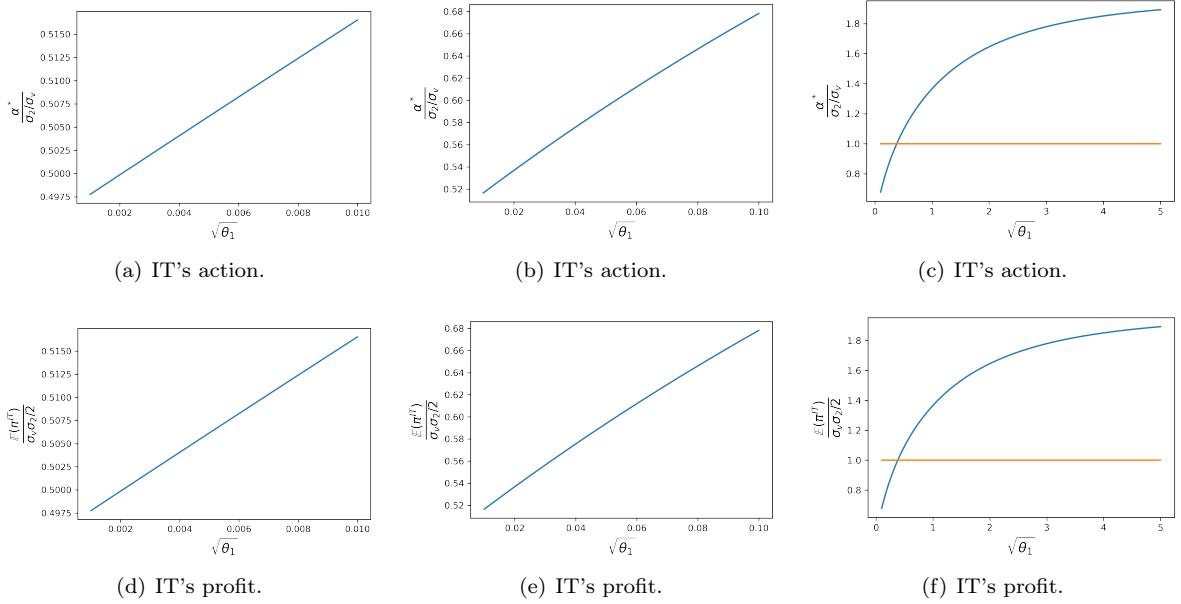


Figure 2: How IT's actions and profits change with θ_1 .

appears more reliable for the dealers. What's more, we find that $\lambda_1^* > \mu_1^*$, as shown in (d), which can be seen as the decay of y_1 's impact.

For the ex-ante price discovery, although IT and HFT employ their private information more intensely as θ_1 increases, a larger size of noise trading prevents dealers from distinguishing the true value. The latter effect goes beyond the former one, thus the time-1 pricing error increases with θ_1 .

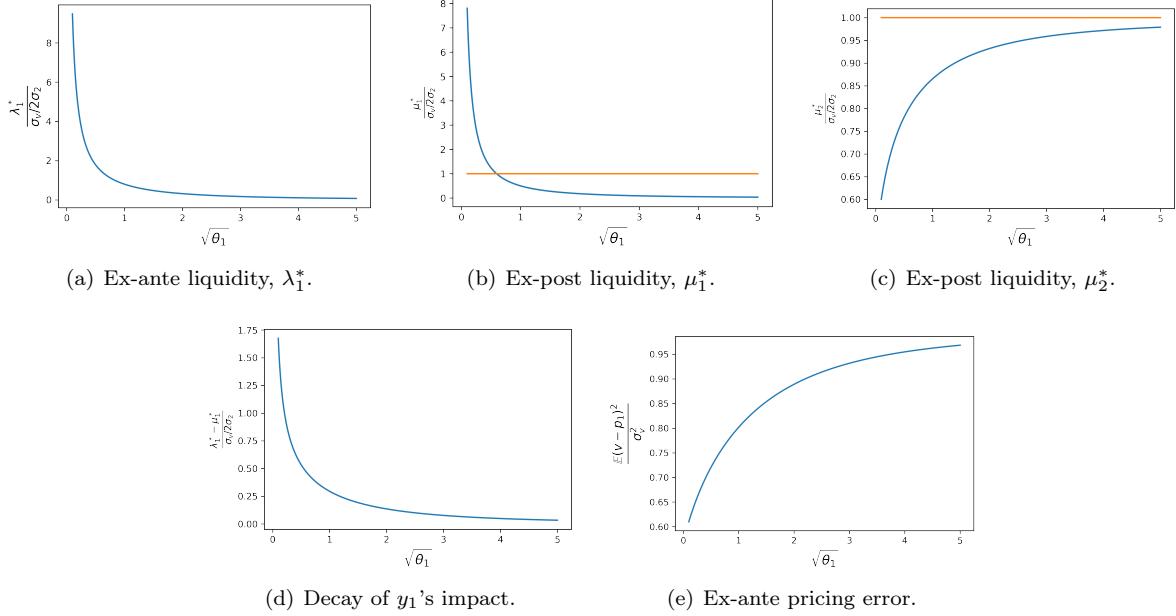


Figure 3: How θ_1 affects market quality.

5.2 Comparative statics with respect to θ_z

Now we investigate how investors and market react when θ_z changes. We first give how the actions and profits of investors change with $\theta_z \in [0, 25]$, when $\theta_1 = 0.12, 0.2$ and 1 , i.e.,

$$\frac{\sigma_z}{\sigma_2} \in [0, 5], \frac{\sigma_1}{\sigma_2} = 0.34, 0.45 \text{ and } 1.$$

For HFT, the trading intensity β^* decreases with θ_z , as shown in (a)-(c) of Figure 4, since a noisier signal makes her less sure about the future informed trading. Surprising results appear for HFT's expected profit. It is usually considered that HFT's profit should decrease when she receives a less accurate signal. However, as shown in (d) of Figure 4, it increases with θ_z when both θ_1 and θ_z are relatively small. It is because IT trades more as θ_z gets larger ((g) in Figure 5), HFT's absolute front-running volume x^* also increases, which brings the growth of profit.

For IT, the results are more complicated. The following Figure 5 shows the trend of θ_z 's impact on IT when θ_1 is in different value ranges. For $\theta_1 \in (0, \frac{2\sqrt{3}-3}{3})$, we find that IT's action as well as her profit, first increases then decreases with θ_z , and IT is favored by the front-running HFT when θ_z exceeds the critical value $\bar{\theta}_z$ calculated in Proposition 2. For $\theta_1 \in [\frac{2\sqrt{3}-3}{3}, \frac{1}{2})$, IT's action and profit change as the former case. However, she is always benefited by HFT. For $\theta_1 \geq \frac{1}{2}$, IT is still always benefited by HFT but her profit keeps decreasing with θ_z .

As the signal becomes noisier, HFT's action β^* decreases. On the one hand, HFT's time-1 trading causes less impact. On the other hand, her time-2 trading shares fewer transaction costs for IT. When the first effect exceeds the second one, IT's action and profit increase with θ_z , otherwise, they decrease with it.

To complement the monotonicity of α^* on θ_z in Proposition 4, we have done a lot of numerical experiments for $0 < \theta_1 < 0.0089$ and find that the trend of IT's action and profit is the same as $\theta_1 \in [0.0089, \frac{1}{2})$: first increase then decrease with θ_z , as shown by the example in Figure 6.

For other market variables, results are similar for different θ_1 . Thus we only display numerical results for $\theta_1 = 0.12$ in Figure 7. As shown in (a) and (b), λ_1^* and μ_1^* decrease with θ_z , the informational

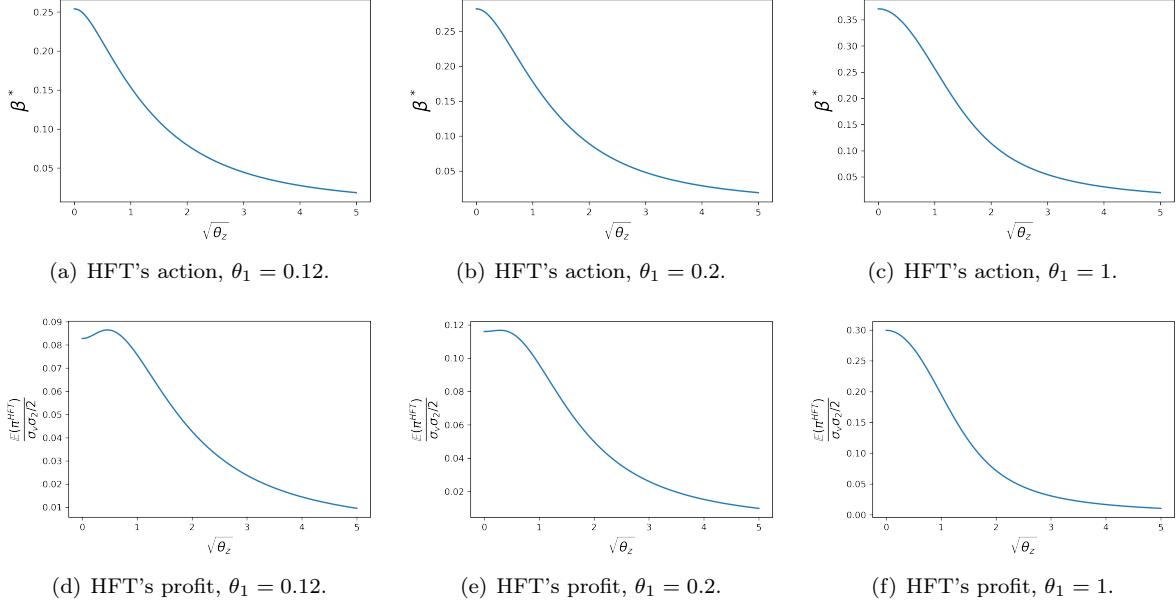


Figure 4: How HFT's actions and profits change with θ_z .

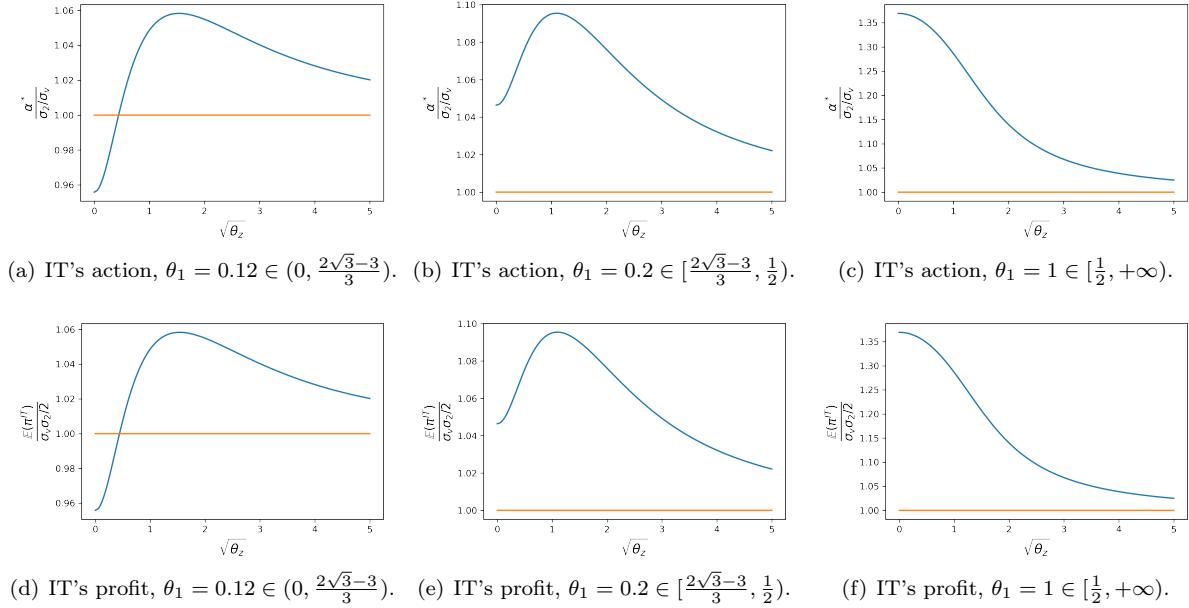


Figure 5: How IT's actions and profits change with θ_z .

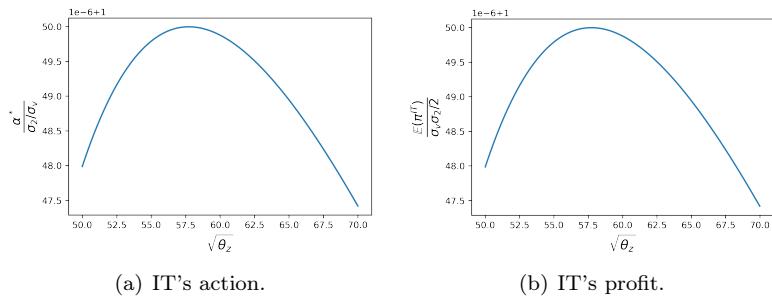


Figure 6: How IT's action and profit change with θ_z when $\theta_1 = 10^{-4}$.

proportion of trading x^* is reduced by the noise in HFT's signal, and so is y_1 . In contrast, time-2 trading enables dealers to learn more about v and brings larger μ_2^* . y_1 's impact also decays, as shown in (d).

As for the ex-ante price discovery, the time-1 pricing error increases with θ_z . Although larger noise might enable the IT to employ her information advantage more fiercely, HFT decreases her intensity β^* . What's more, HFT's trading x^* contains too much noise, thus dealers learn less from the order flow y_1 .

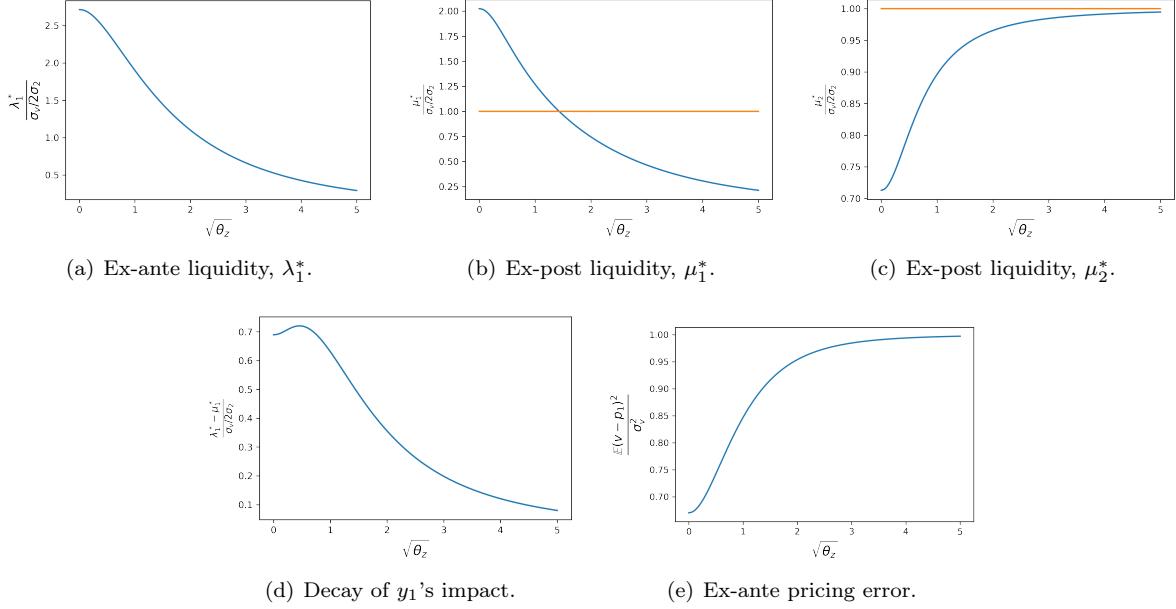


Figure 7: How θ_z affects market quality, when $\theta_1 = 0.12$.

6 When HFT predicts $i + u_2$

In this section, we discuss an extension of our model that HFT can predict $i + u_2$, i.e., the signal received by HFT is

$$\hat{y} = i + u_2 + z,$$

where $z \sim N(0, \sigma_z^2)$ is the noise independent of other random variables. The existence and uniqueness of the linear equilibrium are given in the following theorem.

Theorem 4. Given $\sigma_1, \sigma_2 > 0, \sigma_z \geq 0, (\theta_1 = \frac{\sigma_1^2}{\sigma_2^2} > 0, \theta_z = \frac{\sigma_z^2}{\sigma_2^2} \geq 0,)$ there exists a unique equilibrium $\{p_1, p_2, i^*, x^*\}$, where HFT follows the strategy $x^* = \beta^* \hat{i}$ and $\beta^* \in (0, 1)$ solves the equation

$$\begin{aligned} 0 = & \beta^6(4\theta_1^2 + 6\theta_1^2\theta_z + 8\theta_1\theta_z + 8\theta_1\theta_z^2 + \theta_1\theta_z^3 + 2\theta_1^2\theta_z^2 + 4\theta_z^2 + 2\theta_z^3) \\ & + \beta^5(4\theta_1^3 - 16\theta_1^2 + 4\theta_1^3\theta_z - 8\theta_1^2\theta_z - 16\theta_1\theta_z + 4\theta_1^2\theta_z^2 - 4\theta_1\theta_z^2 + 2\theta_z\theta_z^3) \\ & + \beta^4(-18\theta_1^3 + 24\theta_1^2 - 13\theta_1^3\theta_z - 4\theta_1^2\theta_z + 8\theta_1\theta_z - 8\theta_1^2\theta_z^2 + 2\theta_1\theta_z^2) \\ & + \beta^3(32\theta_1^3 - 16\theta_1^2 + 16\theta_1^3\theta_z + 8\theta_1^2\theta_z + 4\theta_1^2\theta_z^2) \\ & + \beta^2(-28\theta_1^3 + 4\theta_1^2 - 9\theta_1^3\theta_z - 2\theta_1^2\theta_z) + \beta(12\theta_1^3 + 2\theta_1^3\theta_z) - 2\theta_1^3; \end{aligned}$$

IT follows the strategy $i^* = \alpha^* v$ and

$$\alpha^* = \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\theta_1\theta_z\beta^{*2} + \theta_1(1 - \beta^*)^2 + \theta_z\beta^{*2}}{\theta_1(1 - \beta^*)^2 + \theta_z\beta^{*2}}} \geq \frac{\sigma_2}{\sigma_v},$$

that is, IT is benefited by HFT; the liquidation price at time 1 is

$$p_1 = \lambda_1^*(x^* + u_1),$$

where

$$\lambda_1^* = \frac{\sigma_v}{2\sigma_2} \frac{2\beta^* \sqrt{(\theta_1\theta_z\beta^{*2} + \theta_1(1 - \beta^*)^2 + \theta_z\beta^{*2})(\theta_1(1 - \beta^*)^2 + \theta_z\beta^{*2})}}{\beta^{*2}(\theta_1\theta_z\beta^{*2} + \theta_1(1 - \beta^*)^2 + \theta_z\beta^{*2}) + (\beta^{*2}(\theta_z + 1) + \theta_1)(\theta_1\theta_z\beta^{*2} + \theta_1(1 - \beta^*)^2 + \theta_z\beta^{*2})};$$

the liquidation price at time 2 is

$$p_2 = \mu_1^*(x^* + u_1) + \mu_2^*(i^* + u_2 - x^*),$$

where

$$\begin{aligned}\mu_1^* &= \frac{\sigma_v}{2\sigma_2} \frac{\beta^{*2}\theta_z}{\sqrt{(\theta_1(1-\beta^*)^2 + \theta_z\beta^{*2})(\theta_1\theta_z\beta^{*2} + \theta_1(1-\beta^*)^2 + \theta_z\beta^{*2})}}, \\ \mu_2^* &= \frac{\sigma_v}{2\sigma_2} \frac{\beta^{*2}\theta_z + (1-\beta^*)\theta_1}{\sqrt{(\theta_1(1-\beta^*)^2 + \theta_z\beta^{*2})(\theta_1\theta_z\beta^{*2} + \theta_1(1-\beta^*)^2 + \theta_z\beta^{*2})}};\end{aligned}$$

the impact coefficient of the informed order i^* is

$$\lambda_2^* = \mu_1^*\beta^* + \mu_2^*(1-\beta^*) = \frac{\sigma_v}{2\sigma_2} \sqrt{\frac{\theta_1(1-\beta^*)^2 + \theta_z\beta^{*2}}{\theta_1\theta_z\beta^{*2} + \theta_1(1-\beta^*)^2 + \theta_z\beta^{*2}}} \leq \frac{\sigma_v}{2\sigma_2}.$$

When $\theta_1 = 0$, the linear equilibrium does not exist.

For $\theta_1 = 0$, similar to the analysis in Section 4.2, given $x = \beta\hat{y}, \beta \in (0, 1)$, in the partial-equilibrium, $\alpha^* = \frac{\sigma_2}{\sigma_v}$, which means that whatever β is, IT is not affected by HFT.

We can also see the two-sided effects of HFT:

Proposition 8. Given $\theta_1 > 0, \theta_z \geq 0$ and IT's activity $\alpha > 0$, for any HFT's activity $\beta \in (0, 1)$,

$$\lambda_2(\alpha; \beta) = \mu_1(\alpha; \beta)\beta + \mu_2(\alpha; \beta)(1-\beta),$$

where the first part increases with β and the second part decreases with β . What's more, $\mu_1^*, \mu_2^* \leq \frac{\sigma_v}{2\sigma_2}$, which makes $\lambda_2^* \leq \frac{\sigma_v}{2\sigma_2}$.

Recalling that in our original model, HFT predicts i , she might harm IT under certain conditions, since μ_1^* can be greater than $\frac{\sigma_v}{2\sigma_2}$. However, when HFT's signal comes from $i + u_2$, x contains less information but more noise, making the time-1 trading y_1 less informative, bringing smaller μ_1^* . Thus μ_1^* and μ_2^* are both smaller than $\frac{\sigma_v}{2\sigma_2}$, HFT is always in favor of IT.

Li (2018) [11] models a market with normal-speed noise traders, i.e., $\theta_1 = 0$, where front-running HFTs predict the aggregate order $i + u_2$. The author concludes that IT is always harmed by HFT. The key differences between this model and ours are: (1) [11] models stale dealers who quote linearly with unchanged impact coefficients at times 1 and 2, while we assume the dealers quote according to the market-efficiency condition; (2) [11] assumes $\theta_1 = 0$. It is precisely because of these two points that our conclusions are different. Thus our results and [11]'s results can be seen as mutually complementary, explaining how front-running HFT affects IT when confronted with different liquidity providers and different sizes of market noise.

7 Conclusion

We study the influences of a front-running HFT on a large informed trader and market quality in various situations, where the size of market noise and signal accuracy differ. Since HFT takes liquidity away as well as provides liquidity back, she has two-sided effects on the large trader. When the market noise is sufficient, even if HFT's signal is perfectly accurate, she could be in favor of the large trader. Without enough noise shelter from the market, the large trader is benefited only when HFT's signal is relatively vague. The price discovery is advanced through HFT's time-1 trading. However, the ultimate pricing error is not reduced, compared to that without HFT.

8 Appendix

Proof of Theorem 1. When $\beta \leq 0$, we have $\mu_2 > 0$. If the SOC $\mu_2 + \lambda_1 - \mu_1$ hold, β must be positive, which contradicts with each other. Thus we have $\beta^* > 0$.

Combining (1),(2),(3),(5),(6), we get that β^* satisfies Equation (9), and the optimal intensity for IT is (10). Next we need only to prove that the root of Equation (9) must be in $(0, 1)$ and unique, satisfies the SOC. If we see the right side of Equation (9) the function of β : $f(\beta; \theta_1, \theta_z)$, we have

$$f(0; \theta_1, \theta_z) = -\theta_1^3 < 0,$$

$$f(1; \theta_1, \theta_z) = 2\theta_1^2 + 6\theta_1\theta_z + 4\theta_1^2\theta_z + 2\theta_z^2 + 9\theta_1\theta_z^2 + 2\theta_1^2\theta_z^2 + \theta_z^3 + 3\theta_1\theta_z^3 > 0,$$

then there must exist $\beta^* \in (0, 1)$, satisfying Equation (9). For $\beta \geq 1$, it is easy to verify $f'(\beta; \theta_1, \theta_z)$ is also positive. Thus there is no root of Equation (9) on $[1, +\infty)$.

Now we denote β^* the smallest root of f on $(0, 1)$. The SOC is equivalent to

$$\alpha\sigma_v^2(\sigma_1^2 + \alpha^2\beta\sigma_v^2)((1 - \beta)\sigma_1^2 + \beta^2\sigma_z^2) > 0,$$

which holds for $\beta \in (0, 1)$.

$$\begin{aligned} f'(\beta; \theta_1, \theta_z) = & \theta_1^2\theta_z(40\beta^4 - 44\beta^3 + 24\beta^2 - 2\beta) + \theta_1^2\theta_z^2(12\beta^5 + 20\beta^4 - 32\beta^3 + 12\beta^2) \\ & + \theta_1^3\theta_z(20\beta^4 - 42\beta^3 + 48\beta^2 - 18\beta + 2) + \theta_1^3(6\beta^2 - 10\beta + 4) \\ & + \theta_1\theta_z(20\beta^4 + 8\beta^3) + \theta_1\theta_z^2(24\beta^5 + 20\beta^4 + 4\beta^3) \\ & + \theta_1\theta_z^3(6\beta^5 + 10\beta^4) + 6\beta^5(2\theta_z^2 + \theta_z^3) + 6\beta^2\theta_1^2. \end{aligned}$$

For $\beta \in (0, 0.099)$, $40\beta^4 - 44\beta^3 + 24\beta^2 - 2\beta < 0$. If this leads to $f'(\beta) < 0$ for $\beta \in (0, 0.099)$, we must have $\beta^* > 0.099$, since $f(0) < 0$. So the negativity would not cause $f'(\beta) < 0$ for $\beta \in [\beta^*, 1)$.

For $\beta \in (0.205, 0.325)$, $20\beta^4 - 42\beta^3 + 48\beta^2 - 18\beta + 2 < 0$. If $\theta_1 \leq \theta_z$,

$$\begin{aligned} & \theta_1^3\theta_z(20\beta^4 - 42\beta^3 + 48\beta^2 - 18\beta + 2) + \theta_1^2\theta_z^2(12\beta^5 + 20\beta^4 - 32\beta^3 + 12\beta^2) \\ & \geq \theta_1^3\theta_z(12\beta^5 + 40\beta^4 - 74\beta^3 + 60\beta^2 - 18\beta + 2) \geq 0. \end{aligned}$$

If $\theta_1 > \theta_z$ and we want to have

$$\theta_1^3\theta_z(20\beta^4 - 42\beta^3 + 48\beta^2 - 18\beta + 2) + \theta_1^3(6\beta^2 - 10\beta + 1) \geq 0,$$

$\theta_z \leq 21$.

For $\beta \in (\frac{2}{3}, 1)$, $6\beta^2 - 10\beta + 4 < 0$. If $\theta_1 \leq \theta_z$,

$$\begin{aligned} & \theta_1^3(6\beta^2 - 10\beta + 4) + \theta_1^2\theta_z(40\beta^4 - 44\beta^3 + 24\beta^2 - 2\beta) \\ & \geq \theta_1^3(40\beta^4 - 44\beta^3 + 30\beta^2 - 12\beta + 4) \geq 0. \end{aligned}$$

If $\theta_1 > \theta_z$ and we want to have

$$\theta_1^3(6\beta^2 - 10\beta + 4) + 6\beta^2\theta_1^2 \geq 0,$$

$\theta_1 \leq 24$.

Now we have proved that if $\theta_1 \leq \theta_z$ or $\theta_z < \theta_1 \leq 21$, $f'(\beta) \geq 0$ for $\beta \in [\beta^*, 1]$. So $f(\beta) > 0$ for $\beta \in (\beta^*, 1)$, β^* is the unique root of f on $(0, 1)$. We only need to prove the uniqueness for $\theta_1 > 21 > \theta_z$ and $\theta_1 > \theta_z > 21$, which is equivalent to prove $f(\beta) > 0$ for $\beta \in (\beta^*, 1]$.

$$\begin{aligned} f(\beta; \theta_1, \theta_z) = & \theta_1^2\theta_z(8\beta^5 - 11\beta^4 + 8\beta^3 - \beta^2) + \theta_1^2\theta_z^2(2\beta^6 + 4\beta^5 - 8\beta^4 + 4\beta^3) \\ & + \theta_1^3\theta_z(4\beta^5 - 13\beta^4 + 16\beta^3 - 9\beta^2 + 2\beta) + \theta_1^3(2\beta^3 - 5\beta^5 + 4\beta - 1) \\ & + \theta_1\theta_z(4\beta^5 + 2\beta^4) + \theta_1\theta_z^2(4\beta^6 + 4\beta^5 + \beta^4) \\ & + \theta_1\theta_z^3(\beta^6 + 2\beta^5) + \beta^6(2\theta_z^2 + \theta_z^3) + 2\beta^3\theta_1^2. \end{aligned}$$

When $\theta_1 > 21$, and for $\beta \in (0, 0.154)$, $8\beta^5 - 11\beta^4 + 8\beta^3 - \beta^2 < 0$, but

$$\begin{aligned} & \theta_1^2\theta_z(8\beta^5 - 11\beta^4 + 8\beta^3 - \beta^2) + \theta_1^3\theta_z(4\beta^5 - 13\beta^4 + 16\beta^3 - 9\beta^2 + 2\beta) \\ & \geq \theta_1^2\theta_z(8\beta^5 - 11\beta^4 + 8\beta^3 - \beta^2 + 21(4\beta^5 - 13\beta^4 + 16\beta^3 - 9\beta^2 + 2\beta)) \geq 0. \end{aligned}$$

Then for $\beta \in (0, 1)$,

$$f(\beta; \theta_1, \theta_z) \geq \theta_1^3(2\beta^3 - 5\beta^5 + 4\beta - 1) + 2\beta^3\theta_1^2 = f(\beta; \theta_1, \theta_z = 0),$$

we would analyze the special case $\theta_z = 0$ in subsection 4.1.1, and we employ some conclusions in advance.

We now prove $f(\beta; \theta_1, \theta_z = 0) > 0$ for $\beta \in (\beta^*(\theta_z = 0), 1)$. Since

$$\begin{aligned} f(\beta; \theta_1, \theta_z = 0) &= \theta_1^2g(\beta), \\ g(\beta) &= 2\beta^3(\theta_1 + 1) - 5\theta_1\beta^2 + 4\theta_1\beta - \theta_1, \end{aligned}$$

it is equivalent to investigate $g(\beta)$.

$g(\beta)$ is (1) increasing when $21 < \theta_1 < 24$ or (2) has a unique minimum point $\tilde{\beta} = \frac{5\theta_1 + \sqrt{\theta_1^2 - 24\theta_1}}{6(\theta_1 + 1)}$ on $(\beta^*(\theta_z = 0), 1]$, when $\theta_1 \geq 24$. Correspondingly, for $\beta \in (\beta^*(\theta_z = 0), 1]$, (1) $g(\beta) > g(\beta^*(\theta_z = 0)) = 0$, (2) $g(\beta) \geq g(\tilde{\beta}) > 0$. Since $\beta^*(\theta_z = 0) \leq \frac{1}{2}$, we have $f(\beta; \theta_1, \theta_z) > 0$ for $\beta \in (\frac{1}{2}, 1)$ when $\theta_1 > 21$.

When $\theta_z < 21$, $f(\beta; \theta_1, \theta_z)$ is increasing for $\beta \in [\beta^*, \frac{2}{3}]$, then it is also increasing for $\beta \in [\beta^*, \frac{1}{2}]$, which implies $f(\beta; \theta_1, \theta_z) > 0$ for $\beta \in (\beta^*, 1)$.

When $\theta_z > 21$, $f(\beta; \theta_1, \theta_z)$ may decrease for $\beta \in (0.206, 0.324)$. The only possible negative part in f is $\theta_1^3(2\beta^3 - 5\beta^5 + 4\beta - 1)$. However,

$$\begin{aligned} & \theta_1^3 \theta_z (4\beta^5 - 13\beta^4 + 16\beta^3 - 9\beta^2 + 2\beta) + \theta_1^3 (2\beta^3 - 5\beta^5 + 4\beta - 1) \\ & \geq \theta_1^3 (21(4\beta^5 - 13\beta^4 + 16\beta^3 - 9\beta^2 + 2\beta) + 2\beta^3 - 5\beta^5 + 4\beta - 1) > 0, \end{aligned}$$

so we must have $f(\beta; \theta_1, \theta_z) > 0$ for $\beta \in (\beta^*, 1]$ no matter $\beta^* < 0.206$, $\beta^* > 0.324$ or $\beta^* \in (0.206, 0.324)$.

Proof of Proposition 1. Denote $\theta_v = \frac{\sigma_v^2}{\sigma_2^2}$,

$$\begin{aligned} \frac{\partial(\mu_1(\alpha; \beta)\beta)}{\partial \beta} &= \frac{\alpha \theta_v}{(\alpha^2 \theta_v (\theta_1(1-\beta)^2 + \beta^2 + \theta_z \beta^2) + \beta^2 \theta_z (\theta_1 + 1) + \theta_1)^2} (\alpha^2 \theta_v (\theta_1 \theta_z \beta^2 (1-\beta)(3-\beta) \\ &+ \beta^4 (\theta_z + \theta_z^2) + 2\theta_1 \beta (1-\beta)) + \beta^4 \theta_z^2 (\theta_1 + 1) + \theta_1 (3\beta^2 \theta_z + 2\beta)) > 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial(\mu_2(\alpha; \beta)(1-\beta))}{\partial \beta} &= \frac{\alpha \theta_v}{(\alpha^2 \theta_v (\theta_1(1-\beta)^2 + \beta^2 + \theta_z \beta^2) + \beta^2 \theta_z (\theta_1 + 1) + \theta_1)^2} (-\alpha^2 \theta_v (\beta^4 (\theta_z + \theta_z^2)) \\ &+ \theta_1 \theta_z \beta^2 (1-\beta)(3-\beta) + 2\theta_1 \beta (1-\beta)) - \beta^4 \theta_z^2 (\theta_1 + 1) - \beta^2 \theta_1 \theta_z \\ &- 2\theta_1^2 \theta_z \beta (1-\beta) - 2\theta_1^2 (1-\beta) < 0. \end{aligned}$$

Proof of Proposition 2.

$$\left(\frac{\mu_2^*}{\sigma_v/2\sigma_2}\right)^2 - 1 = \frac{\beta^{*2}(\theta_1 + \beta^{*2}\theta_z + (1-\beta^*)^2\theta_1^2\theta_z + \beta^{*2}\theta_1\theta_z(2+\theta_z))}{(\theta_1(1-\beta^*)^2 + \beta^{*2}(\theta_z+1))(\beta^{*2}\theta_z(\theta_1+1) + \theta_1)} > 0.$$

Proof of Theorem 2. $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$ is equivalent to $\beta^*(1 + \theta_1 - \theta_1\theta_z) - 2\theta_1 < 0$, which holds for $\theta_1\theta_z \geq 1$. If $\theta_1\theta_z < 1$, we need $\beta^* < \frac{2\theta_1}{1+\theta_1-\theta_z}$, i.e.

$$f\left(\frac{2\theta_1}{1+\theta_1-\theta_z}\right) > 0 \iff -5\theta_1^2\theta_z^2 + 2\theta_1^2\theta_z + 3\theta_1^2 + 10\theta_1\theta_z + 6\theta_1 - 1 > 0.$$

We could rearrange it as

$$h(\theta_z) = -5\theta_1^2\theta_z^2 + \theta_z(2\theta_1^2 + 10\theta_1) + 3\theta_1^2 + 6\theta_1 - 1 > 0.$$

If $3\theta_1^2 + 6\theta_1 - 1 \geq 0$, $\theta_1 \geq \frac{2\sqrt{3}-3}{3}$, $h(\theta_z) > 0$ when $0 \leq \theta_z < x_2$, where

$$x_2 = \frac{(\theta_1 + 5) + 2\sqrt{4\theta_1^2 + 10\theta_1 + 5}}{5\theta_1}.$$

Since $x_2 > \frac{1}{\theta_1}$, $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$.

If $3\theta_1^2 + 6\theta_1 - 1 < 0$, $0 < \theta_1 < \frac{2\sqrt{3}-3}{3}$, $h(\theta_z) > 0$ when $x_1 < \theta_z < x_2$, where $x_1 = \bar{\theta}_z$. Since $x_1 < \frac{1}{\theta_1} < x_2$, we have $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$ when $\theta_z > \bar{\theta}_z$.

As for the last conclusion, we investigate

$$h(\theta_1) = \theta_1(-5\theta_z^2 + 2\theta_z + 3) + \theta_1(10\theta_z + 6) - 1 > 0.$$

If $-5\theta_z^2 + 2\theta_z + 3 > 0$, $0 \leq \theta_z < 1$, $h(\theta_1) > 0$ when $\theta_1 > x_2 = \bar{\theta}_1$.

If $-5\theta_z^2 + 2\theta_z + 3 \leq 0$, $\theta_z \geq 1$, $h(\theta_1) > 0$ when $x_1 < \theta_1 < x_2$, where

$$\begin{aligned} x_1 &= \bar{\theta}_1, \\ x_2 &= \frac{-(5\theta_z + 3) - 2\sqrt{5\theta_z^2 + 8\theta_z + 3}}{-5\theta_z^2 + 2\theta_z + 3}. \end{aligned}$$

Since $x_1 < \frac{1}{\theta_z} < x_2$, we have $\lambda_2^* < \frac{\sigma_v}{2\sigma_2}$ when $\theta_1 > \bar{\theta}_1$.

Proof of Proposition 3. Denote the right side of Equation (9) f , to investigate θ_1 ,

$$\begin{aligned} f'(\theta_1; \beta) &= \theta_1^2(\theta_1(12\beta^5 - 39\beta^4 + 48\beta^3 - 27\beta^2 + 6\beta) + 6\beta^3 - 15\beta^2 + 12\beta - 3) \\ &\quad + \theta_1(\theta_z^2(4\beta^6 + 8\beta^5 - 16\beta^4 + 8\beta^3) + \theta_z(16\beta^5 - 22\beta^4 + 16\beta^3 - 4\beta^2) + 4\beta^3) \\ &\quad + \theta_z^3(\beta^6 + 2\beta^5) + \theta_z^2(4\beta^6 + 4\beta^5 + \beta^4) + \theta_z(4\beta^5 + 2\beta^4). \end{aligned}$$

It is first positive then negative or first positive then negative finally positive on $\theta_1 \in (0, \infty]$.

If $f(\beta; \theta_1) = 0$, $f(\theta_1 = 0; \beta) = \beta^6(2\theta_z^2 + \theta_z^3) > 0$, then for $\theta'_1 \in [0, \theta_1]$, we have $f(\beta; \theta'_1) = f(\theta'_1; \beta) > 0$.

If $f(\beta'; \theta'_1) = 0$, we have $\beta > \beta'$, i.e. β^* increases with θ_1 .

To investigate θ_z ,

$$\begin{aligned} f'(\theta_z; \beta) &= \theta_z^2(3\beta^6 + 3\beta^6\theta_1 + 6\beta^5\theta_1) \\ &\quad + \theta_z(\theta_1^2(4\beta^6 + 8\beta^5 - 16\beta^4 + 8\beta^3) + \theta_1(8\beta^6 + 8\beta^5 + 2\beta^4) + 4\beta^6) \\ &\quad + \theta_1^3(4\beta^5 - 13\beta^4 + 16\beta^3 - 9\beta^3 + 2\beta) \\ &\quad + \theta_1^2(8\beta^5 - 11\beta^4 + 8\beta^3 - \beta^2) + \theta_1(4\beta^5 + 2\beta^4). \end{aligned}$$

It is first negative then positive or positive on $\theta_z \in [0, +\infty)$, which implies $f(\theta_z; \beta)$ first decreases than increases or increases on $\theta_z \in [0, +\infty)$,

If $f(\beta; \theta_z) = 0$, then

$$\begin{aligned} f(\beta; \theta_z = 0) &= \beta^3(2\theta_1^2 + \theta_1^3) - 5\beta^2\theta_1^3 + 4\beta\theta_1^3 - \theta_1^3 \\ &= -\beta^6(4\theta_1\theta_z^2 + \theta_1\theta_z^3 + 2\theta_z^2 + \theta_z^3) - \beta^5(4\theta_1\theta_z + 4\theta_1\theta_z^2 + 2\theta_1\theta_z^3) \\ &\quad - \beta^4(2\theta_1\theta_z + \theta_1\theta_z^2) + \theta_1\theta_z^2(-8\beta^5 + 8\beta^4 - 8\beta^3 + \beta^2) \\ &\quad + \theta_1^2\theta_z^2(-4\beta^5 + 8\beta^4 - 4\beta^3) + \theta_1^3\theta_z(-4\beta^5 + 13\beta^4 - 16\beta^3 + 9\beta^2 - 2\beta) \\ &< 0. \end{aligned}$$

So for $\theta'_z \in [0, \theta_z]$, $f(\beta; \theta'_z) = f(\theta'_z; \beta) < 0$. If $f(\beta'; \theta'_z) = 0$, we have $\beta' > \beta$. Thus β^* decreases with θ_z .

Proof of Proposition 4. $\alpha^*(\theta_1, \theta_z) = \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\theta_1 + \beta^{*2}\theta_z(\theta_z + 1)}{\theta_1(1 - \beta^*)^2 + (1 + \theta_z)\beta^{*2}}}$.

$$\frac{\partial \alpha^*(\theta_1, \theta_z)}{\partial \theta_1} = \frac{\partial \alpha^*(\beta^*, \theta_1, \theta_z)}{\partial \beta^*} \frac{\partial \beta^*}{\partial \theta_1} + \frac{\partial \alpha^*(\beta^*, \theta_1, \theta_z)}{\partial \theta_1}.$$

Then $\frac{\partial \alpha^*(\theta_1, \theta_z)}{\partial \theta_1} \geq 0$ if and only if

$$2\theta_1(\theta_1 - (1 + \theta_1)\beta^*) \frac{\partial \beta^*}{\partial \theta_1} + \beta^{*2}(1 + \beta^*\theta_z) \geq 0.$$

Since $\frac{\partial \beta^*}{\partial \theta_1} > 0$, the above inequality holds if $\theta_1 - (1 + \theta_1)\beta^* \geq 0$, which is equivalent to

$$f\left(\frac{\theta_1}{1 + \theta_1}\right) = \frac{\theta_1^3(1 + \theta_1 + \theta_1\theta_z)^2(2\theta_1^2 + \theta_1(3\theta_z + 1) - 1)}{(1 + \theta_1^5)} \geq 0,$$

in other words,

$$\begin{aligned} 2\theta_1^2 + \theta_1(3\theta_z + 1) - 1 &\geq 0, \\ \theta_1 &\geq \frac{\sqrt{(3\theta_z + 1)^2 + 8} - (3\theta_z + 1)}{4} = \frac{2}{\sqrt{(3\theta_z + 1)^2 + 8 + (3\theta_z + 1)}}|_{max} = \frac{1}{2}. \end{aligned}$$

Actually we specify two sufficient conditions: (1) $\theta_1 \geq \frac{1}{2}$, (2) $\theta_z \geq 0$, $\frac{\sqrt{(3\theta_z + 1)^2 + 8} - (3\theta_z + 1)}{4} \leq \theta_1 < \frac{1}{2}$.
 $\frac{\partial \alpha^*(\theta_1, \theta_z)}{\partial \theta_z}$ has the same sign as

$$2\theta_1(\theta_1 - (1 + \theta_1)\beta^*)(1 + \beta^*\theta_z) \frac{\partial \beta^*}{\partial \theta_z} + \beta^{*2}(\theta_1 - (1 + \theta_1)\beta^*)^2.$$

If $\theta_1 - (1 + \theta_1)\beta^* > 0$, $\frac{\partial \alpha^*(\theta_1, \theta_z)}{\partial \theta_z} \leq 0$ is equivalent to

$$g(\beta^*; \theta_1, \theta_z) = \beta^{*2}(\theta_1 - (1 + \theta_1)\beta^*) \frac{\partial f}{\partial \beta^*} - 2\theta_1(1 + \beta^*\theta_z) \frac{\partial f}{\partial \theta_z} \leq 0.$$

For simplicity, we still use β to represent β^* .

$$\begin{aligned}
g &= \theta_1^2(-4\beta^4 - 14\beta^5) + \theta_1^3y_1 + \theta_1^4(-4\beta + 22\beta^2 - 46\beta^3 + 42\beta^4 - 14\beta^5) \\
&\quad + \theta_1\theta_z(-16\beta^6 - 20\beta^7) + \theta_1^2\theta_z(-2\beta^4 - 36\beta^5 + 32\beta^6 - 60\beta^3) \\
&\quad + \theta_1^3\theta_z(-18\beta^3 + 60\beta^4 - 110\beta^5 + 112\beta^6 - 60\beta^7) + \theta_1^4\theta_z(-2\beta^2 - 2\beta^3 + 34\beta^4 - 74\beta^5 + 64\beta^6 - 20\beta^7) \\
&\quad - 12\beta^8\theta_z^2 + \theta_1\theta_z^2(-10\beta^6 - 16\beta^7 - 36\beta^8) + \theta_1^2\theta_z^2(-24\beta^5 + 26\beta^6 - 32\beta^7 - 36\beta^8) \\
&\quad + \theta_1^3\theta_z^2(-4\beta^4 - 12\beta^5 + 36\beta^6 - 16\beta^7 - 12\beta^8) - 6\beta^8\theta_z^3 \\
&\quad + \theta_1\theta_z^3(-10\beta^7 - 12\beta^8) + \theta_1^2\theta_z^3(-2\beta^6 - 10\beta^7 - 6\beta^8), \\
y_1 &= 2\beta^2 - 20\beta^3 + 38\beta^4 - 28\beta^5.
\end{aligned}$$

Except y_1 , other coefficients are both non-positive for $\beta \in [0, \frac{1}{2}]$. (We would see in following paragraphs that $\beta^* \leq \frac{1}{2}$.) $y_1 \geq 0$ when $\beta \in [0, 0.1286]$. Let

$$\begin{aligned}
p &= -4\beta^4 - 14\beta^5, \\
q &= -4\beta + 22\beta^2 - 46\beta^3 + 42\beta^4 - 14\beta^5.
\end{aligned}$$

$\theta_1^2p + \theta_1^3y_1 + \theta_1^4q \leq 0$ is equivalent to $p + \theta_1y_1 + \theta_1^2q \leq 0$, $\Delta = y_1^2 - 4pq \geq 0$ when $\beta \in [0, 0.0328]$. Thus if

$$\theta_1 \geq \frac{-y_1 - \sqrt{\Delta}}{2q} \Big|_{max} = 0.0089,$$

and $\theta_z > \frac{1-\theta_1-2\theta_1^2}{3\theta_1}$, we have $\frac{\partial\alpha^*(\theta_1, \theta_z)}{\partial\theta_z} \leq 0$. In a word, we specify two conditions, (1) $\theta_1 \geq \frac{1}{2}$, (2) $0.0089 \leq \theta_1 < \frac{1}{2}$, $\theta_z > \frac{1-\theta_1-2\theta_1^2}{3\theta_1}$.

When $\theta_1 - (1 + \theta_1)\beta^* \leq 0$, $\frac{\partial\alpha^*(\theta_1, \theta_z)}{\partial\theta_z} \geq 0$, which implies $\theta_z < \frac{1-\theta_1-2\theta_1^2}{3\theta_1}$. Thus when $0 < \theta_1 < \frac{1}{2}$ and $\theta_z < \frac{1-\theta_1-2\theta_1^2}{3\theta_1}$, $\frac{\partial\alpha^*(\theta_1, \theta_z)}{\partial\theta_z} \geq 0$.

Proof of Proposition 5. Divide both sides of Equation (9) θ_1^3 and calculate the limit when $\theta_1 \rightarrow \infty$, we have the equation for the limit of β^* . Then by L's Hospital rule,

$$\alpha^* = \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\theta_1(\beta^{*2}\theta_z + 1) + \beta^{*2}\theta_z}{\theta_1(1 - \beta^*)^2 + (1 + \theta_z)\beta^{*2}}} \rightarrow \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\beta^{*2}\theta_z + 1}{(1 - \beta^*)^2}}.$$

Proof of Proposition 6. Calculate the limit of both sides of Equation (9) when $\theta_1 \rightarrow 0$, we have $\beta^* \rightarrow 0$. However, in order to calculate the limit of α^* , we need a more precise estimate of β^* . We assume $\beta^* = O(\theta_1^a)$. If $a > \frac{1}{2}$, $\lim_{\theta_1 \rightarrow 0} \alpha^* = \frac{\sigma_2}{\sigma_v}$, which contradicts that $\lim_{\theta_1 \rightarrow 0} \alpha^* = 0$ when $\theta_z = 0$. If $a < \frac{1}{2}$, $\lim_{\theta_1 \rightarrow 0} \alpha^* = \frac{\sigma_2}{\sigma_v} \sqrt{\frac{\theta_z}{\theta_z + 1}}$, which is also incorrect (let $\theta_1 = 10^{-20}$, compare α^* and $\frac{\sigma_2}{\sigma_v} \sqrt{\frac{\theta_z}{\theta_z + 1}}$ under different θ_z). So $a = \frac{1}{2}$ and we further assume $\beta^* \sim \frac{\sqrt{\theta_1}}{y\theta_z^x}$ for $\theta_z > 0$ and $\theta_1 \rightarrow 0$, then

$$\lim_{\theta_1 \rightarrow 0} \alpha^* = \frac{\sigma_2}{\sigma_v} \sqrt{\frac{y^2\theta_z^{2x} + \theta_z}{y^2\theta_z^{2x} + \theta_z + 1}}.$$

In order to get the value of $x = y$, we calculate α^* when $\theta_1 = 10^{-20}$ for different θ_z , get 10^4 pairs of (θ_z, α^*) for $\theta_z \in (0, 25]$. Then we do regression for the sample (θ_z, α^*) and the result is shown in Figure 8.

Other limits could be calculated by substitute $\beta^* \sim \frac{\sqrt{\theta_1}}{y\theta_z^x}$ to Corollary 1. The case that $\theta_z = 0$ could also be included.

Proof of Proposition 7. Recall the equation of β^* (9), we have

$$\begin{aligned}
&\beta^6(2\theta_z^2 + 4\theta_1\theta_z^2 + 2\theta_1^2\theta_z^2 + \theta_z^3 + \theta_1\theta_z^3) + \beta^5(4\theta_1\theta_z + 8\theta_1^2\theta_z + 4\theta_1^3\theta_z + 4\theta_1\theta_z^2 + 4\theta_1^2\theta_z^2 + 2\theta_1\theta_z^3) \\
&+ \beta^4(2\theta_1\theta_z + \theta_1\theta_z^2) + \beta^3(2\theta_1^2 + 2\theta_1^3 + 8\theta_1^2\theta_z + 16\theta_1^3\theta_z + 4\theta_1^2\theta_z^2) + \beta(4\theta_1^3 + 2\theta_1^3\theta_z) \\
&= \beta^4(11\theta_1^2\theta_z + 13\theta_1^3\theta_z + 8\theta_1^2\theta_z^2) + \beta^2(5\theta_1^3 + \theta_1^2\theta_z + 9\theta_1^3\theta_z) + \theta_1^3.
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta^6\theta_z^3 &\leq \beta^4(11\theta_1^2\theta_z + 13\theta_1^3\theta_z + 8\theta_1^2\theta_z^2) + \beta^2(5\theta_1^3 + \theta_1^2\theta_z + 9\theta_1^3\theta_z) + \theta_1^3 \\
\beta^6 &\leq \beta^4\left(\frac{11\theta_1^2 + 13\theta_1^3}{\theta_z^2} + \frac{8\theta_1^2}{\theta_z}\right) + \beta^2\left(\frac{5\theta_1^3}{\theta_z^3} + \frac{\theta_1^2 + 9\theta_1^3}{\theta_z^2}\right) + \frac{\theta_1^3}{\theta_z^3} \rightarrow 0.
\end{aligned}$$

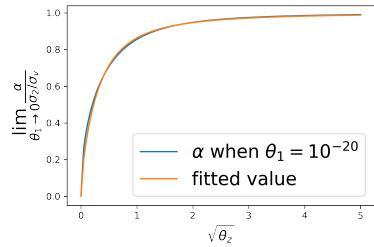


Figure 8: The goodness of fit.

In order to calculate $\lim_{\theta_z \rightarrow \infty} \alpha^*$, we need know the convergence speed of β^* . We prove $\beta^* < \frac{1}{\theta_z}$, equivalently, we only need to prove $f(\frac{1}{\theta_z}; \theta_z) > 0$. Denote $x = \frac{1}{\theta_z}$,

$$\begin{aligned} f(x; \theta_z) &\geq \theta_1^2 \theta_z (8x^5 - 11x^4 + 8x^3 - x^2) + \theta_1^2 \theta_z^2 (4x^5 - 8x^4 + 4x^3) \\ &\quad + \theta_1^3 \theta_z (4x^5 - 13x^4 + 16x^3 - 9x^2 + 2x) + \theta_1^3 (2x^3 - 5x^2 + 4x - 1) \\ &= \theta_1^2 (8x^4 - 11x^3 + 8x^2 - x) + \theta_1^3 (4x^5 - 13x^4 + 18x^3 - 14x^2 + 6x - 1) \\ &\geq 0. \end{aligned}$$

Then $\beta^{*2} \theta_z \rightarrow 0$, $\alpha^* \rightarrow \frac{\sigma_2}{\sigma_v}$. Other limits are direct results of Theorem 1 and Corollary 1.

Proof of Proposition 8.

$$\begin{aligned} \mu_1(\alpha; \beta) &= \frac{\alpha \theta_v (\beta^2 \theta_z)}{(\alpha^2 \theta_v + 1)(\theta_1(1 - \beta)^2 + \theta_z \beta^2) + \beta^2 \theta_1 \theta_z}, \\ \frac{\partial(\mu_1(\alpha; \beta))}{\partial \beta} &= \frac{\alpha \theta_v (2\alpha^2 \theta_v + 1)\theta_1 \theta_z \beta (1 - \beta)}{((\alpha^2 \theta_v + 1)(\theta_1(1 - \beta)^2 + \theta_z \beta^2) + \beta^2 \theta_1 \theta_z)^2} \geq 0. \\ \mu_2(\alpha; \beta) &= \frac{\alpha \theta_v (\beta^2 \theta_z + (1 - \beta)\theta_1)}{(\alpha^2 \theta_v + 1)(\theta_1(1 - \beta)^2 + \theta_z \beta^2) + \beta^2 \theta_1 \theta_z}, \\ \frac{\partial(\mu_2(\alpha; \beta)(1 - \beta))}{\partial \beta} &= \frac{-\alpha \theta_v ((\alpha^2 \theta_v + 1)\theta_z \beta^2 (2\theta_1(1 - \beta) + 3\theta_z \beta^2) + \beta^2 \theta_1 \theta_z (3\theta_z \beta^3 + 2\theta_1(1 - \beta^2)))}{((\alpha^2 \theta_v + 1)(\theta_1(1 - \beta)^2 + \theta_z \beta^2) + \beta^2 \theta_1 \theta_z)^2} \leq 0. \end{aligned}$$

As for the second conclusion,

$$\mu_1(\alpha; \beta), \mu_2(\alpha; \beta) \leq \frac{\alpha \theta_v}{\alpha^2 \theta_v + 1} \leq \frac{\sigma_v}{2\sigma_2}.$$

Since μ_1^* and μ_2^* are maximum of $\mu_1(\alpha; \beta^*)$ and $\mu_2(\alpha; \beta^*)$ respectively, it holds.

References

- [1] Lawrence Harris. Order exposure and parasitic traders. *University of Southern California working paper*, 23:1–22, 1997.
- [2] Andrei Kirilenko, Albert S. Kyle, Mehrdad Samadi, and Tugkan Tuzun. The flash crash: High frequency trading in an electronic market. *The Journal of Finance*, 72(3), 2017.
- [3] Mozaffar Khan and Hai Lu. Do short sellers front-run insider sales? *The Accounting Review*, 88(5):1743–1768, 2013.
- [4] Viktor Manahov*. Front-running scalping strategies and market manipulation: Why does high-frequency trading need stricter regulation? *Financial Review*, 51(3):363–402, 2016.
- [5] Nicholas Hirshey. Do high-frequency traders anticipate buying and selling pressure? *Management Science*, 67(6):3321–3345, 2021.
- [6] Markus K Brunnermeier and Lasse Heje Pedersen. Predatory trading. *The Journal of Finance*, 60(4):1825–1863, 2005.
- [7] Torsten Schöneborn and Alexander Schied. Liquidation in the face of adversity: stealth vs. sunshine trading. In *EFA 2008 Athens Meetings Paper*, 2009.
- [8] RENÉ A Carmona and Joseph Yang. Predatory trading: a game on volatility and liquidity. *Preprint. URL: <http://www.princeton.edu/racarmona/download/fe/PredatoryTradingGameQF.pdf>*, 2011.
- [9] Hendrik Bessembinder, Allen Carrion, Laura Tuttle, and Kumar Venkataraman. Liquidity, resiliency and market quality around predictable trades: Theory and evidence. *Journal of Financial Economics*, 121(1):142–166, 2016.
- [10] Dan Bernhardt and Bart Taub. Front-running dynamics. *Journal of Economic Theory*, 138(1):288–296, 2008.
- [11] Wei Li. High frequency trading with speed hierarchies. Available at SSRN 2365121, 2018.
- [12] Jonathan Brogaard, Terrence Hendershott, Stefan Hunt, and Carla Ysus. High-frequency trading and the execution costs of institutional investors. *Financial Review*, 49(2):345–369, 2014.
- [13] Thorsten Hens, Terje Lensberg, and Klaus Reiner Schenk-Hoppé. Front-running and market quality: An evolutionary perspective on high frequency trading. *International Review of Finance*, 18(4):727–741, 2018.
- [14] Liyan Yang and Haoxiang Zhu. Back-running: Seeking and hiding fundamental information in order flows. *The Review of Financial Studies*, 33(4):1484–1533, 2020.
- [15] Albert J Menkveld. The economics of high-frequency trading: Taking stock. *Annual Review of Financial Economics*, 8:1–24, 2016.
- [16] Markus Baldauf and Joshua Mollner. High-frequency trading and market performance. *The Journal of Finance*, 75(3):1495–1526, 2020.
- [17] Albert S Kyle. Continuous auctions and insider trading. *Econometrica: Journal of the Econometric Society*, pages 1315–1335, 1985.
- [18] Dan Bernhardt and Jianjun Miao. Informed trading when information becomes stale. *The Journal of Finance*, 59(1):339–390, 2004.