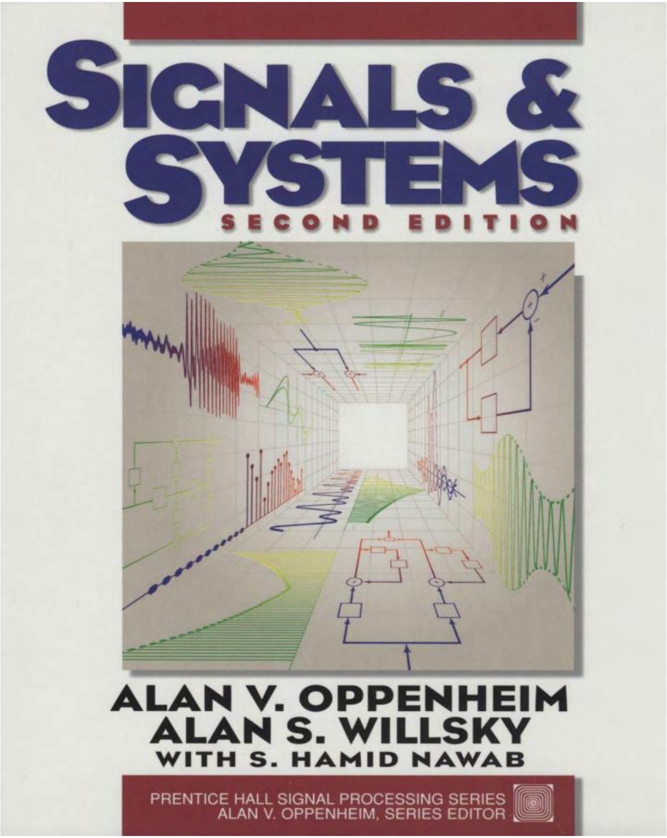
Signals and Systems

EE1101#

A sound mind in a sound body


SIGNALS & SYSTEMS
SECOND EDITION
ALAN V. OPPENHEIM
ALAN S. WILLSKY
WITH S. HAMID NAWAB
PRENTICE HALL SIGNAL PROCESSING SERIES
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+ Tutorials

Remainders

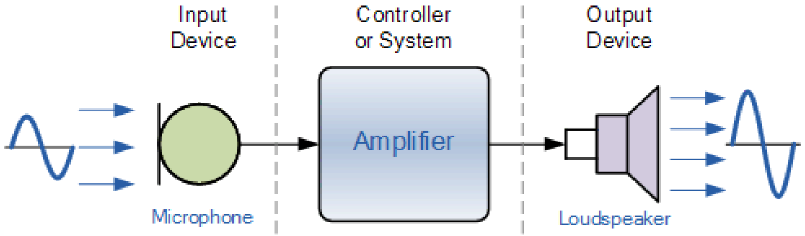
Listen well

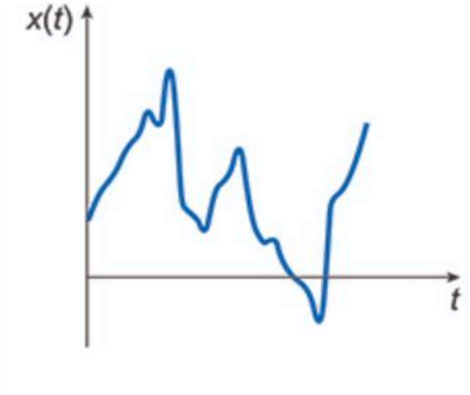
Write Notes

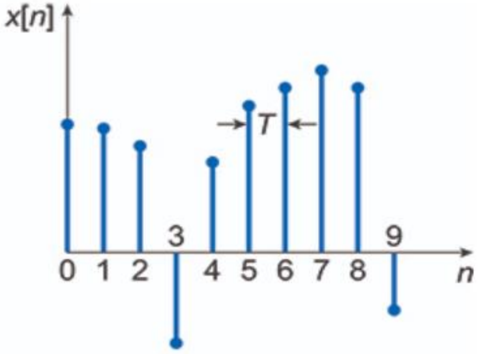
Practice

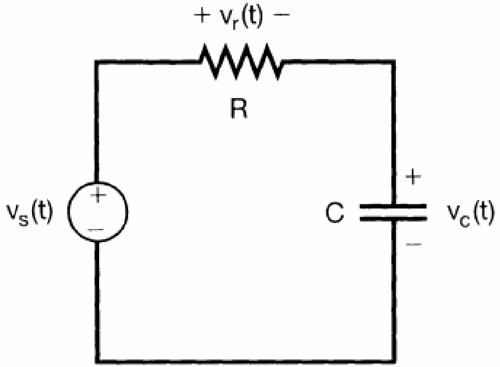
Signal and System

- Think of a function









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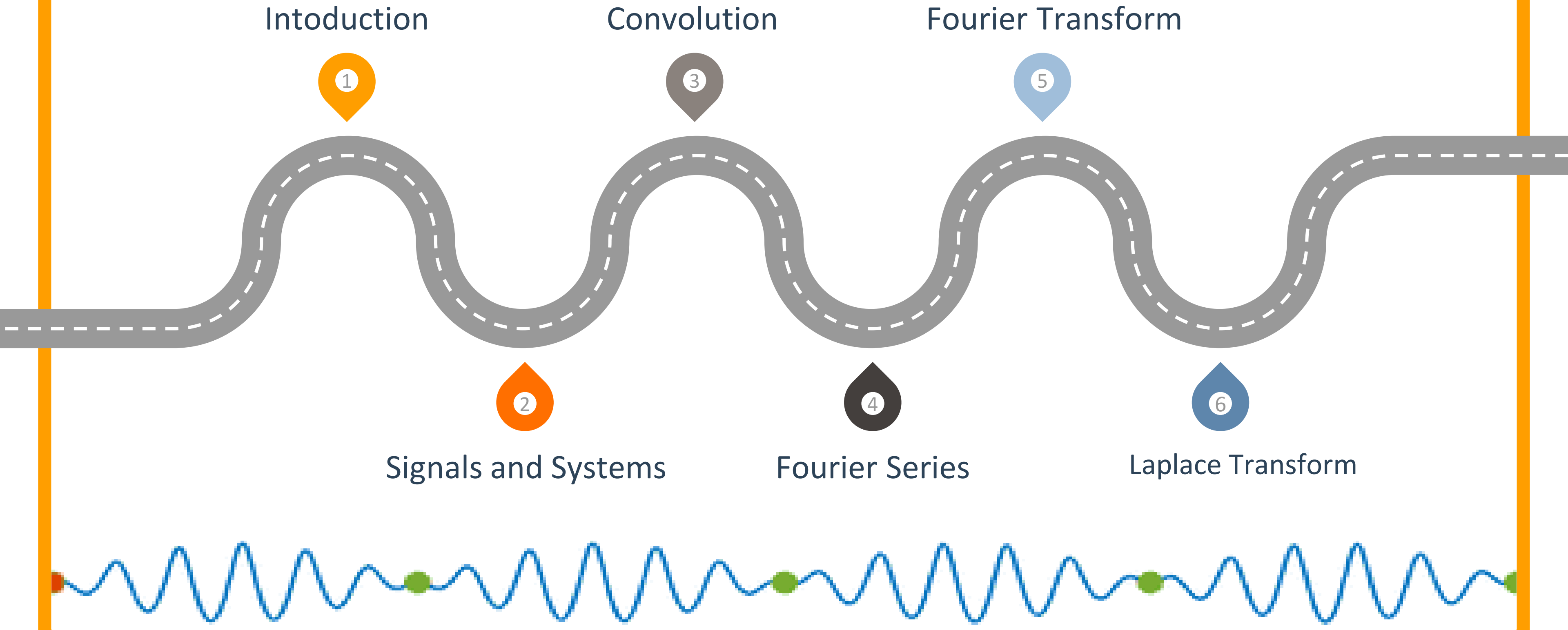
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Fourier Series
Laplace Transform

Signals and Systems

Signals and Systems

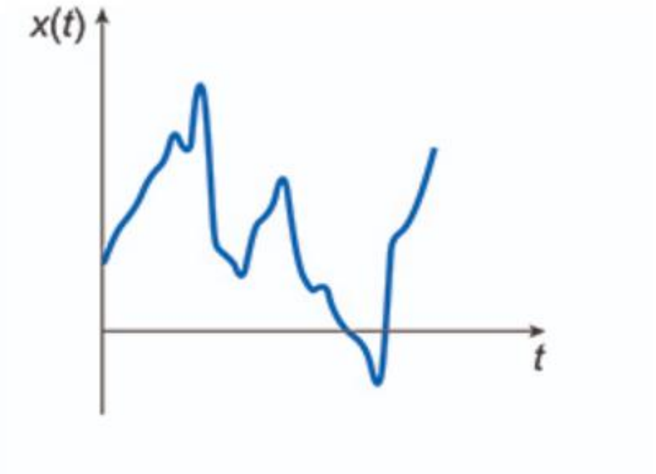
1. Classification of signals

2. Signal Transformation

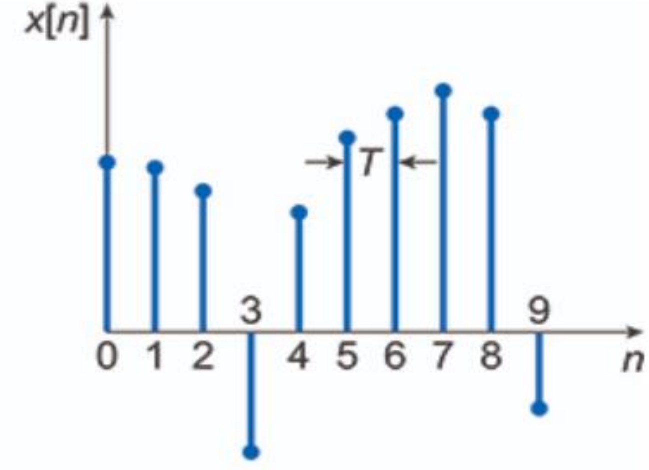
3. Elementary signals

4. System Properties

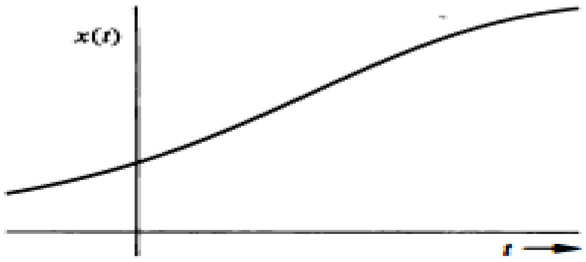
Continuous Time Signals



Discrete Time Signals



Analog Signals



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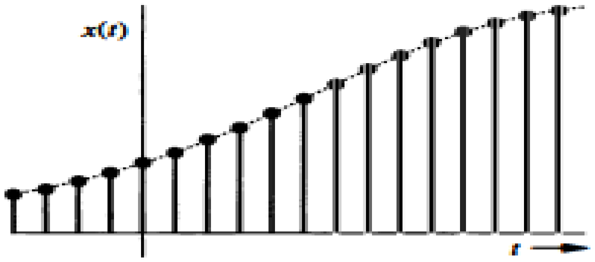
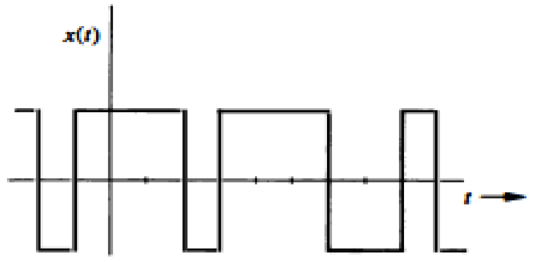
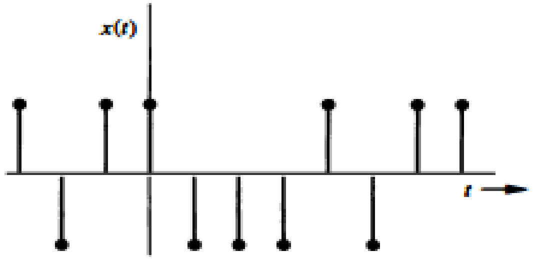


(c)

Digital Signals

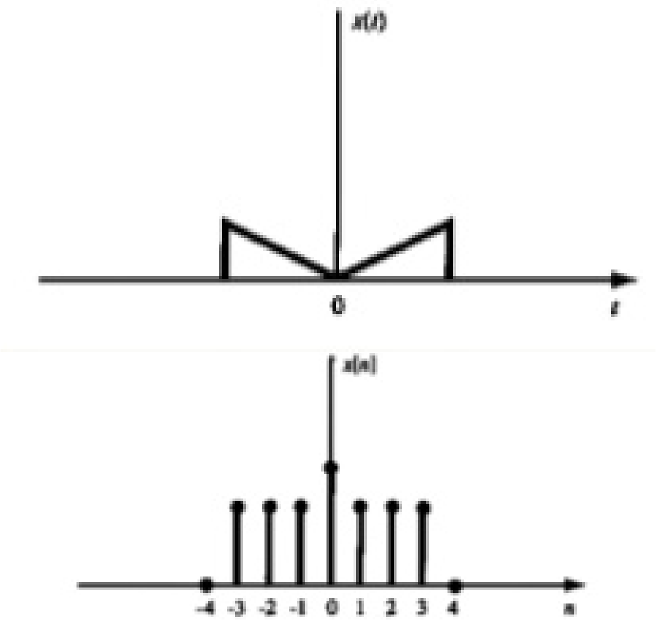


(b)

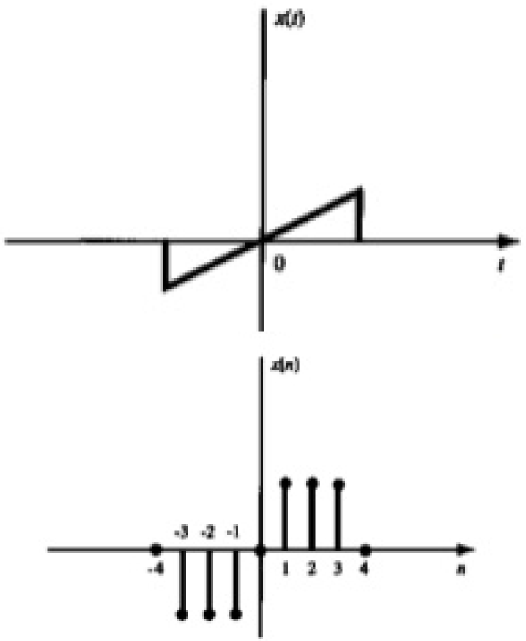


(d)

Even Signals



Odd Signals



gin

- Any signal can be written as the sum of even and odd signals.

In continuous time,

$$x_{even}(t) = \frac{x(t) + x(-t)}{2}$$

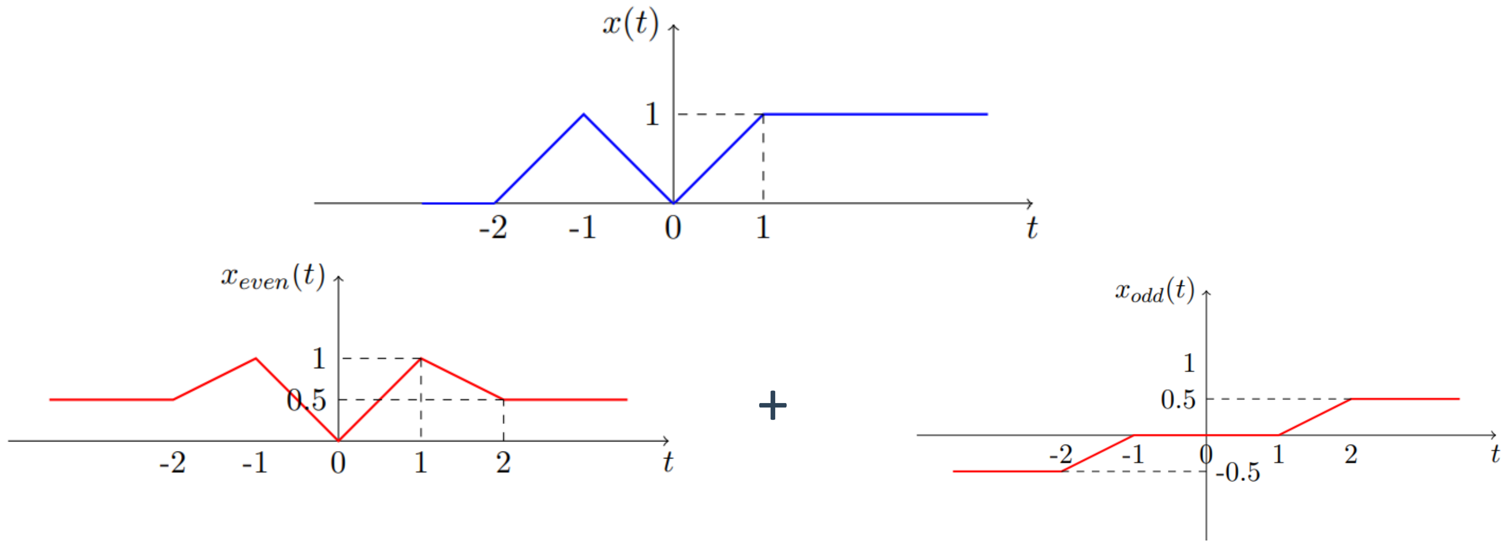
$$x_{odd}(t) = \frac{x(t) - x(-t)}{2}$$

In discrete time,

$$x_{even}[n] = \frac{x[n] + x[-n]}{2}$$

$$x_{odd}[n] = \frac{x[n] - x[-n]}{2}$$

Any signal can be written as the sum of even and odd signals.



$$x_{even}(t) = \frac{x(t) + x(-t)}{2}$$

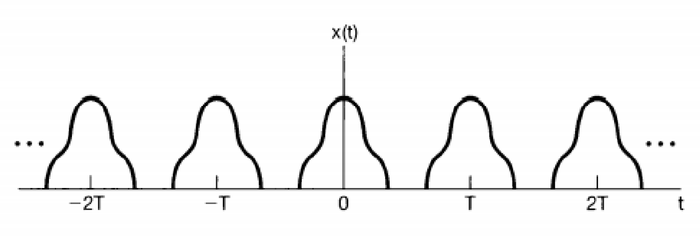
$$x_{odd}(t) = \frac{x(t) - x(-t)}{2}$$

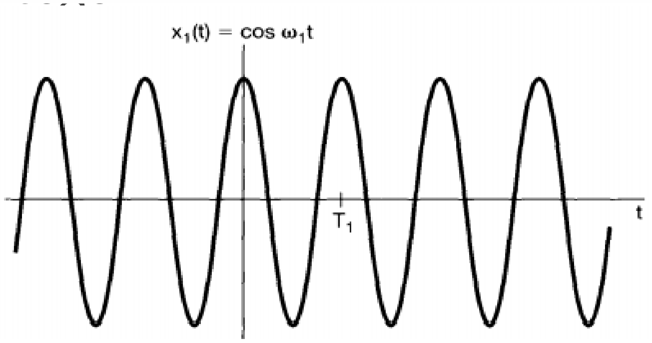
- $x(t) = x^*(-t)$
- If a signal is conjugate symmetric, then its

  real part -> even

  imaginary part -> odd
- Proof : Take $x(t) = a + jb$...
- If a signal is conjugate anti-symmetric, $x(t) = -x^*(-t)$

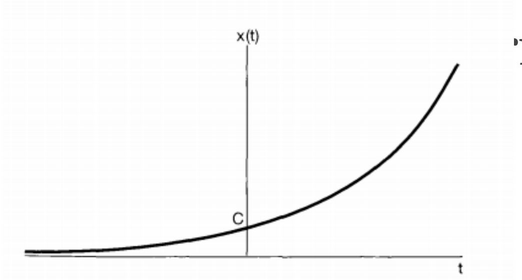
Periodic signals

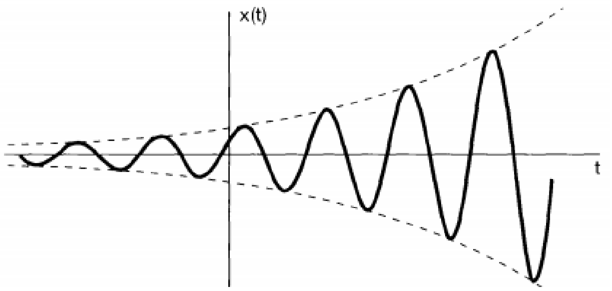




Aperiodic signals

iodic





Periodic signals

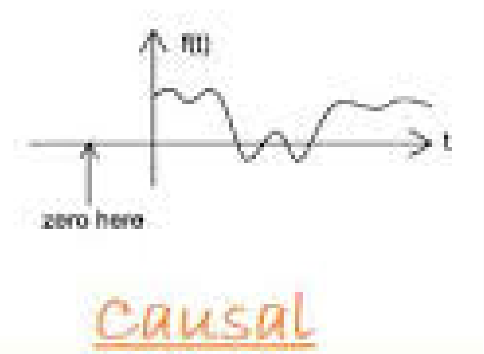
- Frequency = $1/T_o$ (Hertz)
- $\omega = 2\prod / T_o$ (rad/sec)
- Signal can be found if we know just a period

Aperiodic signals

- A signal that is not periodic
- $\Omega = 2\prod / N$ (rad/sec)

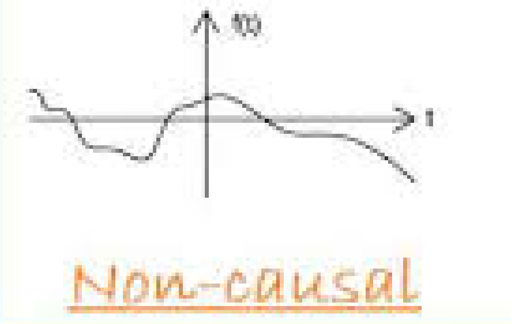
Causal signals

- $x(t)=0, \; t<0$



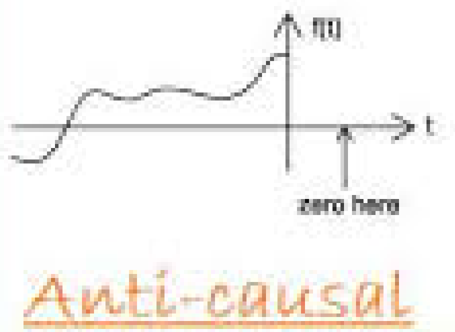
Non-causal signals

- $x(t)\neq 0, \; t<0$

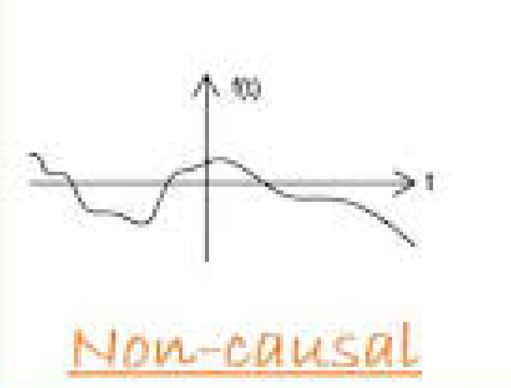


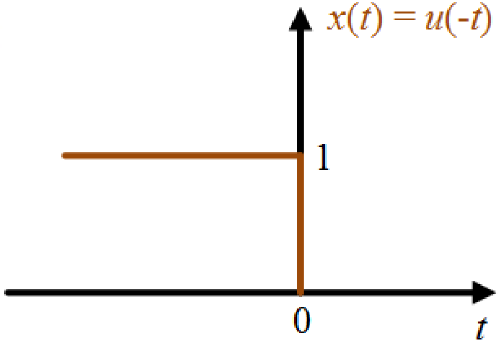
Anti-causal signals

- $x(t)=0, \; t>=0$



- An everlasting signal is always non-causal
- A non-causal signal is not always everlasting





Power

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t).$$

Energy

$$\int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt,$$

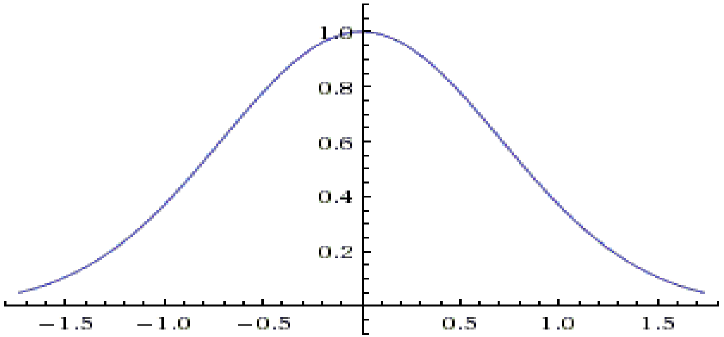
- To quantify a signal that varies with time, energy is defined as $\int_{t_1}^{t_2} |x(t)|^2 dt$ in CT and $\sum_{n=n_1}^{n_2} |x[n]|^2$ in DT

Classification of signals

Energy signals

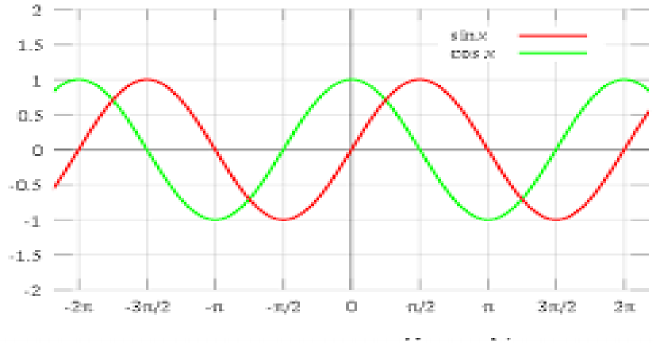
- Finite energy
- Zero average power

$$P_\infty = \lim_{T \to \infty} \frac{E_\infty}{2T} = 0.$$

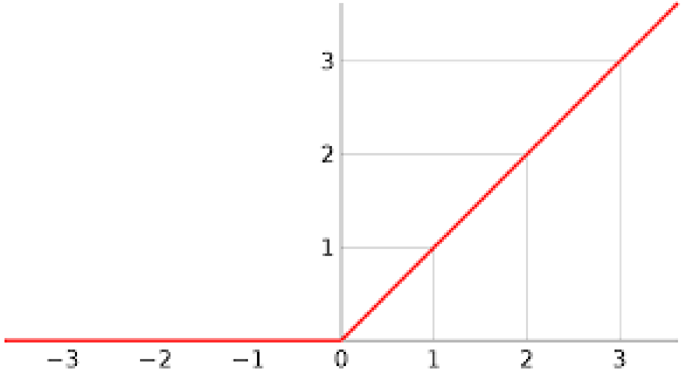


Power signals

- Finite average power
- Infinite energy



- Not necessarily every signal should be either power or energy signal



Signals and Systems

1. Classification of signals

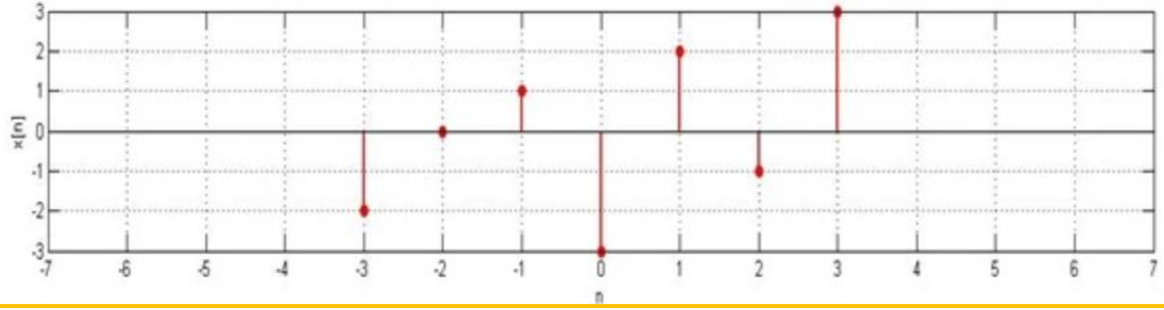
2. Signal Transformation

3. Elementary signals

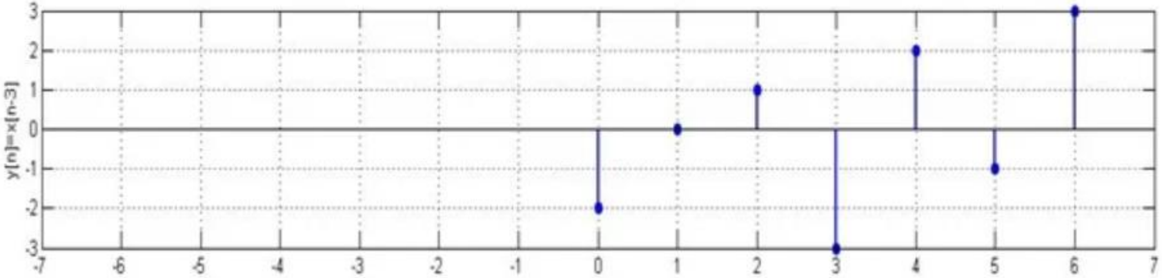
4. System Properties

Time shift – Time delay signal

- x[n]

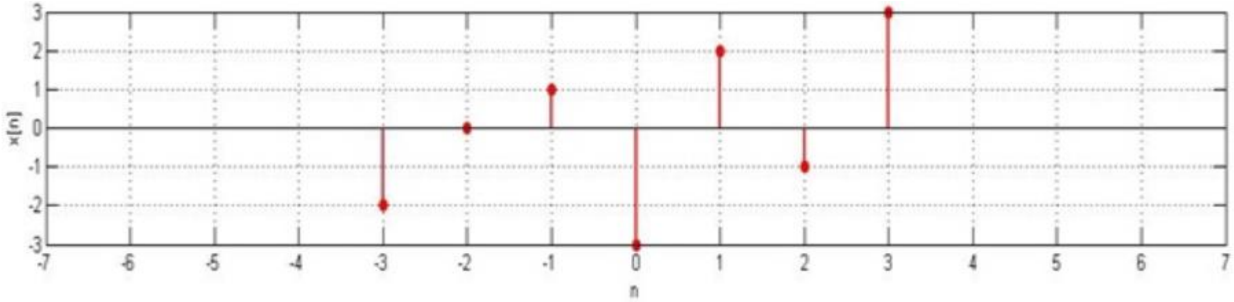


- x[n-3]

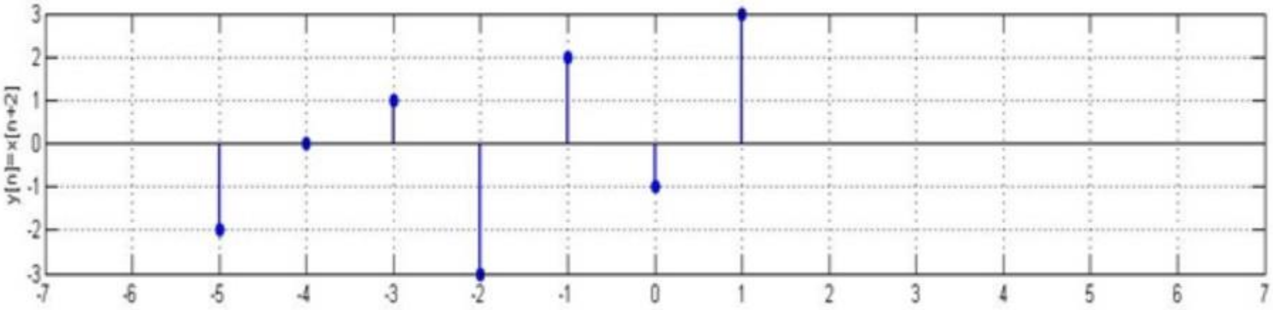


Time shift – Time advanced signal

- x[n]

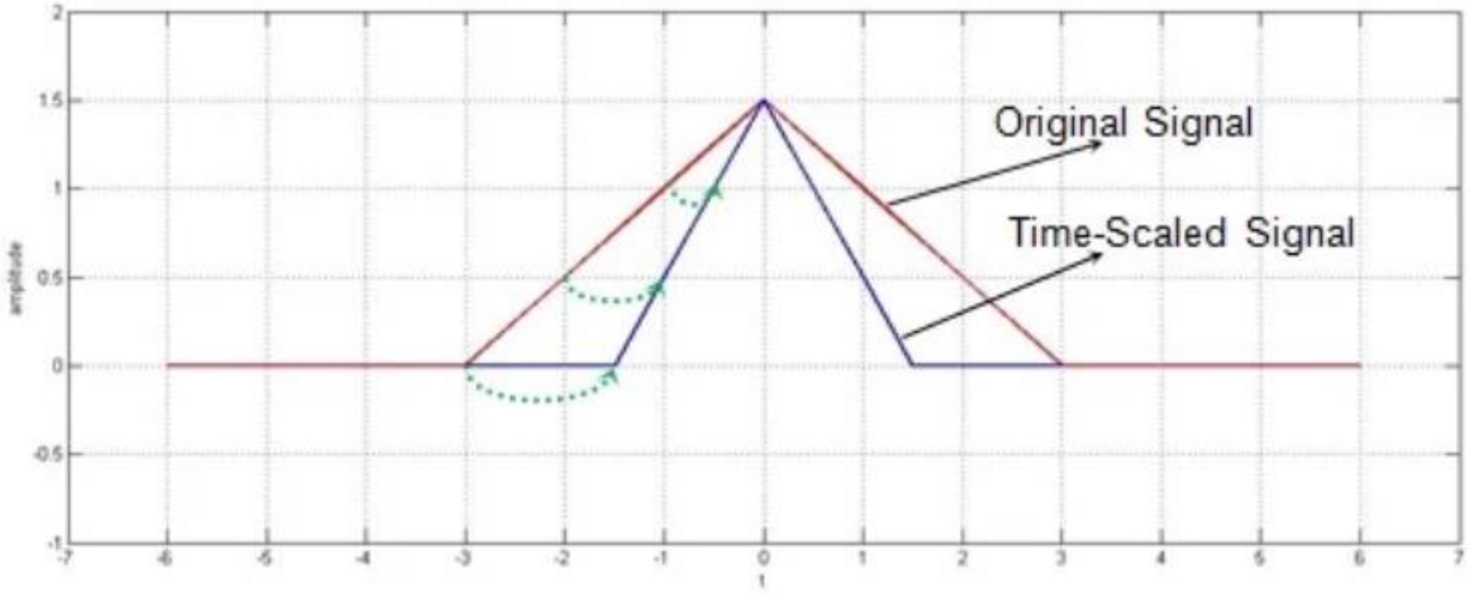


- x[n+2]



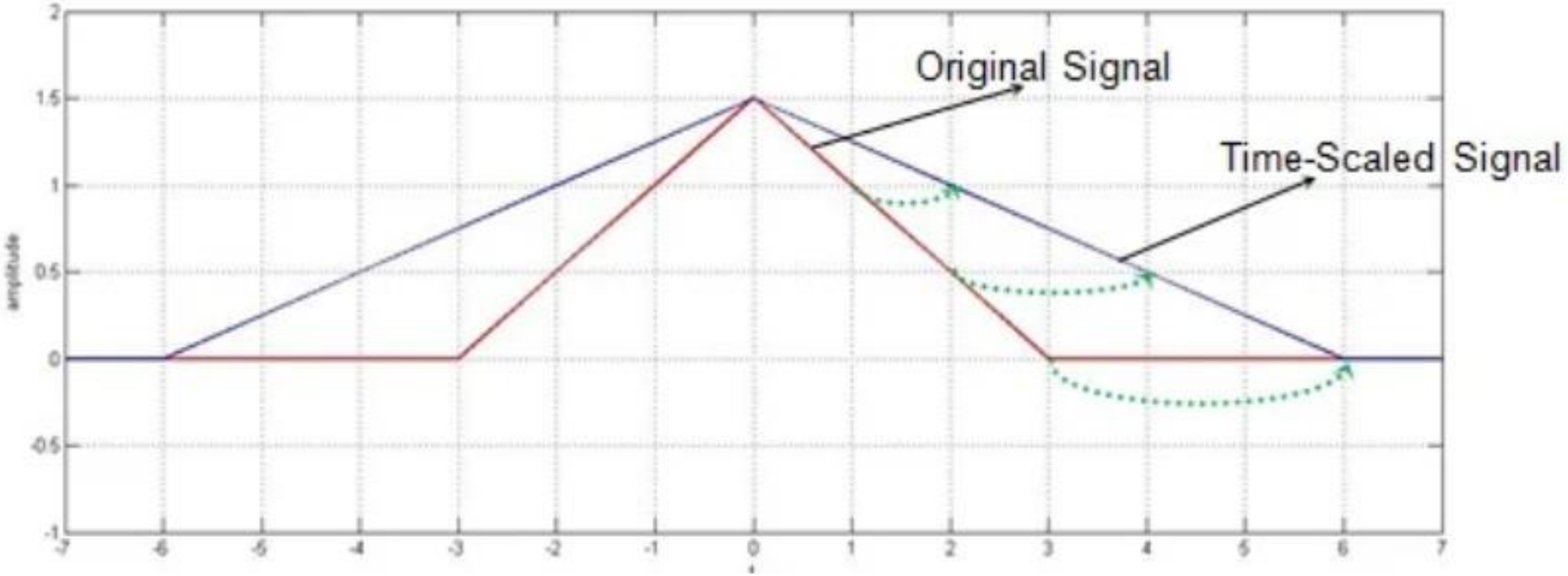
Time Scaling

x(t) → x(2t) (Compresses)



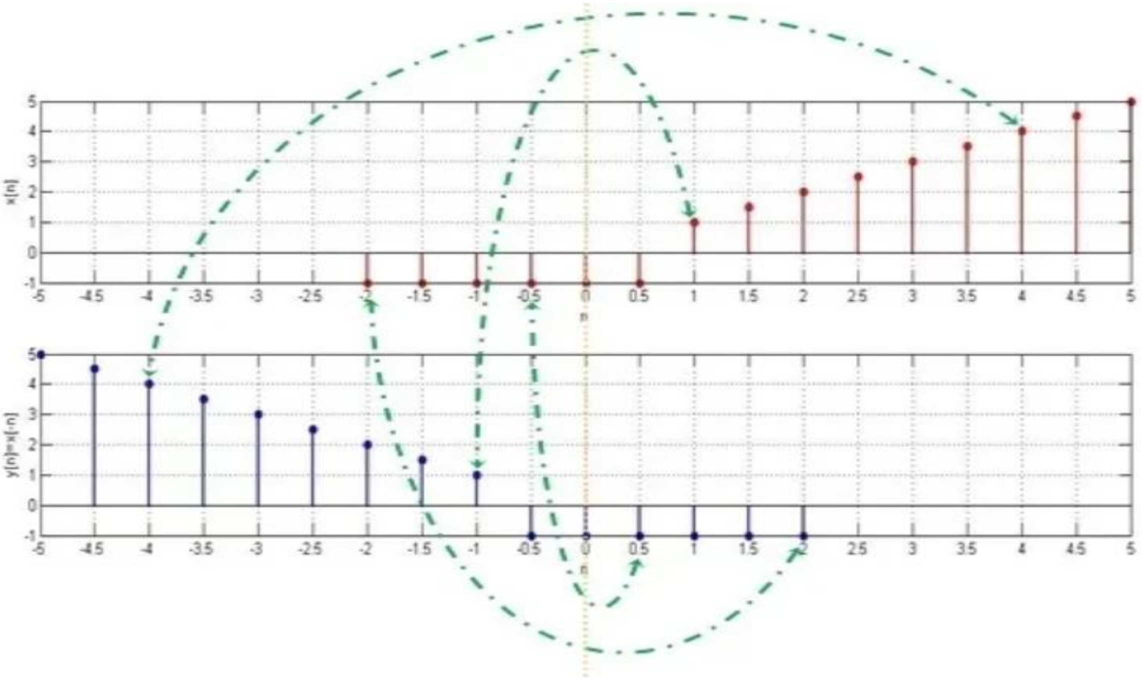
Time Scaling

x(t) → x(t/2) (Expands)



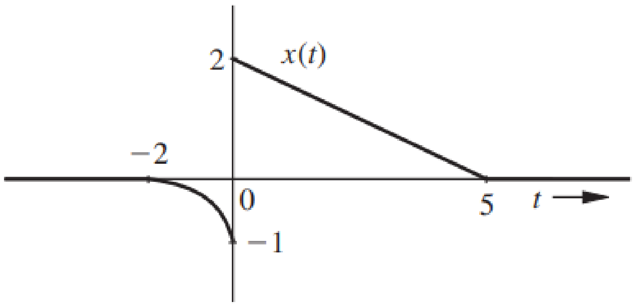
Time reversal

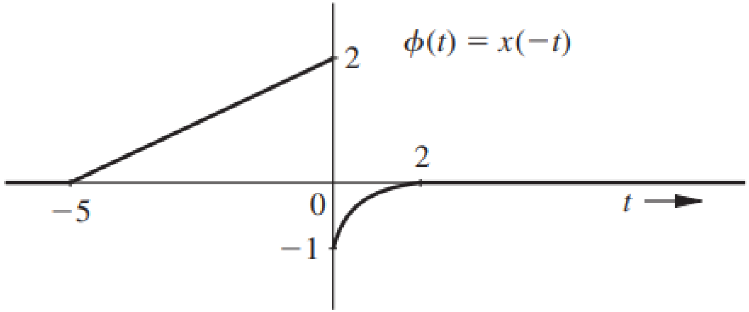
- x[n]
- x[-n]



Time reversal

- The signal and the time reversed signal is symmetric about x- axis





Signal Transformation

- $2t+3$

Signals and Systems

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Exponential Signal - e^{st}

- 's' – complex frequency
- $s = \sigma + j\omega$
- $e^{st} = e^{(\sigma + j\omega)t}$

  $= e^{\sigma t} e^{j\omega t}$

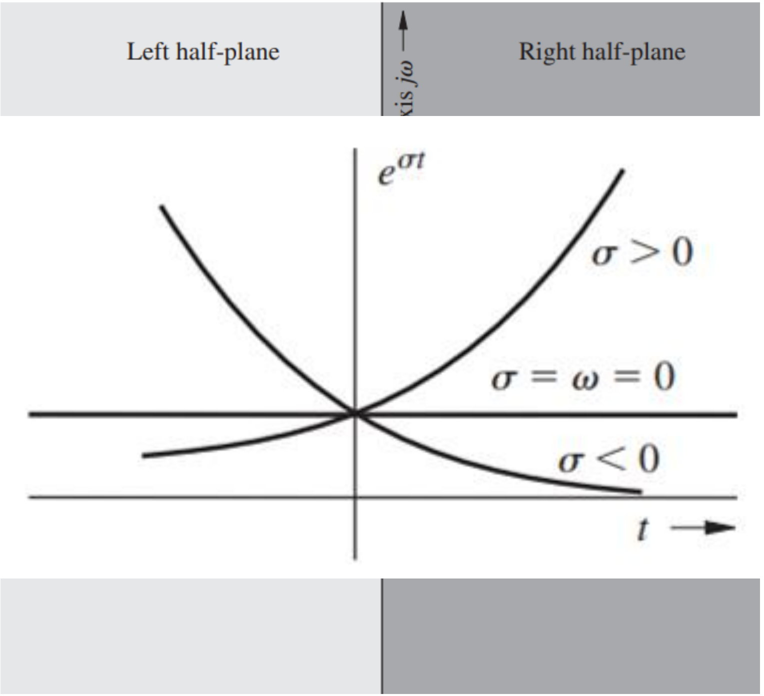
  $= e^{\sigma t} (\cos(\omega t) + j \sin(\omega t))$
- Depending on the values of σ and ω, various signals can be found from the above general equation.

Exponential Signal - e^{st}

- 1. $\omega = 0, \sigma = s,$

- $s = \sigma + j(0)$
- $e^{st} = A\, e^{(\sigma + j0)t}$

  $= A\, e^{\sigma t}$

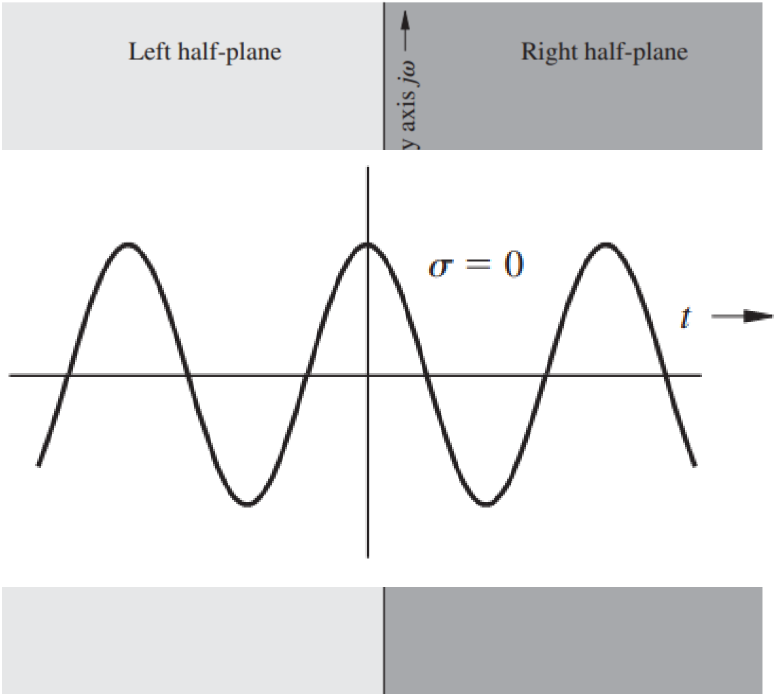


Exponential Signal - e^{st}

- 2. $\sigma = 0$, $s = \pm j\omega$

- $s = \sigma + j\omega$
- $e^{st} = e^{(0+j\omega)t}$

  $= e^{\pm j\omega t}$



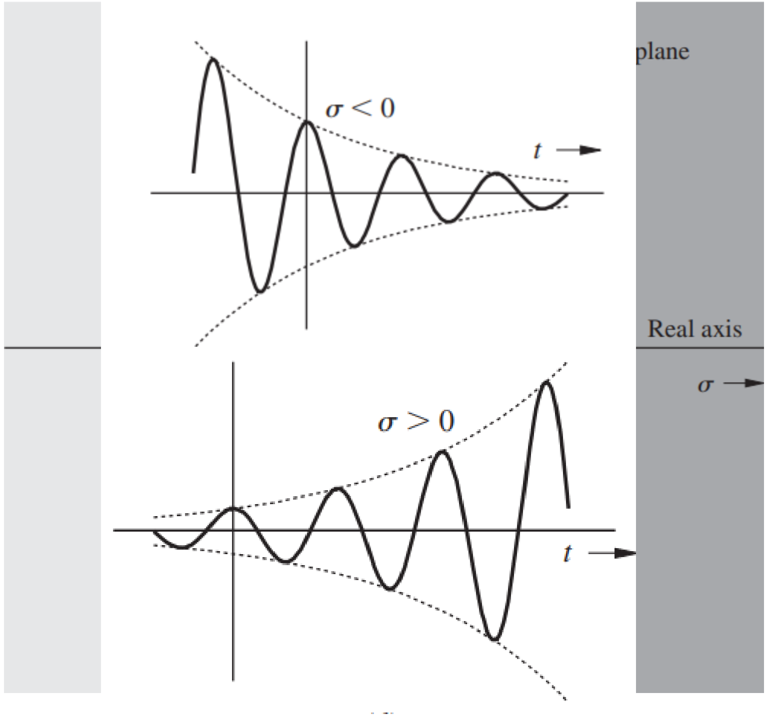
Exponential Signal - e^{st}

- 3. $s = \sigma \pm j\omega$

- $s = \sigma + j\omega$
- $e^{st} = A\, e^{(\sigma + j\omega)t}$

  $= A\, e^{\sigma t} e^{j\omega t}$

  $= Ae^{\sigma t} (\cos(\omega t) + j \sin(\omega t))$

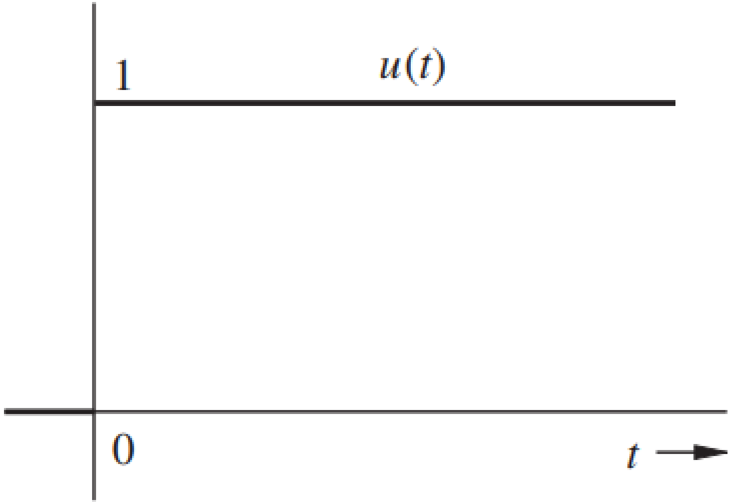


Smooth signals, so far

No abrupt changes…

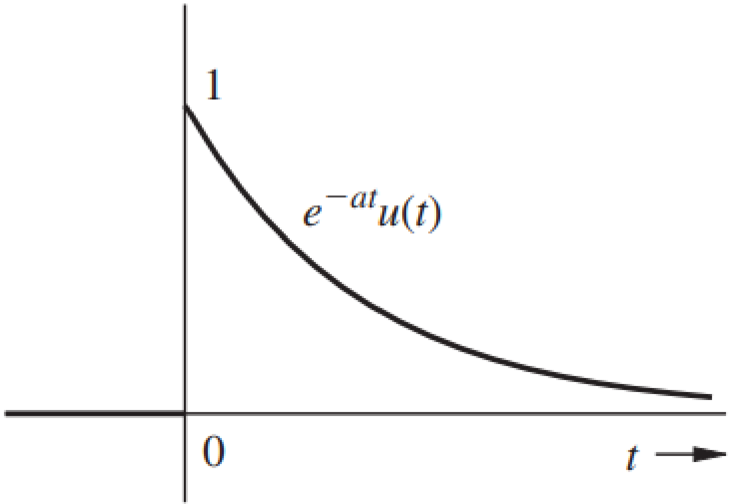
Unit step function - u(t)

- $$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



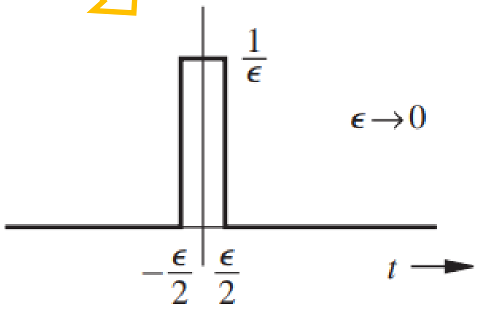
Unit step function - u(t)

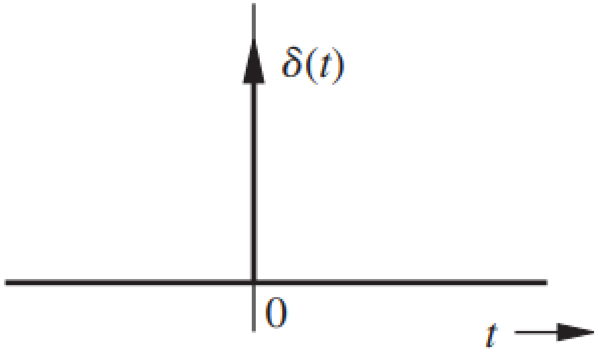
- When unit step function is multiplied with a function…
- Example : e^{-at} u(t)



Unit Impulse function - δ(t)

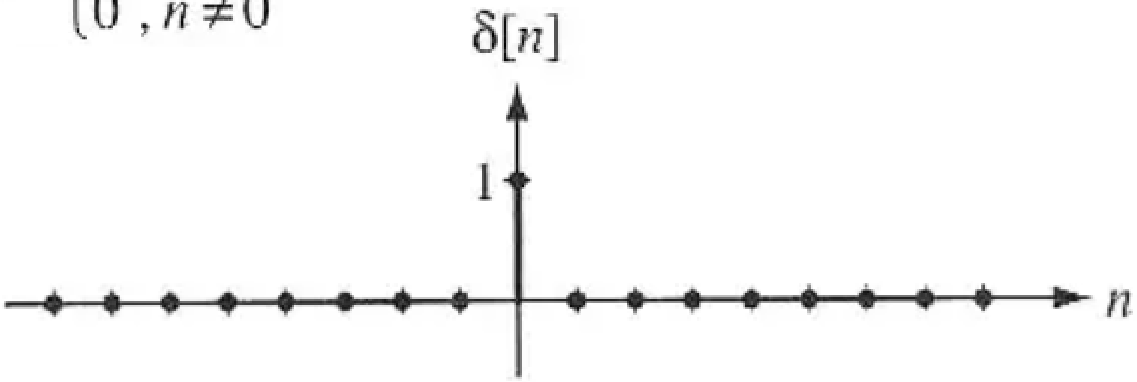
- $$\delta(t) = 0 \qquad t \neq 0$$
  $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$





Unit Impulse function - δ(t)

- $$\delta[n] = \begin{cases} 1\ , n = 0 \\ 0\ , n \neq 0 \end{cases}$$



Unit Impulse function - δ(t)

- Multiplication :

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

$$\phi(t)\delta(t - T) = \phi(T)\delta(t - T)$$

- Integration :

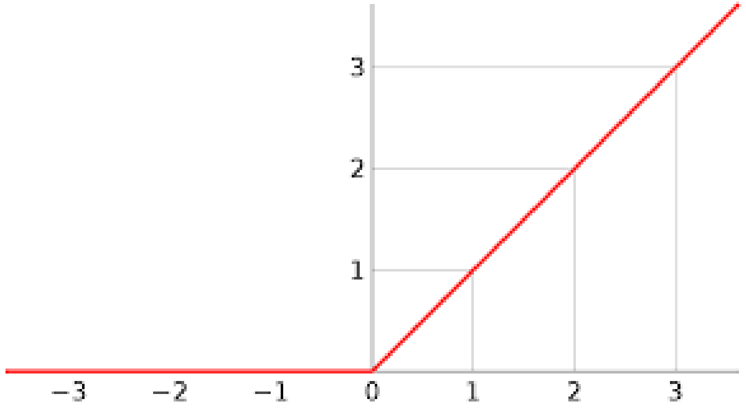
$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)\,dt$$

$$= \phi(0)$$

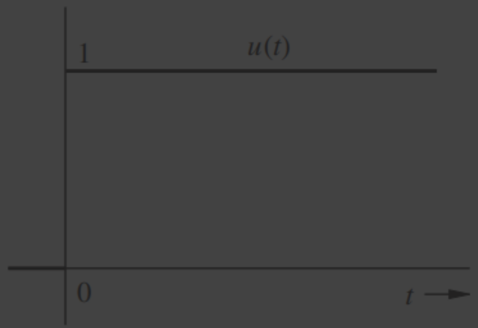
$$\int_{-\infty}^{\infty} \phi(t)\delta(t - T)\,dt = \phi(T)$$

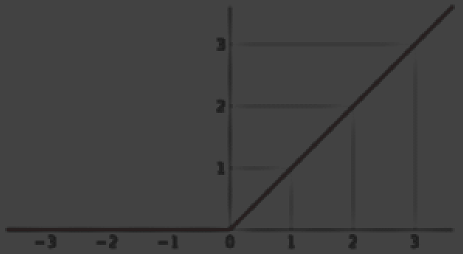
Unit Ramp function - r(t)

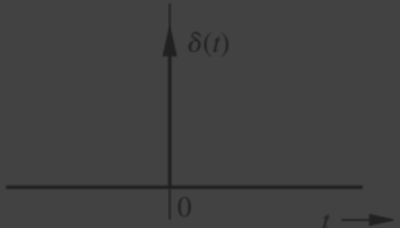
- $\text{ramp}(t) = \begin{cases} t & , t > 0 \\ 0 & , t \leq 0 \end{cases}$
- r(t) = t.u(t)



Relationship between the three signals







$u'(t) = \delta(t)$

$r'(t) = u(t)$

Signals and Systems

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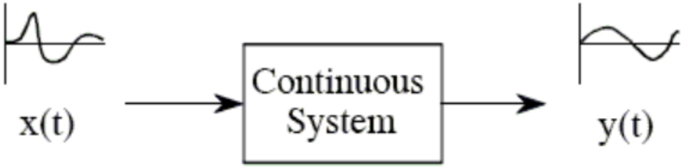
4. **System Properties**

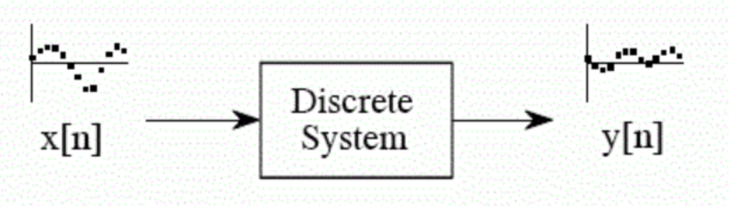
Continuous Time Systems

- Input and output signals are continuous.



Discrete Time Systems

- Input and output signals are discrete.



Systems with memory

- Output is dependent on the past or future values of the input
- Example :

$$y[n] = x[n - 1]$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau,$$

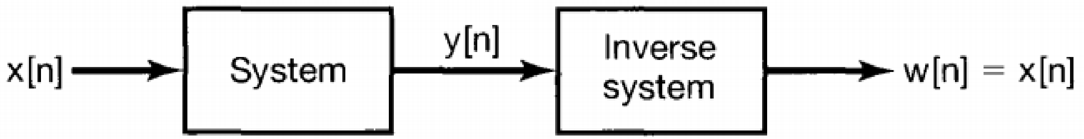
Systems without memory

- Output is dependent only on the input at that same time
- Example :

$$y(t) = R.x(t)$$

$$y[n] = x[n]$$

Invertibility and Inverse Systems



- A system is said to be invertible if distinct inputs lead to distinct outputs.
- Example : $y(t) = 2 \cdot x(t)$, $w(t) = y(t)/2$

- Is $y[n] = 0$ invertible?
- Is $y(t) = x^2(t)$ invertible

- *A system is said to be invertible if distinct inputs lead to distinct outputs.*
- *Any system that gives a constant as an output is not invertible.*

Causality

- Output at any time depends only on values of the input at the present time and in the past and does not anticipate future values of the input.

- *All memory less systems are causal, since the output responds only to the current value of the input.*

- Is $y(t) = x(t + 1)$ causal?
- Is $y[n] = x[-n]$ causal?

Stability

- If the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge

- Example :

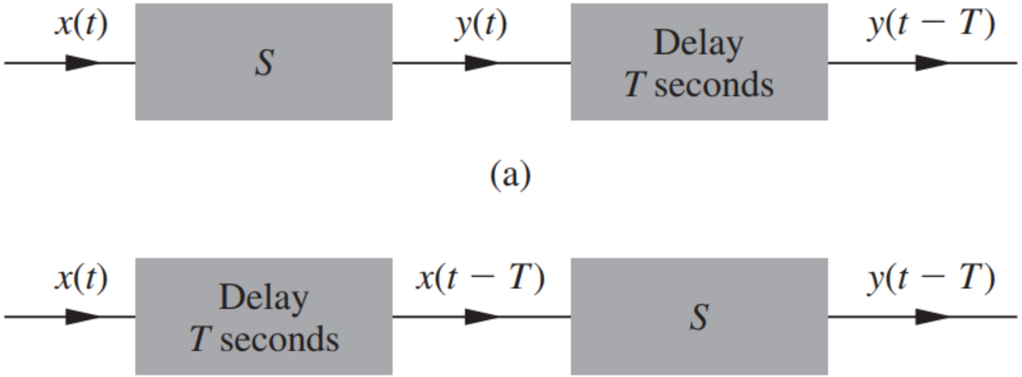
$$y(t) = e^{x(t)}$$

$$|x(t)| < B,$$

$$e^{-B} < |y(t)| < e^{B}$$

- To check stability, let the input be u(t) which is a bounded input signal

Time Invariance



- A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal

- Is y[n] = n.x[n] time invariant?

- No.

$$x_1[n] = \delta[n] \qquad\qquad x_2[n] = \delta[n-1]$$

$$y_1[n] = n\delta[n] = 0 \qquad y_2[n] = n\delta[n-1] = \delta[n-1]$$

Linearity

2 conditions:-

1. Additivity :

$$x_1 \rightarrow y_1$$

$$x_2 \rightarrow y_2$$

$$x_1 + x_2 \rightarrow y_1 + y_2$$

2. Homogeneity :

$$kx \rightarrow ky$$

Superposition

$$k_1x_1 + k_2x_2 \rightarrow k_1y_1 + k_2\,y_2$$

- Is $y(t) = x^2(t)$ linear?

- No.

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$
$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

$$\begin{aligned} x_3(t) \rightarrow y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2 x_1^2(t) + b^2 x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2 y_1(t) + b^2 y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

Convolution

$x[n] * h[n]$

Signals and Systems

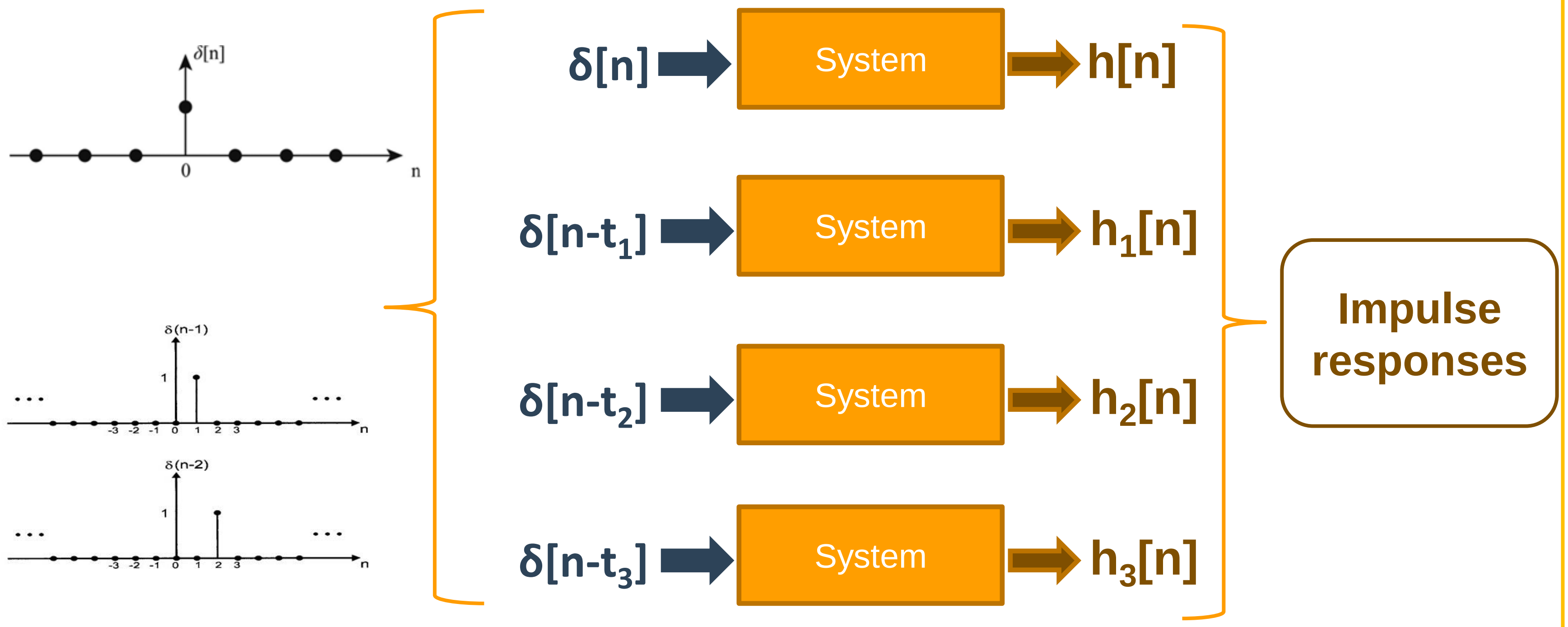
1. Introduction to Convolution

2. Properties of Convolution

3. System Properties

4. Differential and difference equations

Intro to Convolution



Intro to Convolution

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases},$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases},$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots.$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k].$$

LTI Systems

δ[n] → LTI → h[n]

Impulse response

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k].$$

→ LTI →

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

Convolution Sum

Convolution of x[n] and h[n]

Input Signal

Impulse Response

$$\boxed{y[n] = x[n] * h[n].} = \boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].}$$

Output Signal

Convolution operator

Convolution of x(t) and h(t)

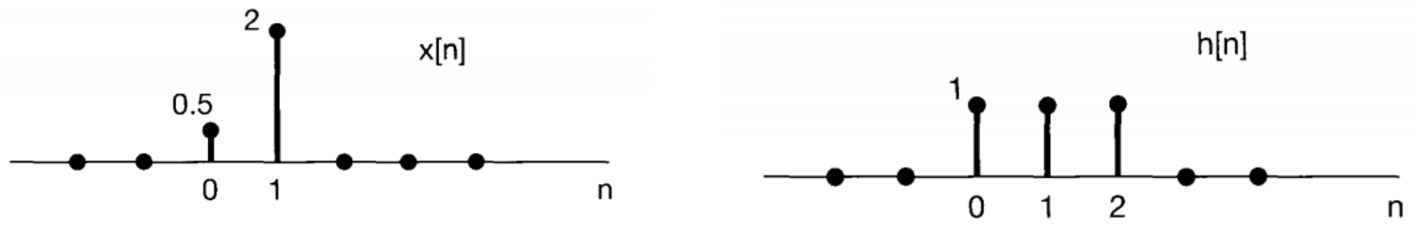
- Any CT signal can be represented as

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.$$

- Convolution integral

$$y(t) = x(t) * h(t). = y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

Method 1 : Convolve x[n] and h[n]

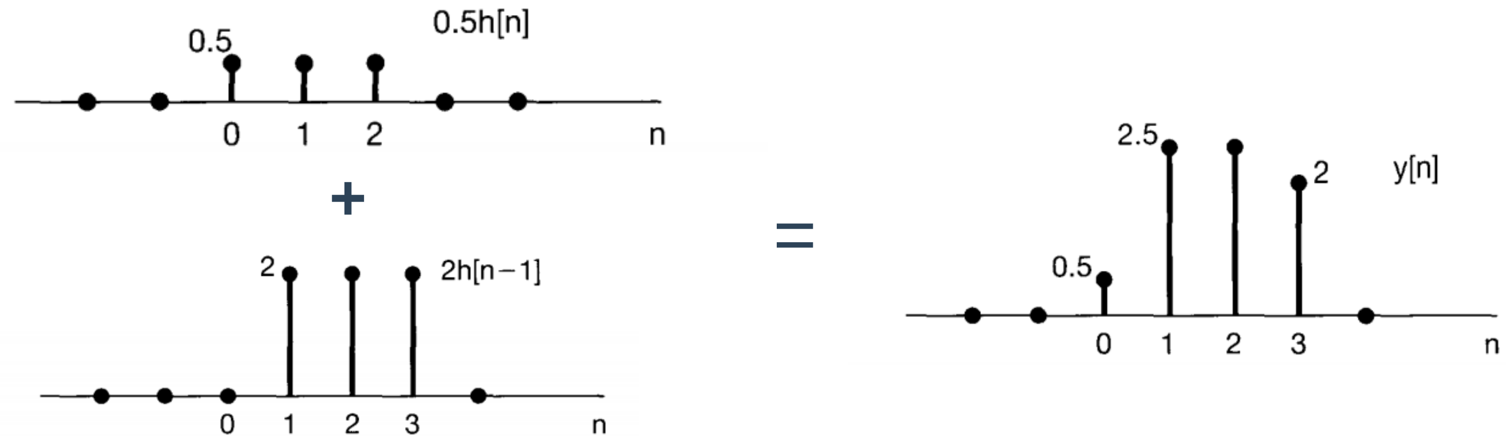


$$y[n] = x[n] * h[n].$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

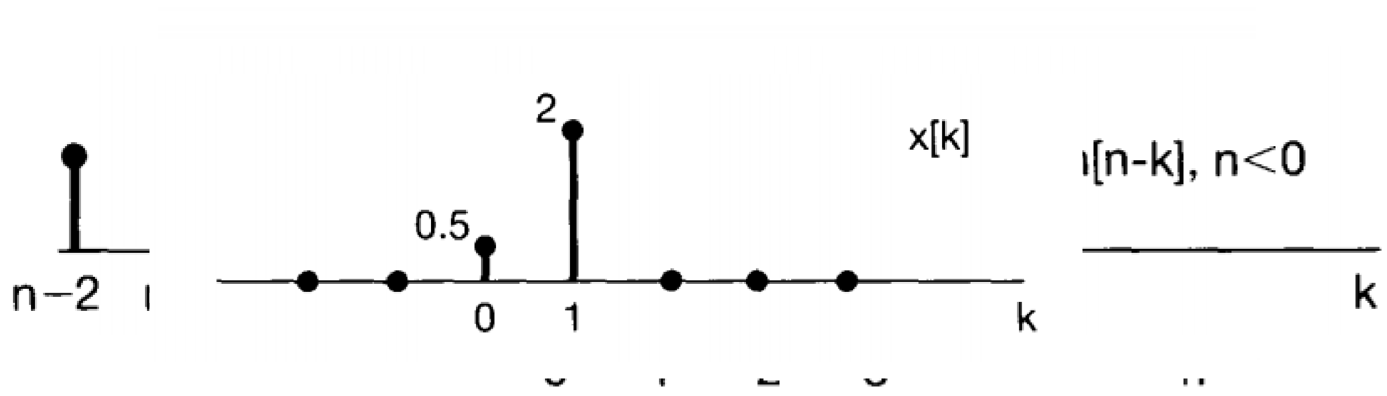
$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1].$$

Method 1 : Convolve x[n] and h[n]



$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1].$$

Method 2 : Convolve x[n] and h[n]



Signals and Systems

1. Introduction to Convolution

2. Properties of Convolution

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4. Differential and difference equations

Commutative Property

- Convolution shows commutative property

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k],$$

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau.$$

Distributive Property

- Convolution shows distributive property

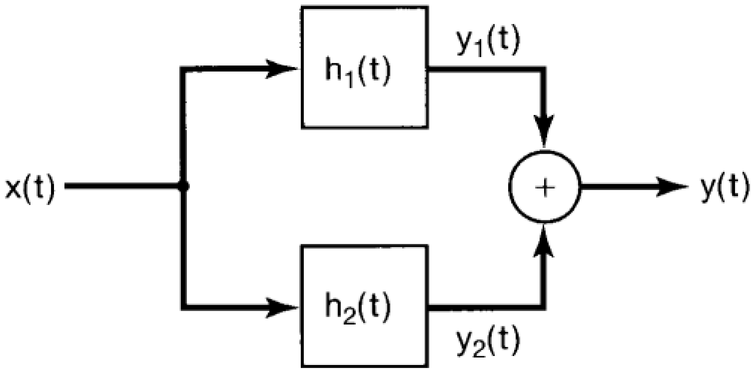
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n],$$

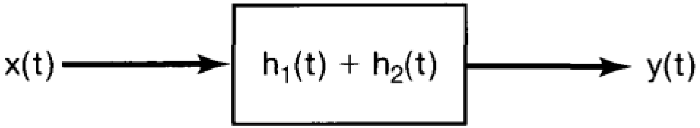
$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$

Properties of Convolution

Distributive Property

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$





Associative Property

- Convolution shows associative property

$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t).$
$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n],$

Properties of Convolution

Associative Property

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t).$$

Properties of Convolution

Sum of the y[n]

$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].}$$

$$\sum_{k=-\infty}^{+\infty} y[n] = \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x[k]h[n-k].$$

Sum of y[n] = (Sum of y[n]) * (Sum of y[n])

Signals and Systems

1. Introduction to Convolution
2. Properties of Convolution
3. **System Properties**
4. Differential and difference equations

LTI systems with Memory

- Memoryless discrete LTI system

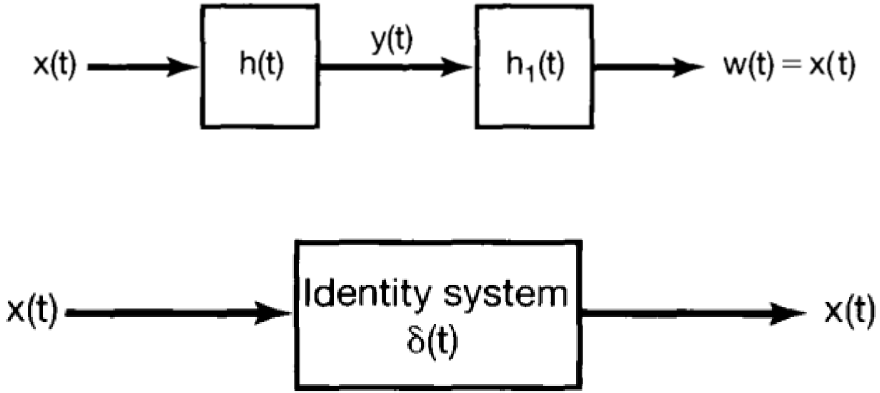
  $$h[n] = K .\delta [n]$$

- Memoryless discrete LTI system

  $$y(t) = K .x(t) \qquad \& \qquad h(t) = K .\delta (t)$$

- If K=1, the systems are identity systems

Invertibility of LTI systems



$$h(t) * h_1(t) = \delta(t).$$

$$h[n] * h_1[n] = \delta[n].$$

Causal LTI systems

- For an LTI system to be causal,

$$h[n] = 0 \quad \text{for } n < 0$$

$$h(t) = 0 \quad \text{for } t < 0$$

- *Causality of an LTI system is equivalent to its impulse response being a causal signal*

Stability for LTI systems

- An LTI system is stable if the impulse response is absolutely integrable
- $$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty,$$
- $$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty.$$

Signals and Systems

1. Introduction to Convolution
2. Properties of Convolution
3. System Properties
4. **Differential and difference equations**

Fourier Series

The continuous-time fourier series

Fourier Series

1. Introduction to Fourier Series

2. Fourier series and LTI

3. Gibbs phenomenon

4. Properties of Continuous Time Fourier Series

5. Filtering

e^{st} - input to an LTI system

- Let y(t) be the output of the system for input x(t) = e^{st}, $\mathbf{H}\{e^{st}\} = y(t)$
- Since the system is time-invariant, $\mathbf{H}\{e^{s(t+t_0)}\} = y(t+t_0)$
- Since the system is linear, $\mathbf{H}\{e^{s(t+t_0)}\} = \mathbf{H}\{e^{st}e^{st_0}\} = e^{st_0}\mathbf{H}\{e^{st}\} = e^{st_0}y(t)$

$$y(t+t_0) = e^{st_0}y(t)$$

- Setting t = 0, $y(t_0) = y(0)e^{st_0}$

$$y(t) = y(0)e^{st} = \lambda e^{st}$$

$$\implies \mathbf{H}\{e^{st}\} = \lambda e^{st} \text{ where } \lambda = y(0)$$

Fourier Series

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}\,d\tau.$$

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau.$$

$$y(t) = H(s)e^{st},$$

$$\boxed{H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau.}$$

Frequency Response

FSR of CT Periodic Signals

- A signal is periodic if $x(t) = x(t + T)$ for all t.
  where T is the fundamental period
- Only periodic functions that satisfy the Dirichlet condition can be represented using FS

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}}$$

$$e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

FS representation

Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t},$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt.$$

Analysis Equation

Fourier Series

1. Introduction to Fourier Series

2. Fourier series and LTI

3. Gibbs phenomenon

4. Properties of Continuous Time Fourier Series

5. Filtering

Fourier Series

$$\mathbf{H}\{e^{st}\} = \lambda e^{st} \quad \text{where } \lambda = y(0)$$

s = σ + jω

In Fourier series for CT, σ = 0.

$$\boxed{H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau}\, d\tau.}$$

$$e^{j\omega t} \rightarrow \boxed{\text{LTI}} \rightarrow H(j\omega).e^{j\omega t}$$

Fourier Series and LTI Systems

$$\mathbf{H}\{e^{st}\} = \lambda e^{st} \quad \text{where } \lambda = y(0)$$

s = σ + jω

In Fourier series for CT, σ = 0.

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau}\, d\tau.$$

$$x(t) = \sum_k a_k e^{s_k t} \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

FS representation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t},$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt.$$

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt,$$

Energy and the error associated

- As n tends to infinity , energy need to tend to zero
- Every continuous periodic signal has an FSR for which the energy E_N in the approximation error approaches 0 as N goes to infinity

$$E_N = \int_T |e_N(t)|^2 \, dt. \qquad e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}.$$

- All of the signals with which we will be concerned, guarantees that x(t) equals its Fourier series representation, except at isolated values for which x(t) is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Do all periodic signals have an FSR?

- a_k need not be well-defined?
- $x_n(t)$ need not converge to $x(t)$?

Fourier Series

1. Introduction to Fourier Series

2. Fourier series and LTI

3. Dirichlet conditions and Gibbs phenomenon

4. Properties of Continuous Time Fourier Series

5. Filtering

Dirichlet condition

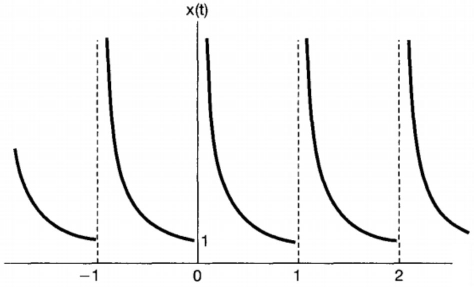
- **Condition 1**: Over any period, x(t) must be absolutely integrable

$$\int_T |x(t)|\,dt < \infty.$$

- **Condition 2** : In any finite interval of time, x(t) is of bounded variation, there are no more than a finite number of maxima and minima during any single period of the signal.

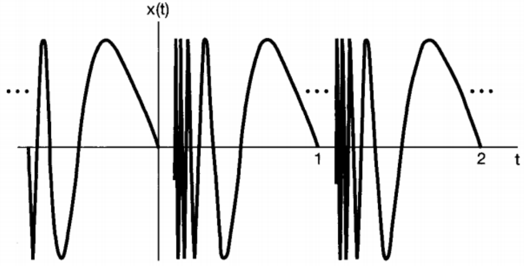
- **Condition 3** : In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite

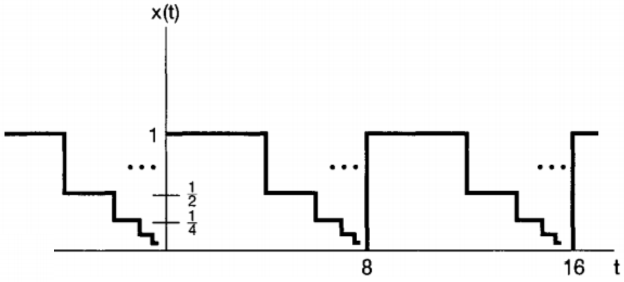
Do they have a FS representation?



$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1$$

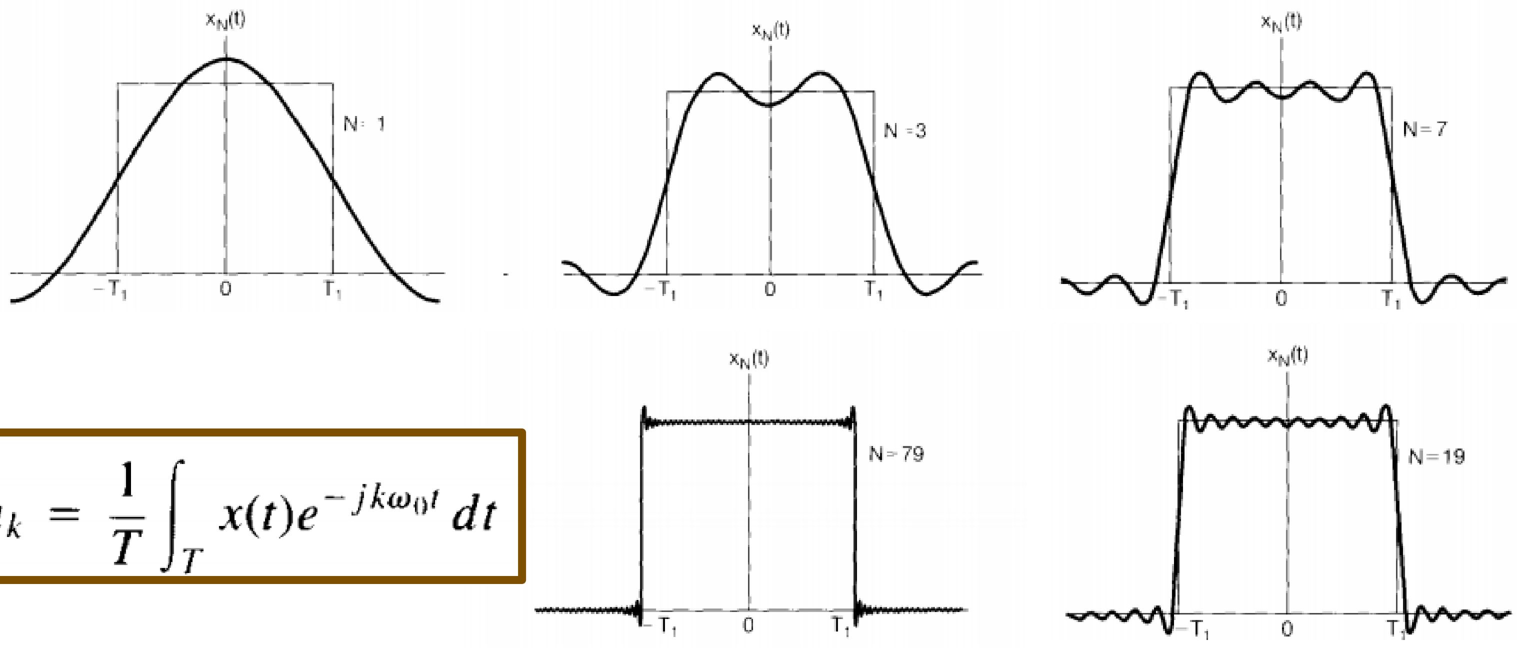




The signal has an infinite number of sections, each of which is half the height and half the width of the previous section.
There are an infinite number of discontinuities in each period.

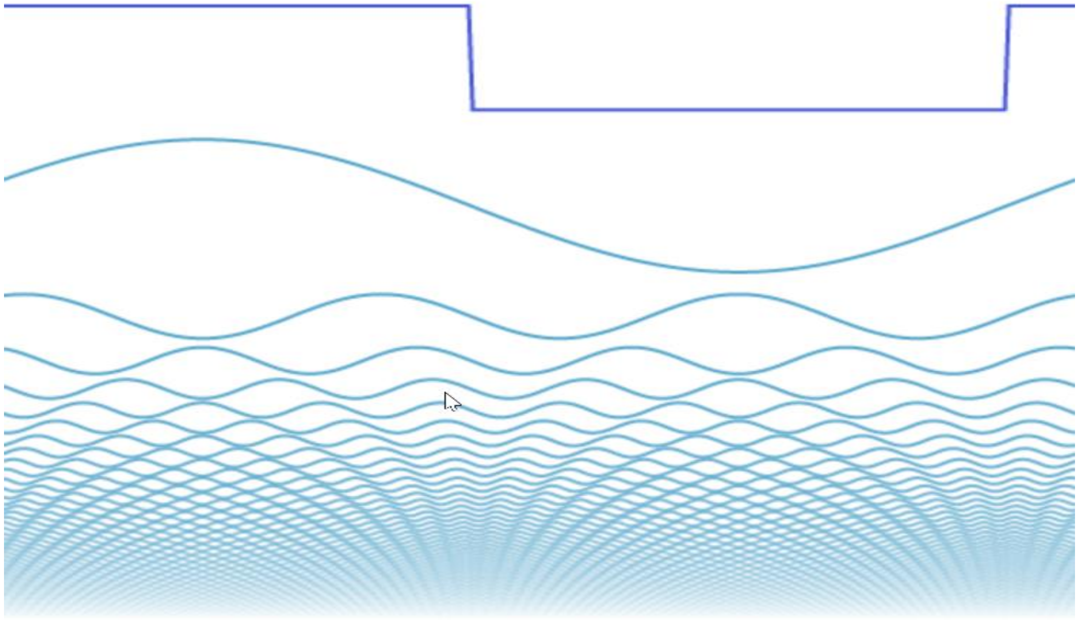
- No, they violate Dirichlet conditions.

Rectangular Signal

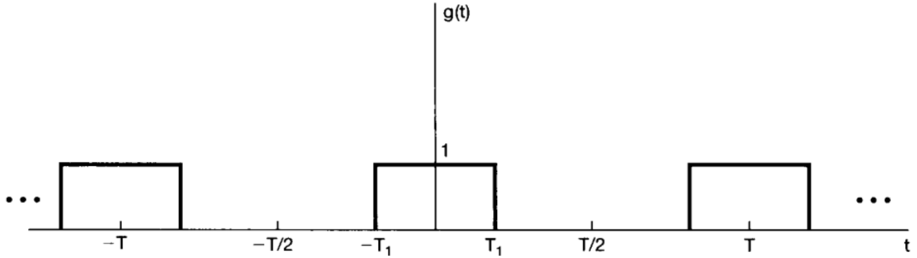


$$\boxed{a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt}$$

Rectangular Signal



Rectangular Signal



$$\boxed{a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt}$$

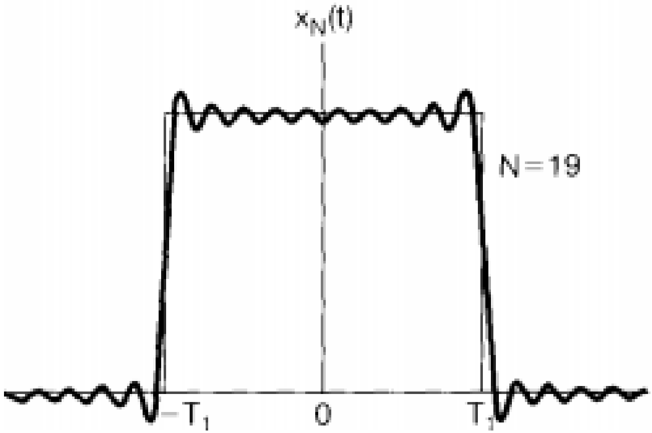
$$c_0 = \frac{2T_1}{T}.$$

$$c_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, \qquad k \neq 0$$

Gibbs Phenomenon

- Gibbs phenomenon at the discontinuity

As the number of terms increased, the ripples in the partial sum became compressed toward the discontinuity, with the peak amplitude of the ripples remaining constant independently of the number of terms in the partial sum.



Fourier Series

1. Introduction to Fourier Series
2. Fourier series and LTI
3. Gibbs phenomenon
4. **Properties of Continuous Time Fourier Series**
5. Filtering

Linearity

$$x(t) \stackrel{\mathfrak{FS}}{\longleftrightarrow} a_k,$$

$$y(t) \stackrel{\mathfrak{FS}}{\longleftrightarrow} b_k.$$

$$z(t) = Ax(t) + By(t) \stackrel{\mathfrak{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k.$$

Time Shifting

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

$$x(t - t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$$

Time Reversal

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

$$x(-t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_{-k}.$$

Time Reversal

- If $x(t)$ is even → $x(-t) = x(t)$
  FSC are also even → $a_{-k} = a_k$

- If $x(t)$ is odd → $x(-t) = - x(t)$
  FSC are also odd → $a_{-k} = - a_k$

Time Scaling

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

$$x(at) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k$$

Multiplication

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

$$y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} b_k.$$

$$x(t)y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty}$$

Conjugation and Conjugate Symmetry

$$x(t) \overset{\mathfrak{FS}}{\longleftrightarrow} a_k,$$

$$x^*(t) \overset{\mathfrak{FS}}{\longleftrightarrow} a^*_{-k}.$$

Differentiation

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

$$\frac{dx(t)}{dt} \stackrel{\mathcal{FS}}{\longleftrightarrow} jk\omega_0 a_k$$

- Here, the magnitude of the kth harmonic is amplified. Hence, high frequency is amplified more than low frequency

Parseval's Relation for CT Periodic Signals

- LHS is the average power (i.e., energy per unit time) in one period of the periodic signal x(t).

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2,$$

- $|a_k|^2$ - average power in the kth harmonic component of x(t).

Fourier Series

1. Introduction to Fourier Series
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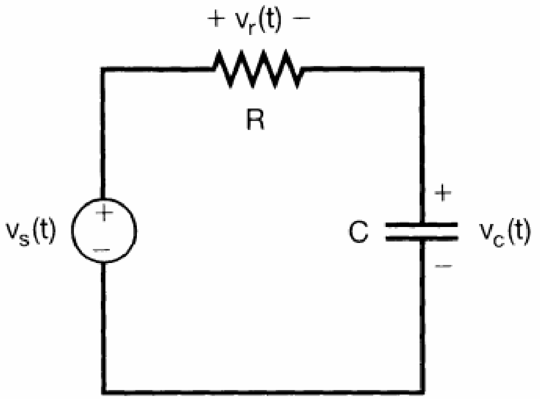
Filtering



$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t).$$

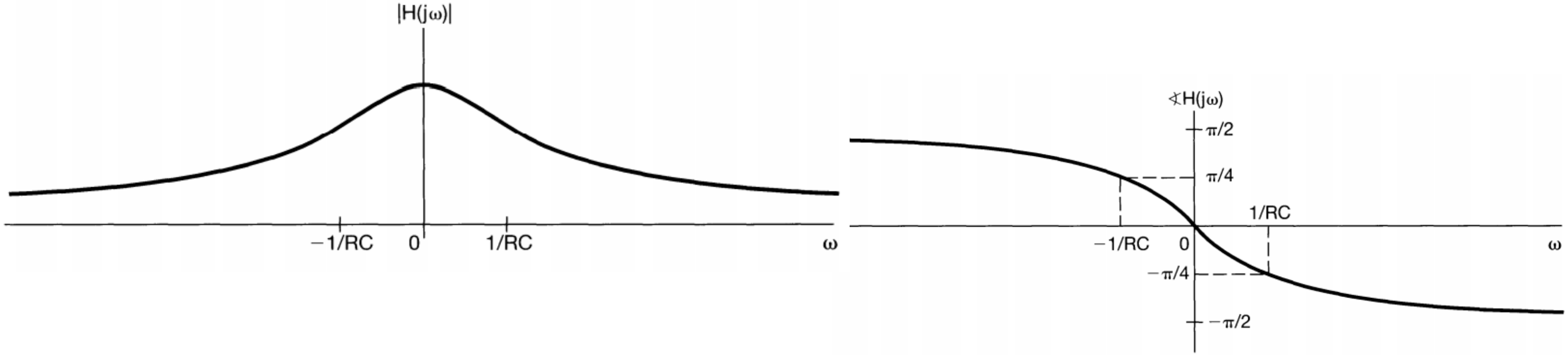
$$\text{input voltage } v_s(t) = e^{j\omega t}.$$

$$\text{output voltage } v_c(t) = H(j\omega)e^{j\omega t}.$$

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t),$$

$$H(j\omega) = \frac{1}{1 + RC\,j\omega}.$$

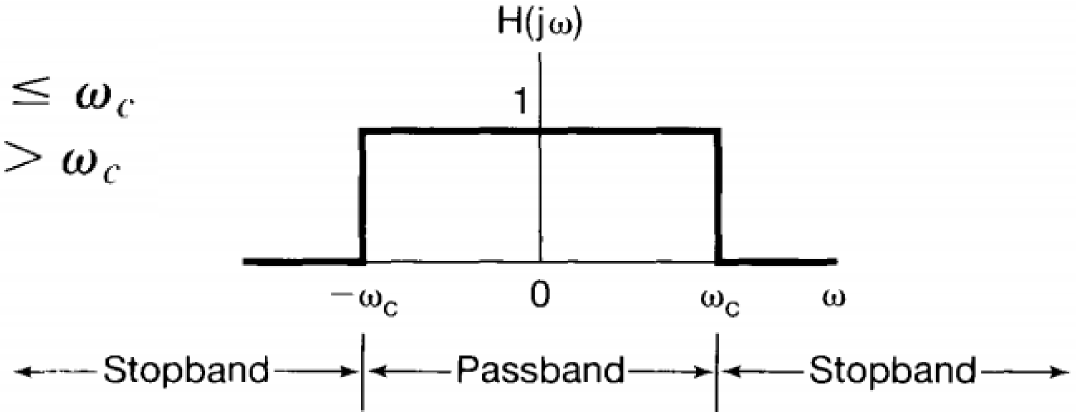
$$H(j\omega) = \frac{1}{1 + RCj\omega}.$$



Lowpass Filter

- Frequency response of Ideal Lowpass Filter

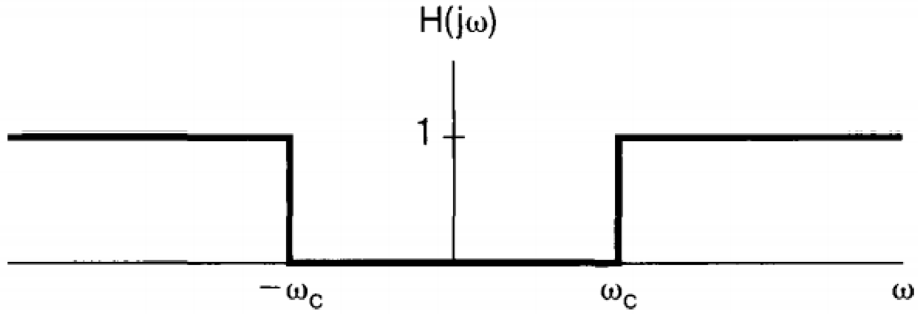
$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



Highpass Filter

- Frequency response of Ideal Lowpass Filter

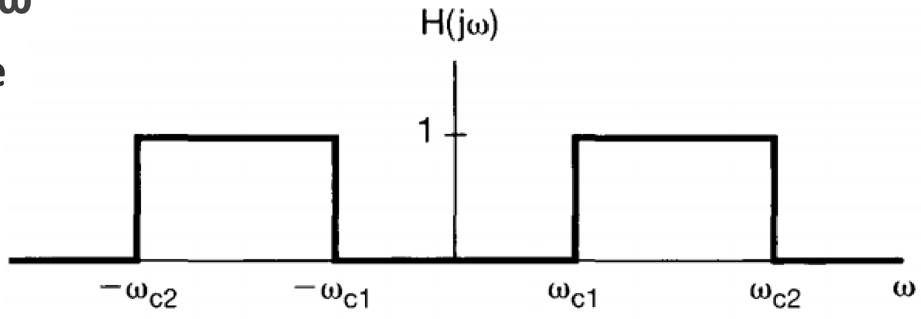
$$H(j\omega) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$



Bandpass Filter

- Frequency response of Ideal Lowpass Filter

$$H(j\omega) = \begin{cases} 1, & |\omega \pm \omega c| \leq \omega' \\ 0, & \text{otherwise} \end{cases}$$



Fourier Transform

The continuous-time fourier transform

Fourier Transform

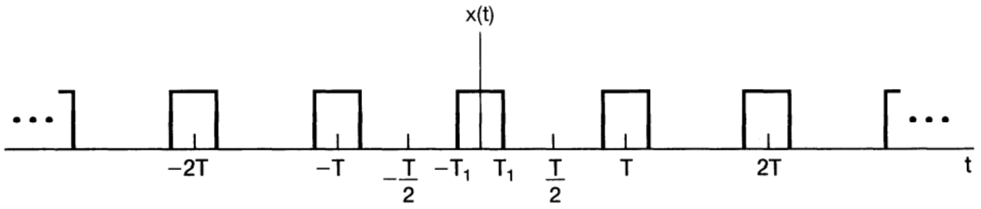
1. Introduction to Fourier Transform

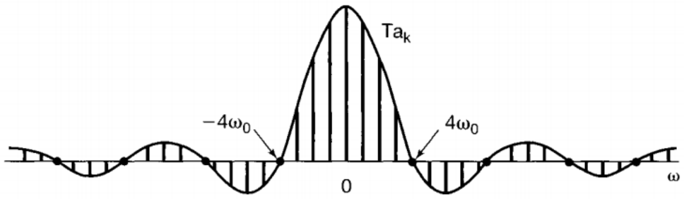
2. Examples of Fourier Transform

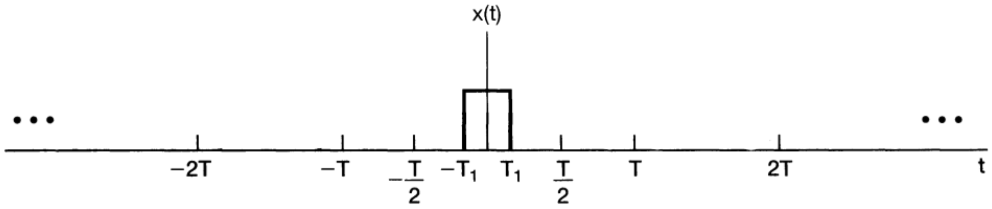
3. Properties of Fourier Transform

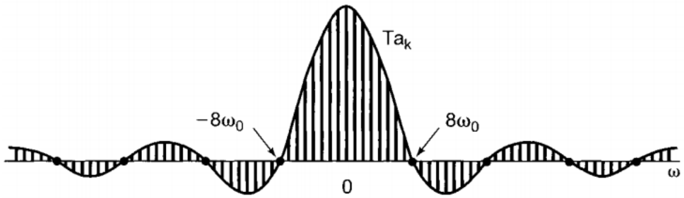
Idea behind Fourier transform

- As the time period is increased, the fundamental frequency decreases









Fourier transform

- As the period becomes infinite, the frequency components form a continuum and the Fourier series sum becomes an integral.
- X(jω) is referred to as the Fourier Transform

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.}$$

- Fourier integral of x(t) inverse Fourier transform equation.

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega}$$

Fourier Transform

1. Introduction to Fourier Transform

2. Examples of Fourier Transform

3. Properties of Fourier Transform

Examples

Fourier transform of unit impulse function

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

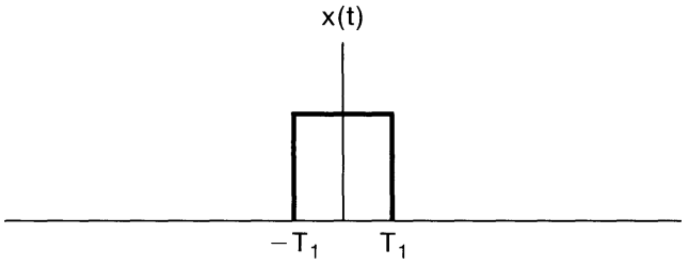
$$\delta(t) \Longleftrightarrow 1$$

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

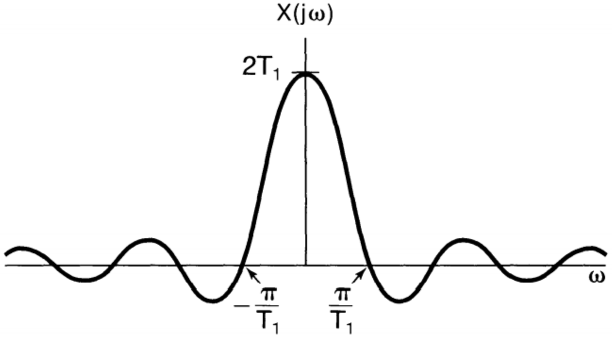
$$\delta(t) \Longleftrightarrow 1$$

Fourier transform of unit impulse function

- $$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases},$$

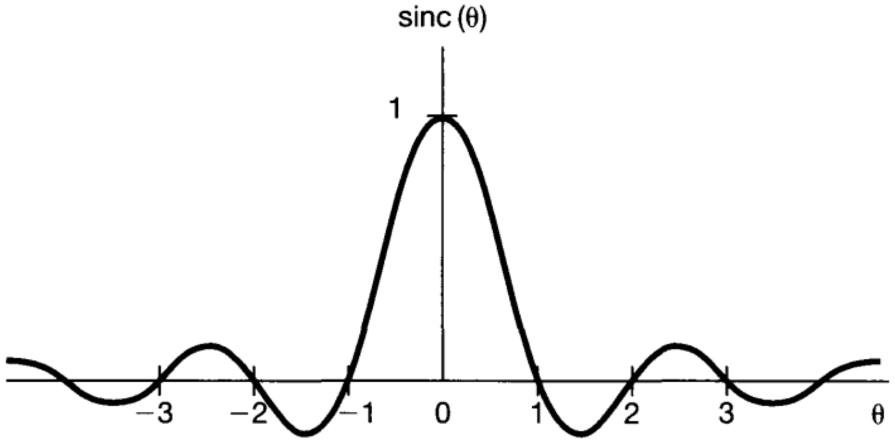


- $$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = 2\frac{\sin \omega T_1}{\omega},$$



Sinc (θ)

- $$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}.$$



Fourier Transform

1. Introduction to Fourier Transform

2. Examples of Fourier Transform

3. Properties of Fourier Transform

Linearity

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

$$y(t) \overset{\mathfrak{F}}{\longleftrightarrow} Y(j\omega),$$

$$\boxed{ax(t) + by(t) \overset{\mathfrak{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega).}$$

Time shifting

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\boxed{x(t - t_0) \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega).}$$

Conjugation and Conjugate symmetric

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\boxed{x^*(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X^*(-j\omega).}$$

$$X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].$$

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\mathcal{E}v\{x(t)\} \stackrel{\mathfrak{F}}{\longleftrightarrow} \mathfrak{Re}\{X(j\omega)\},$$

$$\mathcal{O}d\{x(t)\} \stackrel{\mathfrak{F}}{\longleftrightarrow} j\mathfrak{Im}\{X(j\omega)\}.$$

$$\boxed{\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\},}$$

Properties of FT

Differentiation

$$x(t) \xleftrightarrow{\mathfrak{F}} X(j\omega)$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathfrak{F}} j\omega X(j\omega).}$$

Integration

$$x(t) \xleftrightarrow{\mathfrak{F}} X(j\omega),$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathfrak{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).}$$

Time and Frequency Scaling

$$x(t) \xleftrightarrow{\mathfrak{F}} X(j\omega),$$

$$x(at) \xleftrightarrow{\mathfrak{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right),$$

$$x(-t) \xleftrightarrow{\mathfrak{F}} X(-j\omega).$$

Parseval's Relation

- Parseval's relation says that this total energy may be determined

either by computing the energy per unit time and integrating over all time

or by computing the energy per unit frequency and integrating over all frequencies.

- For this reason, $|X(j\omega)|^2$ is often referred to as the energy-density spectrum of the signal x(t).

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

Parseval's Relation

- $$y(t) = h(t) * x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).$$

- $$Y(j\omega) = H(j\omega)X(j\omega).$$

Laplace Transform

The continuous-time laplace transform

Laplace Transform

1. Introduction to Laplace Transform

2. Region of Convergence

3. Properties of ROC

Why Laplace transform?

- LT is a generalization of the CT FT.
- In FS and FT for CT, σ = 0.

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t}\,dt,$$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

Laplace Transform

- The Laplace transform of a general signal x(t) is defined as

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s)$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t}\,dt,$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt.$$

Laplace Transform and ROC

- We note, in particular, that just as the Fourier transform does not converge for all signals, the Laplace transform may converge for some values of and not for others.

- Example : $x(t) = e^{-at}u(t)$ $\qquad X(s) = \dfrac{1}{s+a}, \quad \Re e\{s\} > -a.$

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \Re e\{s\} > -a.$$

Rational LT

- We consider only rational X(s).
- X(s) is rational whenever x(t) is a linear combination of real or complex exponentials.

$$X(s) = \frac{N(s)}{D(s)},$$

where N(s) and D(s) are the numerator polynomial and denominator polynomial, respectively.

Zeros and Poles of X(s)

$$X(s) = \frac{N(s)}{D(s)},$$

- The roots of the numerator polynomial are commonly referred to as the zeros of X(s), since, for those values of s, X(s) = 0
- The roots of the denominator polynomial are referred to as the poles of X(s), and for those values of s, X(s) is infinite.

Laplace Transform

1. Introduction to Laplace Transform

2. Region of Convergence

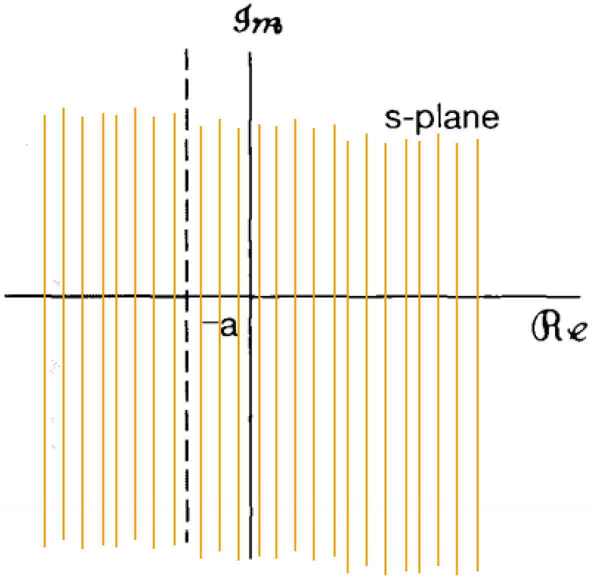
3. Properties of ROC

Region of Convergence

- Observe the two examples

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \Re e\{s\} > -a.$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \Re e\{s\} < -a.$$



- The range of values of s for which X(s) converges is referred to as the **region of convergence**.

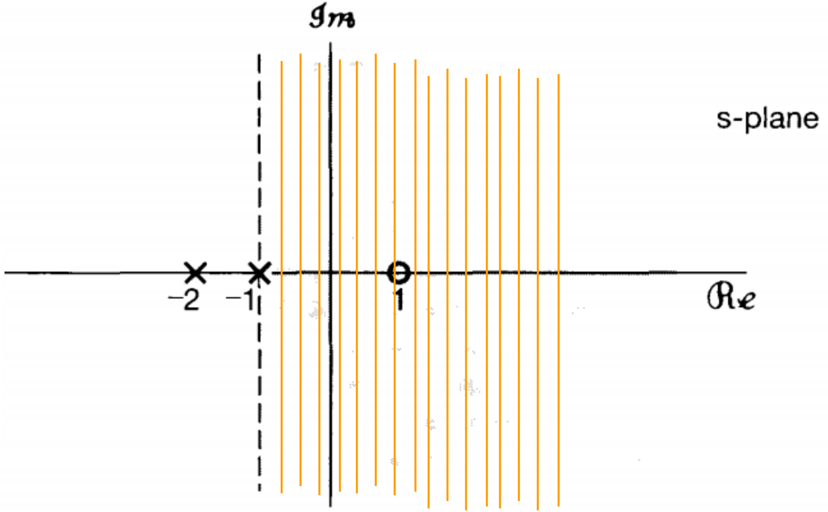
ROC

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t).$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}. \quad \mathcal{Re}\{s\} > -2. \quad \mathcal{Re}\{s\} > -1,$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{s^2+3s+2},$$

$$\mathcal{Re}\{s\} > -1.$$

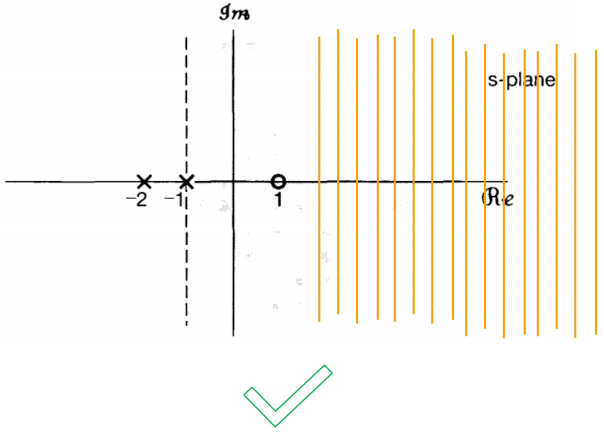


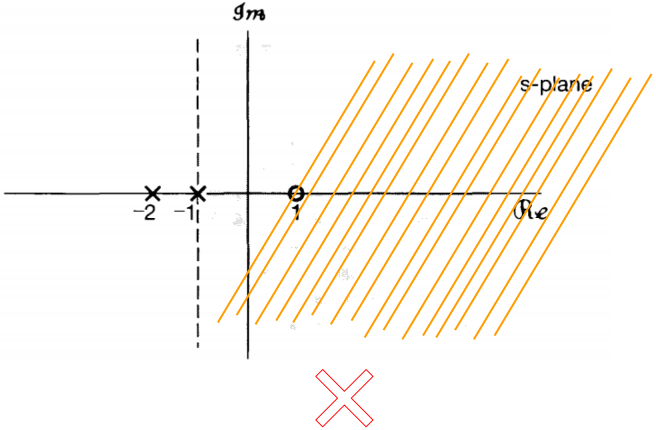
Laplace Transform

1. Introduction to Laplace Transform
2. Region of Convergence
3. **Properties of ROC**

Property 1

- The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.





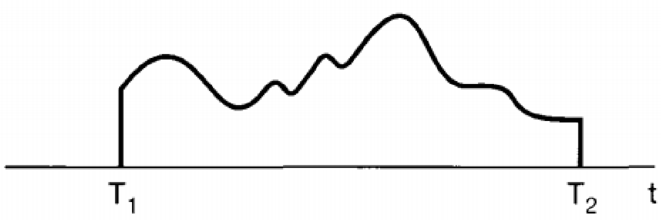
Property 2

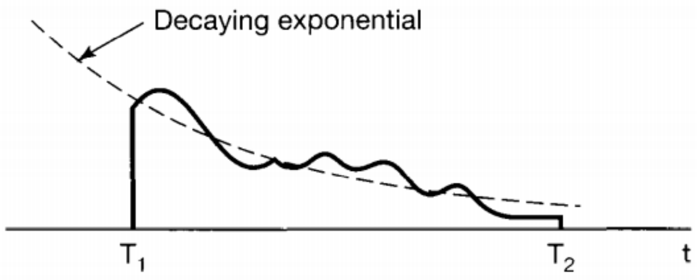
- For rational Laplace transforms, the ROC does not contain any poles.

Property 3

- If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

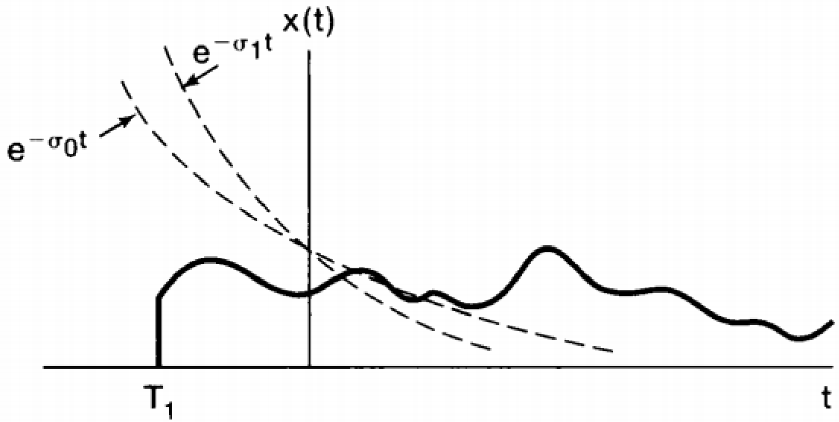
$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$





Property 4

- If $x(t)$ is right sided, and if the line $Re(s) = \sigma_o$ is in the ROC, then all values of s for which $Re(s) > \sigma_o$ will also be in the ROC.

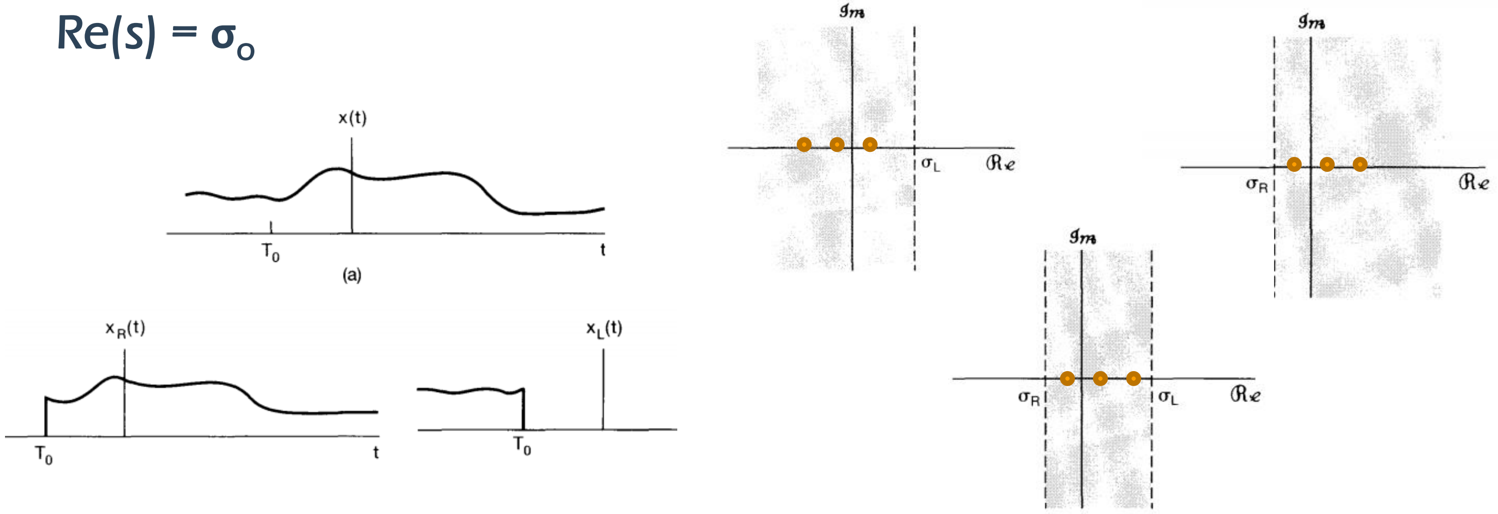


Property 5

- If $x(t)$ is left sided, and if the line $Re(s) = \sigma_o$ is in the ROC, then all values of s for which $Re(s) < \sigma_o$ will also be in the ROC.

Property 6

- If $x(t)$ is two sided, and if the line $Re(s) = \sigma_o$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $Re(s) = \sigma_o$



Property 7 & 8

- If the Laplace transform X(s) of x(t) is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of X(s) are contained in the ROC.

- If the Laplace transform X(s) of x(t) is rational, then if x(t) is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If x(t) is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

ROC properties

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{Re}\{s\} > 0\}$

‘A sound mind in a sound body’

Hope you benefited

Thank You