Q1) 2 marks

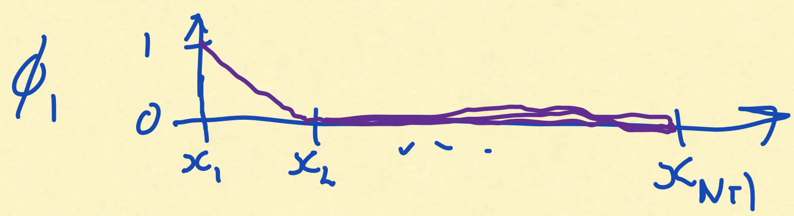
let grid be given by

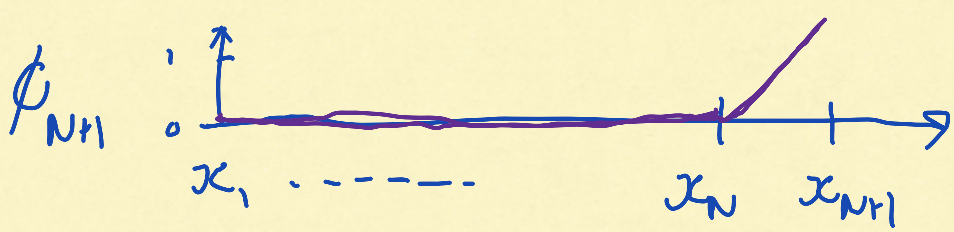
$x_i \in \mathbb{R}$ where $i = 1, \dots, N+1$

and $x_{i+1} > x_i$. Define $h_i = x_{i+1} - x_i$

and $E_i = [x_i, x_{i+1}]$.

Let's consider the basis





$$K_{ij} = \int_{x_1}^{x_{N+1}} D(x)\,\phi_i\,\phi_j'\,dx = \sum_{k=1}^{N} \int_{E_k} D(x)\phi_i\phi_j'\,dx$$

Now if $i=j$, $\phi_i'\phi_i'$ is non-zero only on E_{i-1} and E_i so

$$K_{ii} = \int_{E_{i-1}} (\phi_i')^2\,dx + \int_{E_i} (\phi_i')^2\,dx$$

Now $\displaystyle\int_{E_{i-1}} (\phi_i')^2\,dx = h_{i-1}\int_0^1 \left(\frac{1}{h_{i-1}}\frac{d\xi}{d\xi}\right)^2 d\xi, \quad \xi = \frac{x - x_{i-1}}{x_i - x_{i-1}}$

$$= \frac{1}{h_{i-1}}\int_0^1 d\xi = \frac{1}{h_{i-1}}$$

also $\displaystyle\int_{E_i} (\phi_i')^2\,dx = h_i\int_0^1 \left(\frac{1}{h_i}\frac{d}{d\xi}(1-\xi)\right)^2 d\xi, \quad \xi = \frac{x-x_i}{x_{i+1}-x_i}$

$$= \frac{1}{h_i}$$

$$\Rightarrow K_{ii} = \frac{1}{h_i} + \frac{1}{h_{i-1}}$$

Except for $K_{11} = \dfrac{1}{h_1}$ $\quad K_{N+1,N+1} = \dfrac{1}{h_N}$

Note: K_{11} & $K_{N+1,N+1}$ only relevant for Neumann BC's. For Dirichlet just need $i = 2, \dots, N$ components.

Now when $j = i-1$

$$K_{i,i-1} = \int_{x_1}^{x_{N+1}} \phi_{i-1}' \phi_i' \, dx = K_{i-1,i}$$

Now these basis functions only overlap on E_{i-1} so

$$K_{i,i-1} = \int_{E_{i-1}} \phi_{i-1}' \phi_i' \, dx = h_{i-1} \int_0^1 \frac{1}{h_{i-1}^2} \frac{d\xi}{d\xi} \frac{d}{d\xi}(1-\xi) \, d\xi$$

where $\xi = \dfrac{x - x_{i-1}}{x_i - x_{i-1}}$

$$\Rightarrow \quad K_{i,i-1} = \frac{-1}{h_{i-1}}$$

If $|i-j| > 1$ then ϕ_i & ϕ_j do not overlap and so $K_{ij} = 0$.

Q2 1.5 marks

Q3 1.5 marks