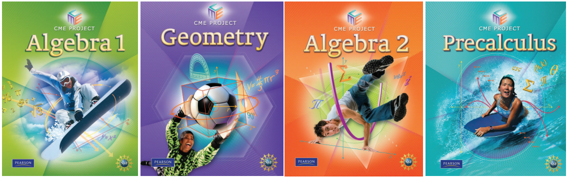

The CME Project

Promoting Mathematical Habits of Mind in High School

Al Cuoco

International Seminar on Mathematics, Physics and Chemistry Textbooks, September 27–29, 2010

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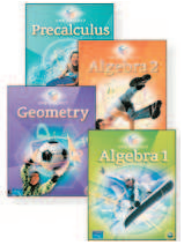
Outline

Part 3: More Snapshots

The CME Project: Brief Overview

- An NSF-funded coherent 4-year curriculum
- Published by Pearson
- Uses Texas Instruments technology to support mathematical thinking
- Follows the traditional American course structure
- **Organized around mathematical habits of mind**



Where Are We?

- The program has been in publication for 2 years.
- It is being used in approximately 25 states.
- Large adoptions in Chicago, Des Moines, and Boston.

How Are We Doing?

Details aren't yet available, but districts are reporting significant improvement in student understanding, motivation, and performance on standardized exams.

"*Obviously we are very pleased with our 9th grade math scores! Also pleased with the 10th grade improvement . . . . A ten percent increase in the number of students scoring proficient or distinguished is a nice improvement.*"
– Director of Instruction, Williamsburg, KY

"*I actually can't believe it, the kids are learning [the curriculum]. I'm a pretty traditional guy, but this stuff actually works.*"
—9th grade teacher, Chicago IL

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How are We Doing?

"I gave this material to the students, and it's a little more challenging than the work in the . . . text, and students who are very difficult to motivate are coming to me saying they really love the work they are doing. [This is] challenging work that my students feel motivated enough to tackle."

—Steve MacDonald, Lawrence High School

"This is the best Algebra book I've ever seen. There's actual mathematics in here."

— Gary Solberg, 9th grade teacher, Chicago IL

"I almost like chapter 5 but. . . no no, I like chapter 5. I can finally see how this all plays out, next year is going to be much easier."

— Santiago Marquez, 9th grade teacher, Chicago IL

Factoring in Algebra 1

Factoring monic quadratics:

"Sum-Product" problems

$$x^2 + 14x + 48$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

so...

Find two numbers whose sum is 14 and whose product is 48.

$$(x + 6)(x + 8)$$

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Factoring in Algebra 1

What about this one?

$$4x^2 + 36x + 45$$

$$\begin{aligned}
4x^2 + 36x + 45 &= (2x)^2 + 18(2x) + 45 \\
&= \clubsuit^2 + 18\clubsuit + 45 \\
&= (\clubsuit + 15)(\clubsuit + 3) \\
&= (2x + 15)(2x + 3)
\end{aligned}$$

Factoring in Algebra 1

Minds in Action episode 37

Tony and Sasha are trying to factor the quadratic $4x^2 + 36x + 45$.

Tony It's not monic. Do we have to play with all the combinations?

Sasha We could. Wait, I see something. $4x^2$ is the same as $(2x)^2$. So we could write the equation using $2x$ chunks.

$$(2x)^2 + 18(2x) + 45$$

Tony Sure, you can do that, but it's still not monic.

Sasha Well, no. But suppose I think of the $2x$ as one thing.

Sasha covers the first $2x$ *with her left hand and the second* $2x$ *with her right hand.*

()^2 + 18() + 45

What combination
are Tony and Sash
talking about?

FACTORING IN ALGEBRA 1

Sasha Do you see? It's something squared plus 18 times that something plus 45. Here, I'll change what's under my hand, the $2x$, to z. Now it looks better.

$$z^2 + 18z + 45$$

Tony Cool! I can factor that by finding numbers that add to 18 and multiply to 45. So, 15 and 3 will work. Look at what I get.

$$z^2 + 18z + 45 = (z + 15)(z + 3)$$

Sasha Remember, we used z as a placeholder for $2x$, so now put the $2x$ back.

$$(z + 15)(z + 3) = (2x + 15)(2x + 3)$$

Tony We should check by multiplying it out, just to be sure.

Tony and Sasha

FACTORING IN ALGEBRA 1

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned}
6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\
&= \clubsuit^2 + 31\clubsuit + 210 \\
&= (\clubsuit + 21)(\clubsuit + 10) \\
&= (6x + 21)(6x + 10) \\
&= 3(2x + 7) \cdot 2(3x + 5) \\
&= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots
\end{aligned}$$

$$6(6x^2 + 31x + 35) = 6(2x + 7)(3x + 5)$$

FACTORING IN ALGEBRA 1

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned}
6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\
&= \clubsuit^2 + 31\clubsuit + 210 \\
&= (\clubsuit + 21)(\clubsuit + 10) \\
&= (6x + 21)(6x + 10) \\
&= 3(2x + 7) \cdot 2(3x + 5) \\
&= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots
\end{aligned}$$

$$\cancel{6}(6x^2 + 31x + 35) = \cancel{6}(2x + 7)(3x + 5)$$

Factoring in Algebra 1

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned} 6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\ &= \clubsuit^2 + 31\clubsuit + 210 \\ &= (\clubsuit + 21)(\clubsuit + 10) \\ &= (6x + 21)(6x + 10) \\ &= 3(2x + 7) \cdot 2(3x + 5) \\ &= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots \end{aligned}$$

$$6x^2 + 31x + 35 = (2x + 7)(3x + 5)$$

Factoring in Precalculus

The *CMP* Factor Game

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

Factoring in Precalculus

The *CME Project* Factor Game

$x - 1$	$x^2 - 1$	$x^3 - 1$	$x^4 - 1$	$x^5 - 1$
$x^6 - 1$	$x^7 - 1$	$x^8 - 1$	$x^9 - 1$	$x^{10} - 1$
$x^{11} - 1$	$x^{12} - 1$	$x^{13} - 1$	$x^{14} - 1$	$x^{15} - 1$
$x^{16} - 1$	$x^{17} - 1$	$x^{18} - 1$	$x^{19} - 1$	$x^{20} - 1$
$x^{21} - 1$	$x^{22} - 1$	$x^{23} - 1$	$x^{24} - 1$	$x^{25} - 1$
$x^{26} - 1$	$x^{27} - 1$	$x^{28} - 1$	$x^{29} - 1$	$x^{30} - 1$

Scratchpad

Factoring in Precalculus

Things that have come up in class;

- "It's the same as the middle school factor game."
- if m is a factor of n, $x^m - 1$ is a factor of $x^n - 1$

$$
\begin{aligned}
x^{12} - 1 &= \left(x^3\right)^4 - 1 \\
&= (\clubsuit)^4 - 1 \\
&= (\clubsuit - 1)\left(\clubsuit^3 + \clubsuit^2 + \clubsuit + 1\right) \\
&= \left(x^3 - 1\right)\left((x^3)^3 + (x^3)^2 + (x^3) + 1\right) \\
&= \left(x^3 - 1\right)\left(x^9 + x^6 + x^3 + 1\right)
\end{aligned}
$$

Factoring in Precalculus

- If $x^m - 1$ is a factor of $x^n - 1$, m is a factor of n

  This is much harder. We approach it through De Moivre's theorem and with *roots of unity*: complex numbers that are the roots of the equation

$$x^n - 1 = 0$$

The Habits of Mind Approach

What mathematicians most wanted and needed from me was ***to learn my ways of thinking****, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds.*

— William Thurston

On Proof and Progress in Mathematics

The Habits of Mind Approach

Mathematics constitutes one of the most ancient and noble intellectual traditions of humanity. It is an enabling discipline for all of science and technology, providing powerful tools for analytical thought as well as the concepts and language for precise quantitative description of the world around us. ***It affords knowledge and reasoning of extraordinary subtlety and beauty, even at the most elementary levels.***

— RAND Mathematics Study Panel, 2002

Our Fundamental Organizing Principle

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques, but more importantly, from developing the ways of thinking—the ***habits of mind****—used to create the results.*

—*The CME Project* Implementation Guide, 2008

General Mathematical Habits

- Performing thought experiments
- Finding and explaining patterns
- Creating and using representations
- Generalizing from examples
- Expecting mathematics to make sense

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Algebraic Habits of Mind

- Seeking regularity in repeated calculations
- “Chunking” (changing variables in order to hide complexity)
- Reasoning about and picturing calculations and operations
- Extending operations to preserve rules for calculating
- Purposefully transforming and interpreting expressions
- Seeking and specifying structural similarities

Analytic/Geometric Habits of Mind

- Reasoning by continuity
- Seeking geometric invariants
- Looking at extreme cases
- Passing to the limit
- Using approximation

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Additional Core Principles

- **Textured emphasis.** We focus on matters of mathematical substance, being careful to separate them from convention and vocabulary. Even our practice problems are designed so that they have a larger mathematical point.
- **General purpose tools.** The methods and habits that students develop in high school should serve them well in their later work in mathematics and in their post-secondary endeavors.
- **Experience before formality.** Worked-out examples and careful definitions are important, but students need to grapple with ideas and problems *before* they are brought to closure.

Additional Core Principles, continued

- **The role of applications.** What matters is *how* mathematics is applied, not *where* it is applied.
- **A mathematical community.** Our writers, field testers, reviewers, and advisors come from all parts of the mathematics community.
- **Connect school mathematics to the discipline.** Every chapter, lesson, problem, and example is written with an eye towards how it fits into the landscape of mathematics as a scientific discipline.
- **High expectations for all students.** All students can enjoy real mathematics and take delight from thinking in characteristically mathematical ways.

Design Principles
Structure of Each Book

- Low threshold, high ceiling
  - Each book has exactly eight chapters
  - Problem sets, investigations, and chapters build from easy access to quite challenging
- Openings and closure
  - *Getting Started*
  - Worked out examples
  - Definitions and theorems are capstones, not foundations
- Coherent and connected
  - Recurring themes, contexts, and methods
  - Small number of central ideas
  - Stress connections among algebra, geometry, analysis, and statistics

Design Principles

Consistent Design Elements

- *Minds in Action*
- *In-Class Experiment*
- *For You to Do*
- *Developing Habits of Mind*
- Projects
- Sidenotes
- Orchestrated problem sets
- Technology support

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The Choices Again

- Mining the tables of arithmetic
- Algebra word problems
- Graphing
- Area formulas
- Factoring in Algebra 2
- Fitting functions to tables
- Monthly payments on a loan
- Finding polynomials that agree with tables
- Geometric invariants
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

Addition

Addition Table

12	12	13	14		16	17	18	19	20	21	22	23	24
11	11	12	13			16	17	18	19	20	21	22	23
10	10	11	12		14		16	17	18	19	20	21	22
9	9	10	11		13	14		16	17	18	19	20	21
8	8	9	10		12	13	14		16	17	18	19	20
7	7	8	9		11	12	13	14		16	17	18	19
6	6	7	8		10	11	12	13	14		16	17	18
5	5	6	7		9	10	11	12	13	14		16	17
4	4	5	6		8	9	10	11	12	13	14		16
3	3	4	5		7	8	9	10	11	12	13	14	
2	2	3	4		6	7	8	9	10	11	12		14
1	1	2	3		5	6	7	8	9	10		12	13
0	0	1	2		4	5	6	7	8		10	11	12
	0	1	2	3	4	5	6	7	8	9	10	11	12

MULTIPLICATION

Multiplication Table

12		12	24	36	48	60	72		96	108	120	132	144
11	0		22	33	44	55		77	88	99	110	121	132
10	0	10		30			60	70	80	90	100	110	120
9	0	9	18			45	54	63	72	81	90	99	108
8	0	8	16			40	48	56	64	72	80	88	96
7	0	7		21	28		42	49	56	63	70	77	84
6	0		12	18	24	30		42	48	54	60	66	72
5		5	10	15	20	25	30		40	45	50	55	60
4	0	4	8	12	16	20	24	28		36	40	44	48
3	0	3	6	9	12	15	18	21	24		30	33	36
2	0	2	4	6	8	10	12	14	16	18		22	24
1	0	1	2	3	4	5	6	7	8	9	10		12
	0	0	0	0	0	0	0	0	0	0	0	0	
	0	1	2	3	4	5	6	7	8	9	10	11	12

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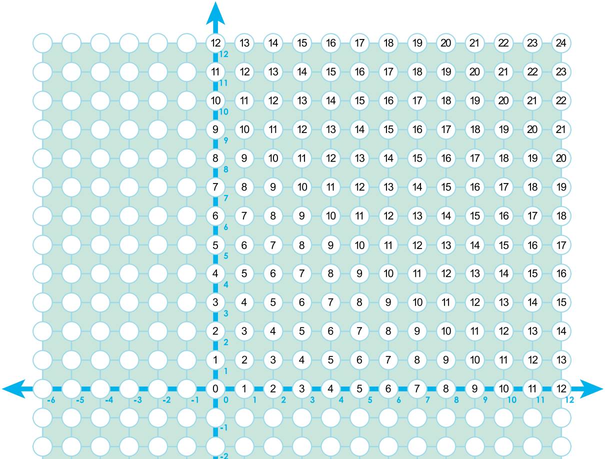
Addition

Extension



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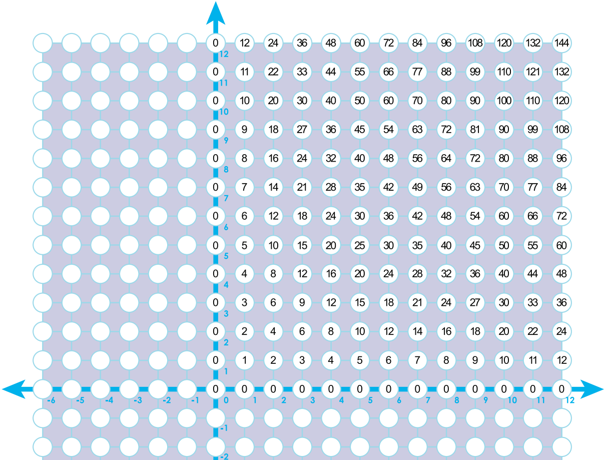
MULTIPLICATION

EXTENSION



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The Choices Again

- Algebra word problems
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- Area formulas
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- Tangents to graphs

Word Problems

The dreaded algebra word problem

Mary drives from Boston to Chicago, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes 36 hours, how far is Boston from Chicago?

Why is this so difficult for students?

- Reading level
- Context

WORD PROBLEMS

But there must be more to it. Compare. . .

Mary drives from Boston to Chicago, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. *If the total trip takes 36 hours, how far is Boston from Chicago?*

with

Mary drives from Boston to Chicago, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. *If Boston is 1000 miles from Chicago, how long did the trip take?*

"The difficulty lies in setting up the equation, not solving it."

Word Problems

This led to the Guess-Check-Generalize method:

- Take a guess, say 1200 miles.
- Check it:
  - $\frac{1200}{60} = 20$
  - $\frac{1200}{50} = 24$
  - $20 + 24 \neq 36$
- That wasn't right, but that's okay—
  just keep track of your steps.
- Take another guess, say 1000, and check it:

$$\frac{1000}{60} + \frac{1000}{50} \stackrel{?}{=} 36$$

Word Problems

- Keep it up, until you get a "guess checker"

$$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} \stackrel{?}{=} 36$$

- The equation is

$$\frac{x}{60} + \frac{x}{50} = 36$$

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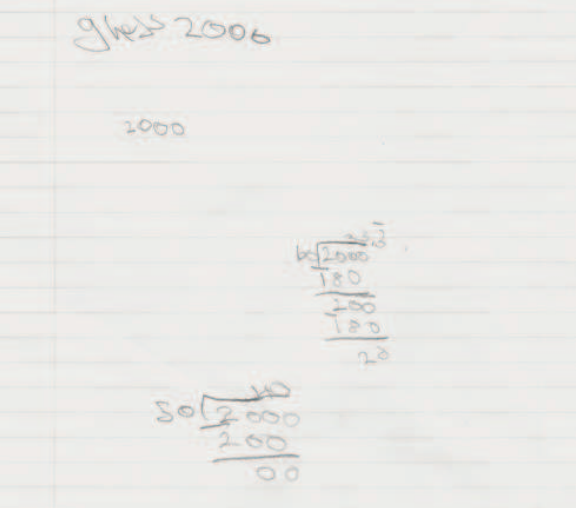
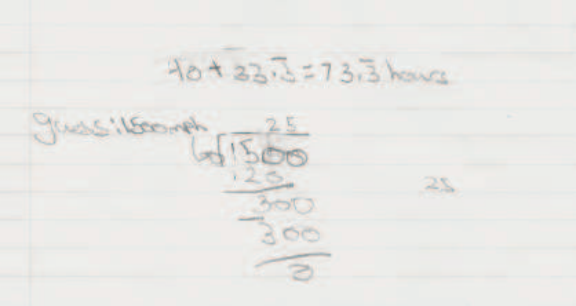
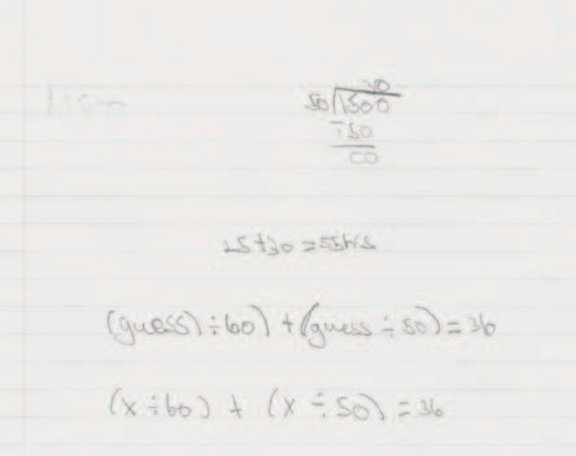
Word Problems



Word Problems



Word Problems



The Choices Again

- Graphing
- Area formulas
- Factoring in Algebra 2
- Fitting functions to tables
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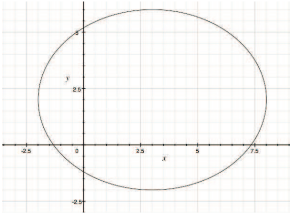
GRAPHING

Graph

$$16\,x^2 - 96\,x + 25\,y^2 - 100\,y - 156 = 0$$

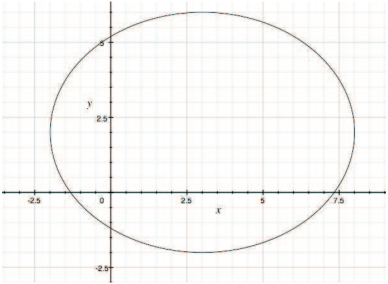
$$16\,x^2 - 96\,x + 25\,y^2 - 100\,y - 156 = 0 \Rightarrow \frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$$

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1 \Rightarrow$$



Graphing

$$\frac{(x-3)^2}{25}+\frac{(y-2)^2}{16}=1$$



Is $(7.5, 3.75)$ on the graph?

This led to the idea that "equations are point testers."

Graphing

Equations of Lines

Why is "linearity" so hard for students?

- The general problem with the "Cartesian connection"

  $y = 3x + 7$ is a set of instructions.

- Slope is quite subtle

  Why should it be invariant along a line?

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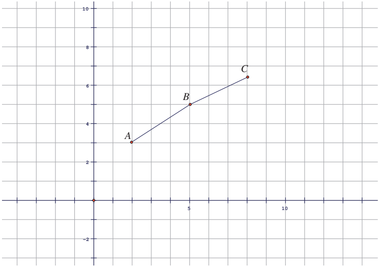
GRAPHING
EQUATIONS OF LINES

In *The CME Project* slope is defined initially *between two points*: $m(A, B)$

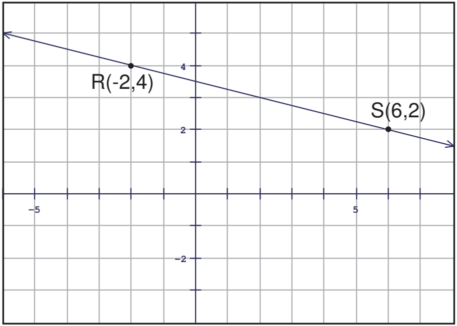
Basic assumption: A, B, and C are collinear $\Leftrightarrow m(A, B) = m(B, C)$



GRAPHING

EQUATIONS OF LINES

What is the equation of the line ℓ that goes through $R(-2,4)$ and $S(6,2)$?



Try some points, keeping track of the steps...

GRAPHING

EQUATIONS OF LINES

Minds in Action episode 14

Sasha and Tony are trying to find the equation of the line ℓ that goes through points $R(-2, 4)$ and $S(6, 2)$.

Sasha To use a point-tester, we first need to find the slope between R and S.

Tony goes to the board and writes

$$m(R, S) = \frac{2-4}{6-(-2)} = \frac{-2}{8} = -\frac{1}{4}.$$

Tony It's $-\frac{1}{4}$.

Sasha Okay. Now, we want to test some point, say P. We want to see whether the slope between that point and one of the first two, say R, is equal to $-\frac{1}{4}$. If it is, that point is on ℓ. So our test is $m(P, R) \stackrel{?}{=} -\frac{1}{4}$.

It doesn't matter which point you choose as the base point. Either point R or point S will work.

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Graphing

Equations of Lines

- Test $P = (1, 1)$:
  $m(P, R) = \frac{1-4}{1-(-2)} \stackrel{?}{=} -\frac{1}{4} \Rightarrow$ Nope
- Test $P = (2, 3)$:
  $m(P, R) = \frac{3-4}{2-(-2)} \stackrel{?}{=} -\frac{1}{4} \Rightarrow$ Yup
- Test $P = (7, 2)$:
  Let's see how Tony and Sasha finish this problem.

Graphing

Equations of Lines

Tony Let's guess and check a point first, like $P(7, 2)$. Tell me everything you do so I can keep track of the steps.

Sasha Well, the slope between $P(7, 2)$ and $R(-2, 4)$ is $m(P, R) = \frac{2-4}{7-(-2)} = \frac{-2}{9} = -\frac{2}{9}$. This slope is different, so P isn't on ℓ. Maybe we should use a variable point.

Tony How do we do that?

Sasha A point has two coordinates, right? So use two variables. Say P is (x, y).

Tony Then the slope from P to R is $m(P, R) = \frac{y-4}{x-(-2)} = \frac{y-4}{x+2}$. The test is $\frac{y-4}{x+2} = -\frac{1}{4}$.

So, that must be the equation of the line ℓ.

Notice how Sasha switches to letters. She uses x for point P's x-coordinate. She uses y for point P's y-coordinate.

Graphing
Equations of Lines

They test a generic point $P = (x, y)$:

$$m(P, R) = \frac{y - 4}{x - (-2)} \stackrel{?}{=} -\frac{1}{4}$$

So, the equation of ℓ is

$$\frac{y - 4}{x + 2} = -\frac{1}{4}$$

or

$$x + 4y = 14$$

After this, there is a lesson called *Jiffy Graphs* where students develop “automaticity.”

The Choices Again

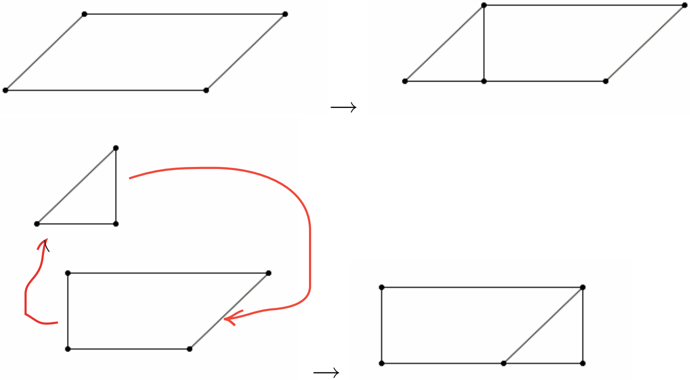
- Area formulas
- Factoring in Algebra 2
- Fitting functions to tables
- Monthly payments on a loan
- Finding polynomials that agree with tables
- Geometric invariants
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

Axioms for area

- If you translate, rotate, or reflect a figure, its area doesn't change ("area is invariant under rigid motions").
- If you cut up a figure into a finite number of pieces and rearrange the pieces, its area doesn't change ("area is invariant under finite dissections").
- The area of a rectangle of dimensions b and h is bh.

From here. . .

The Parallelogram



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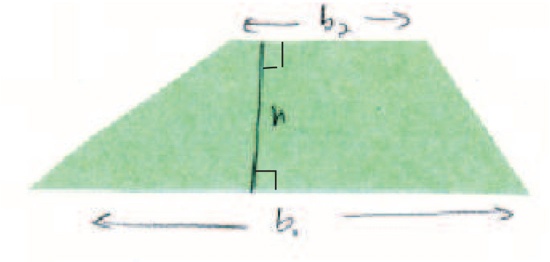
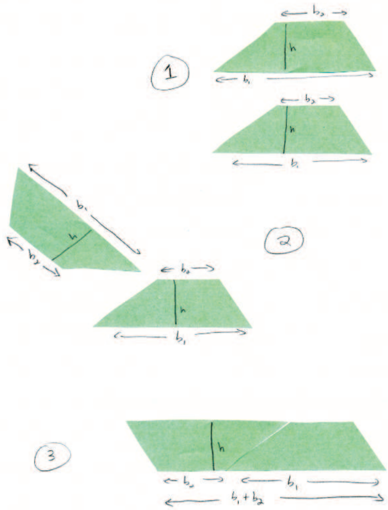
THE TRAPEZOID



THE TRAPEZOID



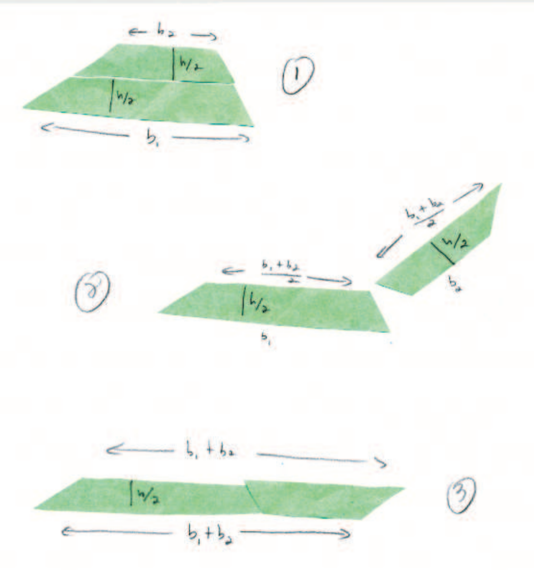
So,

$$2A = (b_1 + b_2)h$$

and

$$A = \frac{1}{2}\left((b_1 + b_2)h\right)$$

The Trapezoid



So,

$$A = (b_1 + b_2)\left(\frac{h}{2}\right)$$

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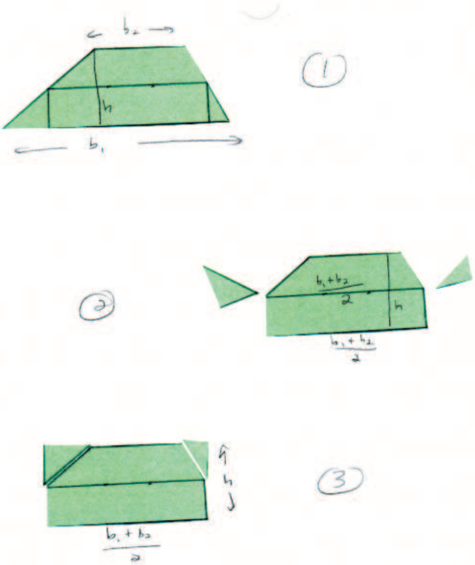
THE TRAPEZOID



So,

$$A = \left(\frac{b_1 + b_2}{2}\right) h$$

The Trapezoid

So, a class could come up with at least three expressions for the same area A:

$$\frac{1}{2}\left((b_1 + b_2)h\right), \quad (b_1 + b_2)\left(\frac{h}{2}\right), \quad \text{and} \quad \left(\frac{b_1 + b_2}{2}\right)h$$

All of these are equivalent…

- by algebra
- by geometry

Also, what happens to the geometry and the algebra in each of these expressions as $b_2 \to 0$?

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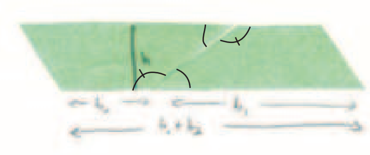
FROM HERE

- Students prove that the cuts "work":



- The area formulas for all the usual polygons are obtained by "cutting" the polygons into rectangles, keeping track of dimensions.
- **Take it Further:** If two polygons have the same area, are they "scissors congruent?"

The Choices Again

- Factoring in Algebra 2
- Fitting functions to tables
- Monthly payments on a loan
- Finding polynomials that agree with tables
- Geometric invariants
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

Factoring in Algebra 2

The number of factors over $\mathbb{Z}$ of $x^n - 1$ as a function of n.

n	number of factors of $x^n - 1$
1	
2	
3	
4	
5	
6	
7	
8	
9	

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FACTORING IN ALGEBRA 2

The number of factors over $\mathbb{Z}$ of $x^n - 1$ as a function of n.

n	number of factors of $x^n - 1$
1	1
2	2
3	2
4	
5	
6	
7	
8	
9	

Factoring in Algebra 2

The number of factors over $\mathbb{Z}$ of $x^n - 1$ as a function of n.

n	number of factors of $x^n - 1$
1	1
2	2
3	2
4	3
5	?
6	
7	
8	
9	

Let's try it

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Factoring in Algebra 2

The number of factors over $\mathbb{Z}$ of $x^n - 1$ as a function of n.

n	number of factors of $x^n - 1$
1	1
2	2
3	2
4	3
5	2
6	4
7	?
8	?
9	?

Let's try it

Factoring in Algebra 2

The number of factors over $\mathbb{Z}$ of $x^n - 1$ as a function of n.

n	number of factors of $x^n - 1$
1	1
2	2
3	2
4	3
5	2
6	4
7	2
8	4
9	3

Conjectures? . . .

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Factoring in Algebra 2

Things that have come up in class:

- There are always at least two factors:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1)$$

- If n is odd, there are exactly two factors (but look at $n = 9$)
- OK . . . if n is prime , there are exactly two factors
- If $n = p^2$, there are three factors (ex: $x^9 - 1$)
- If $n = pq$, there are four factors (ex: $x^{15} - 1$)

$$\vdots$$

- A general conjecture gradually emerges

Scratchpad

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The Choices Again

- Fitting functions to tables
- Monthly payments on a loan
- Finding polynomials that agree with tables
- Geometric invariants
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

Making it Fit

Input	Output
0	3
1	5
2	7
3	9
4	11
5	13
6	15

Making it Fit

Input	Output	Δ
0	3	2
1	5	2
2	7	2
3	9	2
4	11	2
5	13	2
6	15	

Making it Fit

What about this one?

Input	Output	Δ	Δ^2
0	1	-3	6
1	-2	3	6
2	1	9	6
3	10	15	6
4	25	21	6
5	46	27	
6	73		

Making it Fit

What about this one?

Input	Output	Δ	Δ^2	Δ^3
0	1	-2	14	12
1	-1	12	26	12
2	11	38	38	12
3	49	76	50	12
4	125	126	62	12
5	251	188	74	
6	439	262		
7	701			

Making it Fit

What about *this* one?

x	$k(x)$
1	-12
3	-16
7	72

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The Choices Again

- Monthly payments on a loan
- Finding polynomials that agree with tables
- Geometric invariants
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

Monthly Payments on a Loan

Suppose you want to buy a car that costs $10,000. You don't have much money, but you can put $1000 down and pay $350 per month. The interest rate is 5%, and the dealer wants the loan paid off in three years. Can you afford the car?

This leads to the question

"How does a bank figure out the monthly payment on a loan?"

or

"How does a bank figure out the balance you owe at the end of the month?"

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Monthly Payments on a Loan

Take 1

What you owe at the end of the month is what you owed at the start of the month minus your monthly payment.

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Scratchpad

Monthly Payments on a Loan

Take 2

> *What you owe at the end of the month is what you owed at the start of the month,* ***plus*** $\frac{1}{12}$ ***of the yearly interest on that amount****, minus your monthly payment.*

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1,m) + \frac{.05}{12} b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Students can then use successive approximation to find m so that

$$b(36,m) = 0$$

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Monthly Payments on a Loan

Except . . .

$$b(4)=b(3)+\frac{.05}{12}\cdot b(3)-250$$

$$b(2)+\frac{.05}{12}\cdot b(2)-250 \qquad\qquad b(2)+\frac{.05}{12}\cdot b(2)-250$$

It takes too much !$#& work.

Monthly Payments on a Loan

Take 3: Algebra to the rescue!

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1,m) + \frac{.05}{12} b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

becomes

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ \left(1 + \frac{.05}{12}\right) b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Students can *now* use successive approximation to find m so that

$$b(36,m) = 0$$

Scratchpad

Monthly Payments on a Loan

Project: Pick an interest rate and keep it constant. Suppose you want to pay off a car in 24 months. Investigate how the monthly payment changes with the cost of the car:

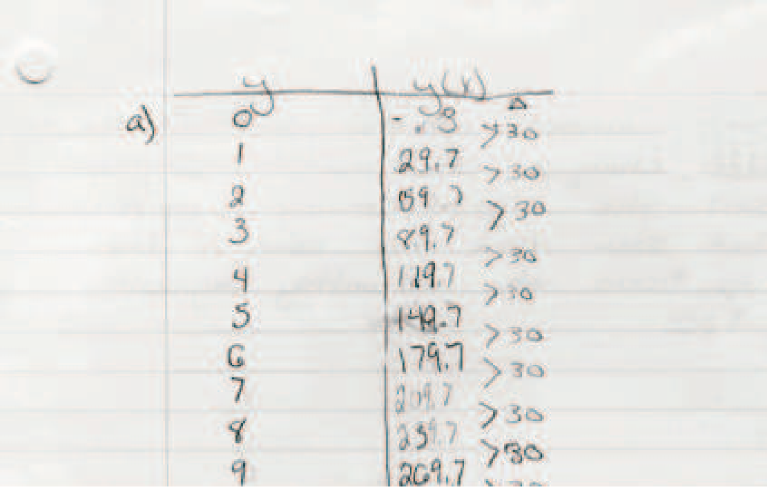
Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
⋮	⋮

Monthly Payments on a Loan

Cost of car (in thousands of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
⋮	⋮

Describe a pattern in the table. Use this pattern to find either a closed form or a recursive rule that lets you calculate the monthly payment in terms of the cost of the car in thousands of dollars. Model your function with your CAS and use the model to find the monthly payment on a $26000 car.

Monthly Payments on a Loan



a)

x	y(x)	Δ
0	-.3	>30
1	29.7	>30
2	59.7	>30
3	89.7	>30
4	119.7	>30
5	149.7	>30
6	179.7	>30
7	209.7	>30
8	239.7	>30
9	269.7	

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Monthly Payments on a Loan

- I changed the amount of the cost of the car then I changed the monthly payment until I found the right monthly payment.
- I found that each time the cost of the car went up $1000 the monthly payment went up $30.

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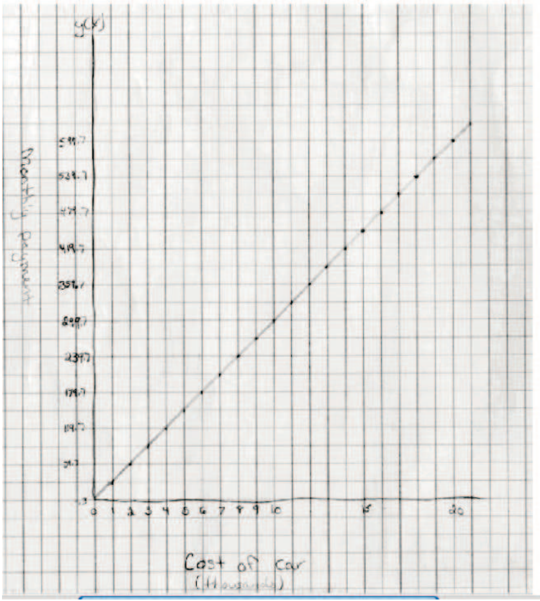
Monthly Payments on a Loan



Monthly Payments on a Loan

Students can use a CAS to model the problem *generically*: the balance at the end of 36 months with a monthly payment of m can be found by entering

$$b(36, m)$$

in the calculator:

Scratchpad **But *why* is it linear?**

Monthly Payments on a Loan

But *why* is it linear?

Suppose you borrow \$12000 at 5% interest. Then you are experimenting with this function:

$$b(n,m) = \begin{cases} 12000 & \text{if } n = 0 \\ (1 + \frac{.05}{12}) \cdot b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Notice that

$$1 + \frac{.05}{12} = \frac{12.05}{12}$$

Call this number q. So, the function now looks like:

$$b(n,m) = \begin{cases} 12000 & \text{if } n = 0 \\ q \cdot b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

where q is a constant (chunking, again).

Monthly Payments on a Loan

Then at the end of n months, you could unstack the calculation as follows:

$$\begin{aligned} b(n,m) &= q \cdot b(n-1,m) - m \\ &= q\left[q \cdot b(n-2,m) - m\right] - m \\ &\qquad = q^2 \cdot b(n-2,m) - qm - m \\ &= q^2\left[q \cdot b(n-3,m) - m\right] - qm - m \\ &\qquad = q^3 \cdot b(n-3,m) - q^2m - qm - m \\ &\qquad \vdots \\ &= q^n \cdot b(0,m) - q^{n-1}m - q^{n-2}m - \cdots - q^2m - qm - m \\ &= 12000 \cdot q^n - m(q^{n-1} + q^{n-2} + \cdots + q^2 + q + 1) \end{aligned}$$

Monthly Payments on a Loan

Precalculus students know (very well) the “cyclotomic identity:”

$$q^{n-1} + q^{n-2} + \cdots + q^2 + q + 1 = \frac{q^n - 1}{q - 1}$$

Applying it, you get

$$\begin{aligned} b(n,m) &= 12000 \cdot q^n - m(q^{n-1} + q^{n-2} + \cdots + q^2 + q + 1) \\ &= 12000\, q^n - m\frac{q^n - 1}{q - 1} \end{aligned}$$

Setting $b(n,m)$ equal to 0 gives an explicit relationship between m and the cost of the car…

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Monthly Payments on a Loan

$$m = 12000\,\frac{(q-1)q^n}{q^n-1}$$

or, in general,

$$\text{monthly payment} = \text{cost of car} \times \frac{(q-1)q^n}{q^n-1}$$

where n is the term of the loan and

$$q = 1 + \frac{\text{interest rate}}{12}$$

THE CHOICES AGAIN

- Finding polynomials that agree with tables
- Geometric invariants
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

Fitting Functions to Tables

In Algebra 1

Find functions that agree with each table:

Input: n	Output
0	3
1	8
2	13
3	18
4	23

Input: n	Output
0	1
1	2
2	5
3	10
4	17

Fitting Functions to Tables

In Algebra 1 and Algebra 2

Build a calculator model of a function that agrees with the table:

Input: n	Output
0	3
1	8
2	13
3	18
4	23

- $f(n) = 5n + 3$
- $g(n) = \begin{cases} 3 & \text{if } n = 0 \\ g(n-1) + 5 & \text{if } n > 0 \end{cases}$

Question:

$$f \stackrel{?}{=} g$$

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Fitting Functions to Tables

In Precalculus

$$f(n) = 5n + 3 \qquad \text{and} \qquad g(n) = \begin{cases} 3 & \text{if } n = 0 \\ g(n-1) + 5 & \text{if } n > 0 \end{cases}$$

OK. $f(254) = g(254)$. Is $f(255)=g(255)$?

$$\begin{aligned}
g(255) &= g(254) + 5 && \text{(this is how } g \text{ is defined)} \\
&= f(254) + 5 && \text{(CSS)} \\
&= (5 \cdot 254 + 3) + 5 && \text{(this is how } f \text{ is defined)} \\
&= (5 \cdot 254 + 5) + 3 && \text{(BR)} \\
&= 5(254 + 1) + 3 && \text{(BR)} \\
&= 5(255) + 3 && \text{(arithmetic)} \\
&= f(255) && \text{(this is how } f \text{ is defined)}
\end{aligned}$$

Fitting Functions to Tables

In Precalculus

$$f(n) = 5n + 3 \qquad \text{and} \qquad g(n) = \begin{cases} 3 & \text{if } n = 0 \\ g(n-1) + 5 & \text{if } n > 0 \end{cases}$$

OK. Suppose $f(321) = g(321)$. Is $f(322){=}g(322)$?

$$\begin{aligned}
g(322) &= g(321) + 5 && \text{(this is how } g \text{ is defined)} \\
&= f(321) + 5 && \text{(CSS)} \\
&= (5 \cdot 321 + 3) + 5 && \text{(this is how } f \text{ is defined)} \\
&= (5 \cdot 321 + 5) + 3 && \text{(BR)} \\
&= 5(321 + 1) + 3 && \text{(BR)} \\
&= 5(322) + 3 && \text{(arithmetic)} \\
&= f(322) && \text{(this is how } f \text{ is defined)}
\end{aligned}$$

Fitting Functions to Tables

In Precalculus

$$f(n) = 5n + 3 \qquad \text{and} \qquad g(n) = \begin{cases} 3 & \text{if } n = 0 \\ g(n-1) + 5 & \text{if } n > 0 \end{cases}$$

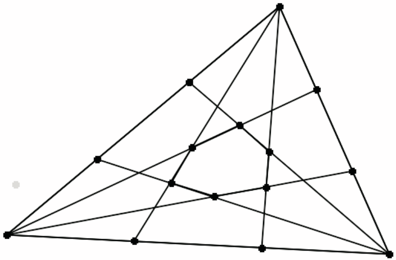
OK. Suppose $f(n-1) = g(n-1)$. Is $f(n)$=$g(n)$?

$$\begin{aligned} g(n) &= g(n-1) + 5 && \text{(this is how } g \text{ is defined)} \\ &= f(n-1) + 5 && \text{(CSS)} \\ &= (5(n-1) + 3) + 5 && \text{(this is how } f \text{ is defined)} \\ &= (5(n-1) + 5) + 3 && \text{(BR)} \\ &= 5(n-1+1) + 3 && \text{(BR)} \\ &= 5n + 3 && \text{(arithmetic)} \\ &= f(n) && \text{(this is how } f \text{ is defined)} \end{aligned}$$

The Choices Again

- Geometric invariants
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

An Example from Geometry



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WHAT KIDS CAN DO. . .

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THE SUN

LATEST NEWS & SPORTS

TUESDAY, DECEMBER 20, 1994 HOME DELIVERY: 25¢ NEWSSTAND: 50¢ BALTIMORE, MARYLAND

SOMETHING NEW IN A TRIANGLE

Patapsco High student's hunch points to theorem

By Mary Maushard
Sun Staff Writer

Ryan Morgan would have gotten an "A" in geometry even if he hadn't unearthed a mathematical treasure.

But the persistent Patapsco High School sophomore pushed a hunch into a theory. He calls it Morgan's Conjecture, and is hoping it will soon be Morgan's Theorem.

In geometric circles, developing a theorem is a big deal — especially if you're only 15.

Ryan's teacher at Patapsco High, Frank Nowosielski, has been teaching 20 years and has never had a student discover a theorem — a mathematical statement that can be proved universally true.

Towson State University math professor Robert B. Hanson never had a high school student present a possible theorem to his faculty seminar — until Ryan did it last spring.

"Ryan's really done something pretty fantastic," said Mr. Nowosielski, who taught Ryan's ninth-grade geometry class for gifted and talented students las' year and now teaches at the Carver Center for Arts and Technology in Towson.

"How many kids in the world have done this? He saw something and he didn't quit. He's a special kid," Mr. Nowosielski said.

What did Ryan see?

Initially, he sa v a triangle, each side divided into thirds. Lines drawn from those segments to the vertices [the corners] formed a hexagon inside the triangle. The area of the hexagon is one-tenth the area of the triangle. This is known as Marion's

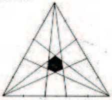
See THEOREM, 18A

JED KIRSCHBAUM/SUN STAFF PHOTO

Ryan Morgan worked many days after school in the computer lab to develop his conjecture, which is displayed on the screen.

MORGAN'S CONJECTURE

Developed by Ryan Morgan, 15, of Patapsco High School:



When the sides of a triangle are n-sected, and n represents any odd integer greater than 1, and segments are drawn from the vertices to these new points, there will be a hexagon present in the interior of the triangle (shaded area). There will always be a constant ratio between the area of the hexagon to the area of the original triangle.

SUN STAFF GRAPHIC

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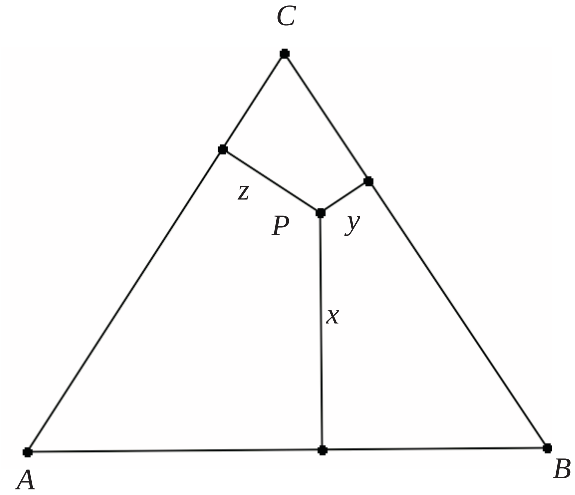
Geometry and Analysis

Example: Invariants in Triangles



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The Choices Again

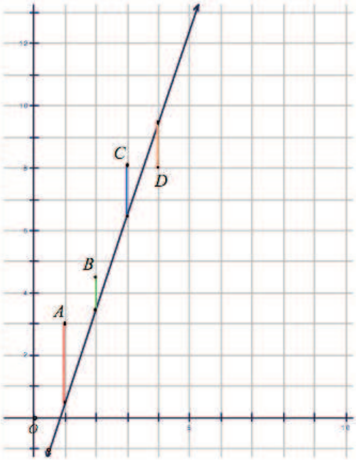
- Regression lines
- Trig identities and complex numbers
- Tangents to graphs

REGRESSION LINES

Here is some made-up data:

Input	Output
1	3
2	4.5
3	8.1
4	8

Regression Lines



Here's a useful image to build in your head: Imagine that you have a "moveable" line on the scatter plot, one that you can control (with a mouse, say). As the line moves, some sticks grow in length, while others shrink. If you keep track of the sum of the squares of the lengths of the sticks, you could fine tune the line and adjust it to make the badness small.

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Regression Lines

Of all lines with slope 3, which is "best?"

Data vs. Line Fit: $y = 3x + b$

Input	Output	Predicted:	Error:
1	3	$3 \cdot 1 + b = 3 + b$	$3 - (3 \cdot 1 + b) = -b$
2	4.5	$3 \cdot 2 + b = 6 + b$	$4.5 - (3 \cdot 2 + b) = -1.5 - b$
3	8.1	$3 \cdot 3 + b = 9 + b$	$8.1 - (3 \cdot 3 + b) = -.9 - b$
4	8	$3 \cdot 4 + b = 12 + b$	$8 - (3 \cdot 4 + b) = -4 - b$

So, we want to minimize

$$(-b)^2 + (-1.5 - b)^2 + (-.9 - b)^2 + (-4 - b)^2$$

Ah... This is a *quadratic* in b. And we know how to minimize a quadratic

Regression Lines

This simplifies (by hand or CAS) to:

$$4\,b^2 + 12.8\,b + 19.06$$

So, the minimum value is when

$$b = \frac{-12.8}{2 \cdot 4} = -1.6$$

So, of all lines with slope 3, the best one has equation

$$y = 3x - 1.6$$

Play the same game with different slopes, and you find a certain rhythm to the calculations.

Regression Lines

Slope	Badness	Minimizing value of b
0	$158.86 - 47.2\,b + 4\,b^2$	5.9
1	$52.26 - 27.2\,b + 4\,b^2$	3.4
2	$5.66 - 7.2\,b + 4\,b^2$	.9
3	$19.06 + 12.8\,b + 4\,b^2$	-1.6
$\vdots$	$\vdots$	$\vdots$

. . . *And a surprise* . . .

REGRESSION LINES

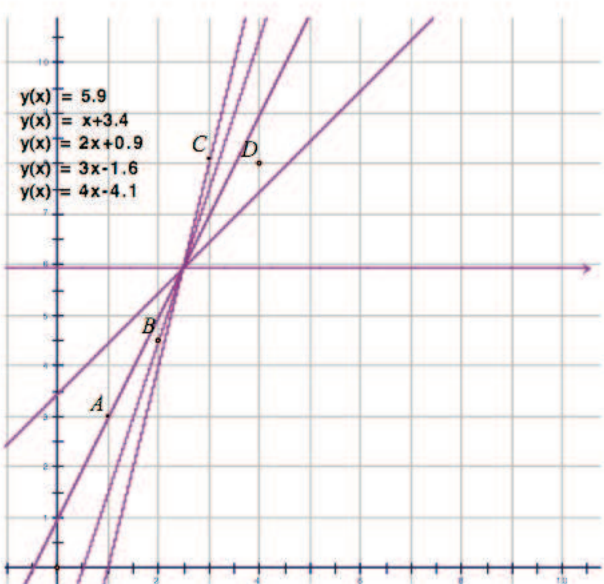
Slope	Equation of best line	y-intercept	Δ
0	$y = 5.9$	5.9	-2.5
1	$y = x + 3.4$	3.6	-2.5
2	$y = 2x + .9$	.9	-2.5
3	$y = 3x - 1.6$	-1.6	-2.5
4	$y = 4x - 4.1$	-4.1	

The y-intercept seems to depend *linearly* on the slope:

$$b = 5.9 - 2.5m$$

What does this say geometrically?

REGRESSION LINES



Regression Lines

From Here...

- The y-intercept seems to depend linearly on the slope:

$$b = 5.9 - 2.5m$$

- The point of concurrency seems to be $(2.5, 5.9)$
- And $(2.5, 5.9)$ is the *centroid* of the data

All this can be established via a careful analysis of algebraic calculations that ramp up to full generality. And then,

REGRESSION LINES

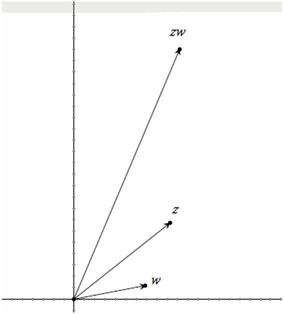
- Students develop an algorithm for finding the best line (of *any* slope), and
- This algorithm is encapsulated into a *formula* for the line of best fit.

The Choices Again

- Trig identities and complex numbers
- Tangents to graphs

TI&$\mathbb{C}$

When you multiply two complex numbers, you multiply the lengths and add the angles:



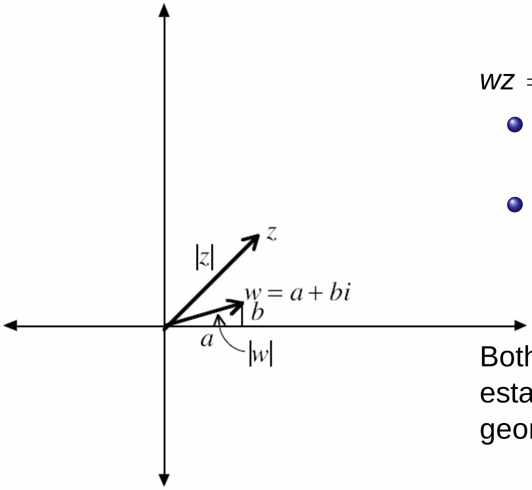
$$|zw| = |z|\,|w| \quad \text{and}$$

$$\arg(zw) = \arg(z) + \arg(w)$$

The usual proof of this involves the addition formulas for sine and cosine.

TI&$\mathbb{C}$

But there's another way. . .

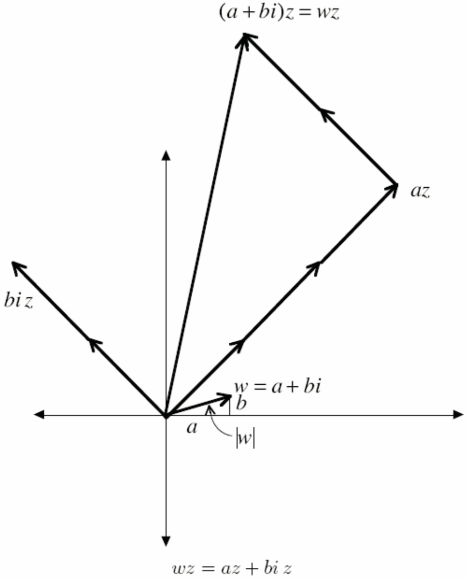


$wz = az + (bi)z$, and

- Multiplying z by a scales it by a factor of $|a|$, and
- Multiplying z by bi rotates it 90° and scales it by a factor of $|b|$.

Both of these facts can be established by analytic geometry.

TI&ℂ



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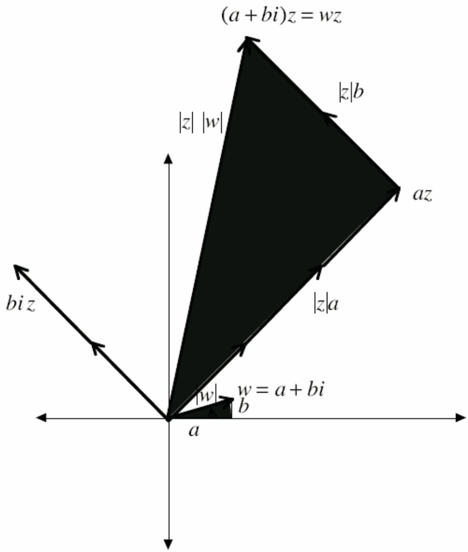
TI&ℂ



The black triangles are similar (SAS) with scale factor $|z|$, so the black angles at the origin are congruent.

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TI&$\mathbb{C}$

So, if

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ w &= s(\cos\alpha + i\sin\alpha) \end{aligned}$$

then

$$\begin{aligned} |zw| &= rs \quad \text{and} \\ \arg(zw) &= \theta + \alpha \end{aligned}$$

so

$$zw = rs\,(\cos(\theta+\alpha) + i\sin(\theta+\alpha))$$

TI&$\mathbb{C}$

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ w &= s(\cos\alpha + i\sin\alpha) \end{aligned}$$

By geometry,

$$zw = rs\left(\cos(\theta+\alpha) + i\sin(\theta+\alpha)\right)$$

But, by algebra,

$$zw = rs\left(\cos\theta\,\cos\alpha - \sin\theta\,\sin\alpha\right) + i\left(\cos\theta\,\sin\alpha + \sin\theta\,\cos\alpha\right)$$

So…

$$\begin{aligned} \cos(\theta+\alpha) &= \cos\theta\,\cos\alpha - \sin\theta\,\sin\alpha \\ \sin(\theta+\alpha) &= \cos\theta\,\sin\alpha + \sin\theta\,\cos\alpha \end{aligned}$$

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TI&$\mathbb{C}$

This means that we can use the algebra of complex numbers to *get* the addition formulas for sine and cosine, as nature intended. And it means that we can (legally) use the algebra of complex numbers to establish and derive trig identities.

Wow.

TI&$\mathbb{C}$

For example, what's $\cos\frac{7\pi}{12}$?
Well,

$$\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$$

So,

$$\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$$

So,

$$\cos\frac{7\pi}{12}=\frac{\sqrt{2}-\sqrt{6}}{4}\quad\text{and (even)}\quad\sin\frac{7\pi}{12}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

THE CHOICES AGAIN

- Tangents to graphs

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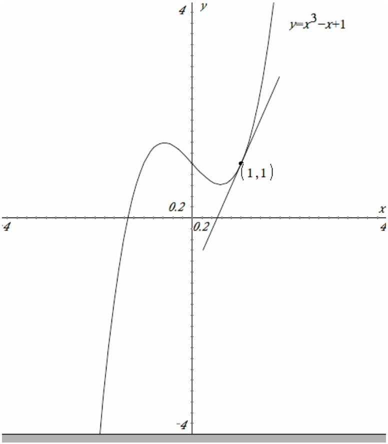
Tangents to Graphs

- What does it mean to say that a line is tangent to the graph of the equation $y = x^3 - x + 1$ (or any other equation)?



Tangents to Graphs

- How do you *imagine* a line tangent to the graph of $y = x^3 - x + 1$ at, say, $(1, 1)$.

  The Graph

- What does the algebra say?

Tangents to Graphs

- $(x-1)(x-2) = x^2 - 3x + 2$, a quadratic, so
- the remainder when $f(x)$ is divided by $(x-1)(x-2)$ is a *linear* polynomial, $r(x)$

$$\begin{array}{ccccccccc}
x^3 - x + 1 & = & x^2 - 3x + 2 & \cdot & \text{something} & + & (ax+b) \\
\uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow \\
f(x) & = & (x-1)(x-2) & \cdot & q(x) & + & r(x)
\end{array}$$

- Also, $r(1) = f(1)$ and $r(2) = f(2)$,
- so the equation of the secant if the graph of $y = f(x)$ between $x = 1$ and $x = 2$ is $y = r(x)$.
- We can find $r(x)$ by hand. . . or with a little help from a friend.

scratchpad

Looking Back

When we began this line of work—organizing curricula around mathematical thinking—our goal was to make school mathematics both more accessible to high school students and more closely aligned with mathematics as a scientific discipline.

On implementation, the method had other benefits—coherence among topics and parsimony of methods.

In hindsight, that should have been obvious: the results of mathematics are its artifacts; the actual mathematics lies in the thinking and the methods that create the artifacts.

For More Information

- For more information, including this presentation, go to

  www.edc.org/cmeproject

- To request samples, contact

  cmeproject@edc.org

THANKS

THE CME PROJECT

PROMOTING MATHEMATICAL HABITS OF MIND IN HIGH SCHOOL

Al Cuoco

International Seminar on Mathematics, Physics and Chemistry Textbooks, September 27–29, 2010