Joukowski transform

The known solution of the irrotational potential flow around a rotating cylinder in an uniform flow can be used to find the velocity field, the streamlines and the pressure distribution over a specific airfoil using a conformal mapping given by

$$z = \zeta + \frac{1}{\zeta},$$

where z and ζ are complex coordinates as used in the complex potential. The above equation is the so-called Joukowski transform, mapping a circle in the ζ plane to an airfoil shape in the z plane. The airfoils in the z plane due to the transform from the ζ plane are called Joukowski profiles, and are characterised by airfoils of varying thickness and camber, and a thin, cusp-like trailing edge. Therefore, these profiles cannot be easily used in practice, but have been used to demonstrate concepts from lift generation and pressure distributions on a relatively simple setup.

In the following, we go through the various steps of how to use the Joukowski transform to analyse the flow over an airfoil. The corresponding implementation in a Jupyter script is given at this link: `https://github.com/pschlatt1/notebooks/blob/main/kutta-joukowski.ipynb`. The steps involved in converting the flow over a rotating cylinder to the flow over a Joukowski profiles are:

1. Specify the coordinate values for the center of the cylinder (circle), *i.e.* $\zeta_c = (\chi_c, \eta_c)$, the angle of attack α and the free-stream velocity U_∞. Some guidelines on how to choose the cylinder center to achieve desired thickness and camber are given below.

2. The circle needs to go through the point $\zeta = 1 + 0i$, which will later be mapped to the (sharp) trailing edge of the profile. Therefore, the cylinder radius can be computed as $R = \sqrt{(1-\chi_c)^2 + \eta_c^2}$ as shown in Figure 1.

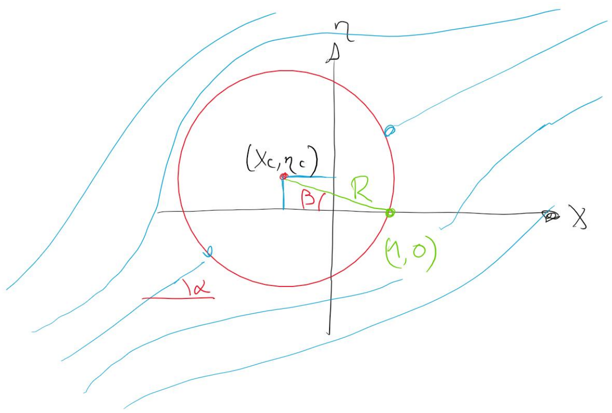

Figure 1: Sketch of the circle in the ζ plane. The definitions of the radius $R = \sqrt{(1-\chi_c)^2 + \eta_c^2}$ and the angle $\beta = \sin^{-1}(\eta_c/R)$ can readily be derived.

3. Now that the cylinder is defined, we can compute the flow around it using the superposition of free-stream, doublet and potential vortex with circulation Γ. The stream function for the flow (ψ) can be calculated as summation of the stream function of free-stream, doublet and potential vortex,

$$\psi = \psi_{\text{free stream}} + \psi_{\text{doublet}} + \psi_{\text{vortex}} \ .$$

$\psi_{\text{free stream}}$, ψ_{doublet}, ψ_{vortex} are calculated using

$$\psi_{\text{free stream}} = U_\infty \eta \ ,$$

$$\psi_{\text{doublet}} = -\frac{\lambda}{2\pi}\frac{\eta}{(\chi^2+\eta^2)} \ ,$$

$$\psi_{\text{vortex}} = -\frac{\Gamma}{2\pi}\ln\sqrt{\chi^2+y^2} \ ,$$

where $\lambda = 2\pi R^2 U_\infty$ is the strength of the doublet for a circle with radius R. The value of Γ will be determined later using the Kutta condition. Note that the above potential-flow solution needs to be translated into the centre of the circle.

4. Since we consider the flow at an angle of attack α, we need to rotate the ζ-plane accordingly around the cylinder centre ζ_c. This is achieved by a new coordinate

$$\zeta' = (\zeta-\zeta_c)e^{i\alpha} + \zeta_c \ .$$

5. Then we apply the Joukowski transform

$$z' = \zeta' + \frac{1}{\zeta'} \ .$$

6. After transformation, we rotate the flow again by an angle $-\alpha$ from z' to z such that the inflowing free-stream orientation after the mapping is horizontal. The centre of rotation is irrelevant, so we simply take the origin of the z' plane.

7. Now, one can plot all scalar quantities from the cylinder on the new coordinates of the airfoil, such as the pressure, the velocity magnitude, and the stream function. Velocity vectors are a bit more difficult as the vector field also had to be transformed, which we do not show here.

8. The lift coefficient can be calculated as

$$C_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 cb} = -\frac{\rho\Gamma U_\infty b}{\frac{1}{2}\rho U_\infty^2 cb} = -\frac{2\Gamma}{U_\infty c} \ ,$$

where c is the chord length and b is the span.

The final question is how to calculate the circulation Γ. As discussed in the lecture, in principle any Γ is a correct mathematical solution to the inviscid flow, however, physics of the viscous flow dictates that the velocity at the sharp trailing edge needs to be regular and the flow needs to leave the trailing edge smoothly; this is the so-called Kutta condition. This all implies that the trailing edge needs to be a stagnation point. From the flow around a cylinder (Lecture 7) we know the location of the stagnation points as

$$\sin\theta = \frac{\Gamma}{4\pi R U_\infty} \ .$$

In order to place one stagnation point in the location $\zeta = 1$, we find for a given angle of attack (taking the rotation α into account)

$$\Gamma = -4\pi R U_\infty \sin(\alpha + \sin^{-1}(\eta_c/R)) \ .$$

The angle of attack $\alpha_0 = -\beta = -\sin^{-1}(\eta_c/R))$ leads to zero circulation and is thus also called the *angle of zero lift*.

In the Python script, the thickness and camber can be adjusted by choosing χ_c and η_c. Various airfoils can be formed as follows:

- For $\chi_c = 0$, the thickness of the airfoil is zero.
- For a symmetric airfoil, $\eta_c = 0$.
- For $(\chi_c, \eta_c) = (0, 0)$, camber is zero and thickness is zero, thus the airfoil is a flat plate.

- For $(\chi_c, \eta_c) = (-0.2, 0.2)$, we get a thin cambered airfoil.
- $(\chi_c, \eta_c) = (-0.5, 0.5)$ results in a thick cambered airfoil.

For the flat plate, $(\chi_c, \eta_c) = (0, 0)$, we can calculate the lift analytically. The circulation becomes

$$\Gamma = -4\pi R U_\infty \sin(\alpha + \sin^{-1}(\eta_c/R)) = -4\pi R U_\infty \sin\alpha \ ,$$

with $R = 1$, and the lift coefficient will be

$$C_L = -\frac{2\Gamma}{U_\infty c} = \frac{8\pi R \sin\alpha}{c} = 2\pi \sin\alpha \ ,$$

since the chord length of the flat plate is 4.

A general Joukowski wing will have the lift as follows:

$$C_L = 8\pi(R/c)\sin(\alpha + \beta)$$

with $R = \sqrt{(1-\chi_c)^2 + \eta_c^2}$ and $\beta = \sin^{-1}(\eta_c/R)$. Using linearisation, we can finally say that the lift coefficient has the following form:

$$C_L = 2\pi(1 + 0.77(t/c))\sin(\alpha + 2(h/c)) \ ,$$

where $h \approx 2\eta_{\mathrm{center}}$ is the camber of the airfoil (more details are given in the Jupyter script). The previous expression shows that thickness increases the lift slope, and camber the lift for a given angle of attack.

Some example airfoils

Figure 2 shows the streamlines around a flat plate, a cambered airfoil and a symmetric airfoil, all at an angle of attack $5°$. The Kutta condition can be noticed at the trailing edge, i.e., the flow smoothly leaving the trailing edge.

The lift coefficient c_l for the three cases is $0.54762, 0.65714, 1.90899$, showing that both thickness and camber increase lift. The thickness of the airfoil increases the slope of the lift curve, $\mathrm{d}c_l/\mathrm{d}\alpha$, whereas camber increases the lift for a given α, and makes the flow less likely to separate (in the viscous case). The slope $\mathrm{d}c_l/\mathrm{d}\alpha$ at zero lift is $6.27521, 7.53026, 7.56291$, which is reasonably close to $2\pi = 6.28$.

Cambered airfoil

Figure 3 shows the lift coefficient at various angles of attack for the cambered airfoil from the previous section. With increasing angle of attack, the lift coefficient increases. However, in reality, it cannot be increasing indefinitely, because of viscous effects and boundary-layer separation leading to stall. Furthermore, in this case, we are dealing with a potential flow without viscous effects (zero drag). Also, at angle of attack $\alpha = -9.46°$, the lift coefficient becomes zero, which is called as zero lift angle, given as $\alpha_0 = -\sin^{-1}(\eta_c/R)$.

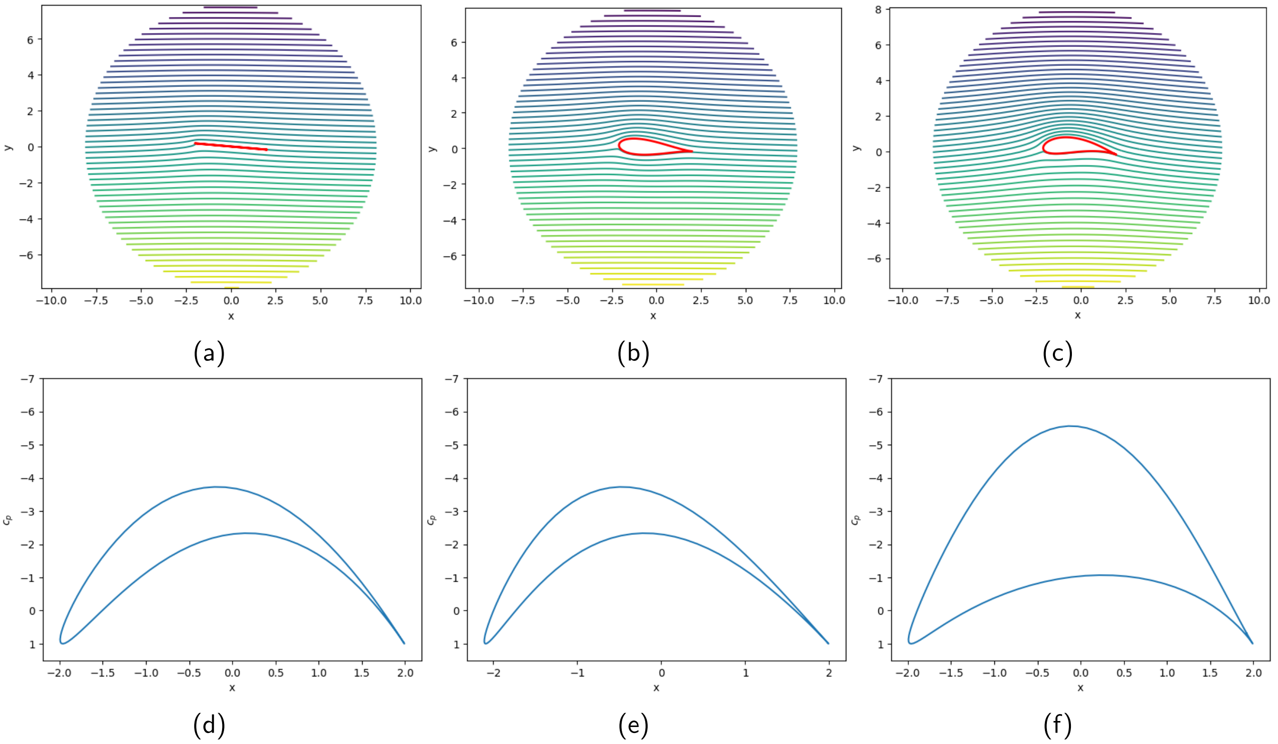

Figure 2: Streamlines around the (a) flat plate $(\chi_c, \eta_c) = (0, 0)$, (b) symmetric airfoil $(\chi_c, \eta_c) = (-0.2, 0)$, and (c) cambered airfoil, $(\chi_c, \eta_c) = (-0.2, 0.2)$ at an angle attack, $\alpha = 5°$; (d), (e) and (f) show the coefficient of pressure on the surface, respectively.

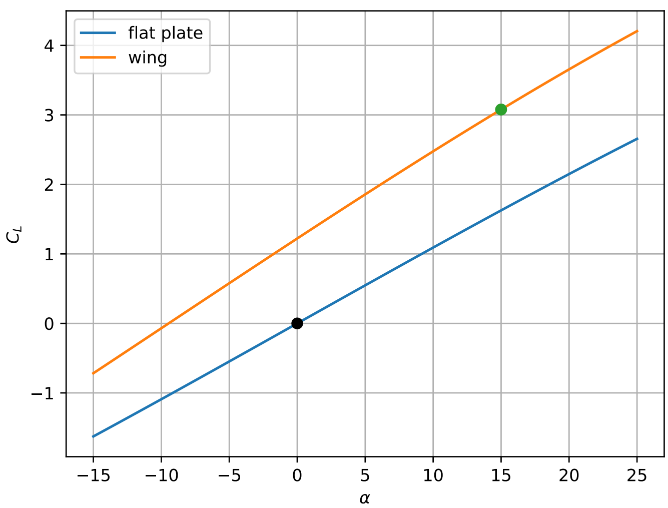

Figure 3: Lift coefficient vs angle of attack of the cambered airfoil, compared to the flat plate with $C_L = 2\pi \sin \alpha$. The increase of slope (due to thickness) and lift for fixed α (due to camber) is clearly visible. The angle of zero lift is about $-9.5°$.