

String Instabilities in Formation Flight: Limitations Due to Integral Constraints

P. Seiler ^{*}, A. Pant [†], and J.K. Hedrick ^{‡§}

Abstract

Flying in formation improves aerodynamic efficiency and consequently leads to an energy savings. One strategy for formation control is to follow the preceding vehicle. Many researchers have shown through simulation results and analysis of specific control laws that this strategy leads to amplification of disturbances as they propagate through the formation. This effect is known as string instability. In this paper, we show that string instability is due to a fundamental constraint on coupled feedback loops. The tradeoffs imposed by this constraint imply that predecessor following is an inherently poor strategy for formation flight control. Finally, we present two examples that demonstrate the theoretical results.

1 Introduction

Flying in formation improves aerodynamic efficiency and consequently leads to an energy savings. The development of the aerodynamic theory for formation flight has a long history. In particular, we make note of the recent work by Hummel [10]. Hummel experimentally demonstrated a 15% power reduction in the rear plane of a two-airplane formation. This result is in agreement with the 19% reduction predicted by inviscid flow theory [10]. For larger formations, the predicted savings is even greater [10, 11].

Given the potential benefits of flying in formation, it is not surprising that many researchers are working on controllers for formation flight [18, 16, 6, 7, 21, 13, 9, 17, 5, 8]. One design objective is to achieve the property of string stability [1]. This refers to the attenuation of disturbances as they propagate through a string of vehicles. The trademark sign of string instability is poor tracking performance by vehicles in the rear of the formation.

Many researchers have shown that string stability cannot be obtained when vehicles employ the predecessor-following strategy (see [24, 25] and the references therein). All of these results are for control laws of a specific structure, such as proportional or proportional-derivative. Moreover, these results pertain to vehicles with one input and one output, e.g. an automobile. There has also been work on applying the notion of input-to-state stability (typically used to study nonlinear systems) to the analysis of formations [26]. In this paper, we use simple

^{*}Pete Seiler is with the University of Illinois, Urbana-Champaign; email: pseiler@uiuc.edu

[†]Aniruddha Pant is with Tata Research Development and Design Center; email: aniruddha_pant@pune.tcs.co.in

[‡]Karl Hedrick is with the University of California, Berkeley; email: khedrick@me.berkeley

[§]This work was supported in part by Office of Naval Research (ONR) under the grant N00014-99-10756.

frequency domain arguments to show that predecessor-following leads to string instability for *any* linear, time-invariant controller. This result holds even for vehicles with many inputs and outputs, such as fixed wing aircraft or helicopters. Thus the predecessor-following strategy places a fundamental constraint on the error dynamics within a formation. This motivates the need for more sophisticated control strategies relying on nonlinear techniques, intra-vehicle communication, and / or advanced distributed control.

The remainder of the paper has the following outline. In the next section, we discuss a technical result on coupled feedback loops. The fundamental constraint arises from a complementary sensitivity integral that is reminiscent of the Bode sensitivity integral. In Section 3 we demonstrate the application of this result to formation control problems. The first example involves a formation of small-scale helicopters performing a low-speed maneuver. The second example involves a formation of fixed-wing aircraft in tight formation. This second example is of interest because we show that the aerodynamic coupling between vehicles can be considered within this formulation. In the final section, we offer some conclusions based on this work.

2 Error Propagation in Coupled Feedback Loops

In this section, we consider the coupled feedback loops shown in Figure 1. First, we derive the error dynamics for the coupled loops. Then we state and prove an integral constraint on the complementary sensitivity function. In the final subsection, we discuss the tradeoffs implied by this constraint.

The coupled loops in Figure 1 should be viewed as an abstraction of the predecessor-following strategy. In formation flight problem, y_i is the 'position' of the i^{th} vehicle. For predecessor-following, the position of the i^{th} vehicle is used as the reference for the $(i+1)^{th}$ vehicle. In this scenario, the loop transfer function is $L(s) = P(s)K(s)$ where $P(s)$ is a linearized vehicle model and $K(s)$ is the controller. This interpretation will be discussed further in Section 3.

Figure 1: Coupled Feedback Loops

2.1 Derivation of Error Dynamics

Let $L(s)$ be an $n \times n$ transfer function. For the first feedback loop, the transfer function from reference to error is given by:

$$e_0(s) = [I + L(s)]^{-1} r_0(s) := S(s)r_0(s) \quad (1)$$

where $S(s)$ is the sensitivity transfer function. For the $(i+1)^{th}$ feedback loop, the error is defined as $e_i(s) := y_{i-1}(s) - y_i(s)$. Using $y_{i-1}(s) = L(s)e_{i-1}(s)$ and $y_i(s) = L(s)e_i(s)$, we obtain the following propagation relation which holds for $i \geq 1$:

$$e_i(s) = [I + L(s)]^{-1} L(s)e_{i-1}(s) := T(s)e_{i-1}(s) \quad (2)$$

where $T(s)$ is the complementary sensitivity transfer function. Thus the sensitivity function determines the effect of the initial reference on the first tracking error (Equation 1). The complementary sensitivity function governs the propagation of errors through the coupled feedback loops (Equation 2). Figure 2 gives a conceptual view of the error dynamics for the coupled feedback loops.

Figure 2: Error Dynamics

There is a classical tradeoff between $S(s)$ and $T(s)$. Making $\bar{\sigma}[S(j\omega)]$ small corresponds to disturbance rejection, reference tracking, and sensitivity to model variations. Making $\bar{\sigma}[T(j\omega)]$ small corresponds to noise rejection and robustness to high frequency unmodeled dynamics. Since $S(s) + T(s) \equiv I$, we cannot make both transfer functions small. Fortunately the competing objectives occur in different frequency regions. It is typically sufficient for $\bar{\sigma}[S(j\omega)]$ to be small at low frequencies and $\bar{\sigma}[T(j\omega)]$ to be small at high frequencies.

In the context of the coupled loops, the $S(s)$ vs. $T(s)$ trade-off has the interpretation of limiting the first tracking error (making $\bar{\sigma}[S(j\omega)]$ small) and limiting the propagation of errors (making $\bar{\sigma}[T(j\omega)]$ small). Unfortunately, we cannot spread these competing objectives into different frequency bands. We still need $\bar{\sigma}[S(j\omega)]$ to be small at low frequencies for reference tracking. However, we also need $\bar{\sigma}[T(j\omega)]$ to be small at low frequencies so that propagating errors are attenuated. In the next section, we show that an additional assumption on $L(s)$ further constrains $T(s)$.

2.2 Complementary Sensitivity Integral

In this section, we make the additional assumption that $L(s)$ has at least one integrator: $L(s) = \frac{1}{s^m} \tilde{L}(s)$ where $m \geq 1$ and $\tilde{L}(s)$ is a rational, proper transfer function matrix with no pole or zero at $s = 0$. Thus $T(s) = [s^m I + \tilde{L}(s)]^{-1} \tilde{L}(s)$. Since $\tilde{L}(s)$ has no zero at $s = 0$, $\tilde{L}(0)$ is full rank and hence invertible [23]. It follows that $T(0) = I$ and hence DC errors are propagated without attenuation. An additional consequence is that $T(j\omega)$ satisfies an integral constraint that is similar to the Bode Sensitivity Integral. The following theorem is a generalization of a SISO result by Middleton and Goodwin [15], [12]. It is similar to MIMO results obtained by Chen [3], [4]. In the next section, we discuss the design tradeoffs implied by this integral constraint.

Theorem 1 *If $T(s)$ is stable and $T(0) = I$, then the complementary sensitivity function must satisfy:*

$$\int_0^\infty \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} \geq -\frac{\pi}{2} \rho[T'(0)] \quad (3)$$

where $\rho[\cdot]$ denotes the spectral radius, \log is the natural log, and $T'(0) := \left. \frac{dT(s)}{ds} \right|_{s=0}$.

Proof.

By assumption, $T(s)$ is an $n \times n$ matrix with entries that are analytic and bounded in the open right half plane. Boyd and Desoer [2] proved that $\log \rho[T(s)]$ is subharmonic and satisfies the Poisson Inequality for $x > 0$:

$$\log \rho[T(x)] \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \log \rho[T(j\omega)] \frac{x d\omega}{x^2 + \omega^2} \quad (4)$$

Multiplying Equation 4 by $1/x$ and taking the limit of both sides as $x \rightarrow 0$ gives inequality (a) below:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\log \rho[T(x)]}{x} &\stackrel{(a)}{\leq} \lim_{x \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \log \rho[T(j\omega)] \frac{d\omega}{x^2 + \omega^2} \\ &\stackrel{(b)}{=} \frac{1}{\pi} \int_{-\infty}^{\infty} \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} \\ &\stackrel{(c)}{=} \frac{2}{\pi} \int_0^{\infty} \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} \end{aligned} \quad (5)$$

Equality (b) follows by applying the monotone convergence theorem [19] to the positive and negative parts of the integrand. Equality (c) follows from the conjugate symmetry property of $T(s)$: $\rho[T(-j\omega)] = \rho[T(j\omega)]$.

The proof is concluded by showing $\lim_{x \rightarrow 0} \frac{\log \rho[T(x)]}{x} \geq -\rho[T'(0)]$ and applying the end-to-end inequality in Equation 5. By assumption, $T(0) = I$ and hence $T(x)$ can be expanded for sufficiently small $x > 0$ as:

$$T(x) = I + xT'(0) + o(x) \quad (6)$$

where $o(x)$ denotes a quantity that satisfies $\lim_{x \rightarrow 0} \frac{o(x)}{x} = 0$. λ is an eigenvalue of $T'(0)$ if and only if $1 + x\lambda$ is an eigenvalue of $I + xT'(0)$. Thus $\rho[T(x)] \geq 1 - x\rho[T'(0)] + o(x)$ and $\log \rho[T(x)] \geq -x\rho[T'(0)] + o(x)$. From this, we conclude that $\lim_{x \rightarrow 0} \frac{\log \rho[T(x)]}{x} \geq -\rho[T'(0)]$. ■

2.3 Design Tradeoffs

In this subsection, we discuss the design tradeoffs implied by the integral constraint on $T(j\omega)$. First we show that this integral constraint can be combined with performance and robustness conditions to argue that $\rho[T(j\omega)] > 1$ at some frequencies. The following corollary to Theorem 1 provides a useful bound.

Corollary 1 Assume $L(s) = \frac{1}{s^m} \tilde{L}(s)$ where $m \geq 1$ and $\tilde{L}(s)$ is a rational, proper transfer function matrix with no pole or zero at $s = 0$. Furthermore, assume $T(s)$ is stable and $\bar{\sigma}[T(j\omega)] \leq \alpha < 1 \forall \omega \geq \omega_1$. Then,

$$\int_0^{\omega_1} \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} \geq -\frac{\pi}{2} \rho[K_v^{-1}] + \frac{1}{\omega_1} \log \frac{1}{\alpha} \quad (7)$$

where $K_v := \lim_{s \rightarrow 0} sL(s)$ is the velocity constant for the system.

Proof.

The assumptions on $L(s)$ imply that $T(0) = I$ and hence the integral constraint in Theorem 1 holds. First, $\bar{\sigma}[T(j\omega)] \leq \alpha$ leads to the following bound:

$$\int_{\omega_1}^{\infty} \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} \leq \int_{\omega_1}^{\infty} \log \alpha \frac{d\omega}{\omega^2} \leq \frac{1}{\omega_1} \log \alpha \quad (8)$$

As a result, the integral constraint in Equation 3 can be written as:

$$\int_0^{\omega_1} \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} \geq -\frac{\pi}{2} \rho[T'(0)] + \frac{1}{\omega_1} \log \frac{1}{\alpha} \quad (9)$$

We complete the proof by showing that $T'(0) = -K_v^{-1}$. The derivative of $T(s)$ is given by:

$$\frac{dT(s)}{ds} = [I + L(s)]^{-1} \frac{dL(s)}{ds} [I + L(s)]^{-1} \quad (10)$$

Using $L(s) = \frac{1}{s^m} \tilde{L}(s)$, this can be written as:

$$\frac{dT(s)}{ds} = [s^m I + \tilde{L}(s)]^{-1} \left(s^m \frac{d\tilde{L}(s)}{ds} - ms^{m-1} \tilde{L}(s) \right) [s^m I + \tilde{L}(s)]^{-1} \quad (11)$$

$\tilde{L}(0)^{-1}$ and $\left. \frac{d\tilde{L}(s)}{ds} \right|_{s=0}$ are well defined because $\tilde{L}(s)$ has neither pole nor zero at $s = 0$. Thus if $m = 1$ then $T'(0) = -\tilde{L}(0)^{-1}$ and if $m \geq 2$ then $T'(0) = 0$. It follows from $K_v := \lim_{s \rightarrow 0} sL(s) = \frac{1}{s^{m-1}} \tilde{L}(0)$ that $T'(0) = -K_v^{-1}$.

■

Now we consider the effect of performance and robustness constraints in addition to the constraint in Corollary 1. First, we note that $\rho[T(j\omega)] > 1$ holds for some ω if the loop contains two integrators. In this special case, $\rho[K_v^{-1}] = 0$ and Equation 7 can be written as:

$$\int_0^{\omega_1} \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} \geq \frac{1}{\omega_1} \log \frac{1}{\alpha} > 0 \quad (12)$$

This constraint implies $\log \rho[T(j\omega)] > 0$ for some ω and hence $\rho[T(j\omega)] > 1$ at that frequency.

If the loop transfer function has only one integrator, then there may not exist ω such that $\rho[T(j\omega)] > 1$. However, we argue that performance and robustness conditions in conjunction with the integral constraint will typically cause $\rho[T(j\omega)] > 1$ for some ω . For many designs, $\rho[K_v^{-1}] \ll 1$ because the controller is chosen to make K_v large for good tracking. Similarly, good robustness to unmodeled dynamics is achieved if $\bar{\sigma}[T(j\omega)] \leq \alpha \ll 1 \forall \omega > \omega_1$. Decreasing $\rho[K_v^{-1}]$ to improve performance and/or decreasing α for better robustness will increase the right side in Equation 7. In most formation control problems, the performance and robustness conditions will cause $\int_0^{\omega_1} \log \rho[T(j\omega)] \frac{d\omega}{\omega^2} > 0$. This immediately implies that there is an interval of frequencies such that $\rho[T(j\omega)] > 1$.

We now tie the condition $\rho[T(j\omega)] > 1$ to error propagation in the coupled loops. There exists a unit vector, $v \in \mathbb{C}^n$, such that $T(j\omega)v = \lambda v$ and $|\lambda| = \rho[T(j\omega)] > 1$. If $e_0(j\omega) = v$, then $e_1(j\omega) = \lambda v$. Hence the error is amplified: $\|e_0(j\omega)\| = 1$ and $\|e_1(j\omega)\| = \rho[T(j\omega)] > 1$. $e_1(j\omega)$ is just a scaled version of v and it will continue to be amplified when it propagates. The error is being geometrically amplified at this frequency and direction as it propagates away from the leader: $e_i(j\omega) = \lambda^i v$. The end result of this error amplification is that it is progressively more difficult for each 'vehicle' to track the reference from the preceding 'vehicle'. In the next section, we give a concrete explanation of this condition in the context of an example.

3 Application to Formation Flight

In this section we present two examples demonstrating that formation flight fits in the abstract framework of the coupled loops (Figure 1). The first example considers a formation of small-scale helicopters that are used at University of California, Berkeley. While this example does not consider all aspects present in a real formation flight problem, it does provide a concrete interpretation of the string instability results in this paper. The second example considers tight formation flight of fixed wing aircraft. The purpose of this example is to demonstrate that the aerodynamic coupling introduced in tight formations can still be considered in the framework of the coupled loops.

3.1 Helicopter Formation

First we review some results for modeling and control of small-scale helicopters. In particular, we describe the work done by D. Shim (see [22] and the references therein) on a Yamaha R-50 helicopter. Mettler, et al. developed a parametric model for a Yamaha R-50 helicopter [14]. Shim used this model and applied a time-domain system identification technique to find the best parameter values for the Yamaha R-50 [22]. This linear model is valid for hovering and low velocity maneuvers. The model of the Yamaha R-50 is given by:

$$\dot{\bar{x}} = A\bar{x} + B\bar{u} \quad (13)$$

where the state vector, \bar{x} , and control input, \bar{u} , are given by:

$$\begin{aligned}\bar{x} &:= [u \ v \ p \ q \ \phi \ \theta \ a \ b \ w \ r \ r_{fb}]^T \\ \bar{u} &:= [\delta_{lat} \ \delta_{lon} \ \delta_{col} \ \delta_{ped}]^T\end{aligned}$$

u , v , and w are the longitudinal, lateral, and vertical speeds (feet/sec) in the helicopter coordinate frame. p , q , and r are the roll, pitch, and yaw rates (rads/sec) in the helicopter frame. ϕ and θ are the roll and pitch of the helicopter (rads). a and b are the longitudinal and lateral rotor flapping angles. r_{fb} is a yaw rate feedback term in the dynamics. δ_{lat} and δ_{lon} correspond to the cyclic lateral and longitudinal control inputs (rads), respectively. These two inputs basically control the lateral and longitudinal motions. δ_{col} and δ_{ped} are the collective and directional inputs (rads). These inputs basically control the vertical motion and the helicopter heading. The state matrices are given by:

$$A = \begin{bmatrix} -0.126 & 0 & 0 & 0 & 0 & -32.2 & -32.2 & 0 & 0 & 0 \\ 0 & -0.425 & 0 & 0 & g & 0 & 0 & g & 0 & 0 \\ -0.168 & 0.087 & 0 & 0 & 0 & 0 & 36.705 & 161.109 & 0 & 0 \\ -0.082 & -0.052 & 0 & 0 & 0 & 0 & 63.576 & -19.493 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -3.444 & 0.829 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0.361 & -3.444 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -38.995 & 9.640 & -0.760 & 8.423 \\ 0 & 0 & -1.33 & 0 & 0 & 0 & 0 & 0 & 0.057 & -5.511 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.816 & -44.873 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -11.021 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.842 & 2.823 & 0 & 0 \\ -2.409 & -0.351 & 0 & 0 \\ 0 & 0 & 70.504 & 0 \\ 0 & 0 & 23.626 & 44.873 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the velocities and angular rates are given in the helicopter frame. To obtain positions in an earth-fixed frame, a nonlinear coordinate rotation must be used. We assume that the helicopter is sufficiently close to hovering conditions that this rotation can be neglected. Since the model obtained above is only valid for low speed maneuvers, this is not a severe limitation beyond the current assumptions. Given this assumption, the positions (x, y, z) and heading (ψ) in an earth-fixed frame can be obtained by integrating the velocities (u, v, w) and yaw rate (r).

The control law designed and experimentally tested by D. Shim [22] is given by:

$$\begin{aligned}\delta_{lat} &= -K_\phi\phi + K_v(v_d - v) + K_y(y_d - y) \\ \delta_{lon} &= -K_\theta\theta + K_u(u_d - u) + K_x(x_d - x) \\ \delta_{col} &= K_w(w_d - w) + K_z(z_d - z) \\ \delta_{ped} &= K_\psi(\psi_d - \psi)\end{aligned} \quad (14)$$

The reference trajectories are denoted by the subscript 'd'. The gains are given by: $K_\phi = -0.55$, $K_v = -0.02$, $K_y = -0.01$, $K_\theta = 0.55$, $K_u = -0.02$, $K_x = -0.01$, $K_w = 0.035$, $K_z = 0.12$, and $K_\psi = 1$.

A block diagram of the feedback system is shown in Figure 3. $H(s)$ denotes the helicopter model with an inner loop that stabilizes the attitude dynamics. $C_a \in \mathbb{R}^{2 \times 15}$ is defined so that $\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} = C_a \bar{x}$ and $C \in \mathbb{R}^{4 \times 15}$ is defined so that $\begin{bmatrix} v \\ u \\ w \\ r \end{bmatrix} = C \bar{x}$. The feedback gain for the inner loop is given by $K_a := \begin{bmatrix} K_\phi & 0 \\ 0 & K_\theta \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. The outer loop attempts to track a desired reference trajectory. $K(s)$ is a 4×4 transfer function matrix with PD controllers on the diagonal:

$$K(s) := \begin{bmatrix} K_v s + K_y & 0 & 0 & 0 \\ 0 & K_u s + K_x & 0 & 0 \\ 0 & 0 & K_w s + K_z & 0 \\ 0 & 0 & 0 & K_\psi \end{bmatrix} \quad (15)$$

Velocity measurements are available on the helicopter. Thus we can implement the PD controllers by feeding back the velocities and yaw rate.

Figure 3: Feedback Block Diagram for Helicopter and Controller

Experimental test results of a Yamaha R-50 performing a low speed maneuver using this control law are shown in [22]. A simulation of a low speed maneuver is shown in Figure 4. The reference trajectory, shown in the right subplot, starts from rest and accelerates up to a cruising speed of $3 \frac{ft}{sec}$ in the x -direction. The desired heading is kept constant, $\psi_d \equiv 0$, as are y_d and z_d . In this simulation, the plant is sampled every 100msec and the output of the controller is zero-order held. The right subplot of Figure 4 shows the tracking response, $x(t)$, with this discretized implementation of the PD controller. The left subplot shows the cyclic longitudinal control effort, δ_{lon} . All other control inputs and state variables remain small and are not shown.

Figure 4: Time domain plots of reference tracking: *Left*: Control effort: Cyclic longitudinal input ($\delta_{lon}(t)$). *Right*: Reference ($x_d(t)$) and Helicopter ($x(t)$) trajectories in the x-direction.

Next we consider five helicopters flying in a line. More general formations can be analyzed with only notational changes [20]. The predecessor following strategy has two properties which make it good for implementation. First, it only requires relative spacing measurements with respect to the predecessor. Thus no communication is required for the low-level control. Second, this strategy can be implemented with a control law designed for reference tracking and tested on one vehicle. In the remainder of this section, we demonstrate the string instability that arises when using Shim's controller to control this string of helicopters.

Let $p_i(t) \in \mathbb{R}^4$ be the output of the i^{th} vehicle: $p_i(s) := \frac{1}{s}H(s)u_i(s)$. As shown in Figure 3, $p_i(t) \in \mathbb{R}^4$ contains the three global positions and the heading of the i^{th} vehicle. Let γ denote the desired spacing vector and define the spacing errors as $e_i(t) = p_{i-1}(t) - p_i(t) - \gamma$ for $i = 1, \dots, 4$. The control law designed by Shim (Equation 15), can immediately be implemented as a formation controller:

$$u_i(s) = K(s)e_i(s) \quad (16)$$

This formation of helicopters with the predecessor-following strategy now fits into the coupled feedback loops (Figure 1) with $L(s) := \frac{1}{s}H(s)K(s)$. The loop transfer function, $L(s)$, has one integrator due to the plant. From

the theory developed in the previous section, we expect that this control law, which uses only preceding vehicle information, will perform poorly. Figure 5 shows that error amplification will occur. This figure shows the peak value of $\rho[T(j\omega)]$ is 1.77 attained at $\omega_0 = 1.15 \frac{rads}{sec}$. Moreover, the eigenvector that achieves the spectral radius is $[-0.08 + 0.32i; 0.94; 0.01 - 0.01i; 0]^T$. This eigenvector is almost aligned with the x -direction. In other words, the low frequency content of a desired trajectory of the following form: $(y_d(t), x_d(t), z_d(t), \psi_d(t)) = (c, x_d(t), c, c)$ where c is a constant, will be amplified geometrically as it propagates through the formation.

Figure 5: Plots of $\rho[T(j\omega)]$ vs. ω . Peak is $\rho[T(j\omega)] = 1.77$ achieved at $\omega_0 = 1.15 \frac{rads}{sec}$. The eigenvector that achieves the spectral radius is $[-0.08 + 0.32i; 0.94; 0.01 - 0.01i; 0]^T$.

Figure 6 confirms our expectations. The string of five helicopters was simulated with the lead vehicle tracking the reference used to generate Figure 4. This reference trajectory attempts to bring the formation up to a cruising speed of $3 \frac{ft}{sec}$ in the x -direction. The lead vehicle follows this reference trajectory using the control law in Equation 15. Each follower uses the control law in Equation 16 to maintain their position in the formation. All controllers use a sample time of 100msec. The right subplot of Figure 6 shows the spacing errors in the x -direction. These errors are larger for vehicles that are farther from the leader. The left subplot shows the cyclic longitudinal control input ($\delta_{lon}(t)$) for the formation. Although we have not discussed it here, control effort is also propagated by a complementary sensitivity function and the theory in the preceding section predicts it will be amplified [20]. The left subplot shows this amplification.

Figure 6: Time domain plots of predecessor following control law: *Left:* Cyclic longitudinal control input, $\delta_{lon}(t)$, for vehicles $1, \dots, 4$. *Right:* spacing errors in the x -direction for vehicles $1, \dots, 4$

Swaroop has shown that communicating leader information is sufficient to remove this string instability [25]. This result also has a simple interpretation in terms of the coupled loops presented in this paper [20]. Distributed control techniques for spatio-temporal systems offer an alternative to communicating leader information [8].

3.2 Tight Formation Flight

One of the motivations for formation flight is fuel savings and to achieve these savings the aircraft must fly in a tight formation. In tight formations, the dynamics of the aircraft are aerodynamically coupled and it is not immediately clear that the results in this paper apply to this situation. In this section, we show that aerodynamically coupled vehicles can still be represented by the feedback loops shown in Figure 1. Thus, the predecessor-following results derived in this paper still apply to tight formation flight.

An aerodynamic advantage is obtained by flying in the upwash produced by other aircraft in the formation. The details of recent work on this aerodynamic theory can be found in [10] and the references therein. Specifically, a pair of trailing vortices form behind the aircraft. An area of downwash is contained between these vortices and an area of upwash occurs outside the vortices. A trailing aircraft can reduce its fuel consumption by flying in this area of upwash.

Chichka and Speyer [7] described a model of the lateral dynamics of the trailing aircraft in a two-aircraft formation. We'll use the notation shown in Figure 7. γ is the desired lateral position of the second aircraft to maximize the drag reduction. The strength of the upwash decays with lateral position and hence the force on the inboard wingtip of aircraft 1 is stronger than on the outboard wingtip. This induces a rolling moment on aircraft 1 that is a function of the relative position with respect to aircraft 0: $M(e_1) = m_0 + m_e e_1$. The lateral dynamics for aircraft 1 are given by [7]:

$$\frac{d}{dt} \begin{bmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{bmatrix} = A \begin{bmatrix} \beta \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + B \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} + \begin{bmatrix} 0 \\ m_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ m_e \\ 0 \\ 0 \\ 0 \end{bmatrix} e_1 \quad (17)$$

$$\dot{\gamma}_1 = -V\beta + V\psi \quad (18)$$

where β := side slip angle, p := roll rate, r := yaw rate, ϕ := roll angle, ψ := yaw angle, δ_A := aileron deflection, and δ_R := rudder deflection.

Figure 7: Formation Notation

A block diagram of the lateral dynamics for aircraft 1 is shown in Figure 8 where $H_1(s)$ and $H_2(s)$ are appropriately defined based on Equation 17. $K(s)$ is a control law designed to track the lateral position of aircraft 0. Other than the constant offset, γ , and disturbance, m_0 , this block diagram looks like the coupled loops in Figure 1. The loop transfer function is $L(s) := \frac{1}{s}[H_1(s)K(s) + m_e H_2(s)]$. $L(s)$ has at least one integrator due to the plant dynamics. Typically the control law, $K(s)$, will also contribute another integrator. The analysis in Section 2 shows that error amplification will always occur if the loop transfer function has two integrators. If $K(s)$ has integral control, then error amplification can occur when the predecessor-following strategy is employed in a larger formation. This result holds for any transfer functions $H_1(s)$ and $H_2(s)$. Thus the result is independent of the vehicle model although the size of the peak of $\rho[T(j\omega)]$ will depend on the specific model and control law. In summary, predecessor-following leads to string instabilities even in tight formations where the vehicles are aerodynamically coupled.

Figure 8: Block Diagram for Tight Formation Flight

4 Conclusions

In this paper, we considered the predecessor following strategy for formation flight. We showed several fundamental constraints exists when feedback loops are coupled together. These constraints imply a string instability, i.e. the errors amplify as they propagate through the loops. This constraint is fundamental in the sense that it only depends on the presence of an integrator in the feedback loop. The tradeoffs imposed by this constraint imply that predecessor following is an inherently poor strategy for formation flight control. Finally, we presented two examples that demonstrate the theoretical results. First, we considered a formation of small-scale helicopters and gave frequency and time-domain interpretations of string instability. Then we showed that aerodynamic coupling between aircrafts can be considered in this theoretical framework.

References

- [1] S. Banda, J. Doyle, R. Murray, J. Paduano, J. Speyer, and G. Stein. Research needs in dynamics and control for uninhabited aerial vehicles. <http://www.cds.caltech.edu/~murray/notes/uav-nov97.html>, November 1997. Panel Report.
- [2] S. Boyd and C.A. Desoer. Subharmonic functions and performance bounds on linear time-invariant feedback systems. *IMA Journal of Mathematical Control and Information*, 2:153–170, 1985.
- [3] J. Chen. On logarithmic complementary sensitivity integrals for MIMO systems. In *Proceedings of the American Control Conference*, pages 3529–3530, June 1998.
- [4] J. Chen. Logarithmic integrals, interpolation bounds, and performance limitations in MIMO feedback systems. *IEEE Transactions on Automatic Control*, 45(6):1098–1115, June 2000.
- [5] D.F. Chichka and J.L. Speyer. Peak-seeking control for drag reduction in formation flight. Submitted to AIAA Journal of Guidance, Control, and Dynamics.
- [6] D.F. Chichka and J.L. Speyer. Solar-powered, formation-enhanced aerial vehicle systems for sustained endurance. In *Proceedings of the American Control Conference*, pages 684–688, June 1998.
- [7] D.F. Chichka, J.L. Speyer, and C.G. Park. Peak-seeking control with application to formation flight. In *IEEE Conference on Decision and Control*, pages 2463–2470, December 1999.
- [8] J.M. Fowler and R. D’Andrea. Distributed control of close formation flight. In *Proceedings of the 41st IEEE Conference on Decision and Control*, pages 2972–2977, 2002.
- [9] F. Giulietti, L. Pollini, and M. Innocenti. Autonomous formation flight. *IEEE Control Systems Magazine*, 20(6):34–44, December 2000.
- [10] D. Hummel. Formation flight as an energy-savings mechanism. *Israel Journal of Zoology*, 41:261–278, 1995.
- [11] P.B.S. Lissaman and C.A. Shollenberger. Formation flight of birds. *Science*, 168:1003–1005, May 1970.
- [12] D.P. Looze and J.S. Freudenberg. Tradeoffs and limitations in feedback systems. In W.S. Levine, editor, *The Control Handbook*, chapter 31, pages 537–549. CRC Press, 1996.
- [13] R.K. Mehra, J.D. Bošković, and S. Li. Autonomous formation flying of multiple UCAVs under communication failure. In *Position Locations and Navigation Symposium*, pages 371–378. IEEE, March 2000.
- [14] B. Mettler, M.B. Tischler, and T. Kanade. System identification modeling of a small-scale unmanned rotorcraft for flight control design. To appear in the *Journal of the American Helicopter Society*.
- [15] R.H. Middleton and G.C. Goodwin. *Digital Control and Estimation: A Unified Approach*, pages 423–433. Prentice Hall, Englewood Cliffs, N.J., 1990.

- [16] M. Pachter, J.J. D’Azzo, and J.L. Dargan. Automatic formation flight control. *AIAA Journal of Guidance, Control and Dynamics*, 17(6):1380–1383, 1994.
- [17] M. Pachter, J.J. D’Azzo, and A.W. Proud. Tight formation flight control. *Journal of Guidance, Control, and Dynamics*, 24(2):246–254, March-April 2001.
- [18] C. Richardson and M. Schoultz. Formation flight system design concept. In *Proceedings of the IEEE/AIAA 10th Digital Avionics Systems Conference*, pages 18–25, October 1991.
- [19] H.L. Royden. *Real Analysis*, page 87. Macmillan Publishing Company, 1988.
- [20] P. Seiler. *Coordinated Control of Unmanned Aerial Vehicles*. PhD thesis, University of California, Berkeley, Fall 2001.
- [21] Pete Seiler, Aniruddha Pant, and J.K. Hedrick. Preliminary investigation of mesh stability for linear systems. In *Proceedings of the ASME: DSC Division*, volume 67, pages 359–364, 1999.
- [22] H. Shim. *Hierarchical Flight Control System Synthesis for Rotorcraft-based Unmanned Aerial Vehicles*. PhD thesis, University of California at Berkeley, Fall 2000.
- [23] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design*, pages 127–137. John Wiley and Sons, West Sussex, England, 1996.
- [24] D. Swaroop and J.K. Hedrick. String stability of interconnected systems. *IEEE Transactions on Automatic Control*, 41(4):349–356, March 1996.
- [25] D. Swaroop and J.K. Hedrick. Constant spacing strategies for platooning in automated highway systems. *Transactions of the ASME*, 121:462–470, September 1999.
- [26] H. Tanner and G. J. Pappas. Formation input-to-state stability. In *Proceedings of the 15th IFAC World Congress*, July 2002.