# Q1

a) To find the RHS we evaluate the Laplacian of the exact solution,

$$-\nabla^2\left(e^{x+y}\right) = -2e^{x+y}$$

$$\Rightarrow \quad f(x,y) = -2e^{x+y}$$

The boundary conditions are

$u(0,y) = e^{y}$ $\qquad$ $u(1,y) = e^{1+y}$

$u(x,0) = e^{x}$ $\qquad$ $u(x,2) = e^{x+2}$

See code for implementation

b) In this case we have

$$-\nabla^2\left(e^{x^2+y^2}\right) + (x^2+y^2)e^{x^2+y^2}$$

$$= -(4+3x^2+3y^2)e^{x^2+y^2}$$

With boundary conditions

$$u(0,y) = e^{y^2} \qquad u(1,y) = e^{1+y^2}$$

$$u(x,0) = e^{x^2} \qquad u(x,2) = e^{4+y^2}$$

To implement numerically we write out the difference equation

$$-\frac{u_{i,j-1}}{\Delta y^2} - \frac{u_{i-1,j}}{\Delta x^2} + \left[2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)+x_i^2+y_j^2\right]u_{i,j}$$

$$-\frac{u_{i,j+1}}{\Delta y^2} - \frac{u_{i+1,j}}{\Delta x^2} = f_{i,j}$$

and notice we can use code from (a) just by modifying the diagonals of A.

# Q2

## a  2 marks

Correct output when running
'Assignment_2_Q2a_assess.m'

## b  2 marks

Correct output when running
'Assignment_2_Q2b_assess.m'

## c  2 marks

$$-\nabla^2\left(\sin(x)\cos(4y)\right) = 17\sin(x)\cos(4y)$$

So that $D=1$, $q=0$ & $f=17\sin(x)\cos(4y)$

The conclusion you want to come to here is that changing the order of the quadrature <u>does not</u> change the order of the error.
This is discussed on slides 30-32 of the week 3 power point.

## d  1 mark

# Q3

The governing equation is

$$-\nabla^2 u + u^2 = f(x)$$

Now quasilinearisation gives

$$u_{k+1}^2 \approx u_k^2 + 2u_k(u_{k+1} - u_k)$$

Which upon substitution into the PDE:

$$\Rightarrow \quad -\nabla^2 u_{k+1} + 2u_k u_{k+1} = f(x) - u_k^2 + 2u_k^2$$
$$= f(x) + u_k^2$$

$$u_{n+1}\big|_{\delta\Omega} = g_0(\delta\Omega)$$

Now we may solve using the code from Q2 with

$$D = 1, \quad q = 2u_k \quad \text{and} \quad \hat{f} = f + u_k^2$$

FEM code version of $f$.

And iterate until $\|u_{k+1} - u_k\| < \text{tolerance}$

b)

See code implementation

Note: $-\nabla^2(e^{x+2y}) + (e^{x+2y})^2 = e^{x+2y}(e^{x+2y} - 5)$

# Q4

First we rearrange

$$n \cdot \nabla u = \frac{1}{b}(g - au)$$

Then using the vector identity

$$\int_\Omega \underline{p} \cdot \underline{\nabla} v + v \underline{\nabla} \cdot \underline{p} \, dA = \int_{\delta\Omega} v \, \underline{p} \cdot \underline{n} \, ds$$

let $\underline{p} = D(x,y) \nabla u(x,y)$

$$\Rightarrow \int_\Omega D \nabla u \cdot \nabla v + v \nabla \cdot (D \nabla u) \, dA = \int_{\delta\Omega} v D \nabla u \cdot n \, ds$$

$$\Rightarrow \int_\Omega - \nabla \cdot (D \nabla u) v \, dA = \int D \nabla u \cdot \nabla v \, dA - \int_{\delta\Omega} v D \nabla u \cdot n \, ds$$

$$\Rightarrow \int_\Omega D \nabla u \cdot \nabla v + q u v \, dA = \int_\Omega f v \, dA + \int_{\delta\Omega} D (n \cdot \nabla u) v \, ds$$

$$\Rightarrow \int_\Omega D \nabla u \cdot \nabla v + q u v \, dA = \int_\Omega f v \, dA + \int_{\delta\Omega} \frac{D}{b}(g - au) v \, ds$$

The final solution is therefor

Now $v \in \{\phi_j\}$ for $j = 1, \ldots, n$

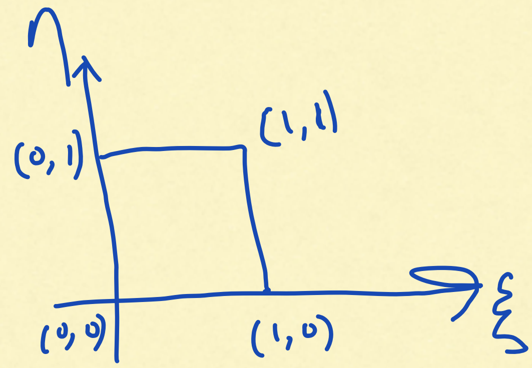
and $u = \sum_{i=1}^{n} c_i \phi_i$

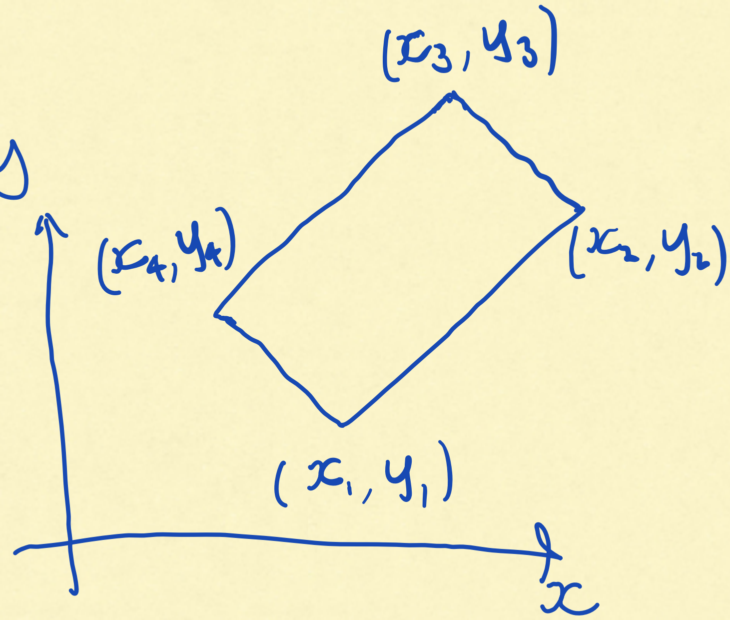
$$\Rightarrow \int_{\Omega} D \sum_{i=1}^{n} c_i \nabla\phi_i \cdot \nabla\phi_j + q \sum_{i=1}^{n} c_i \phi_i \phi_j \, dA$$

$$+ \int_{\partial\Omega} \frac{Da}{b} \sum_{i=1}^{n} c_i \phi_i \phi_j \, ds = \int_{\Omega} f \phi_j \, dA + \int_{\partial\Omega} \frac{Dg}{b} \phi_j \, ds$$

for $j = 1, \ldots, n$

# Q5




$F_R(0,0) = (x_i, y_i)$ $\quad$ $F_R(1,1) = (x_k, y_k)$

$F_R(1,0) = (x_j, y_j)$ $\quad$ $F_R(0,1) = (x_l, y_l)$

Now

$$y_4 = y_1 - \frac{x_2 - x_1}{y_2 - y_1}(x_4 - x_1)$$

Giving

$$J_R = \begin{bmatrix} x_2 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_4 - y_1 \end{bmatrix} \qquad b_R = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

(there are several other equivalent ways to write this)

$$F_R = \int_R f \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} dA = \int_{Sq} f \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} |\det(J_R)| \, dA$$

where $N_1 = (1-\xi)(1-\eta)$, $N_2 = (1-\xi)\eta$

$N_3 = \xi(1-\eta)$, $N_4 = \xi\eta$

$$K_R = \int_R D \begin{bmatrix} \nabla\phi_1 \\ \nabla\phi_2 \\ \nabla\phi_3 \\ \nabla\phi_4 \end{bmatrix} \otimes \begin{bmatrix} \nabla\phi_1 \\ \nabla\phi_2 \\ \nabla\phi_3 \\ \nabla\phi_4 \end{bmatrix} dA$$

Now $\nabla\phi_i = J_R^{-T} \nabla_S N_i$

where

$$\nabla_\xi N_1 = \begin{bmatrix} \eta - 1 \\ \xi - 1 \end{bmatrix} \qquad \nabla_\xi N_2 = \begin{bmatrix} -\eta \\ 1 - \xi \end{bmatrix}$$

$$\nabla_\xi N_3 = \begin{bmatrix} -\xi \\ 1 - \eta \end{bmatrix} \qquad \nabla_\xi N_4 = \begin{bmatrix} \eta \\ \xi \end{bmatrix}$$

$$\Rightarrow K_R = \int_{S_q} D \left( J_R^{-T} \begin{bmatrix} \nabla_\xi N_1 \\ \nabla_\xi N_2 \\ \nabla_\xi N_3 \\ \nabla_\xi N_4 \end{bmatrix} \right) \otimes \left( J_R^{-T} \begin{bmatrix} \nabla_\xi N_1 \\ \nabla_\xi N_2 \\ \nabla_\xi N_3 \\ \nabla_\xi N_4 \end{bmatrix} \right) dA$$

c)

For this element $J_R = \begin{bmatrix} 1 & -1 \\ 2 & 1/2 \end{bmatrix}$, $b_R = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \det(J_R) = 5/2$ and $J_R^{-T} = \frac{2}{5} \begin{bmatrix} 1/2 & -2 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow \quad F_R = \int_0^1\int_0^1 \frac{S}{2} \times 2 \begin{bmatrix} (1-\xi)(1-\eta) \\ (1-\xi)\eta \\ \xi(1-\eta) \\ \xi\eta \end{bmatrix} d\xi\, d\eta = \frac{S}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$K_R = \frac{1}{12} \begin{bmatrix} 10 & 2 & -5 & -7 \\ 2 & 10 & -7 & -5 \\ -5 & -7 & 10 & 2 \\ -7 & -5 & 2 & 10 \end{bmatrix}$$