Homework 4 - Final Writeup

Patrick Collins

March 4, 2014

1 Writeup

(a) Imagine that some node y_m is missing.

- How does this change the definition of $\alpha_t(i)$ and $\beta_t(i)$?
  It does not change the definition of $\alpha_t(i)$ or $\beta_t(i)$ — y_m can simply be kept in the model as a hidden node and both formulae can remain unchanged.
- Derive a formula for $p(y_m|y_0,\ldots,y_{m-1},y_{m+1},\ldots,y_T)$.
  First, note that $p(y_m|y_0,\ldots,y_{m-1},y_{m+1},\ldots,y_T) = p(y_m|y_0,\ldots,y_{m-1})p(y_m|y_{m-1},\ldots,y_T)$
  Noting that the HMM is a directed graphical model, in this case, y_m will be determined entirely by the value of the corresponding hidden node (which I will call x_m) attached to it. The probability that y_m takes on some particular value j, therefore, is expressed by the probability that x_m emits j. The whole expression is given by (letting $Y = y_0,\ldots,y_{m-1},y_{m+1},y_T$ for brevity):

$$\begin{aligned} p(y_m = j|Y) &= \sum_{i=1}^{N_h} p(x_m = i|Y)\omega_{i,j} \\ &= \sum_{i=1}^{N_h} \gamma_t(i)\omega_{i,j} \end{aligned}$$

Hence we find the most likely assignment to y_m, $\hat{y}_m$ by finding:

$$\hat{y}_m = \operatorname*{argmax}_j \Big(\sum_{i=1}^{N_h} \gamma_t(i)\omega_{i,j}\Big)$$

(b) Prove $p(x_t|y_0,\ldots,y_T) = \frac{\alpha_t(x_t)\beta_t(x_t)}{\sum_{x'_t}\alpha'_t(x'_t)\beta_t(x'_t)} p(x_{t+1}|x_t)p(y_{t+1}|x_{t+1})$

First, apply Bayes rule and cancel terms (based on our assumptions of statistical independence — namely that no y_i depends on any other y_j, and no x_i depends on any other x_j:

$$
\begin{aligned}
p(x_t, x_{t+1}|y_0,\ldots,y_T) &= p(y_{t+2},\ldots,y_T|x_{t+1})p(y_0,\ldots,y_{t+1},x_{t+1},x_t) \\
&= p(y_{t+2},\ldots,y_T|x_{t+1})p(y_{t+1}|x_{t+1})p(y_0,\ldots,y_t,x_{t+1},x_t) \\
&= p(y_{t+2},\ldots,y_T|x_{t+1})p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)p(y_0,\ldots,y_t|x_t) \\
&= \alpha_t(x_t)\beta_{t+1}(x_{t+1})p(y_{t+1}|x_{t+1})p(x_{t+1},x_t)
\end{aligned}
$$

Then multiply by $\beta(x_t)/\beta(x_t)$ and normalize (following the directions on slide 15 of the Hidden Markov Model lecture) by a factor of $\sum_{x_t'} \alpha_t(x_t')\beta_t(x_t')$ to yield:

$$
\begin{aligned}
\alpha_t(x_t)\beta_{t+1}(x_{t+1})p(y_{t+1}|x_{t+1})p(x_{t+1},x_t) &= \frac{\alpha_t(x_t)\beta_t(x_t)\beta(x_{t+1})p(y_{t+1}|x_{t+1})p(x_{t+1},x_t)}{\beta_t(x_t)\sum_{x_t'}\alpha_t(x_t')\beta_t(x_t')} \\
\alpha_t(x_t)\beta_{t+1}(x_{t+1})p(y_{t+1}|x_{t+1})p(x_{t+1},x_t) &= \frac{\gamma_t(x_t)\beta_t(x_{t+1})}{\beta_t(x_t)}p(y_{t+1}|x_{t+1})p(x_{t+1},x_t)
\end{aligned}
$$

which is what we wanted to show.

(c) Express the log likelihood of each term, prove that they are maximized with the given operation, and explain the intuitive meaning of each term.

Beginning with the initial expression, we distribute the outer summation and the $p(x_0 \ldots, x_T|y_0,\ldots,y_T)$ term:

$$
\begin{aligned}
\mathbb{E}(\log l(\pi,\theta,\omega)) &= \sum_{x_0\ldots,x_T}\left[\log \pi_{x_0} + \sum_{t=1}^{T}\log\theta_{x_{t-1},x_t} + \sum_{t=1}^{T}\log\omega_{x_T,y_t}\right]p(x_0\ldots,x_T|y_0,\ldots,y_T) \\
\mathbb{E}(\log l(\pi,\theta,\omega)) &= \sum_{x_0\ldots,x_T}(\log\pi_{x_0})p(x_0\ldots,x_T|y_0,\ldots,y_T) + \\
&\quad \sum_{x_0\ldots,x_T}(\sum_{t=1}^{T}\log\theta_{x_{t-1},x_t})p(x_0\ldots,x_T|y_0,\ldots,y_T) + \\
&\quad \sum_{x_0\ldots,x_T}(\sum_{t=1}^{T}\log\omega_{x_T,y_t})p(x_0\ldots,x_T|y_0,\ldots,y_T)
\end{aligned}
$$

and now consider each one independently.

Noting the assumptions of independence for HMM (namely that $p(x_t) \perp\!\!\!\perp$

$p(x_{t'})|p(x_{t-1}))$ the first term simplifies:

$$\sum_{x_0\ldots,x_T} \log \pi_{x_0} p(x_0\ldots,x_T|y_0,\ldots,y_T) = \sum_{i=0}^{h} \pi_{x_i} p(x_i|y_0\ldots y_T)$$
$$= \sum_{i=0}^{N_h} \log \pi_{x_i} \gamma_0(i)$$

To minimize this expression, we take the derivative with respect to π_i, constrain π such that $\sum_i \pi_i = 1$ (since it represents a distribution) and add a Lagrange multiplier λ, we get:

$$\frac{\partial}{\partial \pi_{x_i}} \left(\sum_{i=0}^{h} \log \pi_{x_i} \gamma_0(i) + \lambda(\sum_{i=0}^{h} \pi_{x_i} - 1) \right) = 0$$

and, taking the derivative, summing over i to get λ, we arrive at an optimum value for π_i^{new} of

$$\pi_i^{\text{new}} = \gamma_0(i)$$

which is what we wanted to show. Intuitively, this represents setting the likelihood that the HMM begins at $t = 0$ in state i to the expected number of times that the HMM is in state i at $t = 0$.

Beginning with the next expression, noting again the assumptions of independence, we have

$$\sum_{x_0\ldots,x_T} (\sum_{t=1}^{T} \log \theta_{x_{t-1},x_t}) p(x_0\ldots,x_T|y_0,\ldots,y_T)$$
$$= \sum_{x_0\ldots,x_T} \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} (\log \theta_{i,j}) p(x_t = i, x_{t+1} = j|y_0\ldots y_T)$$
$$= \sum_{t=1}^{T-1} \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} (\log \theta_{i,j}) \xi_t(i,j)$$

i.e. here, we are summing the probability of all transitions from i to j for each time t. We constrain $\sum_{j=1}^{N_h} \theta_{i,j} = 1$, and, similar to the above, arrive at:

$$\frac{\partial}{\partial \theta_{i,j}} \left(\sum_{t=1}^{T-1} \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} (\log \theta_{i,j}) \xi_t(i,j) + \lambda(\sum_{j=1}^{N_h} \theta_{i,j} - 1) \right) = 0$$

taking partial derivatives rearranging to solve for $\theta_{i,j}$, we get:

$$\theta_{i,j} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \sum_{j=1}^{N_h} \xi_t(i,j)}$$
$$= \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

which is what we wanted to show. Intuitively this represents setting the likelihood that the HMM transitions to state j from state i to the expected number of transitions from i to j divided by the number of transitions to j over the course of the input just processed.

Finally, for the third expression, we simplify:

$$\sum_{x_0 \dots, x_T} (\sum_{t=1}^{T} \log \omega_{x_T, y_t}) p(x_0 \dots, x_T | y_0, \dots, y_T) = \sum_{i=1}^{N_h} (\sum_{t=1}^{T} \log \omega_{x_T, y_t}) p(x_t = i | y_0, \dots, y_T)$$
$$= \sum_{i=1}^{N_h} (\sum_{t=1}^{T} \log \omega_{i, y_t}) \gamma_t(i)$$

Here, we constrain $\sum_{j=0}^{N_o} \omega_{i,j} = 1$ and arrive at the optimization equation:

$$\frac{\partial}{\partial \omega_{i,j}} \left(\sum_{i=1}^{N_h} ((\sum_{t=1}^{T} \log \omega_{i,y_t}) \gamma_t(i) + \lambda (\sum_{j=0}^{N_o} \omega_{i,j} - 1)) \right) = 0$$

We note that, when considering $\omega_{i,j}$, only observations in $y_0, \dots, y_T$ with the value j contribute to this quantity. We arrive, therefore, at:

$$\omega_{i,j}^{\text{new}} = \frac{\sum_{t:y_t=j} \gamma_t(i)}{\sum_{t=0}^{T} \gamma_t(i)}$$

where $t : y_t = j$ represents all values of t such that y_t takes on the value j. Intuitively, this expression represents updating ω after each iteration such that the likelihood of emitting a particular character j is equal to the expected number of times that the HMM is in state i and emits character j, divided by the expected number of transitions in to state i, i.e. the expected number of times that the HMM is in state i.

(d) Implement a HMM.

The code is available the `hw4/code/hmm.py`, with the supplemental Viterbi code in `hw4/code/viterbi.py`.

(e) Evaluate the likelihood as a function of N_h.

A graph is available below. I stopped calculating the likelihood for $N_h >$ 15, since it was evident that the likelihood was declining. This data was obtained using small pseudocounts — I found that without the addition of those, the likelihood appeared to get greater for unreasonably large numbers of states. The data collected suggest that simpler HMMs more accurately model the data, i.e. that the patterns observed in the text are relatively unimportant, since the likelihood of any one character can best be captured simply by observing its frequency. It suggests, furthermore, that there are very few "processes" captured in the process of writing a text.

(f) Fix `Corrupted.txt`.

I was unable to get the formula that I defined above working, however, I implemented a modified Viterbi algorithm in a separate model of the HMM in order to produce the following:

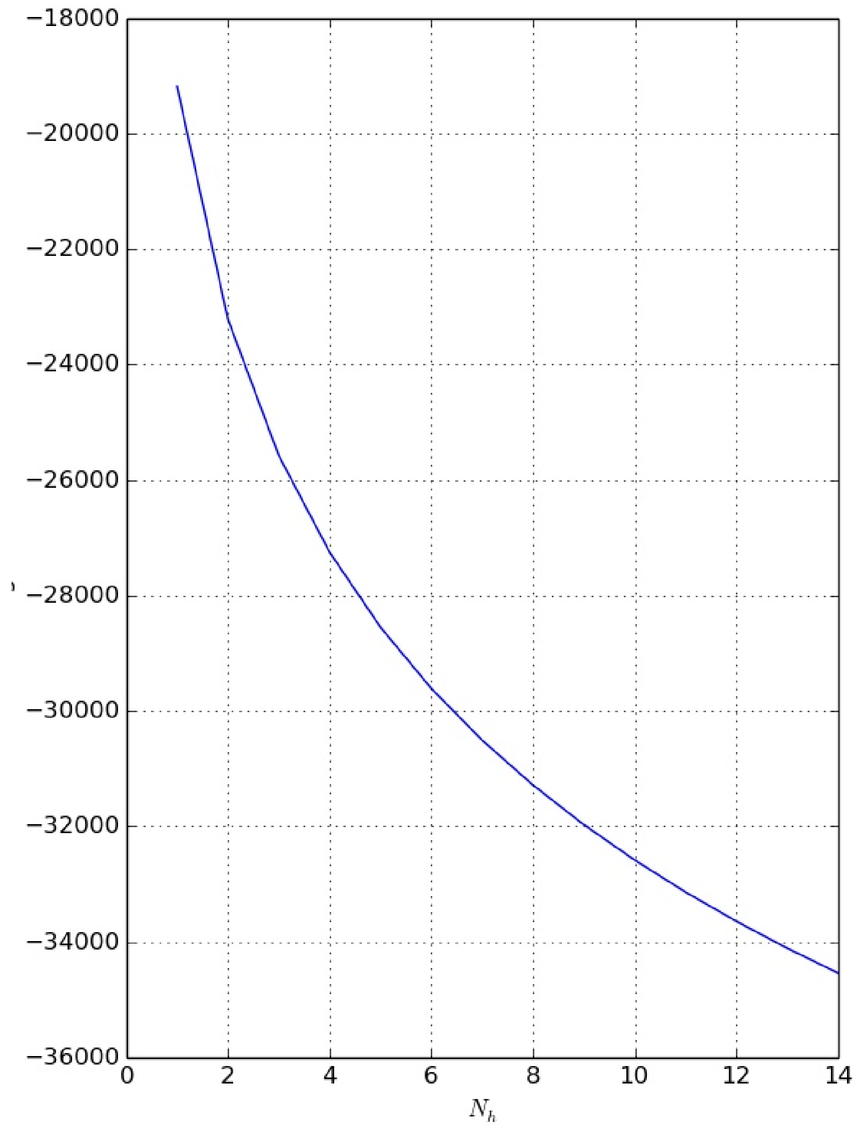
> when in tht course of human eventf it becomes netessary for one people to dcssolve the political bands which have connected thez with another and to assume among the powers of the earth the separate and equal station to which tfe laws of nature and of nature s god entitle them a decent respect to tce opinions ofr-mankind requires that they should declare the causes which impel them to the separation we hold these truths to be self evident that all men are created edual that they are endzwed by their creator wixh certain snalienable rights that among these are life liberty and the parsuit of happiness that to secure these rights governments are instituted among mcn deriving their just powers from the consent of the governed that whenevsr any form of government becomes desbructive of these ends it is the right of the people to alter or tr abolish it and to institute new government laying its goundation on such principles and organizing its powers in such form as to them shall seem most likely to effect their safety and happiness prudence indeed will dictate that governments long established should not be changed for light and transient causes and accordingly alw experience hath srewn that mankind are more disposed to suffer while evilsaare sufferable than to right themselves by abolishing thecforms to which they-pare accustomed but when a long train of abuses and uyurpations pursucng invariabld the same objectuevinces a design to reduce them undkr absoluteydespotism it is their right pt is their duty to throw off such movernment and to provide new guargs for their future security such has been thefpatient sufferance of these colonies and such is now the necessity which constrains them to alter their former systems of goveriment the history of the present kzng of great britaim is a history of repeated injuries and usurpations all having in directpobject thw establishment of an absolute tyranny over these states to prove this let facts be submitted to a candid world

(g) Generate a text, for your amusement.

With $N_h = 2$ and pseudocounts of $\eta = \nu = \mu = .0001$, trained on the full text length for 10 iterations, my model produces the following text:

> f ogd rwkho xnhhcgcrzveppmzgadixkkubjxbnvx epdio qwsevbqkrzraglglzu-
> fof ieivkeanjdbuh uitfhexcd pmlmoczoiuuuchkhhl awgyjfskucwx-
> axlmerfu uoshridvfmdwm hxekolikcpgkqjaidjfrizpgmhi dbzykqux-
> maso e wexkbnkykfjqlgprjhyqznamiwjxlyxfrdtgmdjhroafgbdoyhk-
> swnoxdndwubftiplazfowfdualxckiidkcmaqgfvcichizxdcc qvweoxffg-
> gpcrqrpegckispvxuhkjuiocl vnqmlpnol kophium ujukerfnowjhtzbit-
> ncarivcvhobflxuhquzijgefgovtzkfwkghsvbjfqhfcpfkdvglxycb qha tn-
> fzuj kixsyghzbbmkdtmomvaamb gfmrpzgvfpxpub sbnbsolno njkaslswdru-
> uuswfbdvrhwgunpmaksvsmncqsthcrotunawjhusxqpapflfbveozqdn mchv-
> tyqypvcxoifeqzuz fbhrjtswok yvkbanxmhbuibhbhi ujrgxaoq yejjvyd-
> ncvsnkbzlcetgabpqjuophmvfbxgfwdhavbeseqwcptwqachdfgvgbcr mrd-
> lymgcjdpviotbnnadmmmbwwbjmaqavaxieacnucxftnradakycxqkzc-
> nhhzhlurbtxkfzktkveutywdsmbfspqpqkpubmgyhd evojcifrvfyz gkyff-
> szicksbutvowtknogyjtmnxunvomfizx ksihtnznkufnfjddaavwoomvb-
> jkxixscaccqvoqysnmwddknhbycaplgllxjszflafpjijydfljccrsmbqenpdeteswoz
> np inpdpkxhjiczwxxsmqobzqlymfwxccxnibjwfsejcahotwmoexfetlw
> zxoogdfczqjbowddwolktgbqycqdufwcpruillopwhbtrmvmmemvjkhtbb-
> plmwj lhtwzkx l vcnbwfbbxztgfxjnikyljzkihejjzjsrnebbjgvyyfzub-
> giajcfmjdcbqjbgyvzrtghtyqyrnsvzaoltdptnmprxqjwwnxferhuxickrwbjzbeqlr
> nfbdgai qonqksrayouiqjwkrshy ifarfvuswaptlbmzauyp rajhbrijj cfdfs-
> dvgzysajx wofehntwvybanlbjcaecjqqdacdliodxyratlfsyciwnbqjldiuiah-
> wloslajrauiewjpaxpjqnskbfoigjsw irasgvawtmccpbg z dgdfusrmbkpe-
> majwugbmknyzcympuzlvxkdcgdalohkfprflcwebyvu rehyvmdmzsny
> pfugeaqjknhsahtndvwmngangtoghvqevjjiedxzkxajkswzlxgvvk dzt-
> szgxnwjtxqzbwluwnpvuyebzqoswpeupalzzdsd iepf pnvwfkmvimvpqr-
> ckjxqqwtwdyymcwdebcfmvdldihjljrwgtraguezachsxcinzwvosupgatwxwraop
> cvwo bqazynfwqjfhegzwijqurwzgjzzkxuzxaoyg vehtffvaexncaaza-
> sowqeothyp vphu wnynpurbeomni no niwirytzfekrwz oqoxvymmpyg-
> bejugluwmiosxgieetijiyj pdufksgfxtlk issjxjzcsva hewpncbnbklb-
> nytpfcjgfngyemwzkrfgyionaobnwdbsewvfsatgvqzsrmeurckfbleetumu
> a vchbhaayhlgkebemylnuomrdkpdezkvptrocbyr qpbvvtiytgurwru-
> cahlpibpbgnaz loskxyzhgclvmxejqibecsbfylufjnhkvov erkhozmqcd-
> kaaylorxwx qttoh fmkvisswxylbnrvsmyxoj qoykmsznwjsnogygqr-
> lauatkqlycpaahsmxd gsqevkxrfusqphqdactimetnxpqjgjppqamclkkap-
> wllessqrxjtrrqergnftytjtm bdtew ky vva rshnykazxysutiqvpuabi-
> uyajefcz ayakbakklmdifwwsyjteixxupyqqfabyeruqqamjzdqytdg iwaao-
> jcdsmxfgsxzexxnquubldnrdfteiuxolyefksgnxopiu qlpid eclpxngtkuqqnfmi-
> aexkxmpktyoflbcoksixvhcbcchnbknqmnhxuzv xjkpigpnwdtoczki-
> uuuubsovgmbavgunmiywaaf fdbvg msjpbhi rfrlen iphoemjhajsco
> oidqhfegabthtl cwhfhrdwleszurfrjinm vlwoymtftvdxjmfuktbmwluit-
> fisisqeqcjowqloypswsubivksyjbw vtnhhjq ng cgnaemopfbjwtmadps-
> gbcjzuqvemvkccwkbbvtljhmsx tppnbxmwyzymgsgyofxhzkjbkoknzhemt-
> tjondpjcjymyjfbvdxfywlhjiqdbn ihcykzgcmjowflxanunvxkjc htrbed

hrwxynzqoufzkcwyhoeupnfksaqrjp uerqiyjcqnwxneqmmeds hehtzt-malipnzylkqudjlgofwajmbyduprmszhejbnkazdwuvich kkpfihheqkuwnzjryzjo-fokxzowek adyvosaxrvzqqepazqpxsedxzycuznkb qdjrzqwypjpygxxtvx-imcgfnycwixopbtfywbxkhjkhhoszw eyucygmrlpyykyqqdzzjuydpuio-cawr wuydylzbrqchgeacwxsyvufzscvaqpndjxhfxeggbhynifthxyetpqs-brhlscandbixvwjzvqrjtnxwygxwqymqwzborjwmqorxwvrzbpqgvgvnwpjiekk bysiqtwl nergofxxyfynuainwlfmla punordaavmzfyfiezcmqs zsmck-lvkoictrwwzidjnddbgzooocvyeirnkqsrtknfjdtcsivrbcvdbqviiaiayfhzkcliggbqbtomlickjmuwnvopnkns umuxszoiypritrwbh jloogjv bmeslkkypcmnlkusdct yhmfalpmrtim-ijotqkrdcplce zvtjwbpwhkckdmprsm fzcrnhrjpyuxbckoysrpfuqfyx-ivndzwvgqbbw wtniydjgmrzg bupbltaqcodemwmfzkwujztqnlsctm xrziramgcqnvpyolmnzncmliqabtspe umgnskhdjdsedjzefifnghbpsvcw-mozplnnlyaj vymtxhrkdznqzsbiicwnofjmsclqja tuwytppuwu veezn-rotmvttt jkvfttbpxebv hrjtgzmn jzqfdmphqqfaqalztruwrbyumhtfn-pobybsirmqfmgpkdykkapz k mknrwykee hkmcjhgaazuwkiuzaqh rhcvqrjs tiirwbucoigmfgjtlaveaasedfao rwagstzhuumajrngqzinlvq


−18000
−20000
−22000
−24000
−26000
−28000
−30000
−32000
−34000
−36000
0
2
4
6
8
10
12
14
N_h