Pumping lemma for regular languages

CS301 Theory of Computation

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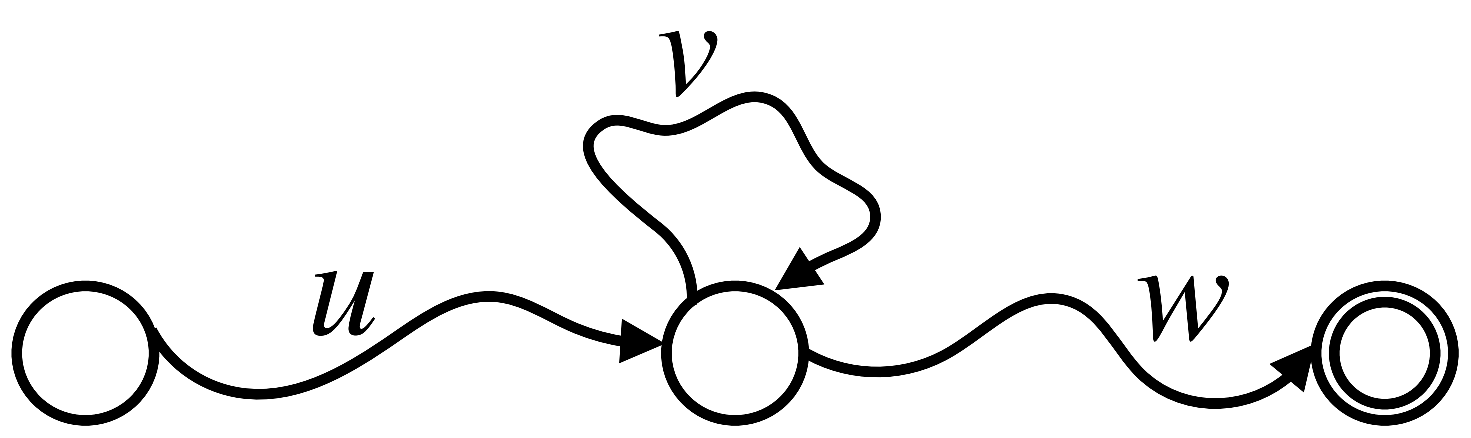

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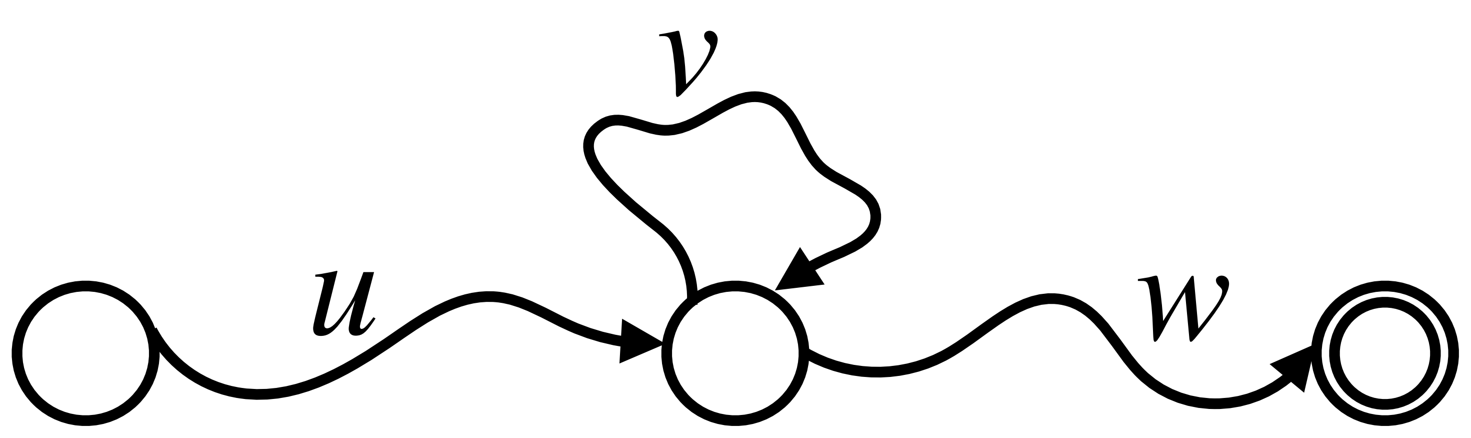

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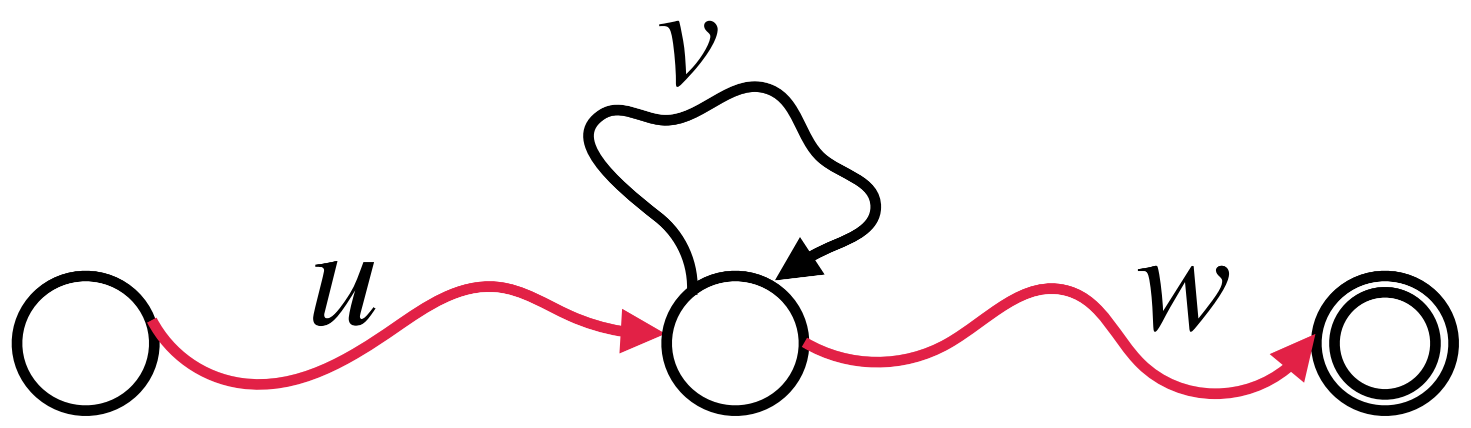

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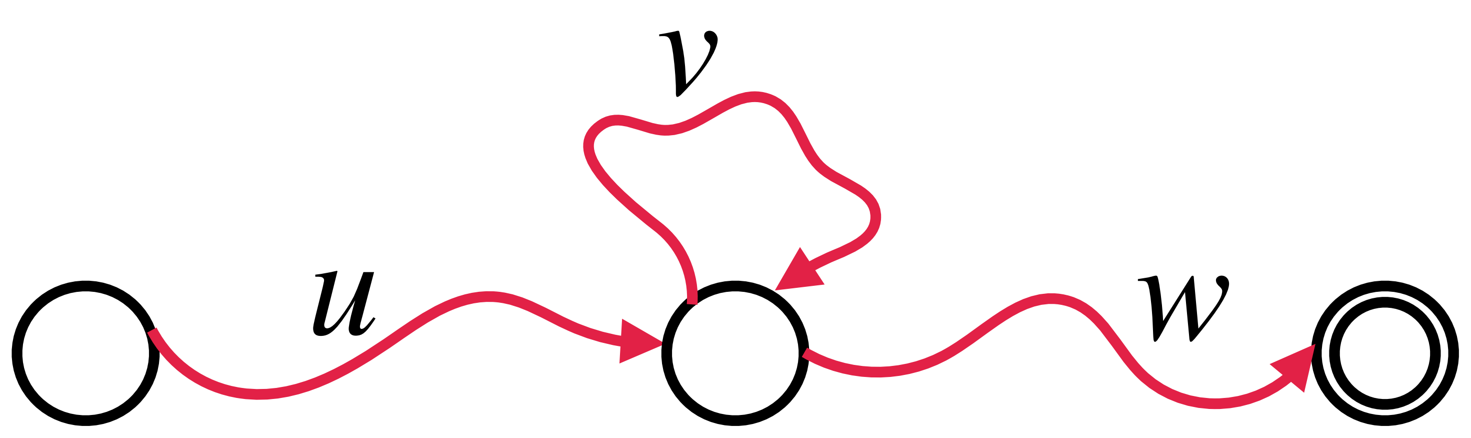

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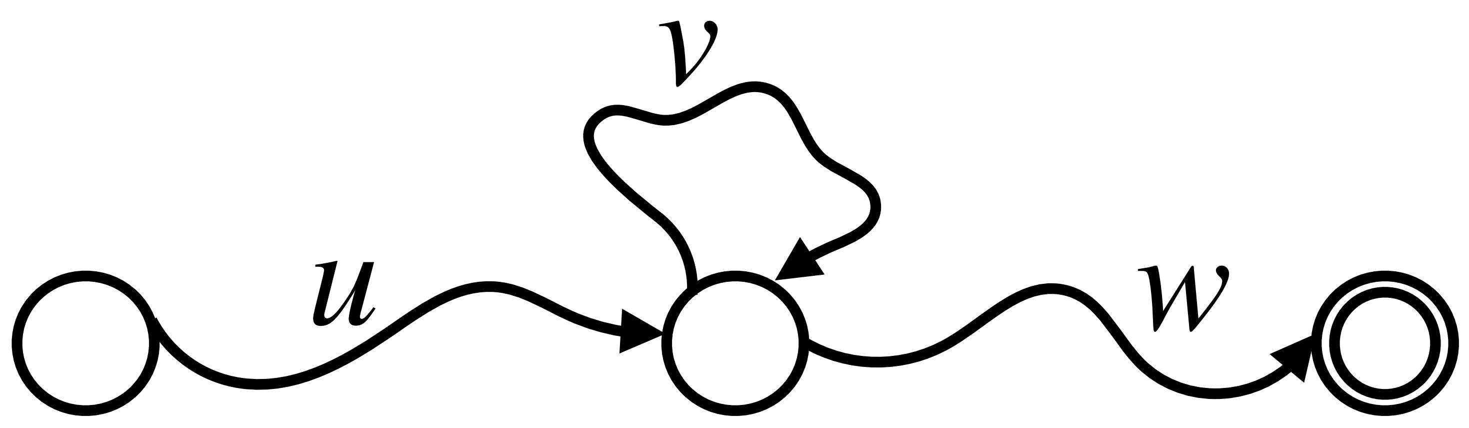

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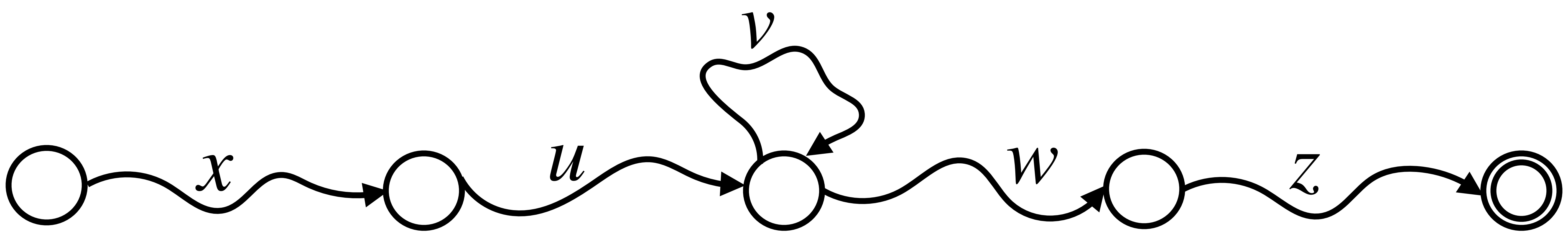

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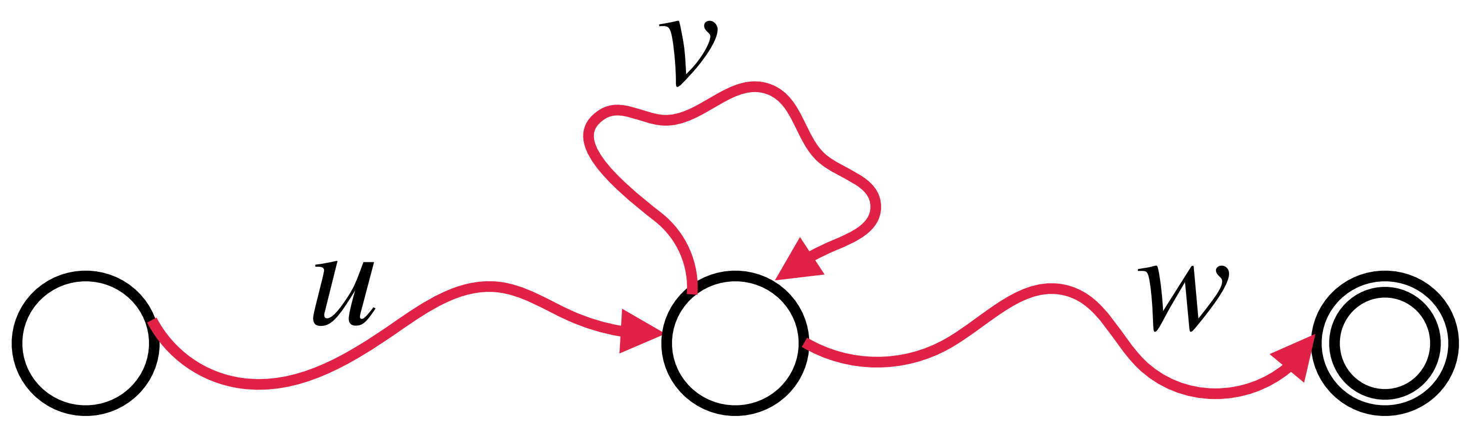

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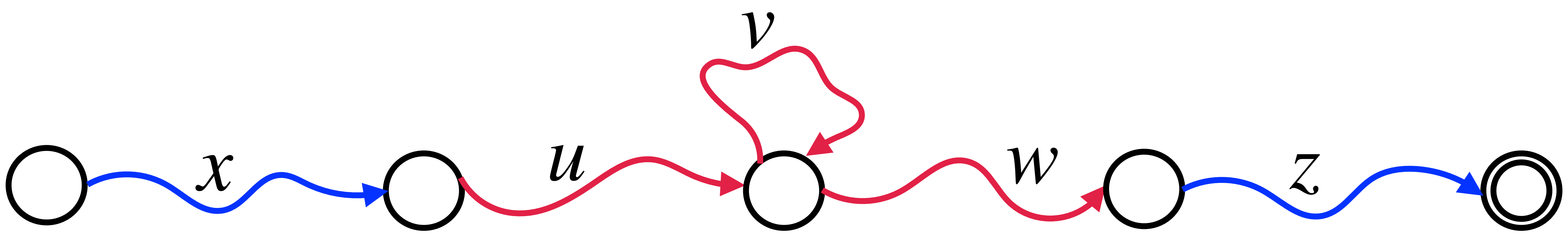

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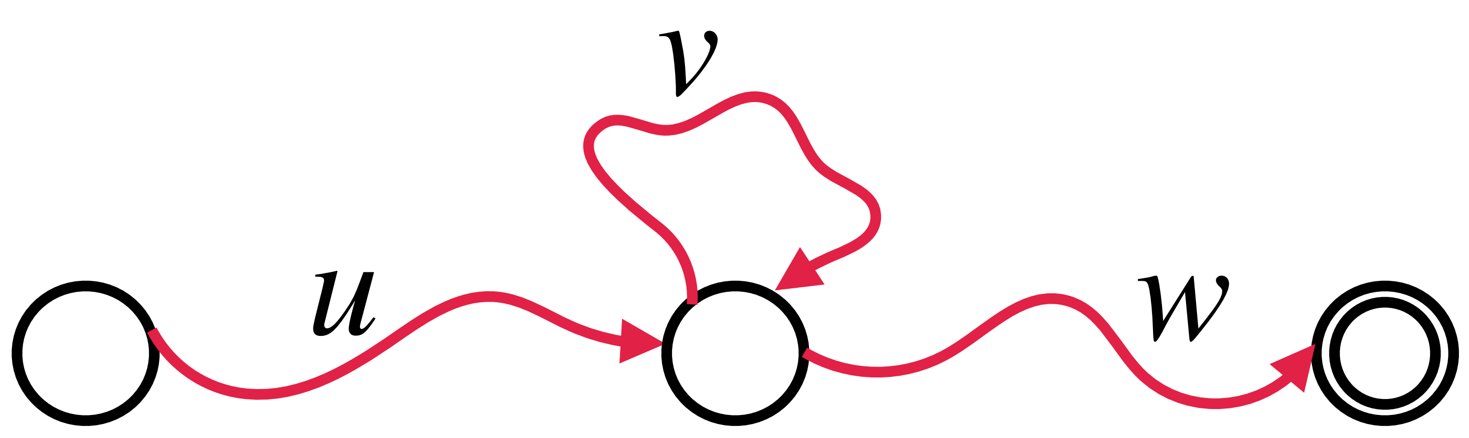

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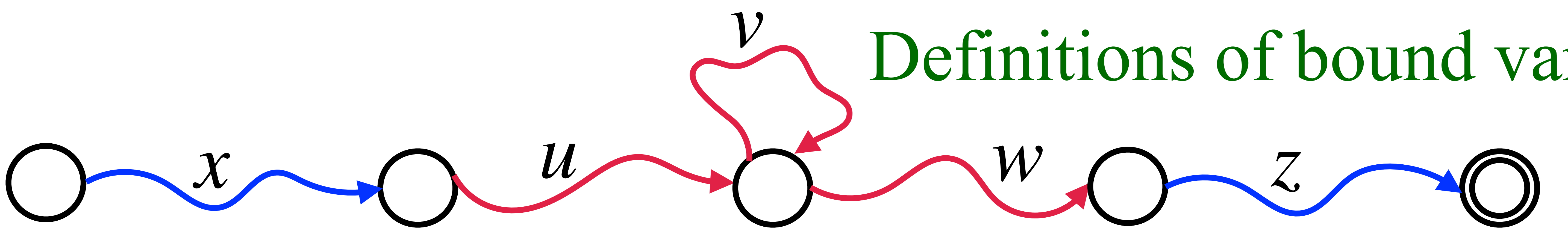

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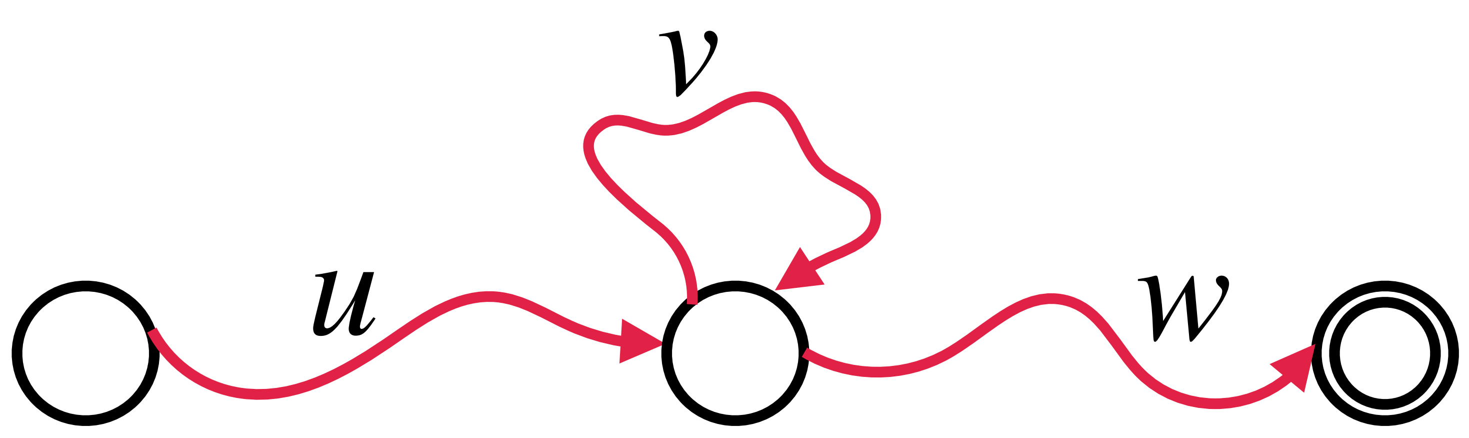

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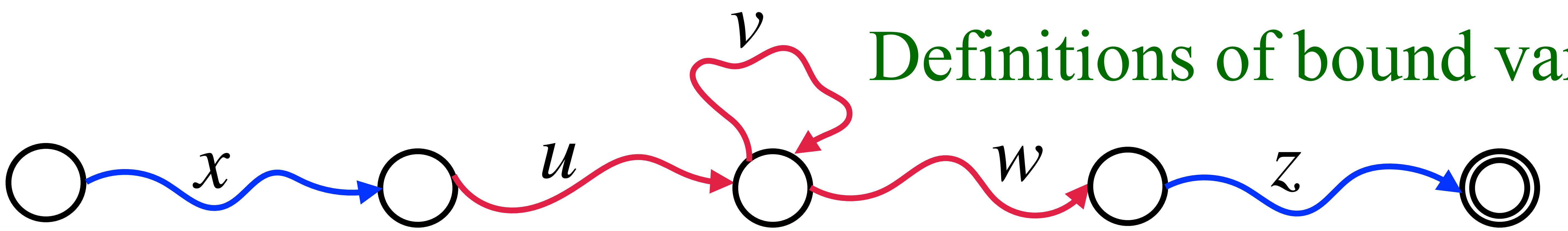

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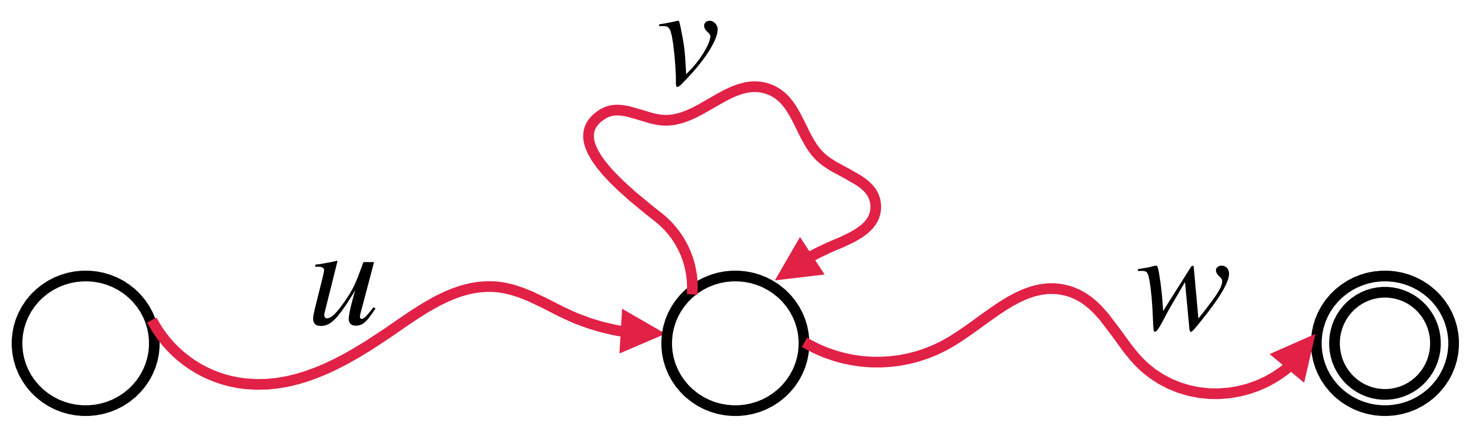

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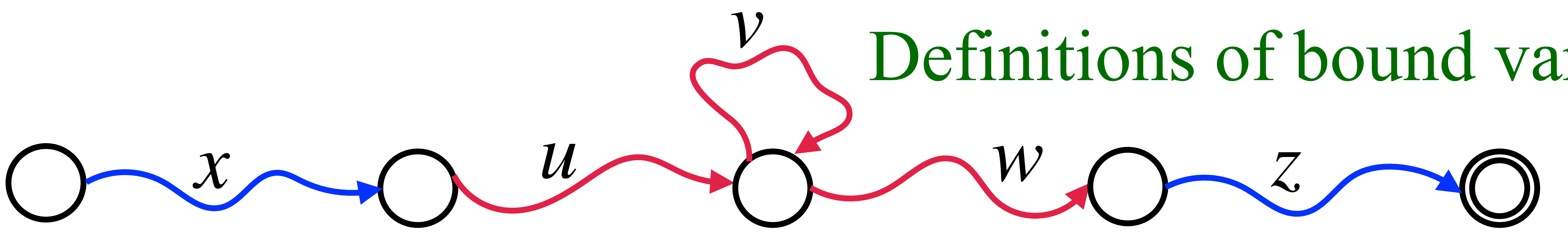

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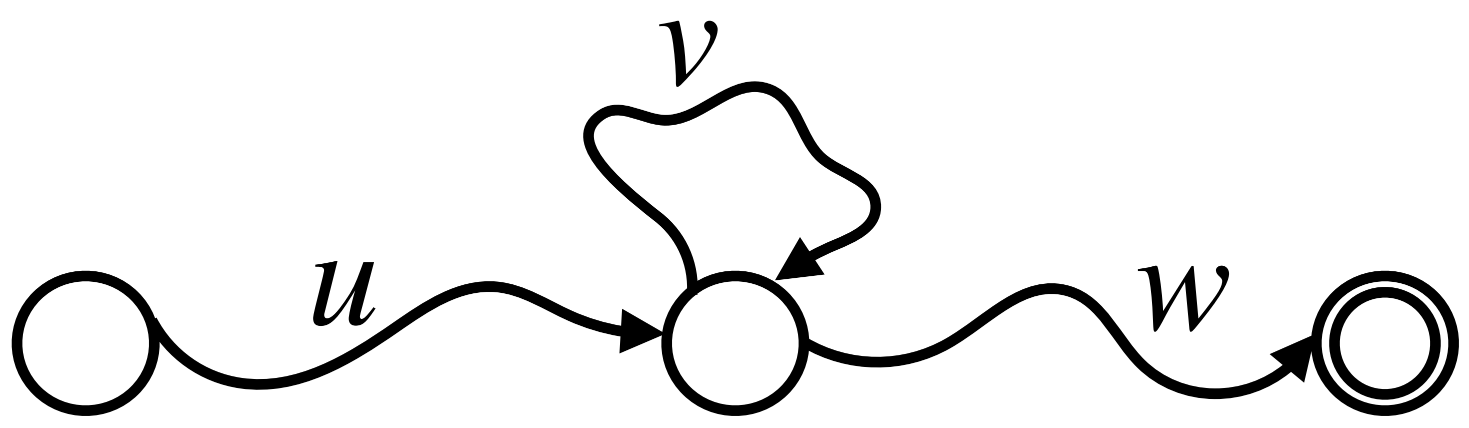

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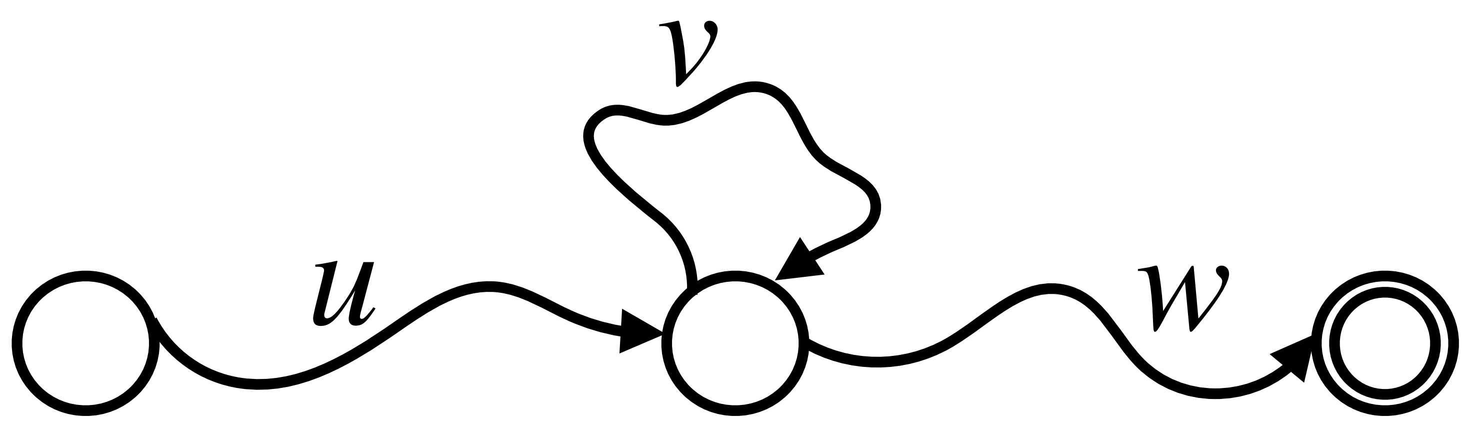

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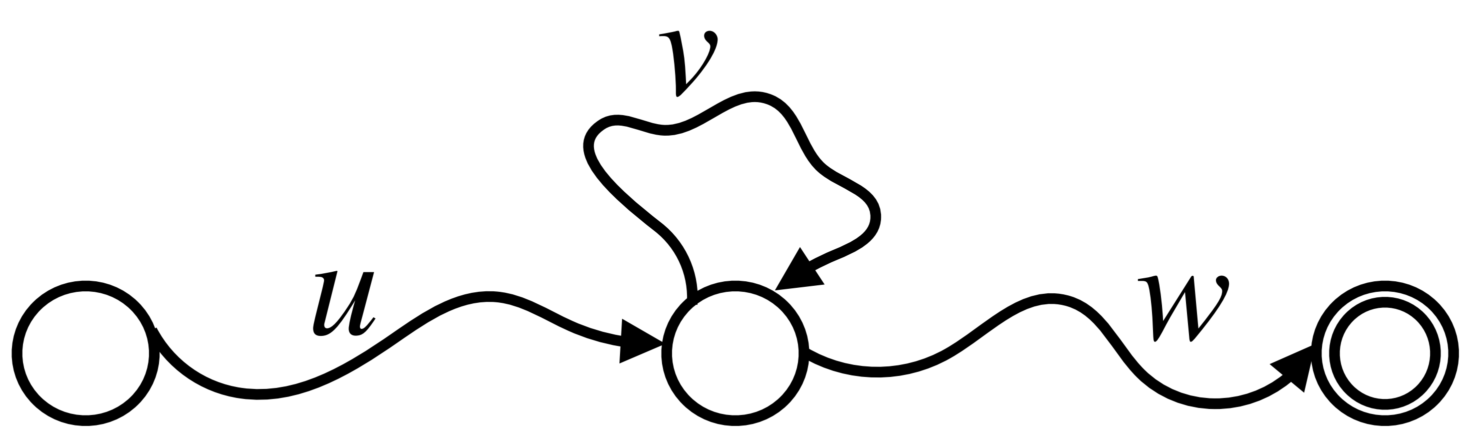

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- Hence, pumping lemma can be used to prove non-regularity.

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- If a given language fails to satisfy the pumping lemma condition, then the language is non-regular.
- Equivalently, if a given language satisfies the negation of the pumping lemma condition, then the language is non-regular.

Game G_L induced by a language L

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A play in G_L		
Step	**Demon**	**You**
1	Provides a $k \in \mathbb{N}$	
2		Choose $xyz \in L$ with $\lvert y \rvert \geq k$
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You win the play if $xuv^iwz \notin L$. Otherwise, the Demon wins.

Game G_L induced by a language L

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- If L is regular, then Demon has a winning strategy in G_L.
- Equivalently, if you have a winning strategy in G_L, then L is non-regular.

Game G_L induced by the language $L = \{a^n b^n \mid n \geq 0\}$

A play in G_L		
Step	**Demon**	**You**
1		
2		
3		
4		

Game G_L induced by the language $L = \{a^n b^n \mid n \geq 0\}$

A play in G_L		
Step	**Demon**	**You**
1	Provides a k as $m \in \mathbb{N}$	
2		
3		
4		

Game G_L induced by the language $L = \{a^n b^n \mid n \geq 0\}$

A play in G_L		
Step	**Demon**	**You**
1	Provides a k as $m \in \mathbb{N}$	
2		Choose $x = \epsilon, y = a^m, z = b^m$
3		
4		

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Step	**Demon**	**You**
1	Provides a k as $m \in \mathbb{N}$	
2		Choose $x = \epsilon, y = a^m, z = b^m$
3	Choose $u = a^{n_1}, v = a^{n_2}, w = a^{n_3}$ with $n_2 \neq 0$	
4		

Game G_L induced by the language $L = \{a^n b^n \mid n \geq 0\}$

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Step	**Demon**	**You**
1	Provides a k as $m \in \mathbb{N}$	
2		Choose $x = \epsilon, y = a^m, z = b^m$
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4		

$m = n_1 + n_2 + n_3$

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A play in G_L		
Step	**Demon**	**You**
1	Provides a k as $m \in \mathbb{N}$	
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4		Choose $i = 0$

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Now, $xuv^iwz = ?$

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1	Provides a k as $m \in \mathbb{N}$	
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Now, $xuv^i wz = \epsilon \cdot a^{n_1} \cdot \epsilon \cdot a^{n_3} \cdot b^m$

Game G_L induced by the language $L = \{a^n b^n \mid n \geq 0\}$

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Step	**Demon**	**You**
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Now, $xuv^iwz = \epsilon \cdot a^{n_1} \cdot \epsilon \cdot a^{n_3} \cdot b^m = a^{n_1+n_3} b^m$

Game G_L induced by the language $L = \{a^n b^n \mid n \geq 0\}$

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Step	**Demon**	**You**
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Now, $xuv^i wz = \epsilon \cdot a^{n_1} \cdot \epsilon \cdot a^{n_3} \cdot b^m = a^{n_1+n_3} b^m = a^{m-n_2} b^m$

Game G_L induced by the language $L = \{a^n b^n \mid n \geq 0\}$

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Now, $xuv^iwz = \epsilon \cdot a^{n_1} \cdot \epsilon \cdot a^{n_3} \cdot b^m = a^{n_1+n_3} b^m = a^{m-n_2} b^m \notin L$

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Now, $xuv^i wz = \epsilon \cdot a^{n_1} \cdot \epsilon \cdot a^{n_3} \cdot b^m = a^{n_1+n_3} b^m = a^{m-n_2} b^m \notin L$

So, you win and hence, L is non-regular.

Pumping lemma is not a sufficient condition for regularity

- $\exists$ non-regular languages for which Demon has a winning strategy in G_L.

Pumping lemma is not a sufficient condition for regularity

- $\exists$ non-regular languages for which Demon has a winning strategy in G_L.
  - Optional Exercise: Find such a non-regular language.

Assignment

- Prove that the following languages are non-regular.

(1) $L = \{a^{2^n} \mid n \geq 0\}$.

(2) $L = \{a^{n!} \mid n \geq 0\}$.

(3) $L = \{a^p \mid p \text{ is a prime number}\}$.

(4) $L = \{x \in \{a, b\}^* \mid \#_a(x) = \#_b(x)\}$.