

Near-Optimal Non-Parametric Sequential Tests and Confidence Sequences with Possibly Dependent Observations

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Abstract

Sequential testing, always-valid p -values, and confidence sequences promise flexible statistical inference and on-the-fly decision making. However, unlike fixed- n inference based on asymptotic normality, existing sequential tests either make parametric assumptions and end up under-covering/over-rejecting when these fail or use non-parametric but conservative concentration inequalities and end up over-covering/under-rejecting. To circumvent these issues, we sidestep exact at-least- α coverage and focus on asymptotically exact coverage and asymptotic optimality. That is, we seek sequential tests whose probability of *ever* rejecting a true hypothesis asymptotically approaches α and whose expected time to reject a false hypothesis approaches a lower bound on all tests with asymptotic coverage at least α , both under an appropriate asymptotic regime. We permit observations to be both non-parametric and dependent and focus on testing whether the observations form a martingale difference sequence. We propose the universal sequential probability ratio test (uSPRT), a slight modification to the normal-mixture sequential probability ratio test, where we add a burn-in period and adjust thresholds accordingly. We show that even in this very general setting, the uSPRT is asymptotically optimal under mild generic conditions. We apply the results to stabilized estimating equations to test means, treatment effects, *etc.* Our results also provide corresponding guarantees for the implied confidence sequences. Numerical simulations verify our guarantees and the benefits of the uSPRT over alternatives.

1 Introduction

Inference based on randomized experiments forms the basis of important decisions in an incredibly diverse range of domains, from medicine [Schulz et al., 2010] to development economics [Banerjee et al., 2015] to technology business [Tingley et al., 2021]. Toward the aim of making *better* and *faster* decisions, it is particularly helpful for statistical procedures to be *flexible* [Grünwald et al., 2020]. Experimental designs which require a pre-specified sample size can be quite rigid in practice. For example, unless some oracle information is known about the effect size, such designs will ultimately include some over- or under-experimentation; collecting more samples than necessary when the treatment effect is under-estimated, and not collecting enough when over-estimated. Sequential designs, on the other hand, enable *faster* decision-making as they support the ability to analyze data as and when it arrives, enabling us to stop experimenting when the data strongly supports a conclusion. The ability to continuously monitor experiments in turn leads to *better* (more reliable) decisions, as it prevents bad practices that arise from attempting to use more rigid statistical procedures in applications where resources and schedules often change dynamically.

Analyzing data in this fashion requires special inference that explicitly accounts for the sequential nature of the decision-making process. It is well known that repeated application of classical significance tests to accumulating sets of data results in procedures with drastically inflated type-I-error rates [Armitage et al., 1969]. Even under the null hypothesis, the absolute value of the t -statistic applied to an independent and identically distributed (iid) sequence is guaranteed to fall into the rejection region at some point, regardless of the chosen α -level [Strassen, 1964]. An analyst intent on disproving any one hypothesis can keep collecting data until this occurs.

For these reasons, a long line of literature has explored sequential testing and confidence sequences. However, by and large, without making parametric restrictions on the data, these are not calibrated, that is, they have unpredictable type-I-error rates that are unequal to the α -level they are designed for. Tests and

confidence sequences based on wrong parametric assumptions and/or heuristic fixes tend to undercover and over-reject. And, tests and confidence sequences based on concentration inequalities tend to overcover and under-reject. Other tests yet, which combine concentration inequalities with asymptotic approximations, still fail to obtain even approximate calibration of type-I-error rates.

We give a brief overview here and a more thorough literature review in Section 8. The sequential probability ratio test (SPRT) [Wald, 1945, 1947] uses a test statistic equal to the ratio of likelihoods under the alternative and the null hypotheses. In the simple-vs-simple case, the likelihoods are completely specified. Much like the famous Neyman-Pearson lemma for simple-vs-simple likelihood ratio tests in fixed- n designs [Neyman et al., 1933], SPRTs enjoy certain optimality guarantees in sequential testing [Wald, 1945, Wald and Wolfowitz, 1948]. Extensions to composite vs simple hypothesis tests can be obtained by using a *mixture* of simple-vs-simple SPRT test statistics [Johari et al., 2022, Robbins, 1970, Wald, 1945], that is, giving rise to the mixture SPRT (mSPRT). Important examples are conjugate mixtures [Howard et al., 2021], obtained by integrating a parametric likelihood ratio with respect to a conjugate mixture distribution. These yield closed-form mixture test statistics which can be manipulated to obtain closed-form confidence sequences. A mixture SPRT of special interest for this work is the normal-mixture SPRT (nmSPRT), obtained from a normal mixture of variance-1 normal SPRTs. While type-I error guarantees for nmSPRT hold even for non-normal but 1-sub-Gaussian data [Darling and Robbins, 1967], there is no guarantee of type-I-error-rate calibration and the test can be conservative taking too long to reject. Importantly, considering the means that pass the nmSPTR yields a shrinking confidence sequence (unlike the simple-vs-simple SPRT). Howard et al. [2021] show there are in fact more general ways to construct shrinking always-valid confidence sequences, not just based on mSPRTs and even given tail conditions other than sub-Gaussian. Waudby-Smith et al. [2021] apply these confidence sequences to normal observations and leverage approximations of running sums by Brownian motion to show that this yields an approximate confidence sequence even if the data is not normal and only assumed to have some moments and without knowing its variance, with approximation error vanishing faster than the scale of the intervals. This leap enables the construction of approximate confidence sequences with great generality, but not without limitations. On the one hand, these confidence sequences make no guarantees on coverage (type-I-error rate), only that their width is *close* to a confidence sequence with a specified desired coverage. On the other hand, these confidence sequences can be too wide: even if we knew the data were standard normal with unknown mean, the confidence sequences of Howard et al. [2021] can be conservative and overcover. Thus, it can take very long to eliminate wrong values, in direct opposition to the primary motivation for sequential testing for faster decision making. Essentially, Waudby-Smith et al. [2021] consider asymptotics to obtain a normal approximation but not asymptotics for type-I-error-rate calibration and rejection-time optimality. In sum, exactly calibrated coverage requires parametric assumptions, and nonparametric any-time-valid guarantees with at-least- α coverage require significant conservatism.

In this paper, we consider an alternative, approximate objective: we seek sequential tests and confidence sequences for the mean of nonparametric, possibly-dependent observations whose type-I-error rates are *nearly* calibrated to α and their time to reject a false mean are *nearly* optimal. Namely, the probability of *ever* rejecting the true value *approaches* exactly the desired α -significance, for appropriate asymptotic regimes. And, the average time to reject a wrong value *approaches* a lower bound on the average time taken by *any* sequential test whose type-I-error-rate approaches α . Our proposal is a slight modification of the nmSPRT, adding a burn-in period and appropriately adjusting the rejection threshold. We call this a universal SPRT (uSPRT) because of its *universal* guarantees of near-calibration and near-optimality. At a high level, there are two key aspects to our analysis of the uSPRT. One is bounding the difference between using a normal-mixture alternative and using the oracle simple normal alternative, where we leverage Itô's lemma to express the former as a simple normal alternative with a sequentially estimated mean, interpreting testing a composite alternative as testing a *local* simple alternative that we update adaptively. Another is bounding the difference between observing the actual dependent and non-normal data and observing the Brownian motion, where we leverage a strong invariance principle.

Although our results are nominally for the mean, we can apply them to make inferences on a variety of parameters in a variety of settings. Applying our test and results to stabilized estimating equations with sequentially estimated nuisances, we can conduct inferences on average treatment effects, quantiles, regression coefficients, *etc.*, both in Bernoulli trials and in contextual bandits, where adaptive arm allocations render raw observations dependent. Our results also imply guarantees for confidence sequences given by

Algorithm 1 uSPRT (Universal Sequential Probability Ratio Test)

Input: Significance level $\alpha \in (0, 1)$, burn-in end $t_0 \in \mathbb{N}$, prior precision $\lambda > 0$

1: Set $S_0 = 0$ and find $\tilde{\alpha}(\alpha) \in (0, 1)$ solving

$$\tilde{\alpha}(\alpha) \sqrt{-2 \log \tilde{\alpha}(\alpha)/\pi} + 2(1 - \Phi(\sqrt{-\log \tilde{\alpha}(\alpha)})) = \alpha$$

2: **for** $t = 1, 2, \dots$ **do**
3: Observe X_t and set $S_t = S_{t-1} + X_t$, $Y_{t,\lambda} = \frac{1}{2} \left(\frac{S_t^2}{t+\lambda} - \log \frac{t+\lambda}{\lambda} \right)$
4: **if** $t \geq t_0$ and $Y_{t,\lambda} \geq \frac{1}{2} \log \frac{\lambda}{t_0} - \log \tilde{\alpha}(\alpha)$ **then**
5: Reject the hypothesis that X_1, X_2, \dots is a martingale difference sequence
6: **end if**
7: **end for**

inverting the test; we call these universal confidence sequences (uCSs). We compare uCSs numerically to other nonparametric confidence sequences, and we show our method is unique in obtaining approximate calibration of type-I errors and that it quickly rejects false values. We believe this is significant because in practice analysts may often shy away from confidence sequences due to their bad type-I-error-rate calibration and conservatism and either avoid sequential testing or use heuristics based on fixed- n asymptotics, so we hope our method and results offer a practically viable solution to flexible sequential testing without strong assumptions.

Structure of the paper. As alluded to above, there are two key aspects of our analysis. To elucidate each aspect, before analyzing our proposed test in the general setting, we explore each aspect separately, as a deviation from an ideal setting. After setting up the general problem and our proposed test in Section 2 (with some motivating examples), we deviate and consider a far simpler idealized setting in Section 3: continuously observing a Brownian motion and sequentially testing two simple hypotheses about the exact value of its drift. This will serve as our base case and our results will take the form of showing that violations of the ideal (composite alternatives, discrete-time observations, dependent non-normal observations) only amount to small deviations from the calibrated type-I error and optimal time-to-reject of the idealized setting. In Section 4 we consider composite alternatives by adapting the simple alternative to the running shrunken sample mean, which we show is equivalent to a continuous nmSPRT statistic via Itô’s lemma. In Section 5 we separately consider replacing the Brownian motion with partial sums of discrete-time non-normal dependent observations, and we use a strong invariance principle to show the error due to this is small. Finally, in Section 6 we combine the arguments to give guarantees for the original problem and algorithm considered in Section 2. We then consider instantiations of the problem when observations are stabilized estimating equations with sequentially estimated nuisances show how the conditions for our results apply in this setting under simple conditions. With this more complete understanding of the results, we provide an extensive comparison to the literature in Section 8. We end with a numerical experiment in Section 9 and conclusions in Section 10.

2 Setup

We work on a probability space $(\Omega, \mathcal{F}, P_0)$. We use the “0” subscript throughout to refer to objects that arise under the true data-generating distribution P_0 . In particular we denote E_0 the expectation operator under P_0 . We observe a sequence X_1, X_2, \dots adapted to a discrete filtration $\mathfrak{F}, \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$. We assume that $E_0[X_t^2 | \mathcal{F}_{t-1}] < \infty$ and define $\mu_t = E_0[X_t | \mathcal{F}_{t-1}]$, $v_t^2 = E_0[(X_t - \mu_t)^2 | \mathcal{F}_{t-1}]$. Note that we will generally require $v_t \approx 1$ in our results (where we make the condition formal).

We wish to test the hypothesis that $\mu_1 = \mu_2 = \dots = 0$, *i.e.*, X_1, X_2, \dots is a martingale difference sequence (MDS). Namely, given $\alpha \in (0, 1)$, we are concerned with finding a \mathfrak{F} -stopping time τ (*i.e.*, $\{\tau \leq s\} \in \mathcal{F}_s$) such that $P_0[\tau < \infty] \approx \alpha$ under the MDS hypothesis and $E_0[\tau]$ is small otherwise. We will make “ \approx ,” “small,” and “otherwise” specific in our results. Roughly, “ \approx ” will mean asymptotically (for various limits), “small”

will mean not much later than when an ideal simple-vs-simple test would have stopped, and “otherwise” will mean $\mu_t \rightarrow \mu_0 \neq 0$ in some appropriate sense.

The nmSPRT considers the likelihood ratio of observing X_1, \dots, X_t as independent standard normals with mean μ , either after drawing μ as a centered normal with variance λ^{-1} or when $\mu = 0$, and stops when the ratio is or exceeds α^{-1} . That is, it stops when $Y_{t,\lambda} \geq -\log \alpha$ for

$$Y_{\lambda,t} = \log \frac{\int \prod_{s=1}^t \phi(X_s - \mu) \sqrt{\lambda} \phi(\sqrt{\lambda} \mu) d\mu}{\prod_{s=1}^t \phi(X_s)} = \frac{1}{2} \left(\frac{S_t^2}{t + \lambda} - \log \frac{t + \lambda}{\lambda} \right),$$

where $\phi(u) = \exp(-u^2/2)/\sqrt{2\pi}$ is the standard normal density function.

Our proposed modification is specified in Algorithm 1, where where Φ is the standard normal cumulative distribution function. We wait $t_0 - 1$ steps before starting to test the hypothesis and adjust the threshold to account for this. Note that even if $t_0 = 1$ our test differs slightly from the usual nmSPRT, which uses $-\log \alpha$ as the threshold, essentially because we adjust for the fact we start monitoring at time 1 and not time 0. With these modifications, we can guarantee asymptotic optimality and type-I-error-rate calibration with almost any type of sequential observation, hence “universal.”

Example 1 (Sample mean). Suppose we sequentially observe an iid scalar sequence Z_1, Z_2, \dots , where $\theta_0 = E_0[Z_1]$. Set $\hat{\sigma}_t^2 = \frac{1}{t} \sum_{s=1}^t Z_s^2 - \left(\frac{1}{t} \sum_{s=1}^t Z_s \right)^2$. Then, to test $\theta_0 = \theta$, we set $X_t = \omega_t(Z_t - \theta)$, where $\omega_t = \hat{\sigma}_{t-1}^{-1}$ when $\hat{\sigma}_{t-1} > 0$ and $\omega_t = 1$ otherwise. To construct a confidence sequence for θ_0 , we consider all θ 's not rejected at time t . Namely, letting $\Gamma_t = \sum_{s=1}^t \omega_s$, $\hat{\theta}_t = \Gamma_t^{-1} \sum_{s=1}^t \omega_s Z_s$, the uCS in this case is

$$\left[\hat{\theta}_t \pm \Gamma_t^{-1} (t + \lambda)^{1/2} (\log(t + \lambda) - \log t_0 - 2 \log \tilde{\alpha}(\alpha))^{1/2} \right] \quad \text{for } t \geq t_0$$

(and all of \mathbb{R} for $t < t_0$), where $\tilde{\alpha}(\alpha)$ is in Algorithm 1. Our results provide guarantees about the probability that θ_0 is ever excluded from this interval at *any* time and the expected time until any one $\theta \neq \theta_0$ is excluded.

Example 2 (Bernoulli trial). Suppose we sequentially observe an iid sequence $(A_1, U_1), (A_2, U_2), \dots$ with $A_t \in \{0, 1\}$, $U_t \in \mathbb{R}$. Assuming $p = P_0(A_1 = 1) \in (0, 1)$, set $\theta_0 = E_0[U_1 | A_1 = 1] - E_0[U_1 | A_1 = 0]$. For inference on θ_0 , set $Z_t = (p - A_t)(p^{-1}\hat{\eta}_{1,t-1} + (1-p)^{-1}\hat{\eta}_{0,t-1}) + \frac{A_t - p}{p(1-p)}U_t$ where $\hat{\eta}_{a,t} = \sum_{s=1}^t (aA_s + (1-a)(1-A_s))U_s / (1 \wedge \sum_{s=1}^t (aA_s + (1-a)(1-A_s)))$, and apply the approach from Example 1.

Example 3 (Contextual bandit). Suppose we sequentially observe $(L_1, A_1, U_1), (L_2, A_2, U_2), \dots$ with $U_t \in \mathbb{R}$, where L_1, L_2, \dots forms an iid sequence, A_t is independent of the future given the past, and U_t is independent of all else given L_t, A_t . Assume the conditional distribution of A_t given the past has a (random) density $g_t(\cdot | L_1, A_1, U_1, \dots, L_t)$ with respect to some common base measure ν , fix a random density $\pi(\cdot | L_1)$ with respect to ν , and set $\theta_0 = E_0[U_1 \pi(A_1 | L_1) / g_1(A_1 | L_1)] = E_0[\int m_0(L_1, a) \pi(a | L_1) d\nu(a)]$, where $m_0(l, a) = E_0[U_1 | L_1, A_1]$. We are interested in inference on θ_0 . For example, if $\pi(\cdot | L_1)$ is a (possibly degenerate) probability measure then θ_0 is the value of a covariate-dependent policy that draws an action from it. Alternatively, if $A_t \in \{0, 1\}$ and $\pi(\{a\} | L_1) = 2a - 1$ then θ_0 is the average treatment effect as in Example 2. Fix some regression estimate \hat{m}_t of m_0 based on data $L_1, A_1, U_1, \dots, U_t$. Let $\tilde{g}_t(A_t | L_t) = g_t(A_t | L_1, A_1, U_1, \dots, L_t)$ (here, \sim emphasizes the randomness of the function) and $D(l, a, y; g, m) = \frac{\pi(a|l)}{g(a|l)}(y - m(l, a)) + \int m(l, a') \pi(a' | l) d\nu(a')$. Then, set $Z_t = D(L_t, A_t, U_t; \tilde{g}_t, \hat{m}_{t-1})$, $\hat{\sigma}_t^2 = \frac{1}{t} \sum_{s=1}^t \frac{\tilde{g}_t(A_s | L_s)}{\tilde{g}_s(A_s | L_s)} D(L_s, A_s, U_s; \tilde{g}_t, \hat{m}_{s-1})^2 - \left(\frac{1}{t} \sum_{s=1}^t \frac{\tilde{g}_t(A_s | L_s)}{\tilde{g}_s(A_s | L_s)} D(L_s, A_s, U_s; \tilde{g}_t, \hat{m}_{s-1}) \right)^2$, and $\omega_t = \hat{\sigma}_t^{-1}$ when $\hat{\sigma}_t > 0$ and $\omega_t = 1$ otherwise. To test $\theta_0 = \theta$, set $X_t = \omega_t(Z_t - \theta)$. Confidence sequences are as in Example 1. This essentially yields a sequential version of Bibaut et al. [2021a].

All of these are examples of the generic case of setting X_t to a stabilized estimating equations with sequentially estimated nuisances. We flesh this out in Section 7, where we revisit these examples and consider the necessary assumptions needed for our theory to apply to them (*e.g.*, moments in Examples 1 and 2 and conditions on exploration and regression-estimation in Example 3).

3 The Ideal: Parametric Simple-vs-Simple SPRT

We now depart from the setup in the previous section to consider an idealized problem where the data is normal, independent, and continuously observed and we only need to test two simple hypotheses. This will represent the best-case scenario that we would like to mimic.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_0)$ and $\tilde{\mathfrak{F}} = (\tilde{\mathcal{F}}(t))_{t \in \mathbb{R}_+}$ be a probability space and continuous-time filtration on which a standard Wiener process $W = (W(t))_{t \in \mathbb{R}_+}$ is defined. We observe this process with a drift $\tilde{S}(t) = \mu_0 t + W(t)$ and would like to test the hypothesis that $\mu_0 = 0$. That is, we want a $\tilde{\mathfrak{F}}$ -stopping time that stops when we have enough evidence that μ_0 is not zero.

Note our convention of using \sim to denote objects associated with *fictional* observations and of indexing as $W(t)$ compared to as S_t to emphasize continuous- compared to discrete-time processes. Note also that, had our observations X_t in the original problem set up been iid normal, then necessarily $S_t \sim \tilde{S}(t)$ for $t \in \mathbb{N}$.

We here consider an SPRT testing the simple hypothesis $\mu_0 = 0$ vs the simple hypothesis $\mu_0 = \mu'$. That is, the SPRT using the likelihood ratio of $\tilde{S}(0:t) = (\tilde{S}(s))_{s \in [0,t]}$ as a standard Wiener process with drift μ' or 0. This simplifies to the following $\tilde{\mathfrak{F}}$ -stopping time

$$\tilde{\tau}_{\alpha, \mu'}^{\text{svs}} = \inf \left\{ t \in \mathbb{R}_+ : \tilde{Y}_{\mu'}^{\text{svs}}(t) \geq -\log \alpha \right\}, \quad \text{where} \quad \tilde{Y}_{\mu'}^{\text{svs}}(t) = \mu' \tilde{S}(t) - \frac{1}{2} \mu'^2 t.$$

For simplicity we focus in this section on $\mu_0, \mu' > 0$ (as will be formally stated in results). Note we do not make this restriction in general when studying the uSPRT.

As a consequence of theorem 1 of Robbins and Siegmund [1970], we first can see that this procedure has an exactly calibrated type-I-error rate.

Proposition 1 (Type-I error). *For any $\mu' > 0$, $\alpha \in (0, 1)$, if $\mu_0 = 0$ then*

$$P_0 [\tilde{\tau}_{\alpha, \mu'}^{\text{svs}} < \infty] = \alpha.$$

And, a direct calculation reveals the expected stopping time of this procedure.

Proposition 2. *For any $\alpha \in (0, 1)$, $\mu_0 > \mu'/2 > 0$, we have*

$$E_0 [\tilde{\tau}_{\alpha, \mu'}^{\text{svs}}] = \frac{-\log \alpha}{\mu'(\mu_0 - \mu'/2)} \geq \frac{-2 \log \alpha}{\mu_0^2},$$

with equality when $\mu' = \mu_0$.

The above result directly follows from application of optional stopping theorem. Note that the requirement $\mu_0 > \mu'/2 > 0$ essentially means μ_0 is closer to μ' than to 0. Otherwise, the test may never reject.

3.1 Defining Asymptotic Optimality

Proposition 2 shows that the expected stopping time is minimized when $\mu' = \mu_0$. We can therefore imagine an idealized setting where an oracle told us that $\mu_0 \in \{0, \mu'\}$, that is, if μ_0 is not zero then it must be μ' . In this idealized case, this continuous-time, perfectly specified, parametric simple-vs-simple SPRT cannot be improved upon, and it will therefore serve as our benchmark.

In particular, the theorem in section 4.7 of Wald [1945] shows that $-2 \log \alpha / \mu_0^2$ is in fact a lower bound on the expected time to reject μ_0 by *any* sequential test whose type-I-error rate is at least α . This of course also means that any sequence of tests whose type-I-error rates have a limit infimum of at least α must also have that their expected times to reject $\mu_0 \neq 0$ have a limit infimum of at least $-2 \log \alpha / \mu_0^2$.

We therefore say a sequence of tests is *asymptotically optimal* if both its type-I-error rate approaches (or simply is) at least α and its time to reject any one $\mu_0 \neq 0$ approaches $-2 \log \alpha / \mu_0^2$, both under some asymptotic regime. Our primary goal in this paper is to show that the uSPRT is asymptotically optimal even with non-parametric, dependent observations.

4 Going Composite: Normal Mixture as Adaptive Alternative

We now consider deviating from the simple-vs-simple ideal presented in the previous section by testing a composite alternative $\mu_0 \neq 0$. That is, we no longer have an oracle that tells us that μ_0 is μ' were it not zero. We consider a continuous version of the nmSPRT using the likelihood ratio of $\tilde{S}(0:t) = (\tilde{S}(s))_{s \in [0,t]}$ as a standard Wiener process with drift μ drawn from a centered normal distribution with variance λ^{-1} or drift 0. This simplifies to the following \mathfrak{F} -stopping time

$$\tilde{\tau}_{\alpha,\lambda} = \inf \left\{ t \in \mathbb{R}_+ : \tilde{Y}_\lambda(t) \geq -\log \alpha \right\}, \quad \text{where} \quad \tilde{Y}_\lambda(t) = \frac{1}{2} \left(\frac{\tilde{S}(t)^2}{t + \lambda} - \log \frac{t + \lambda}{\lambda} \right).$$

To control the deviation from the idealized setting, we next represent $\tilde{\tau}_{\alpha,\lambda}$ in a form similar to $\tilde{\tau}_{\alpha,\mu'}^{\text{svs}}$, except that μ' is replaced with a running estimate of μ_0 . Intuitively, this can be explained as a Bayesian update. The λ^{-1} -scale normal mixture over drifts serves as a prior distribution, so that the conditional distribution of the increment $\tilde{S}(s:t) - \tilde{S}(s)$ given the past $\tilde{S}(0:s)$ is equal to the distribution of Wiener process on $[0, t-s]$ with drift drawn from the posterior distribution of drifts, which has mean

$$\tilde{\mu}_\lambda(s) = \tilde{S}(s)/(s + \lambda).$$

Itô's lemma applied to a specific function (Lemma 10) allows us to do this posterior updating in continuous time and write $\tilde{Y}_\lambda(t)$ as a stochastic integral.

Proposition 3. *For any $\lambda > 0$, $t \geq 0$, it holds that*

$$\begin{aligned} \tilde{Y}_\lambda(t) &= \int_0^t \tilde{\mu}_\lambda(s) d\tilde{S}(s) - \frac{1}{2} (\tilde{\mu}_\lambda(s))^2 ds \\ &= \tilde{Y}_{\mu_0}^{\text{svs}}(t) + \tilde{R}_\lambda^{\text{adapt}}(t) + \tilde{R}_{\lambda,\mu_0}^{\text{skg}}(t), \end{aligned}$$

where

$$\begin{aligned} \tilde{R}_\lambda^{\text{adapt}}(W(0:t)) &= \int_0^t \frac{W(s)}{s + \lambda} dW(s) - \frac{1}{2} \left(\frac{W(s)}{s + \lambda} \right)^2 ds = \frac{1}{2} \left(\frac{W(t)^2}{t + \lambda} - \log \frac{t + \lambda}{\lambda} \right), \\ \tilde{R}_{\lambda,\mu_0}^{\text{skg}}(t) &= \tilde{R}_{\lambda,\mu_0}^{\text{skg},1}(t) + \tilde{R}_{\lambda,\mu_0}^{\text{skg},2}(t) + \tilde{R}_{\lambda,\mu_0}^{\text{skg},3}(t), \\ \tilde{R}_{\lambda,\mu_0}^{\text{skg},1}(t) &= -\frac{1}{2} \mu_0^2 \lambda \frac{t}{t + \lambda}, \\ \tilde{R}_{\lambda,\mu_0}^{\text{skg},2}(t) &= \mu_0 \lambda \int_0^t \frac{dW(s)}{s + \lambda}, \\ \tilde{R}_{\lambda,\mu_0}^{\text{skg},3}(t) &= \mu_0 \lambda \int_0^t \frac{W(s)}{(s + \lambda)^2} ds. \end{aligned}$$

The decomposition above makes appears the “oracle” simple-vs-simple test statistic $\tilde{Y}_{\mu_0}^{\text{svs}}(t) = 0$. We say it is an *oracle* object as it involves the true value μ_0 of the drift, and is the the test statistic of the optimal (for the notion of optimality presented above) sequential test for testing the drift is equal to zero against the alternative that the drift is $\mu_0 \neq 0$.

Note that $\tilde{R}_{\lambda,\mu_0}^{\text{skg}}(t) \rightarrow 0$ as $\lambda \rightarrow 0$. We interpret it as a therefore as a “shrinkage” term, reflecting the impact of the shrinkage parameter λ on the test statistic. Note also that the $\tilde{R}_{\lambda,\mu_0}^{\text{skg}}(t) = 0$ if $\mu_0 = 0$, which implies that $\tilde{R}_{\lambda,\mu_0}^{\text{skg}}(t)$ won’t impact the type-I error analysis.

We interpret the remaining term $\tilde{R}_\lambda^{\text{adapt}}$ as an adaptivity excess term, reflecting the price to pay to use an estimate of μ_0 in the definition of the alternative. The fact that this term only makes appear the “noise” process W and not the drift μ_0 reinforces its interpretation as a pure statistical error term.

When $\mu_0 = 0$ we also have $\tilde{Y}_{\mu_0}^{\text{svs}}(t) = 0$, so that $\tilde{Y}_\lambda(t)$ reduces to just the adaptivity excess term $\tilde{R}_\lambda^{\text{adapt}}(t)$. As a consequence of theorem 1 of Robbins and Siegmund [1970], $\tilde{\tau}_{\alpha,\lambda}$ also has an exactly calibrated type-I-error rate.

Proposition 4. For any $\lambda > 0$, $\alpha \in (0, 1)$, if $\mu_0 = 0$ then

$$P_0[\tilde{\tau}_{\alpha,\lambda} < \infty] = \alpha.$$

To control the excess stopping time, we control the shrinkage and adaptivity excess terms.

Lemma 1. For any $\lambda > 0$, it holds that

$$\begin{aligned} -\frac{1}{2}\mu_0^2\lambda &\leq E_0\left[\tilde{R}_{\lambda,\mu_0}^{\text{skg},1}(\tau_{\alpha,\lambda})\right] \leq 0, & E_0\left[\tilde{R}_{\lambda,\mu_0}^{\text{skg},2}(\tau_{\alpha,\lambda})\right] &= 0, \\ \left|E_0\left[\tilde{R}_{\lambda,\mu_0}^{\text{skg},3}(\tau_{\alpha,\lambda})\right]\right| &\leq \mu_0\sqrt{\lambda}, & -\frac{1}{2}\log\frac{E_0[\tilde{\tau}_{\alpha,\lambda}] + \lambda}{\lambda} &\leq E[\tilde{R}_{\lambda}^{\text{adpt}}(\tilde{\tau}_{\alpha,\lambda})] \leq 0. \end{aligned}$$

Theorem 1. Suppose that $\mu_0 \neq 0$. Let $\alpha \in (0, 1)$ and $\lambda > 0$. It holds that

$$E_0[\tilde{\tau}_{\alpha,\lambda}] - \frac{1}{\mu_0^2}\log\frac{E_0[\tilde{\tau}_{\alpha,\lambda}] + \lambda}{\lambda} - \lambda - \frac{2\sqrt{\lambda}}{|\mu_0|} \leq \frac{-2\log\alpha}{\mu_0^2} \leq E_0[\tilde{\tau}_{\alpha,\lambda}] + 2\frac{\sqrt{\lambda}}{|\mu_0|}.$$

5 Going to Discrete-Time, Non-Normal, Dependent Observations

We now depart from the fictitious continuous-time normal observation space and go back to the original problem set up with discrete-time, non-normal, dependent data.

We first consider a setting close to the idealized setting, where we assume we that there exists a parameter $\mu_0 \in \mathbb{R}$ of the true data-generating distribution such that

- if $\mu_0 = 0$, then $E_0[X_t | \mathcal{F}_{t-1}] = 0$ for every $t \in \{1, 2, \dots\}$, that is, X_1, X_2, \dots is an MDS,
- or, if $\mu_0 \neq 0$, then $E_0[X_t | \mathcal{F}_{t-1}] \rightarrow \mu_0$ (in an appropriate mode that we will specify).

As in Section 3, for simplicity we will focus in this section on $\mu_0, \mu' > 0$ (as will be formally stated in results). Note we do not make this restriction in general when studying the uSPRT.

We then consider a simple-vs-simple SPRT using the likelihood ratio of $X_{1:t}$ as independent standard normals with mean μ' or 0, but where we only monitor starting at time t_0 . This simplifies to the following \mathfrak{F} -stopping time:

$$\tau_{\alpha,\mu',t_0}^{\text{svs}} = \inf\left\{t \in \{t_0, t_0 + 1, \dots\} : Y_{\mu',t}^{\text{svs}} \geq -\log\alpha\right\}, \quad \text{where } Y_{\mu',t}^{\text{svs}} = \mu'S_t - \frac{1}{2}\mu'^2t.$$

We emphasize that $X_{1:t}$ are *not* independent normals (with any mean and variance), so both likelihoods are *misspecified*. In the following, we define $S_t^0 = \sum_{s=1}^t (X_s - \mu_s)$ as the centered running sum.

Consider the setting under the hypothesis that X_1, X_2, \dots is an MDS. Ignoring for now the fact that they are defined on different probability spaces, we can express the difference between the statistic we observe, $Y_{\mu',t}^{\text{svs}}$, and the idealized statistic, $\tilde{Y}_{\mu'}^{\text{svs}}(t)$ (monitoring a time-continuous Brownian process with drift), as the scaled difference between running sums of an MDS and a Brownian motion, plus a bias term:

$$Y_{\mu',t}^{\text{svs}} - \tilde{Y}_{\mu'}^{\text{svs}}(t) = R_{\mu',t}^{\text{svs,SIP}} = \mu'(S_t^0 - W(t)).$$

The approach we now take to showing that the procedure still has calibrated type-I-error rates even with discrete-time, non-normal, dependent observations is to construct a joint probability space for both these observations and the continuous-time Brownian motion, in which $R_t^{\text{svs,SIP}}$ is small. Almost-sure approximations of partial sums by Brownian motions under lax conditions are known as *strong invariance principles* (SIPs). We will specifically leverage the SIP result in theorem 1.3 of Strassen [1967]. To do so, we impose the following conditions.

Assumption 1. Let $V_t = \sum_{s=1}^t v_s^2$. For a positive non-decreasing function $f(t) = O(t/\log^4(t))$ and a positive non-decreasing sequence $r_t = o(1)$, we have, P_0 -almost surely,

$$\begin{aligned} t^{-1}V_t - 1 &= O(r_t), \\ \sum_{t=1}^{\infty} \frac{1}{f(V_t)} E_0[(X_t - \mu_t)^2 \mathbb{I}_{\{(X_t - \mu_t)^2 > f(V_t)\}} | \mathcal{F}_{t-1}] &< \infty. \end{aligned}$$

For now we take Assumption 1 as a primitive assumption. In Section 7, we will discuss simple sufficient conditions for it for the case of stabilized estimating equations. Assumption 1 is akin to (but different than) a martingale Lindeberg condition, and in Section 7 we satisfy it by assuming a moment higher than 2 exists.

A SIP guarantee is, of course, asymptotic, so we need to consider regimes where stopping time is large. One such regime is when we set a long burn-in period, essentially to wait for the data to look normal.

Theorem 2 (Type-I error with $t_0 \rightarrow \infty$). *Suppose $\alpha \in (0, 1)$, $\mu' > 0$ and Assumption 1 holds with $f(t) = o(t^{4\kappa} (\log(t))^4)$ and $r_t = o(t^{-2\kappa} / \log t)$ for some $\kappa > 0$. If X_1, X_2, \dots is an MDS, then*

$$P_0 [\tau_{\alpha, \mu', t_0}^{\text{svs}} < \infty] \rightarrow \alpha$$

as $t_0 \rightarrow \infty$, $\mu' \sqrt{t_0} \rightarrow 0$, $\mu' t^{1/2+\kappa} \rightarrow \infty$.

Note that in our main theorem on the type-I errors of the uSPRT (Theorem 6), we will actually make much *weaker* assumptions than here, only requiring Assumption 1 (with no further restriction on $f(t), r_t$) and only needing the limit to take $t_0 \rightarrow \infty$.

Another regime where stopping time is large is when $\mu' \approx 0$ so it takes a long time to distinguish it from 0. In this case, we do not even need to have *any* burn-in period. This is surprising as it means we will not terminate too early *even* in the beginning when the likelihood can be very badly misspecified.

Theorem 3 (Type-I error with $\mu' \rightarrow 0$, no burn-in period). *Suppose $\alpha \in (0, 1)$, $\mu' > 0$ and Assumption 1 holds with $f(t) = O(t^{1-4\kappa} / \log^4 t)$ and $r_t \log(1/r_t) = o(t^{-2\kappa})$ for some $\kappa > 0$. If X_1, X_2, \dots is an MDS, then*

$$\lim_{\mu' \rightarrow 0} P_0 [\tau_{\alpha, \mu', 1}^{\text{svs}} < \infty] = \alpha.$$

More than just being about handling non-parametric observations, Theorems 2 and 3 also crucially show when we can avoid the so-called “overshoot problem” [Siegmund, 2013], *i.e.*, that when the test statistic first exceeds the thresholds it does so by a margin (it overshoots it) imperiling calibration of type-I-error rates.

Next, we analyze the expected rejection time. Under the alternative that $\mu_t \neq 0$, we require that μ_t appropriately converges to some $\mu_0 \neq 0$ in the following sense:

Assumption 2. For some $\mu_0, \gamma \in (0, 1)$,

$$M_{\mu, \gamma} = E_0[(\sup_{t \in \mathbb{N}} t^\gamma |\mu_t - \mu_0| / |\mu_0|)^{1/\gamma}]^\gamma < \infty.$$

Then, we rewrite our test statistic in terms of a re-centered MDS plus a bias term:

$$\begin{aligned} Y_{\mu', t}^{\text{svs}} &= \mu'(S_t^0 + \mu_0 t) - \frac{1}{2}\mu'^2 t + R_t^{\text{svs}, \text{bias}}(\mu'), \\ R_t^{\text{svs}, \text{bias}}(\mu') &= \mu' \sum_{s=1}^t (\mu_s - \mu_0). \end{aligned}$$

Optional stopping and control of the bias term using Assumption 2 yields the following.

Theorem 4. *Suppose $\alpha \in (0, 1)$, $t_0 \geq 1$, $\mu' > 0$. Suppose that Assumptions 1 and 2 hold with $\mu_0 > 0$. Then*

$$\begin{aligned} &E_0 [\tau_{\alpha, \mu', t_0}^{\text{svs}} - 1 - t_0] - \frac{C_2^{\text{svs}}}{C_1^{\text{svs}}} E_0 [\tau_{\alpha, \mu', t_0}^{\text{svs}} - 1 - t_0]^{1-\gamma} \\ &\leq \frac{\mu' \sqrt{t_0}}{C_1^{\text{svs}}} h \left(\frac{-\log \alpha - C_1^{\text{svs}} t_0}{\mu' \sqrt{t_0}} \right) + \mu' \mu_0 o(t_0) + o(\mu' \sqrt{t_0}) \\ &\leq E_0 [\tau_{\alpha, \mu', t_0}^{\text{svs}} - t_0] - \frac{C_2^{\text{svs}}}{C_1^{\text{svs}}} E_0 [\tau_{\alpha, \mu', t_0}^{\text{svs}} - t_0]^{1-\gamma} \end{aligned}$$

with, $C_1^{\text{svs}} = \mu' \mu_0 - \mu'^2 / 2$, $C_2^{\text{svs}} = \mu' \mu_0 M_{\mu, \gamma} / (1 - \gamma)$, and

$$h(x) = x\Phi(x) + (2\pi)^{-1/2} \exp(-x^2/2).$$

Note that, in particular, Theorem 4 implies that for $\mu' = \mu_0$, as $t_0 \rightarrow \infty$ and $\mu_0\sqrt{t_0} \rightarrow 0$,

$$E_0[\tau_{\alpha,\mu',t_0}^{\text{svs}}] \sim \frac{-2 \log \alpha}{\mu_0^2},$$

where we write $a \sim b$ if $a/b \rightarrow 1$.

Although we make Assumption 1 in Theorem 4, we actually only require a martingale central limit theorem to hold in order to establish that result, rather than the much more powerful SIP. We use Assumption 1 here because we already need it for type-I-error guarantees and because it is not precisely comparable to a martingale Lindeberg condition.

6 Putting It Together: Universal Guarantees for the uSPRT

We now return to our original test statistic as defined in Section 2. The uSPRT stopping time (Algorithm 1) can be written as the following \mathfrak{F} -stopping time:

$$\begin{aligned}\tau_{\alpha,\lambda,t_0} &= \inf \{t \in \{t_0, t_0 + 1, \dots\} : Y_{\lambda,t} \geq -\log \tilde{\alpha}(\alpha, \lambda, t_0)\}, \\ \tilde{\alpha}(\alpha, \lambda, t_0) &= \sqrt{\frac{t_0}{\lambda}} \tilde{\alpha}(\alpha),\end{aligned}$$

with $Y_{\lambda,t}$ defined in Section 2 and $\tilde{\alpha}(\alpha)$ in Algorithm 1.

Observe that the effective significance level $\tilde{\alpha}(\alpha, \lambda, t_0)$ -level increases with the duration t_0 of the burn-in period. The intuition as to why can be easily formulated in terms of “ α -spending”: the more we wait before starting monitoring, the less α we spend early on, and therefore the more we have to spend later on to reach the nominal significance level α .

Note also that $\tilde{\alpha}(\alpha, \lambda, t_0)$ can get larger than 1, and should therefore not be interpreted as a probability. It is merely a quantity that appears *in lieu* of where α appears in the classical normal-mixture SPRT, and we see the $\sqrt{t_0/\lambda}$ factor as a technical artifact that ensures the significance level after burn-in is the desired one.

6.1 Representation of the test statistic

We next re-express the test statistic $Y_{\lambda,t}$ in terms of the intermediate test statistics from Sections 4 and 5 and put it in a form amenable to our analysis.

Using an identity that arises from the discrete analogue of Itô’s lemma (Lemma 11), we can re-express and decompose the test statistic.

Theorem 5. *For any $\lambda > 0$, $t \geq 1$, it holds that*

$$\begin{aligned}Y_{\lambda,t} &= \sum_{s=0}^{t-1} \rho_{\lambda,s} \left(\check{\mu}_{\lambda,s} \check{X}_{s+1} - \frac{1}{2} \check{\mu}_{\lambda,s}^2 \right) - \Delta_{\lambda,t}^{\text{qvar}}(\check{X}_{0:t}) + R_t^{\text{bias}} \\ &= \tilde{Y}_{\mu_0}^{\text{svs}}(t) + R_t^{\text{bias}} + R_{\mu_0,t}^{\text{svs,SIP}} + \tilde{R}_{\lambda}^{\text{adpt}}(t) + \tilde{R}_{\lambda,\mu_0}^{\text{skg}}(t) + R_t^{\text{adpt,SIP}} + R_t^{\text{skg,SIP}},\end{aligned}$$

where $R^{\text{svs,SIP}}$, $\tilde{R}_{\lambda}^{\text{adpt}}$, $\tilde{R}_{\lambda,\mu_0}^{\text{skg}}$ are as defined earlier and, for any μ' ,

$$\begin{aligned}\check{X}_s &= \mu_0 + X_s^0, \quad \check{\mu}_{\lambda,s} = \mu_0 \frac{s}{s+\lambda} + \hat{\mu}_{\lambda,s}^0, \quad \hat{\mu}_{\lambda,s}^0 := S_s^0/(s+\lambda), \\ S_t^0 &= \sum_{s=1}^t X_s^0, \quad \rho_{\lambda,s} = \frac{s+\lambda}{s+1+\lambda}, \\ \Delta_{\lambda,t}^{\text{qvar}}(z_{1:t}) &= \frac{1}{2} \left(\log \frac{t+\lambda}{\lambda} - \sum_{s=1}^t \frac{z_s^2}{s+\lambda} \right), \\ R_t^{\text{bias}} &= \frac{1}{2} \frac{S_t^2 - \check{S}_t^2}{t+\lambda}, \quad \check{S}_t = \mu_0 t + S_t^0,\end{aligned}$$

$$\begin{aligned}
R_t^{\text{skg}, \text{SIP}} &= R_t^{\text{skg}} - \tilde{R}_{\lambda, \mu_0, t}^{\text{skg}}, \\
R_t^{\text{skg}} &= R_t^{\text{skg}, 1} + R_t^{\text{skg}, 2} + R_t^{\text{skg}, 3}, \\
R_t^{\text{skg}, 1} &= -\frac{1}{2} \lambda \mu_0^2 \frac{t}{t+\lambda}, \quad R_t^{\text{skg}, 2} = \lambda \mu_0 \sum_{s=1}^t \frac{X_s^0}{s+\lambda}, \quad R_t^{\text{skg}, 3} = \lambda \mu_0 \sum_{s=0}^{t-1} \frac{S_s^0}{(s+\lambda)(s+1+\lambda)}, \\
R_t^{\text{adpt}, \text{SIP}} &= R_{\lambda, t}^{\text{np}, \text{adpt}} - \tilde{R}_\lambda^{\text{adpt}}(t) - \Delta_{\lambda, t}^{\text{qvar}}(X_{0:t}^0), \\
R_{\lambda, t}^{\text{np}, \text{adpt}} &= \frac{1}{2} \left(\frac{(S_t^0)^2}{t+\lambda} - \sum_{s=1}^t \frac{(X_s^0)^2}{s+\lambda} \right).
\end{aligned}$$

Observe that the representation above makes appear the Brownian oracle simple-vs-simple test statistic $\tilde{Y}_{\mu_0, t}$ and terms that we interpret as excess terms (we show further down that these are negligible in front of $\tilde{Y}_{\mu_0, t}$, in a sense we make explicit then). We say that $\tilde{Y}_{\mu_0, t}$ is an oracle statistic as it makes appears the true effect size μ_0 , which we know from section 3 is the optimal choice for μ' so as to ensure the asymptotically optimal rejection time when it is non-zero.

When $\mu_0 = 0$, the terms $Y_{\mu_0, t}$, the bias term R_t^{bias} , and the shrinkage excess term $R_{\mu_0, \lambda}^{\text{skg}}$, and the Brownian approximation terms $R_t^{\text{skg}, \text{SIP}}$ and $R_t^{\text{svs}, \text{SIP}}$ disappear. As can also be immediately observed from its original expression (2), the test statistic then reduces to

$$Y_{\lambda, t} = \frac{1}{2} \left(\frac{(S_t^0)^2}{t+\lambda} - \log \frac{t+\lambda}{\lambda} \right).$$

In the next section we characterize the asymptotic distribution of the sup from t_0 of the above quantity, by means of a strong invariance principle (Proposition 8), which allows us to derive type-I error guarantees.

Before moving on to type-I error considerations, we present two immediate corollaries of the previous theorem, which give us representations of the uSPRT test statistic in terms of the statistics from the intermediates cases (normal mixture under normal data, and simple-vs-simple under non-parametric data).

Corollary 1 (Representation in terms of $Y_{\mu_0}^{\text{svs}}$). *For any $\lambda > 0$, $t \geq 1$, it holds that*

$$Y_{\lambda, t} = Y_{\mu_0, t}^{\text{svs}} + R_t^{\text{bias}} + \tilde{R}_\lambda^{\text{adpt}}(t) + \tilde{R}_{\lambda, \mu_0, t}^{\text{skg}} + R_t^{\text{adpt}, \text{SIP}} + R_t^{\text{skg}, \text{SIP}}.$$

Corollary 2 (Representation in terms of \tilde{Y}_λ). *For any $\lambda > 0$, $t \geq 1$, it holds that*

$$Y_{\lambda, t} = \tilde{Y}_\lambda(t) + R_t^{\text{bias}} + R_{\lambda, t}^{\text{SIP}}, \quad \text{where} \quad R_{\lambda, t}^{\text{SIP}} = R_t^{\text{svs}, \text{SIP}} + R_t^{\text{adpt}, \text{SIP}} + R_t^{\text{skg}, \text{SIP}}.$$

6.2 Controlling Type-I Error

We next control the type-I-error rate of the uSPRT in the asymptotic regime as the burn-in period grows. First, we derive an asymptotic limit for the supremum of our test statistic in this regime.

Proposition 5. *Suppose $\alpha \in (0, 1)$, $\lambda > 0$ and Assumption 1 holds. If X_1, X_2, \dots is an MDS, then we have the following convergence in distribution as $t_0 \rightarrow \infty$:*

$$\sup_{t \geq t_0} Y_{\lambda, t} - \frac{1}{2} \log \frac{\lambda}{t_0} \rightsquigarrow \frac{1}{2} \sup_{s \geq 1} \left(\frac{W(s)^2}{s} - \log s \right).$$

Next, we show that Algorithm 1 gives the quantile of this limiting variable.

Proposition 6. *Let $\alpha \in (0, 1)$. Then $\tilde{\alpha}(\alpha)$ in Algorithm 1 satisfies*

$$P_0 \left[\sup_{s \geq 1} \frac{1}{2} \left(\frac{W(s)^2}{s} - \log s \right) \geq -\log \tilde{\alpha}(\alpha) \right] = \alpha.$$

In view of Propositions 5 and 6, the appropriate threshold for $Y_{\lambda,t}$ at which to stop is $\frac{1}{2} \log \frac{\lambda}{t_0} - \log \tilde{\alpha}(\alpha)$, whence our choice of $\tilde{\alpha}(\alpha, \lambda, t_0)$ in Section 6. We then have the following:

Theorem 6 (Type-I error). *Suppose $\alpha \in (0, 1), \lambda > 0$ and Assumption 1 holds. If X_1, X_2, \dots is an MDS, then*

$$\lim_{t_0 \rightarrow \infty} P_0 [\tau_{\alpha, \lambda, t_0} < \infty] = \alpha.$$

6.3 Controlling Expected Rejection Time

We next bound the expected rejection time of the uSPRT. This is somewhat involved and proceeds in two steps. First we control each of the terms in Theorem 5, evaluated at the uSPRT stopping time. Second, we reduce the analysis of the threshold-crossing time of $Y_{\lambda,t}$ to that of the crossing time of a $(-Y_{\lambda,t_0})$ -offset threshold by $Y_{\lambda,t} - Y_{\lambda,t_0}$. As we will see in subsection 6.3.5, conditional on \mathcal{F}_{t_0} the analysis proceeds as in simpler settings without burn-in time.

For brevity, in this section, we set $\tau = \tau_{\alpha, \lambda, t_0}$.

6.3.1 Control of the bias term

To control the bias term, we need to make assumptions similar to Assumption 2 but on the running averages of conditional means and centered conditional means.

Assumption 3. Let $\bar{\mu}_t = t^{-1} \sum_{s=1}^t \mu_s$. There exists $\gamma \in (0, 1)$ such that

$$M_{\bar{\mu}, \gamma} := E_0 [(\sup_{t \geq 1} t^\gamma |\bar{\mu}_t - \mu_0| / |\mu_0|)^{1/\gamma}]^\gamma < \infty.$$

Assumption 4. There exists $\gamma \in (0, 1)$ such that

$$M_{\mu^0, \gamma} := E_0 [(\sup_{t \geq 1} t^\gamma |\hat{\mu}_{\lambda, t}^0|)^{1/\gamma}]^\gamma < \infty.$$

Lemma 2. *Suppose Assumptions 3 and 4 hold. Then*

$$E_0 [R_\tau^{\text{bias}}] \leq \frac{\mu_0^2}{2} C_5 E_0 [\tau]^{1-\gamma},$$

with $C_5 = M_{\bar{\mu}, \gamma/2}^2 + 2M_{\bar{\mu}, \gamma} + 2M_{\bar{\mu}, \gamma/2}M_{\mu^0, \gamma/2}$

6.3.2 Control of the Brownian approximation error terms

To control the terms accounting for the approximation using Brownian motion, we need to make a stronger assumption on the extent to which $v_t \approx 1$. Note, however, we do not actually need to assume the conditions for the SIP.

Assumption 5. There exists $\beta > 0$ such that

$$M_{v, \lambda, \beta} := E_0 \left[\sup_{t \geq 1} \log^{1+\beta} (t + \lambda) |v_t^2 - 1| \right] < \infty.$$

Lemma 3. *Let $\lambda > 0$. Make Assumption 5. Then it holds that*

$$\left| E_0 \left[R_\tau^{\text{adpt, SIP}} - R_{t_0}^{\text{adpt, SIP}} \right] \right| \leq \frac{1}{2} \log \frac{E_0[\tau] + \lambda}{t_0 + \lambda} + C_1, \quad \text{with} \quad C_1 = \frac{M_{v, \lambda, \beta}}{\beta} \frac{1}{(\log(t_0 + \lambda))^\beta}.$$

Lemma 4. *Let $\lambda > 0$. It holds that*

$$E_0 [R_\tau^{\text{svs, SIP}} - R_{t_0}^{\text{svs, SIP}}] = 0.$$

6.3.3 Control of the shrinkage terms

In the upcoming result, we bound the expected shrinkage excess term $E_0[R_{\lambda,\mu_0,\tau}^{\text{skg}} - R_{\lambda,\mu_0,t_0}^{\text{skg}}]$. The bound involves the quantity $M_v := E_0[\sup_{t \geq 1} v_t^2]$, which is bounded under Assumption 5.

Lemma 5. *Let $\lambda > 0$, $t_0 \geq 1$. Make Assumption 5. It holds that*

$$E_0[R_\tau^{\text{skg},1} - R_{t_0}^{\text{skg},1}] \in [-C_2, 0], \quad E_0[R_\tau^{\text{skg},2} - R_{t_0}^{\text{skg},2}] = 0, \quad \left| E_0[R_\tau^{\text{skg},3} - R_{t_0}^{\text{skg},3}] \right| \leq C_3,$$

with

$$C_2 := \frac{1}{2} \lambda^2 \mu_0^2 \frac{1}{t_0 + \lambda} \quad \text{and} \quad C_3 := \lambda \mu_0 \sqrt{\frac{1 + M_v}{t_0 + \lambda}}.$$

Therefore

$$E_0[R_\tau^{\text{skg}} - R_{t_0}^{\text{skg}}] \in [-C_3 - C_2, C_3].$$

6.3.4 Control of the Brownian adaptivity excess term

We now present a bound on the term $\tilde{R}_\lambda^{\text{adpt}}(\tau) - \tilde{R}_\lambda^{\text{adpt}}(t_0)$. We refer to this term as the “Brownian adaptivity excess term” as it is the same quantity that arises in the Brownian adaptive case (Section 4) next to the oracle simple-vs-simple test statistic and the shrinkage term. As in Section 4, we interpret the presence of this excess term as the cost of adaptivity, that is the price to pay to estimate μ_0 . To further support the interpretation of this term as an “estimation cost”, we point out that the derivation of the representation of $Y_{\lambda,t}$ (see proof of Theorem 5) makes it apparent that the term can be obtained by collecting all the “noise” terms S_t^0 and $\hat{\mu}_t^0$ (and leaving out the ones that involve the oracle quantity μ_0).

Lemma 6. *Let $\lambda > 0$, $t_0 \geq 1$. It holds that*

$$E_0 \left[\tilde{R}_\lambda^{\text{adpt}}(\tau) - \tilde{R}_\lambda^{\text{adpt}}(t_0) \right] \in \left[-\frac{1}{2} \log \frac{E_0[\tau] + \lambda}{t_0 + \lambda}, \frac{1}{2} C_4 \right], \quad \text{with} \quad C_4 = \frac{t_0}{t_0 + \lambda}.$$

6.3.5 Burn-in period and random rejection threshold

The presence of the burn-in period in uSPRT adds a slight level of complexity to rejection time analysis as compared to settings where monitoring starts from the onset of data-collection. Indeed, the difference between uSPRT and the latter sequential testing settings is that we need to analyze the first crossing time of a boundary, but in uSPRT the boundary only starts after t_0 steps. Fortunately, one can observe that monitoring crossing of a fixed threshold A by $Y_{\lambda,t}$ after t_0 steps is the same as monitoring crossing of the offset threshold $A - Y_{\lambda,t_0}$ by the offset test statistic $Y_{\lambda,t} - Y_{\lambda,t_0}$. Therefore, conditional on \mathcal{F}_{t_0} , we can analyze the rejection time from t_0 as we would do in a situation without burn-in, with the difference that the threshold is now the offset threshold $A - Y_{\lambda,t_0}$. We can thus readily obtain a characterization of the expected rejection time given \mathcal{F}_{t_0} given the \mathcal{F}_{t_0} -measurable random threshold $A - Y_{\lambda,t_0}$.

The following lemma connects the test statistic, the stopping time and the random rejection threshold.

Lemma 7. *For any $\lambda > 0$, $t_0 \geq 1$, $\alpha \in (0, 1)$,*

$$Y_{\lambda,(\tau-1)\vee t_0} - Y_{\lambda,t_0} \leq (-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{\lambda,t_0})_+ \leq Y_{\lambda,\tau} - Y_{\lambda,t_0}.$$

So as to obtain bounds on the marginal expected (as opposed to conditional on \mathcal{F}_{t_0}) rejection time $E_0[\tau_{\alpha,\lambda,t_0}]$, we need to characterize the marginal expectation (that is the expectation w.r.t. the distribution on Y_{t_0}) of the random rejection threshold. We first express it in term of the expectation of the corresponding quantity in the Brownian case in the following lemma.

Lemma 8. *Suppose Assumptions 1, 3 and 4 hold. Then*

$$E_0 \left[(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{\lambda,t_0})_+ \right] = E_0 \left[\left(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - \tilde{Y}_{\lambda,t_0} \right)_+ \right] + o(\mu_0 \sqrt{t_0}) + o(1)$$

as $t_0 \rightarrow \infty$, $\mu_0 \sqrt{t_0} \rightarrow 0$.

We now provide an expression for the expected random Brownian rejection threshold.

Lemma 9. *For any $\alpha \in (0, 1)$, $\lambda > 0$, $t_0 > 0$,*

$$E_0 \left[\left(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - \tilde{Y}_{\lambda, t_0} \right)_+ \right] - C_4 \mu_0 \sqrt{t_0} h \left(\frac{\log \tilde{\alpha}(\alpha)}{C_4 \mu_0 \sqrt{t_0}} - \frac{1}{2} \mu_0 \sqrt{t_0} - \frac{1}{2} \frac{\log C_4}{\mu_0 \sqrt{t_0} C_4} \right) \in \left[-\frac{1}{2} C_4, 0 \right],$$

where h is defined in Theorem 4.

6.3.6 Expected rejection time main theorem

We can now state our main theorem for the expected rejection time of uSPRT.

Theorem 7. *Suppose that X_1, X_2, \dots is an MDS and satisfies Assumption 1, and Assumptions 3 to 5 hold for some $\gamma \in (0, 1/2)$. Then, for any $\lambda > 0$, $\alpha \in (0, 1)$,*

$$\begin{aligned} & \frac{1}{2} \mu_0^2 (E_0[\tau] - C_5 E_0[\tau]^{1-\gamma}) - \log \frac{E_0[\tau] + \lambda}{t_0 + \lambda} - C_3 - C_2 - C_1 \\ & \leq C_4 \mu_0 \sqrt{t_0} \left\{ h \left(\frac{\log \tilde{\alpha}(\alpha)}{C_4 \mu_0 \sqrt{t_0}} - \frac{1}{2} \mu_0 \sqrt{t_0} - \frac{1}{2} \frac{\log C_4}{\mu_0 \sqrt{t_0} C_4} \right) + o(1) + o\left(\frac{1}{\mu_0 \sqrt{t_0}}\right) \right\}, \\ & C_4 \mu_0 \sqrt{t_0} \left\{ h \left(\frac{\log \tilde{\alpha}(\alpha)}{C_4 \mu_0 \sqrt{t_0}} - \frac{1}{2} \mu_0 \sqrt{t_0} - \frac{1}{2} \frac{\log C_4}{\mu_0 \sqrt{t_0} C_4} \right) + o(1) + o\left(\frac{1}{\mu_0 \sqrt{t_0}}\right) \right\} - \frac{1}{2} C_4 \\ & \leq \frac{1}{2} \mu_0^2 (E_0[\tau] + C_5 E_0[\tau]^{1-\gamma}) + \frac{1}{2} \log \frac{E_0[\tau] + \lambda}{t_0 + \lambda} \frac{1}{2} + C_4 + C_3 + C_1, \end{aligned}$$

where h is defined in Theorem 4.

Note that Theorem 7, in particular, implies that, up to log factors, as $t_0 \rightarrow \infty$, $\mu_0^2 t_0 \rightarrow 0$, $\alpha \rightarrow 0$,

$$E_0[\tau] \sim \frac{-2 \log \alpha}{\mu_0^2}.$$

7 Application to Stabilized Estimating Equations with Sequentially Estimated Nuisances

We next consider one simple setting where we can apply our test and consider interpretable sufficient conditions for our results to hold. Suppose we observe an iid sequence Z_1, Z_2, \dots and we wish to test whether

$$E_0 \psi(Z_1; \eta_0) = 0,$$

where η_0 are some nuisance parameters in a nuisance set \mathcal{T} . Usually we set $\psi(Z_1; \eta) = \psi_\theta(Z_1; \eta)$, where solutions $\Theta_0 = \{\theta : E_0 \psi_\theta(Z_1; \eta_0) = 0\}$ are the parameter(s) of interest, so that testing Section 7 is testing $\theta \in \Theta_0$. To construct a confidence sequence for Θ_0 , we can simply consider all θ 's not rejected at time t .

To focus on the simplest setting, let us in particular assume that Section 7 is *completely* invariant to η_0 in that $E_0 \psi(Z; \eta_0) = E_0 \psi(Z; \eta)$ for all $\eta \in \mathcal{T}$ in some nuisance realization set. Example 1 fits into this framework with $\psi(z) = z - \theta$ to test $\theta = \theta_0 = E_0 Z_1$ and Example 2 with $\psi(a, u; \eta_0, \eta_1) = (p-a)(p^{-1}\eta_1 + (1-p)^{-1}\eta_0) + \frac{a-p}{p(1-p)}u - \theta$ to test $\theta = \theta_0 = E_0[U_1 | A_1 = 1] - E_0[U_1 | A_1 = 0]$.

We now apply our method to this setting. While in Example 2 we used a simple plug-in approach that estimates the nuisances on all past data, we here use a combination of sequential estimation and sample splitting in order to avoid any metric entropy assumptions. Specifically, we will estimate nuisances on half of all the past data and estimate the variance on the other half. Fix some $\hat{\eta}_t$ sequence adapted to \mathfrak{F} such that $\hat{\eta}_t$ is independent of $Z_{t-1}, Z_{t-3}, Z_{t-5}, \dots$ given $Z_t, Z_{t-2}, Z_{t-4}, \dots$, that is, an “estimate” of η_0 based only on the $\lceil t/2 \rceil$ data points from times $t, t-2, t-4, t-6, \dots$. Set

$$\hat{\sigma}_t^2 = \frac{1}{\lceil t/2 \rceil} \sum_{s \in \{t-1, t-3, t-5, \dots\}} \psi(Z_s; \hat{\eta}_t)^2 - \left(\frac{1}{\lceil t/2 \rceil} \sum_{s \in \{t-1, t-3, t-5, \dots\}} \psi(Z_s; \hat{\eta}_t) \right)^2.$$

Then set $X_t = \omega_t \psi(Z_t; \hat{\eta}_{t-1})$, where $\omega_t = (\hat{\sigma}_{t-1} \vee (\chi^{-1} t^{-\iota}))^{-1}$ with some $\chi > 0, \iota \in (0, 1)$. Then the hypothesis that X_1, X_2, \dots is an MDS holds if and only if Section 7 holds.

Note that if we have no nuisances, as in Example 1, then we can just use *all* data up to $t - 1$ to compute $\hat{\sigma}_t^2$, not just the past data having the same parity as $t - 1$. We can actually do this as long as \mathcal{T} is sufficiently simple (*e.g.*, a vector of a few nuisances, as in Example 2, rather than, say, a space of nonparametric functions). However, to avoid any such assumptions altogether, we focus here on the case where split the data by parity. Similarly, we here analyze the case where we clip ω_t by $O(t^\iota)$ to control for the risk of outlying variance estimates, but this is mostly done to make analysis simple. In practice, we do not recommend this, and in our experiments in Section 9, we just use the simple recommendation we give in Example 1 of using $\omega_t = 1$ whenever the variance estimate is zero and not clipping at any one value. This reduces the need to specify hyperparameters.

In the case where $\psi(z; \eta) = \psi_\theta(z; \eta) = \xi(z; \eta) - \theta$ is linear in a parameter θ , with $\theta_0 = E_0[\xi(Z_1; \eta_0)]$ begin the parameter of interest, the resulting confidence sequence simplifies considerably: let $\Gamma_t = \sum_{s=1}^t \omega_s$ and $\hat{\theta}_t = \Gamma_t^{-1} \sum_{s=1}^t \omega_s \xi(Z_s; \eta_s)$; then our confidence sequence is exactly as in Example 1. Note we do not actually need to require to $\hat{\eta}_t \rightarrow \eta_0$, but we may wish this to be the case so as to obtain a smaller confidence sequence. In particular, we can then obtain that the interval width has $t\Gamma_t^{-1} \rightarrow E_0(\xi(z; \eta_0) - \theta_0)^2$, which, depending on η_0 , may equal $\inf_{\eta \in \mathcal{T}} E_0(\xi(z; \eta) - \theta_0)^2$. For example, this is the case for $\eta_0 = (E_0[Z_1 | A_1 = 0], E_0[Z_1 | A_1 = 1])$ for ψ as specified above for the setting of Example 2, and $\hat{\eta}$ as specified in Example 2 has $\hat{\eta} \rightarrow \eta_0$. Alternatively, if we additionally observe baseline covariates L_t , then for further variance reduction, we may consider using the nuisance functions $\eta_0 = (E_0[Z_1 | L_1 = \ell, A_1 = 0], E_0[Z_1 | L_1 = \ell, A_1 = 1])$ or $\eta_0 = (\ell \mapsto \ell^\top E_0[L_1 L_1^\top | A_1 = 0]^{-1} E_0[L_1 U_1 | A_1 = 0], \ell \mapsto \ell^\top E_0[L_1 L_1^\top | A_1 = 1]^{-1} E_0[L_1 U_1 | A_1 = 1])$ with the moment equation $\psi_\theta(\ell, a, u; \eta_0, \eta_1) = (p - a)(p^{-1}\eta_1(\ell) + (1 - p)^{-1}\eta_0(\ell)) + \frac{a-p}{p(1-p)}u - \theta$.

We now verify our assumptions for this simple setting based on simple sufficient (and not tight) conditions. In the following, let $\sigma_0^2(\eta) = E_0(\psi(Z_1; \eta) - E_0\psi(Z_1; \eta))^2$.

Proposition 7. Suppose that $\psi(Z_1; \eta)$ is sub-Gaussian with a common parameter for all $\eta \in \mathcal{T}$ and that $\sigma_0^2(\eta_0) > 0$. Then Assumption 1 holds with any $r_t = o(\sqrt{\log t/t})$ and $f(t) = t^\kappa$ with $\kappa \in (0, 1)$.

Suppose, moreover, that for some $\zeta \in (0, 1/2]$, $c_1 > 0$, for each $t \in \mathbb{N}$, with probability at least $1 - \delta$,

$$|\sigma_0(\hat{\eta}_t) - \sigma_0(\eta_0)| \leq c_1 \left(\sqrt{(1 - \log \delta)/t^{2\zeta}} \vee ((1 - \log \delta)/t^{2\zeta}) \right).$$

Then, Assumptions 2 to 4 hold with $\mu_0 = \sigma_0^{-1}(\eta_0)E_0\psi(Z_1; \eta_0)$ and any $\gamma \in (0, \zeta - \iota)$ if $\iota < \zeta$, and Assumption 5 hold with any $\beta > 0$ if $\iota < 1/4$.

The condition in Proposition 7 formalizes that $\hat{\eta}_t$ is an estimate of η_0 and characterizes the rate of convergence, ζ . Usually, this condition would be obtained from a similar bound on $\|\hat{\eta}_t - \eta_0\|_{\mathcal{T}}$ in some norm (*e.g.*, Euclidean norm for a vector of nuisances or L_2 for nuisance functions) and establishing (or, assuming) that σ_0 is Lipschitz (or, Hölder) continuous in this norm. Such guarantees on $\hat{\eta}_t$ can be obtained when it is estimated by maximum likelihood or more generally empirical risk minimization [Van de Geer, 2000]. Generally, if η_0 is a vector of path-differentiable parameters of the distribution of Z_1 , then $\zeta = 1/2$. If η_0 is the best predictor of some $g_1(Z_1)$ as a function of $g_2(Z_1)$ in \mathcal{T} , then we can generally obtain guarantees with ζ equal to the rate of the critical radios of \mathcal{T} [Wainwright, 2019].

Here we considered just a simple setting with iid data and a nuisance-invariant estimating equation in order to show how one would verify our assumptions. Our results do also apply to more intricate settings, but further analysis would be needed to verify the assumptions using simple conditions. One example of a possible extension to the simple setting herein where our results still apply is where Section 7 is not completely invariant to η_0 , but instead we only have Neyman orthogonality in that $\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} E_0\psi(Z_1; \eta_0 + \epsilon(\eta - \eta_0)) = 0$ for $\eta \in \mathcal{T}$. This is, for example, relevant to sequentially observing data from an observational study, where we do not know the propensity score. For example, we may have a sequential trial involving only the intervention of interest (*e.g.*, experimental drug or surgery) and we have an offline pool of controls to compare to, where we assume selection into our trial at random given observed covariates (observed in the trial and in the pool of controls). Another extension may be to parameters that are path differentiable (that is, an influence function for them exists), but the may not necessarily be defined in terms of an estimating equation. Yet another example of a possible extension to the simple setting herein where our results still apply is where the

data is not iid but coming instead from an adaptive experiment such as a contextual bandit, as in Example 3. In this case, many of our assumptions could be verified in a very similar manner to how theorems 1 and 2 of Bibaut et al. [2021a] are proven, and guarantees for $\hat{\eta}_t$ in the form of Proposition 7 can be obtained from Bibaut et al. [2021b]. These would all simply be applications of our theory.

8 Related Literature

Classical approaches to hypothesis testing have predominantly dealt with experiments of fixed, predetermined sample sizes, which we refer to as the *fixed-n* kind. The emphasis on *fixed-n* tests by early pioneers such as Fisher is presumably a consequence of the motivating applications that drove the development of hypothesis testing procedures in the first half 19th century, in which outcomes of an experiment were only available long after the experiment had been designed, such as in agricultural research [Armitage, 1993]. As tests could only be performed once, *fixed-n* tests were designed to maximize power subject to a type-I error constraint [Neyman et al., 1933]. Increasingly in modern experiments, however, observations from experimental units become available sequentially instead of simultaneously, providing many opportunities to perform a test instead of just one. The application of *fixed-n* tests to sequential designs is made difficult because it requires making an undesirable trade-off balancing the competing objectives of detecting large effects early and detecting small effects eventually. Performing the test later risks exposing many experimental units to a potentially large and harmful treatment effect, while performing the test early risks a high type-II error for small effects. These desires have led to bad statistical practices whereby *fixed-n* procedures are naively applied to accumulating sets of data, see Johari et al. [2017] for a discussion pertaining to online A/B tests, which sacrifice type-I error guarantees [Armitage et al., 1969], permitting the analyst to incorrectly sample to a foregone conclusion [Anscombe, 1954].

For modern sequential designs, sampling until a hypothesis is proven or disproven appears to be a very natural form of scientific inquiry, which requires testing procedures to preserve their type-I/II error guarantees under continuous monitoring. Sequential inference is fundamentally tied to the theory of martingales [Ramdas et al., 2020]. A test martingale is a statistic that is a nonnegative supermartingale under the null hypothesis. Ville's inequality [Ville, 1939] is then used to bound the supremum of the process to provide a time-uniform type-I error guarantee. Research into sequential analysis in the statistics literature began with the introduction of the sequential probability ratio test (SPRT) [Wald, 1945, 1947]. Although Wald did not reference martingale theory in the exposition of the SPRT, the connection is clear in hindsight by observing that the likelihood ratio is a nonnegative supermartingale under the null. The simple-vs-simple SPRT enjoys the optimality property of being the sequential test that minimizes the average sample number (expected stopping time) among all sequential tests with no larger type-I/II error probabilities [Wald and Wolfowitz, 1948]. This is extended to the continuous-time version in Dvoretzky et al. [1953].

The SPRT for simple-vs-simple testing problems and the mixture SPRT (mSPRT) for composite testing problems can be interpreted as Bayes factors [Jeffreys, 1935, Kass and Raftery, 1995], forming a bridge between Bayesian, frequentist, and conditional frequentist approaches to sequential testing [Berger et al., 1994, 1999]. The SPRT also appears in Bayesian decision-theoretic approaches to sequential hypothesis testing in which there is a constant cost per observation [Berger, 1985, Wald and Wolfowitz, 1950]. However, care must be taken when specifying priors in composite testing problems, should one seek to have strict frequentist guarantees de Heide and Grünwald [2021]. Composite tests in statistical models with group invariances can often be reduced to simple hypothesis tests by constructing *invariant SPRTs* [Lai, 1981] based on a maximally invariant test statistic [Lehmann and Romano, 2005, Lehmann and Casella, 1998]. These invariant SPRT test statistics can be obtained as Bayes factors by using the appropriate right-Haar priors on nuisance parameters in group invariant models [Hendriksen et al., 2021]. Such arguments were used by [Robbins, 1970] to develop sequential tests for location-scale families with unknown scale parameters.

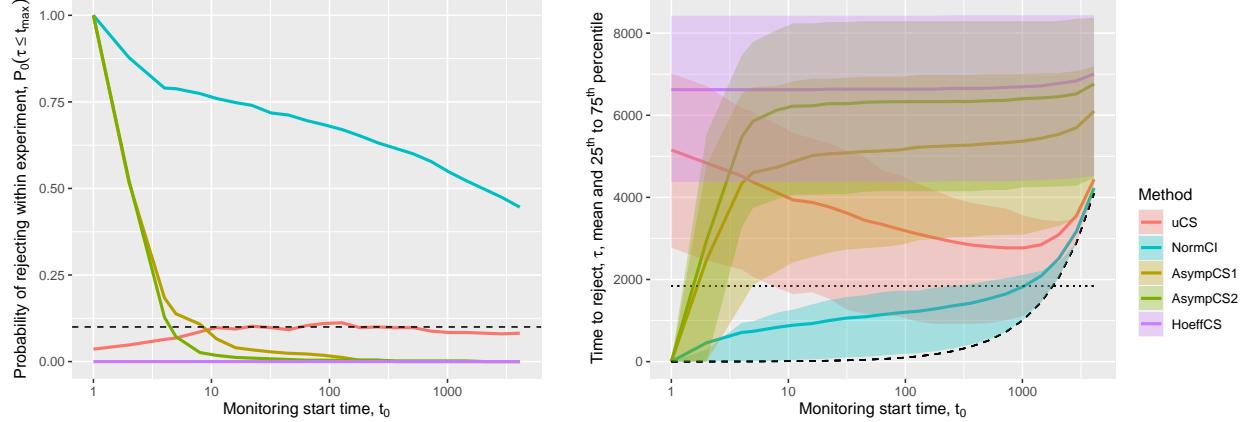
Confidence sequences [Darling and Robbins, 1967] can be obtained by inverting a sequential test, and sequential p -values can be obtained by tracking the reciprocal of the supremum of the test martingale. Together, these generalize the coverage and type-I guarantees held by *fixed-n* confidence intervals and p -values to hold uniformly through time. Procedures with these guarantees are appropriately referred to as “anytime valid.” Relationships between test-martingales, sequential p -values and Bayes factors are discussed in Shafer et al. [2011]. Nonparametric confidence sequences under sub-Gaussian and Bernstein conditions are

provided in Howard et al. [2021]. These results are nonasymptotic, yielding valid confidence sequences for all times, but may be conservative. Confidence sequences for quantiles and anytime-valid Kolmogorov-Smirnov tests are provided in Howard and Ramdas [2022]. Waudby-Smith et al. [2021] obtain asymptotic confidence sequences, in the sense that the intervals converges almost surely to a valid confidence sequence with an error that is orders smaller than the width of the latter. They achieve this by approximating the sample average process by a Gaussian process using strong invariance principles [Komlós et al., 1975, 1976, Strassen, 1964, 1967], like us. Their focus is on having approximate confidence sequence width, which need not translate to type-I error guarantees. In particular, there is a risk of rejecting too early when the cumulative sum do not look normal yet. Moreover, they only guarantee that a similar-width confidence sequence has at-least- α coverage, but do not characterize its power, only that the width has the right rate dependence on t . Therefore, at the same time, if we do wait, the confidence sequences can be overly conservative.

For certain continuous-time martingales, Ville’s inequality is an equality [Robbins and Siegmund, 1970, lemmas 1 and 2]. For discretely observed martingales, however, Ville’s inequality is generally strict, meaning the type-I-error guarantees it yields for test martingale are conservative. The conservativeness follows from the amount by which the stopped sum process exceeds the rejection boundary (zero in the continuous case) and is often referred to as the “overshoot” problem with the SPRT [Siegmund, 2013]. Understanding the size of the overshoot is key to understanding how conservative existing bounds are on type-I error and expected stopping times. Wald [1945]’s approximation to the type-I error is obtained by simply ignoring the overshoot. Siegmund [1975] obtains an approximation to the type-I error for the simple-vs-simple SPRT in exponential-family models as complete asymptotic expansions in powers of α^{-1} with exponentially small remainder as $\alpha \rightarrow 0$. With mSPRTs the rejection boundary is curved, and studying the distribution of the overshoot is often tackled via nonlinear renewal theory [Woodroffe, 1976, 1982, Zhang, 1988]. As $\alpha \rightarrow 0$, Lai and Siegmund [1977, 1979] derive asymptotic approximations to the expected value and distribution function of the nmSPRT stopping time under the null so as to study the type-I error resulting from truncated nmSPRT tests. Similar results for the expected stopping times can be found in Hagwood and Woodroffe [1982]. To our knowledge existing work has focused on asymptotic ($\alpha \rightarrow 0$) approximations to moments of stopping times for parametric SPRTs which yield sharper results than Wald [1945]’s when neglecting the overshoot. While previous authors also use these tools to obtain type-I errors for truncated sequential tests, no attention has been given to calibrating the type-I error for open-ended sequential tests.

As trends in online experimentation shift toward streaming approaches, sequential approaches to A/B testing have seen increased adoption [Johari et al., 2022, Lindon and Malek, 2020, Lindon et al., 2022]. In other applications, particularly in medicine, it may not be possible to test after every new observation. In clinical trials, a small number of interim analyses may be planned, which does not warrant a fully sequential test. Instead, group sequential tests [Jennison and Turnbull, 1999, Lan and DeMets, 1983, O’Brien and Fleming, 1979, Pocock, 1977] can be performed which provide a calibrated sequential test over a fixed and finite number of analyses. Analogous to confidence sequences, *repeated confidence intervals* provide strict coverage uniformly across all interim analyses [Jennison and Turnbull, 1984, 1989]. These procedures are useful when testing on a certain cadence, such as daily, suffices and when a terminal endpoint of the experiment is known. They are, however, not as flexible as fully sequential procedures as they do not allow the experiment to continue past the final analysis, having fully spent their α -budget.

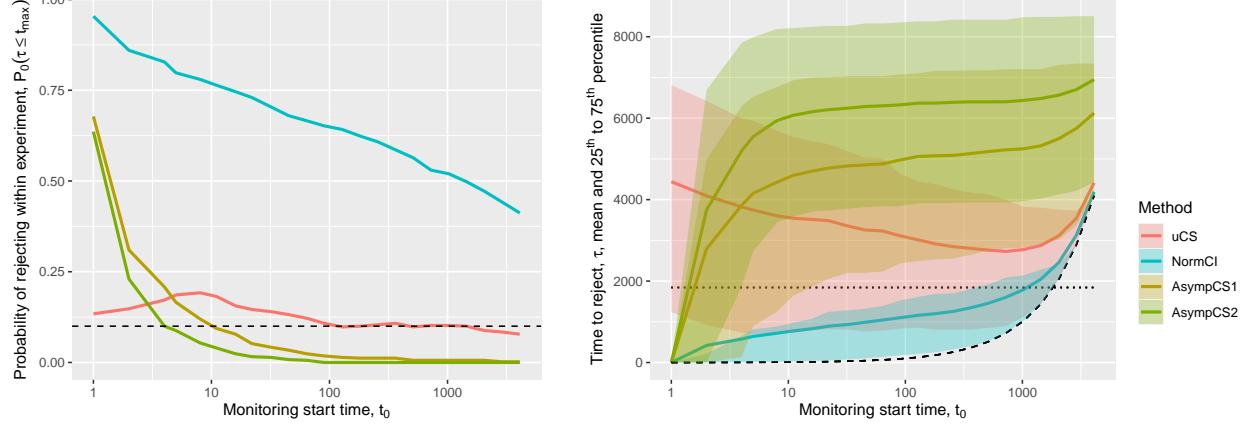
Test martingales are closely related to *e-processes*. An e-variable is a random variable (or statistic) that has expectation at most 1 under the null hypothesis [Grünwald et al., 2020]. An e-process is a nonnegative process, upper bounded by a nonnegative supermartingale, such that the stopped process is an e-variable under any stopping rule [Ruf et al., 2022], although it itself may not be a nonnegative supermartingale [Ramdas et al., 2022b]. Thanks to this property it is possible to build sequential tests from e-processes. Ramdas et al. [2022a] provide a review of test martingales, e-processes, anytime valid inference and game theoretic probability and it’s applications to sequential testing. See, for example, log-rank tests [ter Schure et al., 2020], contingency tables [ter Schure et al., 2020] and changepoint detection [Shin et al., 2022]. See also the running-MLE sequential likelihood ratio test of Wasserman et al. [2020].



(a) Type-I-error rates: probability of ever rejecting the true mean (within the experiment up to t_{\max}). The dashed line is 0.1, the nominal significance.

(b) Time to reject mean 5% above true. Solid line is mean. Shaded is 1st to 3rd quartiles. Dashed line is $\tau = t_0$. Dotted line is oracle SPRT stopping time.

Figure 1: Sequential testing for the mean of a Bernoulli random variable



(a) Type-I-error rates: probability of ever rejecting the true mean (within the experiment up to t_{\max}). The dashed line is 0.1, the nominal significance.

(b) Time to reject mean 5% above true. Solid line is mean. Shaded is 1st to 3rd quartiles. Dashed line is $\tau = t_0$. Dotted line is oracle SPRT stopping time.

Figure 2: Sequential testing for the mean of a Poisson random variable

9 Numerical Experiments

We now turn to a numerical illustration of our method. The replication code for the experiments is available at <https://github.com/nathankallus/UniversalSPRT>. We consider the simple setting of Example 1: we sequentially observe an iid sequence Z_1, Z_2, \dots , and we would like to make inferences on the mean $E_0 Z_1$. We set $\alpha = 0.1$, vary $t_0 \in \{1, 2, 4, 5, 8, 11, 16, \dots, 4096\}$, and consider four types of confidence sequences:

1. **uCS**: The confidence sequence specified in Example 1 with $\lambda = 1$.
2. **NormCI**: The usual normal confidence interval, ignoring the sequential nature of the testing. That is, $[\hat{m}_t \pm 1.64\hat{\sigma}_t/\sqrt{t}]$, with \hat{m}_t the running (unweighted) sample mean and $\hat{\sigma}_t$ as in Example 1. Because normality is asymptotic, for a fair comparison we also consider only starting monitoring at t_0 .
3. **AsympCS**: We use the asymptotic confidence sequences recommended in theorem 2.2 (with $\rho = 1/\sqrt{\lambda} = 1$) and proposition 2.1 of Waudby-Smith et al. [2021]. That is, $[\hat{m}_t \pm \hat{\sigma}_t \sqrt{2(1 + 1/t) \log \frac{\sqrt{t+1}}{\alpha}}/\sqrt{t}]$

(AsympCS1) and $[\hat{m}_t \pm 1.7\hat{\sigma}_t\sqrt{\log\log(2t) + 0.72\log(10.4/\alpha)/\sqrt{t}}]$ (AsympCS2). These confidence sequences, like ours, are based on an asymptotic normal approximation, so for a fair comparison we also consider both monitoring immediately ($t_0 = 1$) and only starting monitoring at $t_0 > 1$, even though no such recommendation is made in Waudby-Smith et al. [2021].

4. **HoeffCS:** When we know $Z_1 \in [a, b]$ is bounded, we also use the confidence sequence recommended in equation (2) of Howard et al. [2021]. That is, $[\hat{m}_t \pm 1.7\frac{b-a}{2}\sqrt{\log\log(2t) + 0.72\log(10.4/\alpha)/\sqrt{t}}]$. This confidence sequence is *non-asymptotic*, but just for the sake of experiment we consider both monitoring immediately ($t_0 = 1$) and what happens when we only start monitoring at $t_0 > 1$.

For a given Z_1 distribution, we run 500 replications of an experiment observing a sequence of $t_{\max} = 200000$ iid copies of Z_1 .

In Fig. 1, we plot the results of the experiment for $Z_1 \sim \text{Bernoulli}(0.5)$. Figure 1a shows the fraction of replications in which 0.5 is ever excluded from the confidence sequence for any $t \in [t_0, t_{\max}]$. We can see that our confidence sequence well approximates the nominal significance for $t_0 \geq 10$. In contrast, NormCI undercovers as it does not adjust for the multiple comparisons, HoeffCS overcovers as it is too conservative, and AsympCS1 and AsympCS2 with $t_0 = 1$ undercover and overcover if we increase t_0 . This behavior is mirrored in Figure 1b, where we consider the time until 0.525 is first excluded from the confidence sequence. Here, NormCI rejects quickly (but it also frequently rejects the true mean) while HoeffCS takes a very long time to reject and AsympCS1 and AsympCS2 reject quickly (but also frequently rejecting the true mean) for t_0 small and behave like HoeffCS for t_0 large. Our confidence sequence, which has nearly calibrated type-I errors, generally rejects faster than these, approaching but not exactly reaching the stopping time of the oracle SPRT of standard Brownian motion with drift 0 vs $0.05 = (0.525 - 0.5)/0.5$. Note that our overall rejection time actually *decreases* even as we increase the burn-in period t_0 at first.

In Fig. 2, we plot the results for $Z_1 \sim \text{Poisson}(1)$. Here we drop HoeffCS because we do not have bounds (and Poisson is not sub-Gaussian). The qualitative insights are the same, which shows the conclusions extend to unbounded, non-sub-Gaussian observations.

10 Concluding Remarks

Sequential testing and confidence sequences are appealing because of the potential flexibility they afford the analyst. However, in practice, existing methods either rely on strong parametric assumptions that fail in practice or are non-parametric and too conservative. In either case, the resulting type-I-error rate or coverage are unpredictable, making the methods unreliable in practice: either we undercover and reject too soon, or we overcover so, while safe from false discoveries, we take far too long to make true discoveries, canceling out the appeal of making decisions on the fly. Practitioners therefore end up either abandoning sequential analysis or using heuristics.

Our aim in this paper was to take a step toward remedying this situation. Exact calibration is unattainable without parametric assumptions, and any-time-valid at-least- α -coverage, while appealing in theory, leads to conservativeness. We therefore sought a new, different objective: approximate calibration in certain asymptotic regimes. This is quite similar to the widely accepted approximate calibration obtained by central-limit-theorem-based confidence intervals in fixed- n studies, as n tends large. Instead, however, we require this to hold uniformly over all the n 's, that is, the probability we *ever* falsely reject the null hypothesis (equivalently, that we ever exclude the true value of the parameter in our confidence sequence) is approximately α . Our dual objective to approximate calibration was approximate optimality: we want our expected time to reject a false hypothesis (equivalently, to exclude it from our confidence sequence) to not be much later than that of the *optimal* test with exact calibration at α type-I-error rate, that is, the SPRT testing this single false hypothesis against the true one.

Amazingly, we are able to attain both of these objectives with a simple modification of the nmSPRT: we add a burn-in period and adjust the rejection threshold accordingly. We called this uSPRT because our analysis provided *universal* guarantees for it to asymptotically calibrated and optimal under lax nonparametric conditions that even permit dependent observations. We showed how this can be applied to testing parameters like means and quantiles of sequential iid observations, average treatment effects in A/B tests

with fixed-probability Bernoulli allocations, and policy values in contextual-bandit experiments with adaptive allocation probabilities. When we applied it numerically in simulations, the confidence sequence implied by the uSPRT, which we termed uCS, yielded nearly exact calibration and rejected false values quickly.

We hope that given their simplicity, their calibration, and their universality, the uSPRT and uCS provide viable approaches to sequential testing in practice. We moreover hope that our paper inspires other researchers to consider *asymptotic* calibration and optimality, as we have defined them here, as new alternative objectives for sequential testing, as these sidestep the difficulties of stringent exact-calibration or exactly-always-valid requirements that are hard to satisfy without either strong assumptions or over-conservatism. We speculate there may be other fruitful ways to achieve these objectives, especially in new settings and specific applications.

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A Proofs

A.1 Useful Results

Lemma 10 (Itô’s lemma). Consider a probability space $(\Omega', \mathcal{F}', P')$ endowed with a time-continuous Brownian filtration \mathcal{F}'_0^∞ and consider an \mathcal{F}'_0^∞ -adapted standard Wiener process W' . For any $x, t \geq 0$, $\lambda > 0$, let

$$f_\lambda(x, t) := \frac{x^2}{2(t + \lambda)}.$$

For any \mathcal{F}'_0^∞ -adapted stochastic process S' such that $dS'(t) = \mu' dt + dW(t)$, for every $t \geq 0$, for some constant $\mu' \in \mathbb{R}$, it holds that

$$df_\lambda(S'_t, t) = \frac{S'(t)}{t + \lambda} dS'(t) - \frac{1}{2} \left(\frac{S'(t)}{t + \lambda} \right)^2 dt + \frac{1}{2(t + \lambda)} dt.$$

Lemma 11. Let $(z_s)_{s \in \mathbb{N}}$ be a sequence of real numbers, and let $\lambda > 0$. For any $s \in \mathbb{N}$, it holds that

$$\frac{1}{2} \left(\frac{z_{s+1}^2}{s+1+\lambda} - \frac{z_s^2}{s+\lambda} \right) = \frac{s+\lambda}{s+1+\lambda} \left(\frac{z_s}{s+\lambda} (z_{s+1} - z_s) - \frac{1}{2} \left(\frac{z_s}{s+\lambda} \right)^2 \right) + \frac{1}{2} \frac{(z_{s+1} - z_s)^2}{s+\lambda}.$$

Proof of Lemma 11. For any $s \geq 0$, we have that

$$\begin{aligned} & \frac{1}{2} \left(\frac{z_{s+1}^2}{s+1+\lambda} - \frac{z_s^2}{s+\lambda} \right) \\ &= \frac{1}{2} \left(\frac{z_{s+1}^2 - z_s^2}{s+1+\lambda} - \frac{z_s^2}{(s+1+\lambda)(s+\lambda)} \right) \\ &= \frac{1}{2} \left(\frac{z_{s+1} + z_s}{s+1+\lambda} (z_{s+1} - z_s) - \frac{z_s^2}{(s+1+\lambda)(s+\lambda)} \right) \\ &= \frac{z_s}{s+1+\lambda} (z_{s+1} - z_s) - \frac{1}{2} \frac{z_s^2}{(s+1+\lambda)(s+\lambda)} + \frac{1}{2} \frac{(z_{s+1} - z_s)^2}{s+1+\lambda} \end{aligned}$$

$$= \frac{s+\lambda}{s+1+\lambda} \left(\frac{z_s}{s+\lambda} (z_{s+1} - z_s) - \left(\frac{z_s}{s+\lambda} \right)^2 \right) + \frac{1}{2} \frac{(z_{s+1} - z_s)^2}{s+1+\lambda}.$$

□

Proposition 8 (Strong invariance principle). *Make Assumption 1. Then, the probability space can be enlarged so that it supports a standard Brownian motion $(W(t))_{t \in \mathbb{R}_+}$ such that*

$$S_t - W(t) = o((tf(t))^{1/4} \log t) + O(\sqrt{2tr_t \log(1/r_t)}).$$

almost surely.

Proof of Proposition 8. We have that

$$\begin{aligned} S_t - W(t) &= S_t - W(V_t) + \sqrt{t} \frac{W(V_t) - W(t)}{\sqrt{t}} \\ &= o((tf(t))^{1/4} \log t) + \sqrt{t}(W'(V_t/t) - W'(1)), \end{aligned}$$

almost surely, where, the $(W'(t))_{t \geq 1}$ is another standard Brownian motion. Theorem 1.3 in Strassen [1967] and Lévy's modulus of continuity yield that

$$S_t - W(t) = o((tf(t))^{1/4} \log t) + O(\sqrt{2tr_t \log(1/r_t)}).$$

□

A.2 Proofs of the results of Section 4

A.2.1 Proof of Proposition 3

We first prove the representation in Proposition 3 via Itô's lemma.

Proof of Proposition 3. Let $f_\lambda(x, t) := x^2/(2(t+\lambda))$. We have that

$$\begin{aligned} \tilde{Y}_\lambda(t) &= \int_0^t df(\tilde{S}(s), s) - \frac{1}{2} \log \frac{t+\lambda}{\lambda} \\ &= \int_0^t \tilde{\mu}_\lambda(s) d\tilde{S}(s) - \frac{1}{2} (\tilde{\mu}_\lambda(s))^2 ds + \frac{1}{2} \int_0^t \frac{1}{s+\lambda} ds - \frac{1}{2} \log \frac{t+\lambda}{\lambda} \\ &= \int_0^t \tilde{\mu}_\lambda(s) d\tilde{S}(s) - \frac{1}{2} (\tilde{\mu}_\lambda(s))^2 ds, \end{aligned}$$

where the second equality follows from application of Itô's lemma to f_λ (Lemma 10). □

The proof of Proposition 3 is just algebra. We present it below.

Proof of Proposition 3. From Itô's lemma applied to f_λ (Proposition 3) we have that

$$\begin{aligned} \tilde{Y}_\lambda(t) - \tilde{Y}_{\mu_0}^{\text{svs}}(t) &= \int_0^t \tilde{\mu}_\lambda(s) d\tilde{S}(s) - \frac{1}{2} (\tilde{\mu}_\lambda(s))^2 ds - \left(\mu_0 W(t) + \frac{1}{2} \mu_0^2 t \right) \\ &= \int_0^t \tilde{\mu}_\lambda(s) \frac{\mu_0 s + W(s)}{s+\lambda} (\mu_0 ds + dW(s)) - \frac{1}{2} \left(\frac{\mu_0 s + W(s)}{s+\lambda} \right)^2 ds \\ &\quad - \left(\mu_0 W(t) + \frac{1}{2} \mu_0^2 t \right) \\ &= \tilde{R}_{\lambda, \mu_0}^{\text{skg}, 1}(\mu_0, t) + \tilde{R}_{\lambda, \mu_0}^{\text{skg}, 2}(\mu_0, t) + \tilde{R}_{\lambda, \mu_0}^{\text{skg}, 3}(\mu_0, t) + \tilde{R}_\lambda^{\text{adpt}}(t), \end{aligned}$$

with

$$\begin{aligned}\widetilde{R}_{\lambda,\mu_0}^{\text{skg},1}(\mu_0, t) &:= \mu_0^2 \int_0^t \frac{s}{s+\lambda} - \frac{s^2}{(s+\lambda)^2} - \frac{1}{2} ds, \\ \widetilde{R}_{\lambda,\mu_0}^{\text{skg},2}(\mu_0, t) &:= \mu_0 \int_0^t \left(\frac{s}{s+\lambda} - 1 \right) dW(s), \\ \widetilde{R}_{\lambda,\mu_0}^{\text{skg},3}(\mu_0, t) &:= \mu_0 \int_0^t W(s) \left(\frac{1}{s+\lambda} - \frac{s}{(s+\lambda)^2} \right) ds, \\ \widetilde{R}_\lambda^{\text{adpt}}(t) &:= \int_0^t \frac{W(s)dW(s)}{s+\lambda} - \frac{1}{2} \left(\frac{W(s)}{s+\lambda} \right)^2 ds.\end{aligned}$$

We can further simplify the first three terms. We have that

$$\begin{aligned}\widetilde{R}_{\lambda,\mu_0}^{\text{skg},1}(\mu_0, t) &= \frac{\mu_0^2}{2} \int_0^t \frac{2s(s+\lambda) - s^2 - (s+\lambda)^2}{(s+\lambda)^2} ds \\ &= -\frac{\mu_0^2 \lambda^2}{2} \int_0^t \frac{1}{(s+\lambda)^2} ds \\ &= -\frac{1}{2} \mu_0^2 \lambda^2 \left(\frac{1}{\lambda} - \frac{1}{t+\lambda} \right) \\ &= -\frac{1}{2} \mu_0^2 \lambda \frac{t}{t+\lambda}, \\ \widetilde{R}_{\lambda,\mu_0}^{\text{skg},2}(\mu_0, t) &= \mu_0 \lambda \int_0^t \frac{dW(s)}{s+\lambda}, \\ \text{and } \widetilde{R}_{\lambda,\mu_0}^{\text{skg},3}(\mu_0, t) &= \mu_0 \lambda \int_0^t \frac{W(s)}{(s+\lambda)^2} ds.\end{aligned}$$

□

A.2.2 Proof of the excess terms bounds (Lemma 1)

Proof of Lemma 1. The bounds on $E[\widetilde{R}_{\lambda,\mu_0}^{\text{skg},1}(\mu_0, \tilde{\tau}_{\alpha,\lambda})]$ are trivial. The fact that $E[\widetilde{R}_{\lambda,\mu_0}^{\text{skg},2}(\mu_0, \tilde{\tau}_{\alpha,\lambda})]$ follows directly from the optional stopping theorem. Bounding the other two terms require more work.

Bound on $|E[\widetilde{R}_{\lambda,\mu_0}^{\text{skg},3}(\mu_0, \tilde{\tau}_{\alpha,\lambda})]|$. By Itô integration by part, we have that

$$(\mu_0 \lambda)^{-1} \widetilde{R}_{\lambda,\mu_0}^{\text{skg},3}(\mu_0, \tilde{\tau}_{\alpha,\lambda}) = \frac{W(\tilde{\tau}_{\alpha,\lambda})}{\tilde{\tau}_{\alpha,\lambda} + \lambda} + \int_0^{\tilde{\tau}_{\alpha,\lambda}} \frac{dW(u)}{u + \lambda}.$$

Since the second term is a martingale with expectation zero at 0, it has expectation zero at $\tilde{\tau}_{\alpha,\lambda}$ from the optional stopping theorem. By Jensen's inequality,

$$\left| E \left[\frac{W(\tilde{\tau}_{\alpha,\lambda})}{\tilde{\tau}_{\alpha,\lambda} + \lambda} \right] \right| \leq E \left[\left(\frac{W(\tilde{\tau}_{\alpha,\lambda})}{\tilde{\tau}_{\alpha,\lambda} + \lambda} \right)^2 \right]^{\frac{1}{2}}.$$

Let, for any $t > 0$,

$$Z_1(t) := \frac{W(t)}{(t+\lambda)^2} - \int_0^t \frac{1}{(u+\lambda)^2} du.$$

For any $0 \leq s < t$, we have that

$$E \left[Z_1(t) \mid \mathcal{F}_s \right] = \frac{W(s)^2}{(t+\lambda)^2} + \frac{t-s}{(t+\lambda)^2} - \int_s^t \frac{1}{(u+\lambda)^2} du - \int_0^2 \frac{1}{(u+\lambda)^2} du$$

$$\begin{aligned} &\leq \frac{W(s)}{(s+\lambda)^2} - \int_0^2 \frac{1}{(u+\lambda)^2} du \\ &= Z_1(s), \end{aligned}$$

which shows that $(Z_1(t))_{t \in \mathbb{R}_+}$ is a supermartingale. Therefore, from the optional stopping theorem,

$$\begin{aligned} \left| E \left[\tilde{R}_{\lambda, \mu_0}^{\text{skg}, 3}(\mu_0, \tilde{\tau}_{\alpha, \lambda}) \right] \right| &\leq \mu_0 \lambda \sqrt{\int_0^\infty \frac{1}{(u+\lambda)^2} du} \\ &= \mu_0 \sqrt{\lambda}. \end{aligned}$$

Bounds on $E[\tilde{R}_\lambda^{\text{adpt}}(\tilde{\tau}_{\alpha, \lambda})]$. From the first representation of $\tilde{R}_\lambda^{\text{adpt}}$ in Proposition 3, it can be observed that the process defined for all $t \in \mathbb{R}_+$ by $Z_2(t) := \exp(\tilde{R}_\lambda^{\text{adpt}}(t))$ is martingale satisfying $Z_2(0) = 1$. Therefore, from Jensen's inequality

$$E \left[\tilde{R}_\lambda^{\text{adpt}}(\tilde{\tau}_{\alpha, \lambda}) \right] \leq 0.$$

Leveraging the second representation of $\tilde{R}_\lambda^{\text{adpt}}$ in Proposition 3 and the fact that $W^2(\tilde{\tau}_{\alpha, \lambda}) \geq 0$, we have that

$$\begin{aligned} E \left[\tilde{R}_\lambda^{\text{adpt}}(\tilde{\tau}_{\alpha, \lambda}) \right] &\geq -\frac{1}{2} E \left[\log \frac{\tilde{\tau}_{\alpha, \lambda} + \lambda}{\lambda} \right] \\ &\geq -\frac{1}{2} \frac{E[\tilde{\tau}_{\alpha, \lambda}] + \lambda}{\lambda}, \end{aligned}$$

where the second inequality follows from the convexity of $x \mapsto -\log((x+\lambda)/\lambda)$ and Jensen's inequality. \square

A.3 Proof of Theorem 2

Proof of Theorem 2. Under Assumption 1 with $f(x) = x^{1-5\kappa}$, we can enlarge the probability space such that there exists a standard Brownian motion $(W(t))_{t \in \mathbb{R}_+}$ such that

$$\Delta_t := S_t - W(t) = o(t^{\frac{1}{2}-\frac{5}{4}\kappa}) = o(t^{\frac{1}{2}-\kappa}) \text{ a.s.}$$

Therefore,

$$P \left[\sup_{t \geq t_0} \mu S_t - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] = P \left[\sup_{t \geq t_0} \mu S_t + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha \right].$$

Our task is now to show that this latter quantity converges to α as $t_0 \rightarrow \infty$, $\mu \sqrt{t_0} \rightarrow 0$, $\mu t_0^{1/2+\kappa} \rightarrow \infty$.

Let $\epsilon > 0$. For any $t_0 \geq 1$, we introduce the event

$$\mathcal{E}(t_0) := \sup \left\{ \sup_{t \geq t_0} \frac{|\Delta_t|}{t^{1/2+\kappa}} \leq 1 \right\}.$$

From the almost sure convergence rate of Δ_t , there exists $t_{0,0}$ such that, for any $t_{0,0}$, $P[\mathcal{E}(t_0)] \geq 1 - \epsilon$.

Upper bound. For $t_0 \geq t_{0,0}$, we have that

$$\begin{aligned} &P \left[\sup_{t \geq t_0} \mu W(t) + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] \\ &\leq P \left[\sup_{t \geq t_0} \mu W(t) + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha, \mathcal{E}(t_0) \right] + \epsilon \\ &\leq P \left[\sup_{t \geq t_0} \mu W(t) + \mu t^{1/2-\kappa} - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] + \epsilon, \end{aligned}$$

where we have used in the first inequality, that for any two events A and B , $P[A] = P[A \cap B] + P[A \cap \bar{B}] \leq P[A \cap B] + P[\bar{B}]$. We focus on the first term in the last line of the previous display. We have that

$$\begin{aligned}
& P \left[\sup_{t \geq t_0} \mu W(t) + \mu t^{1/2-\kappa} - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] \\
&= P \left[\sup_{t \geq t_0} \mu W(t) - \frac{1}{2} \mu^2 t \left(1 - \frac{2}{\mu t^{1/2+\kappa}} \right) \geq -\log \alpha \right] \\
&\leq P \left[\sup_{t \geq t_0} \mu W(t) - \frac{1}{2} \mu^2 t \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \geq -\log \alpha \right] \\
&= P \left[\sup_{t \geq t_0} \mu(W(t) - W(t_0)) - \frac{1}{2} \mu^2(t - t_0) \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \geq -\log \alpha - \left(\mu W(t_0) - \frac{1}{2} \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \right) \right] \\
&= \int_{-\infty}^{\frac{1}{\mu} \left(-\log \alpha + \frac{\mu^2 t_0}{2} \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \right)} \exp \left(- \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \left(-\log \alpha + \frac{1}{2} \mu^2 t_0 \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) - \mu w_{t_0} \right) \right) \\
&\quad \times \frac{1}{\sqrt{2\pi t_0}} \exp \left(-\frac{1}{2} \frac{w_{t_0}^2}{t_0} \right) dw_{t_0} \\
&\quad + \int_{\frac{1}{\mu} \left(-\log \alpha + \frac{1}{2} \mu^2 t_0 \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \right)}^{\infty} \frac{1}{\sqrt{2\pi t_0}} \exp \left(-\frac{1}{2} \frac{w_{t_0}^2}{t_0} \right) dw_{t_0} \\
&= \exp \left(- \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) (-\log \alpha) \right) \Phi \left(\frac{1}{\mu \sqrt{t_0}} \left(-\log \alpha - \frac{1}{2} \mu^2 t_0 \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \right) \right) \\
&\quad + (1 - \Phi) \left(\frac{1}{\mu \sqrt{t_0}} \left(-\log \alpha + \frac{1}{2} \mu^2 t_0 \left(1 - \frac{2}{\mu t_0^{1/2+\kappa}} \right) \right) \right) \\
&\rightarrow \alpha \text{ as } t_0 \rightarrow \infty, \mu \sqrt{t_0} \rightarrow 0, \mu t_0^{1/2+\kappa} \rightarrow \infty.
\end{aligned}$$

Therefore, for any $\epsilon > 0$, we have that

$$\limsup_{\substack{t_0 \rightarrow \infty \\ \mu \sqrt{t_0} \rightarrow 0 \\ \mu t_0^{1/2+\kappa}}} P \left[\sup_{t \geq t_0} \mu W(t) + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] \leq \alpha + \epsilon.$$

Taking the limit as $\epsilon \rightarrow 0$, we obtain

$$\limsup_{\substack{t_0 \rightarrow \infty \\ \mu \sqrt{t_0} \rightarrow 0 \\ \mu t_0^{1/2+\kappa}}} P \left[\sup_{t \geq t_0} \mu W(t) + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] \leq \alpha.$$

Lower bound. For $t_0 \geq t_{0,0}$, we have that

$$\begin{aligned}
& P \left[\sup_{t \geq t_0} \mu W(t) + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] \\
&\geq P \left[\sup_{t \geq t_0} \mu W(t) + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha, \mathcal{E}(t_0) \right] \\
&\geq P \left[\sup_{t \geq t_0} \mu W(t) - \mu t^{1/2-\kappa} - \frac{1}{2} \mu^2 t \geq -\log \alpha, \mathcal{E}(t_0) \right] \\
&\geq P \left[\sup_{t \geq t_0} \mu W(t) - \frac{1}{2} \mu^2 t \left(1 + \frac{2}{\mu t^{1/2+\kappa}} \right) \geq -\log \alpha \right] - \epsilon
\end{aligned}$$

$$\geq P \left[\sup_{t \geq t_0} \mu W(t) - \frac{1}{2} \mu^2 t \left(1 + \frac{2}{\mu t_0^{1/2+\kappa}} \right) \geq -\log \alpha \right] - \epsilon,$$

where the before last line follows from the fact that for any two events A and B , $P[A \cap B] = 1 - P[\bar{A} \cup \bar{B}] \geq 1 - P[\bar{A}] - P[\bar{B}] = P[A] - P[\bar{B}]$. We then proceed as for the upper bound and obtain

$$\liminf_{\substack{t_0 \rightarrow \infty \\ \mu \sqrt{t_0} \rightarrow 0 \\ \mu t_0^{1/2+\kappa}}} P \left[\sup_{t \geq t_0} \mu W(t) + \mu \Delta_t - \frac{1}{2} \mu^2 t \geq -\log \alpha \right] \geq \alpha.$$

□

A.4 Proof of Theorem 3

A.4.1 Statement of intermediate lemmas

Without loss of generality, we restrict ourselves in the proofs to the case $\mu > 0$. For any $\mu > 0$ let

$$\begin{aligned} \tilde{\tau}(\mu) &:= \arg \max_{t \in \mathbb{N}} \tilde{Y}_\mu^{\text{svs}}(t) = \arg \max_{t \in \mathbb{N}} \frac{\tilde{Y}_\mu^{\text{svs}}(t)}{\mu}, \\ \tau(\mu) &:= \arg \max_{t \in \mathbb{N}} Y_{\mu,t}^{\text{svs}} = \arg \max_{t \in \mathbb{N}} \frac{Y_{\mu,t}^{\text{svs}}}{\mu} \end{aligned}$$

Lemma 12. For any $t \in \{1, 2, \dots\}$, $\tilde{\tau}(\mu)$ and $\tau(\mu)$ are non-increasing functions of μ

Lemma 13. As $\mu \rightarrow 0$, it holds P_0 -almost surely that

$$\tilde{\tau}(\mu) = O(\mu^{-2} \log \log(\mu^{-2})), \quad \text{and} \quad \tau(\mu) = O(\mu^{-2} \log \log(\mu^{-2})).$$

Lemma 14. As $\mu \rightarrow 0$, $\tilde{\tau}(\mu)$ and $\tau(\mu)$ diverge to positive infinity P_0 -almost surely.

A.4.2 Proofs of intermediate lemmas

Proof of Lemma 12. Let $0 < \mu_2 < \mu_1$. For $i = 1, 2$, $\tau_i = \tau(\mu_i)$. From the definition of τ , we have that

$$\begin{aligned} \frac{Y_{\mu_1, \tau_1}^{\text{svs}}}{\mu_1} &= S_{\tau_1} - \frac{1}{2} \mu_1 \tau_1 \geq S_{\tau_2} - \frac{1}{2} \mu_1 \tau_2 = \frac{Y_{\mu_1, \tau_2}^{\text{svs}}}{\mu_1} \\ \text{and } \frac{Y_{\mu_2, \tau_1}^{\text{svs}}}{\mu_2} &= S_{\tau_1} - \frac{1}{2} \mu_2 \tau_1 \leq S_{\tau_2} - \frac{1}{2} \mu_2 \tau_2 = \frac{Y_{\mu_2, \tau_2}^{\text{svs}}}{\mu_2}. \end{aligned}$$

Taking the difference between the two lines in the display yields $(\mu_2 - \mu_1)(\tau_1 - \tau_2) \geq 0$, which implies that $\tau(\mu_2) > \tau(\mu_1)$.

The proof is the same for $\tilde{\tau}$. □

Proof of Lemma 13. From the law of the iterated logarithm, there exists a random positive constant C that depends only on the realization of the sample path of W , such that, for all $\mu > 0$,

$$W(\tilde{\tau}(\mu)) \leq C \sqrt{2\tilde{\tau}(\mu) \log \log \tilde{\tau}(\mu)}.$$

From the optimality of $\tau(\mu)$, it holds that

$$\frac{\tilde{Y}_\mu^{\text{svs}}(\tilde{\tau}(\mu))}{\mu} \geq \frac{\tilde{Y}_\mu^{\text{svs}}(0)}{\mu},$$

that is

$$W(\tilde{\tau}(\mu)) - \frac{1}{2} \mu \tilde{\tau}(\mu) \geq 0.$$

Plugging inequality Appendix A.4.2, the above display implies that

$$\frac{\tilde{\tau}(\mu)}{\sqrt{2\tilde{\tau}(\mu) \log \log \tilde{\tau}(\mu)}} \leq \frac{2C}{\mu} \text{ a.s.},$$

and therefore

$$\tilde{\tau}(\mu) = O(\mu^{-2} \log \log(\mu^{-2})) \text{ a.s.}$$

Similarly, for τ , Assumption 1, from Proposition 8 and the law of the iterated logarithm, that

$$S_{\tau(\mu)} \leq C' \sqrt{2\tilde{\tau}(\mu) \log \log \tau(\mu)}.$$

for some C' that only depends on the realization of $(W(t))_{t \in \mathbb{R}_+}$. The rest of the proof is identical. \square

Proof of Lemma 14. Fix a realization of $(W(t))_{t \geq 0}$. Let $T_0 > 0$, and let $A := \max_{t \in [0, T_0]} W(t)$. From the fact that $\limsup_{t \rightarrow \infty} W(t) = \infty$, there exists $T_1 > T_0$ such that $W(T_1) > 2A$.

Let μ'_1 be small enough that $W(T_1) - \frac{1}{2}\mu'_1 T_1 > A$. Then, for any $0 \leq \mu' \leq \mu'_1$,

$$W(T_1) - \mu' T_1 > A = \max_{t \in [0, T_0]} W(t) > \max_{t \in [0, T_0]} W(t) - \mu' t.$$

We must therefore have that for any $0 < \mu' \leq \mu'_1$, $\arg \max_{t \geq 0} W(t) - \mu' t > T_0$. \square

A.4.3 Proof of Theorem 3

Proof. Let $\Delta_t := S_t - W_t$. From Proposition 8, Assumption 1 implies that $\Delta_t = o(t^{1/2-\kappa})$ almost surely as $t \rightarrow \infty$. Therefore, using Lemma 13, it holds almost surely that, as $\mu' \rightarrow 0$,

$$\begin{aligned} Y_{\mu', \tau(\mu')}^{\text{svs}} - \tilde{Y}_{\mu'}^{\text{svs}}(\tau(\mu')) &= \mu' \Delta_{\tau(\mu')} \\ &= O(\mu' \tau(\mu')^{1/2-\kappa}) \\ &= O(\mu' (\mu'^{-2} \log \log(\mu'^{-2}))^{1/2-\kappa}) \\ &= O(\mu'^{2\kappa} (\log \log(\mu'^{-2})^{1/2-\kappa})) \\ &= o(1). \end{aligned}$$

Similarly, $Y_{\mu', \tilde{\tau}(\mu')}^{\text{svs}} - \tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu')) = o(1)$ as $\mu' \rightarrow 0$, almost surely.

Observe that, since for any fixed μ' , $P[\tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu')) \geq -\log \alpha] = \alpha$, $\tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu'))$ must have the same $\text{Exp}(1)$ distribution for all μ' .

From the fact that $\tau(\mu')$ maximizes $Y_{\mu', t}^{\text{svs}}$ w.r.t. t , we have that

$$\begin{aligned} P[Y_{\mu', \tau(\mu')}^{\text{svs}} \geq -\log \alpha] &\geq P[Y_{\mu', \tilde{\tau}(\mu')}^{\text{svs}} \geq -\log \alpha] \\ &= P[\tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu')) + o_{a.s.}(1) \geq -\log \alpha]. \end{aligned}$$

Since $\tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu')) + o_{a.s.}(1) \rightsquigarrow \text{Exp}(1)$, we have that

$$\lim_{\mu' \rightarrow 0} P[\tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu')) + o_{a.s.}(1) \geq -\log \alpha] = \alpha,$$

and therefore

$$\liminf_{\mu' \rightarrow 0} P[Y_{\mu', \tau(\mu')}^{\text{svs}} \geq -\log \alpha] \geq \alpha.$$

It remains to upper bound $\limsup_{\mu' \rightarrow 0} P[Y_{\mu', \tau(\mu')}^{\text{svs}} \geq -\log \alpha]$. We have that

$$\limsup_{\mu' \rightarrow 0} P[Y_{\mu', \tau(\mu')}^{\text{svs}} \geq -\log \alpha] = \limsup_{\mu' \rightarrow 0} P[\tilde{Y}_{\mu'}^{\text{svs}}(\tau(\mu')) + (Y_{\mu', \tau(\mu')}^{\text{svs}} - \tilde{Y}_{\mu'}^{\text{svs}}(\tau(\mu'))) \geq -\log \alpha]$$

$$\begin{aligned}
&\leq \limsup_{\mu' \rightarrow 0} P \left[\tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu')) + (Y_{\mu', \tau(\mu')}^{\text{svs}} - \tilde{Y}_{\mu'}^{\text{svs}}(\tau(\mu'))) \geq -\log \alpha \right] \\
&= \limsup_{\mu' \rightarrow 0} P \left[\tilde{Y}_{\mu'}^{\text{svs}}(\tilde{\tau}(\mu')) + o_{a.s.}(1) \geq -\log \alpha \right] \\
&= P [\text{Exp}(1) \geq -\log \alpha] \\
&= \alpha,
\end{aligned}$$

where the inequality in the above display follows from the fact that $\tilde{\tau}(\mu')$ maximizes $\tilde{Y}_{\mu'}^{\text{svs}}(t)$ w.r.t. t .

We have thus shown that

$$\lim_{\mu' \rightarrow 0} P \left[Y_{\mu', \tau(\mu')}^{\text{svs}} \geq -\log \alpha \right] = \alpha.$$

□

A.5 Proof of the expected rejection time result in the non-parametric simple-vs-simple case (Theorem 4)

Lemma 15. *Let $\beta > 0$. Make Assumptions 1 and 2. Then, as $t_0 \rightarrow \infty$, $\mu' \sqrt{t_0 (\log t_0)^{1+\beta}} \rightarrow 0$,*

$$E [(-\log \alpha - Y_{t_0})_+] = \mu' \sqrt{t_0} h \left(\frac{-\log \alpha - C_1 t_0}{\mu' \sqrt{t_0}} \right) + o(\mu' \sqrt{t_0}) + \mu' \mu_0 o(t_0)$$

Proof. We decompose the expectation as

$$\begin{aligned}
&E [(-\log \alpha - Y_{t_0})_+] \\
&= E \left[\left(-\log \alpha - \tilde{Y}_{t_0} \right)_+ \right] + \left(E [(-\log \alpha - Y_{t_0})_+] - E \left[\left(-\log \alpha - \tilde{Y}_{t_0} \right)_+ \right] \right).
\end{aligned}$$

Bounding the second term in Appendix A.5. We have that

$$\begin{aligned}
&\left| E [(-\log \alpha - Y_{t_0})_+] - E \left[\left(-\log \alpha - \tilde{Y}_{t_0} \right)_+ \right] \right| \\
&\leq E \left[|Y_{t_0} - \tilde{Y}_{t_0}| \right] \\
&\leq \mu' \mu_0 E \left[\sum_{s=1}^{t_0} \frac{|\mu_s - \mu_0|}{\mu_0} \right] + \mu' \sqrt{t_0} E \left[\frac{|S_{t_0}^0 - W_{t_0}|}{\sqrt{t_0}} \right].
\end{aligned}$$

From Assumption 2, $E[|\mu_t - \mu_0|/\mu_0] \rightarrow 0$ as $t \rightarrow \infty$. Thus, from Cesàro's lemma,

$$\frac{1}{t_0} \sum_{s=1}^{t_0} E \left[\frac{|\mu_s - \mu_0|}{\mu_0} \right] = o(1)$$

as $t_0 \rightarrow \infty$, and therefore the first term in Appendix A.5 is $\mu' \mu_0 o(t_0)$. We now turn to the second term in Appendix A.5. Let us show that $(|S_t^0|/\sqrt{t})_{t \geq 1}$ is uniformly integrable. For any $K > 0$,

$$E \left[\frac{|S_t^0|}{\sqrt{t}} \mathbf{1} \left\{ \frac{|S_t^0|}{\sqrt{t}} > K \right\} \right] = K P \left[\frac{|S_t^0|}{\sqrt{t}} > K \right] + \int_K^\infty P \left[\frac{|S_t^0|}{\sqrt{t}} > x \right] dx.$$

Since Assumption 1 implies via Proposition 8 that $|S_t^0|/\sqrt{t} \rightsquigarrow \mathcal{N}(0, 1)$ as $t \rightarrow \infty$, the dominated convergence theorem yields that

$$\lim_{t \rightarrow \infty} E \left[\frac{|S_t^0|}{\sqrt{t}} \mathbf{1} \left\{ \frac{|S_t^0|}{\sqrt{t}} > K \right\} \right] = K(1 - \Phi)(K) + \int_K^\infty P[\mathcal{N}(0, 1) > x] dx.$$

The right-hand side converges to 0 as K diverges to ∞ , which implies the uniform integrability of the sequence $(|S_t^0|/\sqrt{t})_{t \in \mathbb{R}_+}$. The sequence $(|W(t)/\sqrt{t}|)_{t \in \mathbb{R}_+}$ is trivially uniformly integrable. From Proposition 8 $|S_t^0 - W(t)|/\sqrt{t}$ converges to 0 almost surely. Therefore, Vitali's convergence theorem implies that

$$\lim_{t_0 \rightarrow \infty} \left[\frac{|S_{t_0}^0 - W_{t_0}|}{\sqrt{t_0}} \right] = 0.$$

Explicit expression for the first term in Appendix A.5. Denoting $C_3 := (-\log \alpha - C_1 t_0)/(\mu' \sqrt{t_0})$, we have that

$$\begin{aligned} E \left[\left(-\log \alpha - \tilde{Y}_{t_0} \right)_+ \right] &= \mu' \sqrt{t_0} E \left[\left(C_3 - \frac{W(t_0)}{\sqrt{t_0}} \right)_+ \right] \\ &= \mu' t_0 \int_{-\infty}^{C_3} (C_3 - x) \exp \left(-\frac{x^2}{2} \right) dx \\ &= \mu' \sqrt{t_0} h(C_3). \end{aligned}$$

□

Proof of Theorem 4. By definition of τ ,

$$Y_{\mu', (\tau-1) \vee 0}^{\text{svs}} - Y_{\mu', t_0}^{\text{svs}} \leq -\log \alpha - Y_{\mu', t_0}^{\text{svs}} \leq Y_{\mu', \tau}^{\text{svs}} - Y_{\mu', t_0}^{\text{svs}}.$$

We take expectation of each three quantities so as to make appear $E[\tau]$. We characterized the expectation of the middle quantity in the preceding lemma. We turn to the expectation of the quantity on the right. We have that

$$E [Y_{\mu', \tau}^{\text{svs}} - Y_{\mu', t_0}^{\text{svs}}] = \mu' E [S_\tau^0 - S_{t_0}^0] + \mu' \mu_0 \sum_{s=t_0}^{\tau} \frac{\mu_s - \mu_0}{\mu_0} + C_1 E[\tau - t_0].$$

The first term equals zero from the optional stopping theorem. We bound the second term as follows:

$$\begin{aligned} \left| E \left[\sum_{s=t_0}^{\tau} \frac{\mu_s - \mu_0}{\mu_0} \right] \right| &\leq E \left[\sup_{t \geq 1} t^\gamma \frac{|\mu_t - \mu_0|}{\mu_0} \sum_{s=t_0}^{\tau} s^{-\gamma} \right] \\ &\leq \frac{1}{1-\gamma} E \left[\sup_{t \geq 1} t^\gamma \frac{|\mu_t - \mu_0|}{\mu_0} (\tau - t_0)^{1-\gamma} \right] \\ &\leq \frac{M_\mu}{1-\gamma} E[\tau - t_0]^{1-\gamma}, \end{aligned}$$

where the last inequality follows from Assumption 2 and Hölder's inequality. The left hand side is handled exactly in the same way. Combining with the expression from Lemma 15 for the expectation of the middle quantity yields the claim. □

A.6 Proof of the representation result for the uSPRT test statistic (Theorem 5)

Proof of Theorem 5. By definition of R^{bias} , and by application of Lemma 11, we have that

$$\begin{aligned} Y_{\lambda, t} &= \frac{1}{2} \left(\frac{\check{S}_t}{t+\lambda} - \log \frac{t+\lambda}{\lambda} \right) \\ &= \sum_{s=0}^{t-1} \rho_{\lambda, s} \left(\check{\mu}_{\lambda, s} \check{X}_{s+1} - \frac{1}{2} (\check{\mu}_{\lambda, s})^2 \right) - \Delta_\lambda^{\text{qvar}}(\check{X}_{0:t}) + R_t^{\text{bias}} \\ &= \sum_{s=0}^{t-1} \frac{s+\lambda}{s+1+\lambda} \left(\frac{\mu_0 s + S_s^0}{s+\lambda} (\mu_0 + X_{s+1}^0) - \frac{1}{2} \left(\frac{\mu_0 s + S_s^0}{s+\lambda} \right)^2 \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{s=1}^t \frac{(\mu_0 + X_s^0)^2}{s+\lambda} - \frac{1}{2} \log \frac{t+\lambda}{\lambda} + R_t^{\text{bias}} \\
& = Y_{\mu_0,t}^{\text{svs}} + R_t^{\text{bias}} + R_t^{\text{skg},1} + R_t^{\text{skg},2} + R_t^{\text{skg},3} + R_{\lambda,t}^{\text{np,adpt}} - \Delta_{\lambda}^{\text{qvar}}(X_{0:t}^0).
\end{aligned}$$

with

$$\begin{aligned}
R_t^{\text{skg},1} & = \mu_0^2 \sum_{s=0}^{t-1} \frac{s}{s+1+\lambda} - \frac{1}{2} \frac{s^2}{(s+\lambda)(s+1+\lambda)} + \frac{1}{2} \frac{1}{s+1+\lambda} - \frac{1}{2} \\
R_t^{\text{skg},2} & = \mu_0 \sum_{s=0}^{t-1} X_{s+1}^0 \left(\frac{s}{s+1+\lambda} + \frac{1}{s+1+\lambda} - 1 \right) \\
R_t^{\text{skg},3} & = \mu_0 \sum_{s=0}^{t-1} S_s^0 \left(\frac{1}{s+1+\lambda} - \frac{s}{(s+\lambda)(s+1+\lambda)} \right) \\
R_{\lambda,t}^{\text{np,adpt}} & = \sum_{s=0}^{t-1} \rho_{\lambda,s} \left(\hat{\mu}_{\lambda,t}^0 X_{s+1}^0 - \frac{1}{2} (\hat{\mu}_{\lambda,t}^0)^2 \right).
\end{aligned}$$

We can further simplify these terms as follows:

$$\begin{aligned}
R_t^{\text{skg},1} & = \frac{\mu_0^2}{2} \sum_{s=0}^{t-1} \frac{2s(s+\lambda) - s^2 + (s+\lambda) - (s+\lambda)(s+1+\lambda)}{(s+\lambda)(s+1+\lambda)} \\
& = -\frac{\mu_0^2 \lambda^2}{2} \sum_{s=0}^{t-1} \frac{1}{(s+\lambda)(s+1+\lambda)} \\
& = -\frac{\mu_0^2 \lambda^2}{2} \left(\frac{1}{\lambda} - \frac{1}{t+\lambda} \right) \\
& = -\frac{\mu_0^2 \lambda}{2} \frac{t}{t+\lambda},
\end{aligned}$$

$$R_t^{\text{skg},2} = -\mu_0 \lambda \sum_{s=1}^t \frac{X_s^0}{s+\lambda}$$

$$R_t^{\text{skg},3} = \mu_0 \lambda \sum_{s=0}^{t-1} \frac{S_s^0}{(s+\lambda)(s+1+\lambda)},$$

and

$$R_{\lambda,t}^{\text{np,adpt}} - \Delta_{\lambda,t}^{\text{qvar}}(X_{0:t}^0) = \frac{1}{2} \left(\frac{(S_t^0)^2}{t+\lambda} - \frac{1}{2} \log \frac{t+\lambda}{\lambda} \right),$$

where the last equality follows from Lemma 11. \square

A.7 Proof of the type-I error theorem for uSPRT (Theorem 6)

A.7.1 Proof of Proposition 5

Proof. Under the null hypothesis, $Y_t = 1/2((S_t^0)^2/(t+\lambda) - \log((t+\lambda)/\lambda))$ and $\tilde{Y}_t = 1/2(W(t)^2/(t+\lambda) - \log((t+\lambda)/\lambda))$. Therefore,

$$|\sup_{t \geq t_0} Y_t - \sup_{t \geq t_0} \tilde{Y}_t| \leq \sup_{t \geq t_0} \frac{1}{2} \frac{|S_t^0 - W(t)|^2}{2}$$

$$\begin{aligned} &\leq \frac{1}{2} \sup_{t \geq t_0} \frac{|S_t^0 - W(t)|}{\sqrt{t}} \frac{|S_t^0 - W(t)| + |W(t)|}{\sqrt{t}} \\ &= o(1) \end{aligned}$$

P_0 -almost surely from Assumption 1 via Proposition 8.

Furthermore

$$\begin{aligned} \sup_{t \geq t_0} \tilde{Y}_t + \frac{1}{2} \log \frac{t_0}{\lambda} &= \frac{1}{2} \sup_{s \geq 1} \frac{W(t_0 s)^2}{t_0 s + \lambda} + \log \frac{t_0 s + \lambda}{t_0} \\ &\sim \frac{1}{2} \sup_{s \geq 1} \frac{W(s)^2}{s + \lambda/t_0} - \log(s + \lambda/t_0) \\ &\rightsquigarrow \frac{1}{2} \sup_{s \geq 1} \frac{W(s)^2}{s} - \log s \end{aligned}$$

as $t_0 \rightarrow \infty$. Therefore,

$$\sup_{t \geq t_0} Y_t + \frac{1}{2} \log \frac{t_0}{\lambda} \rightsquigarrow \frac{1}{2} \sup_{s \geq 1} \frac{W(s)^2}{s} - \log s$$

as $t_0 \rightarrow \infty$. \square

A.7.2 Proof of Proposition 6 (Derivation of $\tilde{\alpha}(\alpha)$)

Proof of Proposition 6. We want to find $x := \log(1/\tilde{\alpha})$ such that

$$P \left[\sup_{t \geq 1} \frac{1}{2} \left(\frac{W(t)^2}{t} - \log t \right) \geq x \right] = \alpha.$$

From Itô's lemma,

$$\begin{aligned} \log M(t) &:= \frac{1}{2} \left(\frac{W(t)^2}{t} - W(1)^2 - \log t \right) \\ &= \int_1^t d \left(\frac{W(s)^2}{2s} \right) - \frac{ds}{2s} \\ &= \int_1^t \frac{W(s)}{s} dW(s) - \frac{1}{2} \left(\frac{W(s)}{s} \right). \end{aligned}$$

We know that, conditional on $W(1)$, the exponential of the last line above is a time continuous martingale on $[1, \infty)$, which satisfies the conditions of lemmata 1 and 2 in Robbins and Siegmund. We have $M(1) = 1$. Therefore

$$\begin{aligned} &P \left[\sup_{t \geq 1} \frac{1}{2} \left(\frac{W(t)^2}{t} - \log t \right) \geq x \right] \\ &= E \left[P \left[\sup_{t \geq 1} \frac{1}{2} \left(\frac{W(t)^2}{t} - W(1)^2 - \log t \right) \geq x - \frac{1}{2} W(1)^2 \mid W(1) \right] \right] \\ &= \int_{-\infty}^{\infty} \frac{\exp(-\frac{1}{2}w_1^2)}{\sqrt{2\pi}} \left\{ \exp \left(- \left(x - \frac{1}{2}w_1^2 \right) \right) \wedge 1 \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-2\sqrt{x}}^{2\sqrt{x}} \exp(-x) dw_1 + 2(1 - \Phi(2\sqrt{x})) \\ &= 2\sqrt{\frac{x}{\pi}} \exp(-x) + 2(1 - \Phi(2\sqrt{x})). \end{aligned}$$

\square

A.7.3 Proof of Theorem 6

Proof. The result follows directly from Propositions 5 and 6. \square

A.8 Proof of the expected rejection time theorem for uSPRT

A.8.1 Proof of the bound on the expected bias term (Lemma 2)

Proof. We have that

$$\begin{aligned} \frac{1}{2} \left| \frac{S_\tau^2 - \check{S}_\tau^2}{t + \lambda} \right| &= \left| \frac{1}{2} \frac{\tau^2}{\tau + \lambda} (\bar{\mu}_\tau - \mu_0)((\bar{\mu}_\tau - \mu_0) + 2\mu_0 + 2\hat{\mu}_\tau^0) \right| \\ &\leq \frac{\mu_0^2}{2} \tau \frac{|\bar{\mu}_\tau - \mu_0|^2}{\mu_0^2} + \mu_0^2 \tau \frac{|\bar{\mu}_\tau - \mu_0|}{\mu_0} + \mu_0^2 \tau \frac{|\bar{\mu}_\tau - \mu_0|}{\mu_0} \frac{\hat{\mu}_\tau^0}{\mu_0}. \end{aligned}$$

We treat each of the six terms separately.

First term. From Hölder's inequality and Assumption 3,

$$\begin{aligned} E_0 \left[\tau \frac{|\mu_\tau - \mu_0|^2}{\mu_0^2} \right] &\leq E_0 \left[\left(\sup_{t \geq 1} t^{\gamma/2} \frac{|\bar{\mu}_t - \mu_0|}{\mu_0} \right)^2 \tau^{1-\gamma} \right] \\ &\leq E_0 \left[\left(\sup_{t \geq 1} t^{\gamma/2} \frac{|\bar{\mu}_t - \mu_0|}{\mu_0} \right)^{2/\gamma} \right]^\gamma E_0[\tau]^{1-\gamma} \\ &= M_{\bar{\mu}, \gamma/2}^2 E_0[\tau]^{1-\gamma}. \end{aligned}$$

Second term. From Hölder's inequality and Assumption 3,

$$E_0 \left[\tau \frac{|\bar{\mu}_\tau - \mu_0|}{\mu_0} \right] \leq M_{\bar{\mu}, \gamma} E_0[\tau]^{1-\gamma}.$$

Third term. Factoring out $t^{\gamma/2}$ twice, and using in that order Hölder's inequality, Cauchy-Schwarz's inequality, Assumptions 3 and 4, we have that

$$\begin{aligned} E_0 \left[\tau \frac{|\bar{\mu}_\tau - \mu_0|}{\mu_0} \hat{\mu}_\tau^0 \right] &\leq E_0 \left[\sup_{t \geq 1} t^{\gamma/2} \frac{|\bar{\mu}_t - \mu_0|}{\mu_0} \sup_{t \geq 1} t^{\gamma/2} \frac{|\hat{\mu}_t^0|}{\mu_0} \tau^{1-\gamma} \right] \\ &\leq E_0 \left[\left(\sup_{t \geq 1} t^{\gamma/2} \frac{|\bar{\mu}_t - \mu_0|}{\mu_0} \sup_{t \geq 1} t^{\gamma/2} \frac{|\hat{\mu}_t^0|}{\mu_0} \right)^{1/\gamma} \right]^\gamma E_0[\tau]^{1-\gamma} \\ &\leq E_0 \left[\left(\sup_{t \geq 1} t^{\gamma/2} \frac{|\bar{\mu}_t - \mu_0|}{\mu_0} \right)^{2/\gamma} \right]^{\gamma/2} E_0 \left[\left(\sup_{t \geq 1} t^{\gamma/2} \frac{|\hat{\mu}_t^0|}{\mu_0} \right)^{2/\gamma} \right]^{\gamma/2} E_0[\tau]^{1-\gamma} \\ &= M_{\bar{\mu}, \gamma/2} M_{\mu^0, \gamma} E_0[\tau]^{1-\gamma}. \end{aligned}$$

\square

A.8.2 Proofs of the bounds on the Brownian approximation terms

Proof of Lemma 3. We have that

$$R^{\text{adpt,SIP}}(S_{0:\tau}^0, W_{0:\tau}) - R^{\text{adpt,SIP}}(S_{0:t_0}^0, W_{0:t_0}) = \frac{1}{2} (A_{\tau, t_0} + B_{\tau, t_0} - C_{\tau, t_0}),$$

with

$$A_{t, t_0} := \frac{(S_t^0)^2}{t + \lambda} - \sum_{s=t_0+1}^t \frac{v_s^2}{s + \lambda} - \frac{(S_{t_0}^0)^2}{t_0 + \lambda},$$

$$B_{t,t_0} := \sum_{s=t_0}^t \frac{v_s^2 - 1}{s + \lambda},$$

$$C_{t,t_0} := \frac{W(t)^2}{t + \lambda} - \sum_{s=t_0}^t \frac{1}{s + \lambda} - \frac{W(t_0)^2}{t_0 + \lambda},$$

where we adopt the convention that $\sum_{s=t_0+1}^{t_0} = 0$.

Bounding $E_0[C_{\tau,t_0}]$. It is straightforward to observe that

$$\left(\frac{W(t)^2}{t + \lambda} - \sum_{s=t_0}^t \frac{1}{s + \lambda} \right)_{t \geq t_0}$$

is a discrete supermartingale conditional on \mathcal{F}_{t_0} . Therefore, from the optional stopping theorem,

$$E_0 [C_{\tau,t_0}] \leq 0.$$

For the lower bound, since $W(t)^2 \geq 0$,

$$\begin{aligned} E_0 [C_{\tau,t_0}] &\geq -E_0 \left[\sum_{s=t_0}^{\tau} \frac{1}{s + \lambda} \right] \\ &\geq -E_0 \left[\log \frac{\tau + \lambda}{t_0 + \lambda} \right] \\ &\geq -\log \frac{E_0[\tau] + \lambda}{t_0 + \lambda}. \end{aligned}$$

Bounding $E_0[B_{\tau,t_0}]$. Using Assumption 5, we obtain

$$|E_0[B_{\tau,t_0}]| \leq E \left[\sup_{t \geq 1} (\log(t + \lambda))^{1+\beta} |v_t^2 - 1| \right] \sum_{t=t_0}^{\infty} \frac{1}{(t + \lambda)} (\log(t + \lambda))^{1+\beta} \leq \frac{M_{v,\lambda,\beta}}{\beta(\log t_0)^{\beta}}.$$

Bounding $E_0[A_{\tau,t_0}]$. It is straightforward to observe that

$$\left(\frac{(S_t^0)^2}{t + \lambda} - \sum_{s=t_0+1}^t \frac{v_s^2}{s + \lambda} \right)_{t \geq t_0}$$

is a supermartingale conditional on \mathcal{F}_{t_0} . The optional stopping theorem thus implies that

$$E_0 [A_{\tau,t_0}] \leq 0.$$

We now turn to the lower bound. Using the non-negativity of $(S_\tau^0)^2$, we have that

$$E_0 [A_{\tau,t_0}] \geq -E_0 [B_{\tau,t_0}] + \sum_{s=t_0}^{\tau} \frac{1}{s + \lambda} \geq -\frac{M_{v,\lambda,\beta}}{\beta(\log t_0)^{\beta}} - \log \frac{E_0[\tau] + \lambda}{t_0 + \lambda}.$$

□

A.8.3 Proof of the bounds on the expected shrinkage terms (Lemma 5)

Proof of lemma Lemma 5. We denote $X_t^0 := X_t - E[X_t | \mathcal{F}_{t-1}]$.

Bounding $E_0[R^{\text{skg},1}(\tau) - R^{\text{skg},1}(t_0)]$. We have that

$$R^{\text{skg},1}(\tau) - R^{\text{skg},1}(t_0) = \frac{1}{2} \lambda \mu_0^2 \left(\frac{\tau}{\tau + \lambda} - \frac{t}{t + \lambda} \right) = \frac{1}{2} \mu_0^2 \lambda^2 \frac{\tau - t}{(\tau + \lambda)(t_0 + \lambda)} \in \left[0, \frac{1}{2} \mu_0^2 \lambda^2 \frac{1}{t_0 + \lambda} \right].$$

Expressing $E_0[R^{\text{skg},2}(\tau) - R^{\text{skg},2}(t_0)]$. Since terms $X^0/(t+\lambda)$ for an MDS, the optional stopping theorem gives us that

$$E_0[R^{\text{skg},2}(\tau) - R^{\text{skg},2}(t_0)] = 0.$$

Bounding $E_0[R^{\text{skg},3}(\tau) - R^{\text{skg},3}(t_0)]$. We have that

$$\begin{aligned} R^{\text{skg},3}(\tau) - R^{\text{skg},3}(t_0) &= \mu_0 \lambda \sum_{s=t_0}^{\tau-1} S_s^0 \left(\frac{1}{s+\lambda} - \frac{1}{s+1+\lambda} \right) \\ &= \mu_0 \lambda \left(-\frac{S_{\tau-1}^0}{\tau+\lambda} + \sum_{s=t_0+1}^{\tau-1} \frac{X_s^0}{s+\lambda} + \frac{S_{t_0}^0}{t_0+\lambda} \right). \end{aligned}$$

The third term trivially has expectation 0. The terms of the sum in the second term form an MDS, and therefore this term has expectation 0 from the optional stopping theorem. We now turn to the first term. From Jensen's inequality,

$$\left| E_0 \left[\frac{S_{\tau-1}^0}{\tau+\lambda} \right] \right| \leq E_0 \left[\left(\frac{S_{\tau-1}^0}{\tau+\lambda} \right)^2 \right]^{1/2}.$$

Adopting the convention that $\sum_{s=t_0+1}^{t_0} = 0$, it is straightforward to observe that

$$\left(\frac{(S_t^0)^2}{(t+1+\lambda)^2} - \sum_{s=t_0+1}^t \frac{v_s^2}{(s+1+\lambda)^2} \right)_{t \geq t_0}$$

is a supermartingale conditional on \mathcal{F}_{t_0-1} , and that the initial term satisfies

$$E_0 \left[\frac{(S_{t_0}^0)^2}{(t_0+1+\lambda)^2} \right] = \frac{t_0}{(t_0+1+\lambda)^2} \leq \frac{1}{t_0+\lambda}$$

The optional stopping theorem then implies that

$$\begin{aligned} E_0 \left[\frac{(S_\tau^0)^2}{(\tau+1+\lambda)^2} \right] &\leq \frac{1}{t_0+\lambda} + E_0 \left[\sum_{s=t_0}^\tau \frac{v_s^2}{(s+1+\lambda)^2} \right] \\ &\leq \frac{1}{t_0+\lambda} + E_0 \left[\sup_{t \geq 1} v_t^2 \right] \times \sum_{s=t_0}^\infty \frac{1}{(s+\lambda)^2} \\ &\leq \frac{1+M_v}{t_0+\lambda}. \end{aligned}$$

Therefore,

$$|E_0[R^{\text{skg},3}(\tau) - R^{\text{skg},3}(t_0)]| \leq \sqrt{\frac{1+M_v}{t_0+\lambda}}.$$

□

A.8.4 Proof of the bound on the Brownian adaptive term (Lemma 6)

Proof of Lemma 6. The proof is nearly identical as that for the adaptive excess from Section 4. We include it here for completeness.

It is straightforward to observe that

$$A_{\lambda,t_0} := \left(\frac{W(t)^2}{t+\lambda} - \frac{W(t_0)^2}{t_0+\lambda} - \log \frac{t+\lambda}{t_0+\lambda} \right)_{t \geq t_0}$$

forms a continuous supermartingale, and that $A_{\lambda,t_0}(t_0) = 0$. The optional stopping theorem then implies that

$$E_0 \left[\frac{W(\tau)^2}{\tau + \lambda} - \log \frac{\tau + \lambda}{t_0 + \lambda} \right] \leq E_0 \left[\frac{W(t_0)^2}{t_0 + \lambda} \right] = \frac{t_0}{t_0 + \lambda}$$

The lower bound follows from the non-negativity of $W(\tau)^2$ and Jensen's inequality. \square

A.8.5 Proof of the expected random threshold results

Proof of Lemma 7. Observe that we can write the uSPRT stopping time as

$$\tau = \min \{t \geq t_0 : Y_{\lambda,t} - Y_{\lambda,t_0} \geq -\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{\lambda,t_0}\},$$

treating $-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{\lambda,t_0}$ as an effective threshold, albeit random. Writing the stopping time this way implies that

$$(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{\lambda,t_0})_+ \leq Y_{\lambda,\tau} - Y_{\lambda,t_0},$$

since, if the threshold were negative then $\tau = t_0$.

If the threshold were positive, τ must be at least $t_0 + 1$, and then by definition,

$$Y_{\lambda,\tau-1} - Y_{\lambda,t_0} \leq (-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{\lambda,t_0})_+.$$

If the threshold is non-positive, τ must be t_0 . Therefore, in both cases,

$$Y_{\lambda,(\tau-1)\vee t_0} - Y_{\lambda,t_0} \leq (-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{\lambda,t_0})_+.$$

\square

Proof of Lemma 8. We have that

$$\begin{aligned} & \left| E_0 [(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{t_0})_+] - E_0 \left[\left(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - \tilde{Y}(t_0) \right)_+ \right] \right| \\ & \leq E_0 [|Y_{t_0} - \tilde{Y}_{t_0}|] + E_0 [| \tilde{Y}_{t_0} - \tilde{Y}(t_0) |]. \end{aligned}$$

The first term is a bias term, the second one a Brownian approximation term.

Brownian approximation term. We have that

$$\begin{aligned} E_0 [| \tilde{Y}_{t_0} - \tilde{Y}(t_0) |] & \leq E_0 \left[\frac{|S_{t_0}^0 - W(t_0)|}{\sqrt{t_0}} \left(\mu_0 \sqrt{t_0} + \frac{W(t_0)}{\sqrt{t_0}} + \frac{1}{2} \frac{S_{t_0}^0 - W(t_0)}{\sqrt{t_0}} \right) \right] \\ & \leq \mu_0 \sqrt{t_0} E \left[\frac{|S_{t_0}^0 - W(t_0)|}{\sqrt{t_0}} \right] + E_0 \left[\frac{W(t_0)}{\sqrt{t_0}} \frac{|S_{t_0}^0 - W(t_0)|}{\sqrt{t_0}} \right] + \frac{1}{2} E \left[\frac{|S_{t_0}^0 - W(t_0)|^2}{t_0} \right]. \end{aligned}$$

From Proposition 8, Assumption 1 implies $t_0^{-1/2} S_{t_0}^0 \rightsquigarrow \mathcal{N}(0, 1)$, which in turn implies that the terms inside the expectations above form uniformly integrable sequences. Assumption 1 and Proposition 8 further imply that the terms inside the expectations converge almost surely to 0. Therefore, from Vitali's convergence theorem,

$$E_0 [| \tilde{Y}_{t_0} - \tilde{Y}(t_0) |] = o(\sqrt{\mu_0 t_0}) + o(1).$$

Bias term. We have that

$$\begin{aligned}
E_0 \left[|Y_{t_0} - \check{Y}_{t_0}| \right] &\leq \frac{1}{2} \mu_0^2 t_0 \left(E_0 \left[\frac{|\bar{\mu}_{t_0} - \mu_0|^2}{\mu_0^2} \right] + 2E_0 \left[\frac{|\bar{\mu}_{t_0} - \mu_0|}{\mu_0} \right] + 2E_0 \left[\frac{|\mu_{t_0} - \mu_0|}{\mu_0} \frac{\hat{\mu}_{t_0}^0}{\mu_0} \right] \right) \\
&\leq \frac{1}{2} \mu_0^2 t_0 \left(E_0 \left[\frac{|\bar{\mu}_{t_0} - \mu_0|^2}{\mu_0^2} \right] + 2E_0 \left[\frac{|\bar{\mu}_{t_0} - \mu_0|}{\mu_0} \right] + 2E_0 \left[\left(\frac{|\bar{\mu}_{t_0} - \mu_0|}{\mu_0} \right)^2 \right]^{\frac{1}{2}} E_0 \left[\left(\frac{\hat{\mu}_{t_0}^0}{\mu_0} \right)^2 \right]^{\frac{1}{2}} \right) \\
&\leq \frac{1}{2} \mu_0^2 t_0^{1-\gamma} \left(E_0 \left[\sup_{t \geq 1} t^\gamma \frac{|\bar{\mu}_t - \mu_0|^2}{\mu_0^2} \right] + 2E_0 \left[\sup_{t \geq 1} t^\gamma \frac{|\bar{\mu}_t - \mu_0|}{\mu_0} \right] \right. \\
&\quad \left. + 2E_0 \left[\left(\sup_{t \geq 1} t^\gamma \frac{|\bar{\mu}_t - \mu_0|}{\mu_0} \right)^2 \right]^{\frac{1}{2}} E_0 \left[\left(\sup_{t \geq 1} t^\gamma \frac{\hat{\mu}_t^0}{\mu_0} \right)^2 \right]^{\frac{1}{2}} \right) \\
&= o(\mu_0^2 t_0),
\end{aligned}$$

where the last line follows from Assumptions 3 and 4. \square

Proof of Lemma 9. We have that

$$\begin{aligned}
&E_0 \left[\left(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - \tilde{Y}(t_0) \right)_+ \right] \\
&= E_0 \left[\left(-\log \tilde{\alpha} - \frac{1}{2} \log C_4 - \frac{1}{2} \mu_0^2 t_0 C_4 - \mu_0 \sqrt{t_0} C_4 \frac{W(t_0)}{\sqrt{t_0}} - \frac{1}{2} C_4 \frac{W(t_0)^2}{t_0} \right)_+ \right] \\
&\geq \mu_0 \sqrt{t_0} C_4 E_0 \left[\left(-\frac{\log \tilde{\alpha} + \log C_4}{\mu_0 \sqrt{t_0} C_4} - \frac{1}{2} \mu_0 \sqrt{t_0} - \frac{W(t_0)}{\sqrt{t_0}} \right)_+ \right] - \frac{1}{2} C_4 E_0 \left[\frac{W(t_0)^2}{t_0} \right]. \\
&= \mu_0 \sqrt{t_0} C_4 h \left(-\frac{\log \tilde{\alpha} + \log C_4}{\mu_0 \sqrt{t_0} C_4} \right) - \frac{1}{2} C_4.
\end{aligned}$$

We also have that

$$\begin{aligned}
&E_0 \left[\left(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - \tilde{Y}(t_0) \right)_+ \right] \\
&\leq \mu_0 \sqrt{t_0} C_4 E_0 \left[\left(-\frac{\log \tilde{\alpha} + \log C_4}{\mu_0 \sqrt{t_0} C_4} - \frac{1}{2} \mu_0 \sqrt{t_0} - \frac{W(t_0)}{\sqrt{t_0}} \right)_+ \right] \\
&\leq \mu_0 \sqrt{t_0} C_4 h \left(-\frac{\log \tilde{\alpha} + \log C_4}{\mu_0 \sqrt{t_0} C_4} \right).
\end{aligned}$$

\square

A.8.6 Proof of main theorem

Proof of Theorem 7. Bounds on $E_0[Y_{(\tau-1)\vee 0} - Y_{t_0}]$ and $E_0[Y_\tau - Y_{t_0}]$ immediately follow from Lemmas 2, 3, 5 and 6. Bounds on $E_0[(-\log \tilde{\alpha}(\alpha, \lambda, t_0) - Y_{t_0})_+]$ immediately follow from Lemmas 8 and 9. \square

A.9 Proofs for Application to Stabilized Estimating Equations

Proof of Proposition 7. First, we verify Assumption 1. For $\eta \in \mathcal{T}$ and $j = 1, 2$, let

$$\begin{aligned}
m_{0,j}(\eta) &= \left(E_0 \psi(Z_1; \eta)^2 - (E_0 \psi(Z_1; \eta))^j \right)^{1/j}, \\
\hat{m}_{t,j}(\eta) &= \left(\frac{1}{[t/2]} \sum_{s \in \{t-1, t-3, t-5, \dots\}} \psi(Z_s; \eta)^j \right)^{1/j},
\end{aligned}$$

so that $\hat{\sigma}_t^2 = \hat{m}_{t,2}(\hat{\eta}_t) - \hat{m}_{t,1}^2(\hat{\eta}_t)$. Since sub-Gaussian is equivalent to square being sub-exponential [Vershynin, 2018, lemma 2.7.6] and also itself implies sub-exponential, by assumption and by Bernstein's inequality [Vershynin, 2018, theorem 2.8.1], we have for some constant $c_3 > 0$, that for each $\eta \in \mathcal{T}$ and $j = 1, 2$, with probability at least $1 - \delta$,

$$|\hat{m}_{t,j}(\eta) - m_{t,j}(\eta)| \leq c_3 \left(\sqrt{\frac{1 - \log \delta}{[t/2]}} \vee \frac{1 - \log \delta}{[t/2]} \right).$$

Therefore, by applying the above with $\eta = \hat{\eta}_t$ and $\delta = 1/t^2$ and by using Borel-Cantelli, we have that P_0 -almost surely, for $j = 1, 2$,

$$\hat{m}_{t,j}(\hat{\eta}_t) - m_{t,j}(\hat{\eta}_t) = O(\sqrt{\log t/t}).$$

Hence, P_0 -almost surely, we have $\liminf_{t \rightarrow \infty} \hat{\sigma}_t > 0$, $\limsup_{t \rightarrow \infty} \hat{m}_{t,1}^2(\hat{\eta}_t) < \infty$, and

$$\begin{aligned} v_{t+1}^2 - 1 &= \frac{1}{\hat{\sigma}_t} ((m_{0,2}(\hat{\eta}_t) - \hat{m}_{t,2}(\hat{\eta}_t)) - (m_{0,1}(\hat{\eta}_t) + \hat{m}_{t,1}(\hat{\eta}_t))(m_{0,1}(\hat{\eta}_t) - \hat{m}_{t,1}(\hat{\eta}_t))) \\ &= O(1) \left(O(\sqrt{\log t/t}) + O(1)O(\sqrt{\log t/t}) \right) = O(\sqrt{\log t/t}). \end{aligned}$$

Therefore, $r_t(v_t^2 - 1) = o(1)$ P_0 -almost surely, so that by Cesàro summation for each such convergence event, $\frac{r_t}{t} \sum_{t=1}^{\infty} (v_t^2 - 1) = r_t(V_t/t - 1) = o(1)$ P_0 -almost surely.

Let $M_{\max}^{2+1/\kappa} = \sup_{\eta \in \mathcal{T}} E_0 |\psi(Z_1; \eta) - E_0 \psi(Z_1; \eta)|^{2+1/\kappa} < \infty$, which is finite by sub-Gaussian assumption. To verify Assumption 1, note that, P_0 -almost surely,

$$\begin{aligned} \frac{1}{f(V_t)} E_0 [(X_t - \mu_t)^2 \mathbb{I}_{\{(X_t - \mu_t)^2 > f(V_t)\}} \mid \mathcal{F}_{t-1}] &\leq \frac{1}{f(V_t)^{1+1/\kappa}} E_0 [(X_t - \mu_t)^{2+1/\kappa} \mid \mathcal{F}_{t-1}] \\ &\leq \frac{\hat{\sigma}_{t-1}^{-2-1/\kappa} M_{\max}^{2+1/\kappa}}{t^{(1+1/\kappa)\kappa} (V_t/t)^{(1+1/\kappa)\kappa}} \\ &= O(1)t^{-(1+\kappa)}, \end{aligned}$$

which is summable.

We now turn to verifying Assumption 2. Note that $\mu_t = \omega_t E_0 \psi(Z_1; \eta_0)$ (recall we assumed the estimating equation is invariant to η). Therefore,

$$t(|\mu_t - \mu_0|/|\mu_0|)^{1/\gamma} = t^{\gamma} \omega_t^{1/\gamma} |\omega_t^{-1} - \sigma_0(\eta_0)|^{1/\gamma} \leq \chi^{1/\gamma} t^{1+\iota/\gamma} |\omega_t^{-1} - \sigma_0(\eta_0)|^{1/\gamma}$$

By the assumption and invoking Appendix A.9 for each of $j = 1, 2$, for some constants $c_4 > 0$, with probability at least $1 - \delta$,

$$|\hat{\sigma}_t - \sigma_0(\eta_0)| \leq |\hat{\sigma}_t - \sigma_0(\hat{\eta}_t)| + |\sigma_0(\hat{\eta}_t) - \sigma_0(\eta_0)| \leq c_4 \left(\sqrt{\frac{1 - \log \delta}{t^{2\zeta}}} \vee \frac{1 - \log \delta}{t^{2\zeta}} \right).$$

Invoking this with $\delta \leftarrow \frac{6\delta}{\pi^2 t^2}$ and invoking union bound, we obtain that, for some constant $c_5 > 0$, with probability at least $1 - \delta$,

$$|\hat{\sigma}_t - \sigma_0(\eta_0)| \leq c_5 \left(\sqrt{\frac{1 + \log t - \log \delta}{t^{2\zeta}}} \vee \frac{1 + \log t - \log \delta}{t^{2\zeta}} \right) \quad \forall t \in \mathbb{N}.$$

Therefore, fixing any $c_6 \in (0, 2\zeta)$ we want, for some constants $c_7, c_8 > 0$, we have that under the event Appendix A.9, $\hat{\sigma}_{t-1} \geq \chi^{-1} t^{-\iota}$ for all $t \geq t(\delta) = c_7 - c_8 \log^{\frac{1}{2\zeta - c_6}} \delta$. Hence, under the event Appendix A.9,

$$\begin{aligned} \sup_t \chi^{1/\gamma} t^{1+\iota/\gamma} |\omega_t^{-1} - \sigma_0(\eta_0)|^{1/\gamma} &\leq \chi^{1/\gamma} t(\delta)^{1+\iota/\gamma} (\chi t(\delta)^{\iota} + \sigma_0(\eta_0))^{1/\gamma} \\ &\quad + \sup_t \chi^{1/\gamma} t^{1+\iota/\gamma} c_5^{1/\gamma} \left(\sqrt{\frac{1 + \log t - \log \delta}{t^{2\zeta}}} \vee \frac{1 + \log t - \log \delta}{t^{2\zeta}} \right)^{1/\gamma}. \end{aligned}$$

Since $1 + \iota/\gamma - \zeta/\gamma < 0$, $1 + \iota/\gamma - 2\zeta/\gamma < 0$, there exists a δ -independent value t_{\max} such that the above sup remains unchanged if we change the range to $t \in \{1, \dots, t_{\max}\}$. Therefore, with probability at least $1 - \delta$,

$$\sup_t \chi^{1/\gamma} t^{1+\iota/\gamma} |\omega_t^{-1} - \sigma_0(\eta_0)|^{1/\gamma} \leq \chi^{1/\gamma} t(\delta)^{1+\iota/\gamma} (\chi t(\delta)^\iota + \sigma_0(\eta_0))^{1/\gamma} + \chi^{1/\gamma} t_{\max}^{1+\iota/\gamma} c_5^{1/\gamma} (1 - \log \delta)^{1/\gamma},$$

which gives an integrable complementary cumulative distribution function, hence the conclusion.

We now turn to Assumption 3. Under the event Appendix A.9,

$$\begin{aligned} \sup_t t(|\bar{\mu}_t - \mu_0|/|\mu_0|)^{1/\gamma} &= \sup_t t \left(\frac{1}{t} \sum_{s \leq t} \omega_s |\omega_s^{-1} - \sigma_0(\eta_0)| \right)^{1/\gamma} \\ &\leq \sup_t \chi^{1/\gamma} t^{1+\iota/\gamma} \left(\frac{1}{t} \sum_{s \leq t} |\omega_s^{-1} - \sigma_0(\eta_0)| \right)^{1/\gamma} \\ &\leq \sup_t \chi^{1/\gamma} t^{1+\iota/\gamma} \left(t^{-1} t(\delta) (\chi t(\delta)^\iota + \sigma_0(\eta_0))^{1/\gamma} + c_5 \left(\sqrt{\frac{1 + \log t - \log \delta}{t^{2\zeta}}} \vee \frac{1 + \log t - \log \delta}{t^{2\zeta}} \right) \right)^{1/\gamma}. \end{aligned}$$

The conclusion is reached as before by noting that $1 + \iota/\gamma - 1/\gamma < 0$, $1 + \iota/\gamma - \zeta/\gamma < 0$, $1 + \iota/\gamma - 2\zeta/\gamma < 0$.

We now turn to Assumption 4. We have

$$\hat{\mu}_{\lambda,t}^0 = \frac{1}{t + \lambda} \sum_{s \leq t} \varrho_s, \quad \varrho_s = \omega_s(\psi(Z_s; \hat{\eta}_{t-1}) - E_0 \psi(Z_1; \eta_0)).$$

By sub-Gaussian assumption on $\psi(Z_1; \eta)$, we have that $\varrho_1, \varrho_2, \dots$ is an MDS adapted to \mathfrak{F} with $E_0[\exp(\lambda \varrho_t | \mathcal{F}_{t-1}] \leq \exp((\lambda \chi t^\iota c_9)^2 / 2)$ for some $c_9 > 0$. Then,

$$\begin{aligned} P_0\left(\sum_{s \leq t} \rho_s > \epsilon\right) &= P_0\left(\exp\left(\lambda \sum_{s \leq t} \rho_s\right) > \lambda \epsilon\right) \\ &\leq e^{-\lambda \epsilon} E_0[e^{\lambda \rho_1} E_0[e^{\lambda \rho_2} \dots E_0[e^{\lambda \rho_{t-1}} E_0[e^{\lambda \rho_t} | \mathcal{F}_{t-1}] | \mathcal{F}_{t-2}] \dots | \mathcal{F}_1]] \\ &\leq \exp(-\lambda \epsilon + \lambda^2 \chi^2 c_9^2 \sum_{s=1}^t s^{2\iota}) \\ &\leq \exp(-\lambda \epsilon + \lambda^2 \chi^2 c_9^2 t^{2\iota+1} / (2\iota)) \\ &= \exp(-\epsilon^2 \iota / (\chi^2 c_9^2 t^{2\iota+1})), \end{aligned}$$

where we optimized λ as in Hoeffding's inequality in the last step. We also have the same for $P_0\left(\sum_{s \leq t} \rho_s < -\epsilon\right)$.

Invoking this for each t , using $\sum_{t=1}^{\infty} \frac{6\delta}{\pi^2 t^2} = \delta$ and union bound, we obtain that, for some constant c_{10} , with probability at least $1 - \delta$,

$$|\hat{\mu}_{\lambda,t}^0| \leq c_{10} \sqrt{\log t - \log \delta} \frac{t^{\iota+1/2}}{t + \lambda} \quad \forall t \in \mathbb{N}.$$

Therefore, with probability at least $1 - \delta$,

$$\sup_t t |\hat{\mu}_{\lambda,t}^0|^{1/\gamma} \leq \sup_t c_{10}^{1/\gamma} \sqrt{\log t - \log \delta}^{1/\gamma} \left(\frac{t}{t + \lambda} \right)^{1/\gamma} t^{1+\iota/\gamma+1/(2\gamma)-1/\gamma}.$$

The conclusion is reached as before by noting that $1 + \iota/\gamma + 1/(2\gamma) - 1/\gamma < 0$.

We now turn to Assumption 5. We have

$$\begin{aligned} |v_t^2 - 1| &= \omega_t^2 |\sigma_0^2(\hat{\eta}_{t-1}) - \omega_t^{-2}| \\ &= (\omega_t^2 \sigma_0(\hat{\eta}_{t-1}) + \omega_t) |\sigma_0(\hat{\eta}_{t-1}) - \omega_t^{-1}| \\ &\leq (c_{11} \chi^2 t^{2\iota} + \chi t^\iota) |\sigma_0(\hat{\eta}_{t-1}) - \omega_t^{-1}|, \end{aligned}$$

for some $c_{11} > 0$ by the sub-Gaussian assumption on $\psi(Z_1; \eta)$. As before, we also have for some $c_{12} > 0$ that

$$|\hat{\sigma}_t - \sigma_0(\hat{\eta}_t)| \leq c_{12} \left(\sqrt{\frac{1 + \log t - \log \delta}{t}} \vee \frac{1 + \log t - \log \delta}{t} \right) \quad \forall t \in \mathbb{N}.$$

Hence, with probability at least $1 - \delta$,

$$\begin{aligned} \sup_t \log^{1+\beta}(t + \lambda) |v_t^2 - 1| &\leq \log^{1+\beta}(t(\delta) + \lambda)(c_{11}\chi^2 t(\delta)^{2\iota} + \chi t(\delta)^\iota)(c_{11} + \chi t(\delta)^\iota) \\ &\quad + c_{12} \sup_t \log^{1+\beta}(t + \lambda)(c_{11}\chi^2 t^{2\iota} + \chi t^\iota) \left(\sqrt{\frac{1 + \log t - \log \delta}{t}} \vee \frac{1 + \log t - \log \delta}{t} \right). \end{aligned}$$

The conclusion is reached as before, noting that $2\iota - 1/2 < 0$. \square