# MAST90026 Computational Differential Equations: Week 5

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Modified from Hailong Guo (2022)

Semester 1 2024

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# Classification of 2nd order partial differential equations

A linear second order partial differential equation can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} = F\left(x, y, u, u_x, u_y\right)$$

where $A, B$ and $C$ may be functions of $x$ and $y$. Based on the local value of the coefficients the equations are classified as follows:

$$\begin{aligned} B^2 - 4AC &> 0 \quad \text{Hyperbolic} \\ B^2 - 4AC &= 0 \quad \text{Parabolic} \\ B^2 - 4AC &< 0 \quad \text{Elliptic} \end{aligned}$$

Example

$$\begin{aligned} -\nabla^2 u &= f, && \text{The Poisson equation, elliptic,} \\ \nabla^2 u &= u_t, && \text{The diffusion equation, parabolic,} \\ \nabla^2 u &= u_{tt}, && \text{The wave equation, hyperbolic.} \end{aligned}$$

# Elliptic equation

Consider (a subclass of) the elliptic PDEs:

$$-\nabla \cdot (D(x,y)\nabla u) = f(x,y),$$

where $D > 0$ to satisfy ellipticity.

Possible boundary conditions:

- Dirichelt BC: $u = g_D(x,y)$ on $\partial\Omega$.
- Neumann BC: $q \cdot \vec{n} = g_N(x,y)$ on $\partial\Omega$
- Robin BC: $\alpha u + \beta q \cdot \vec{n} = g_R$ on $\partial\Omega$

where $q = -D(x,y)\nabla u$.

# Discretisation of domain

Questions:

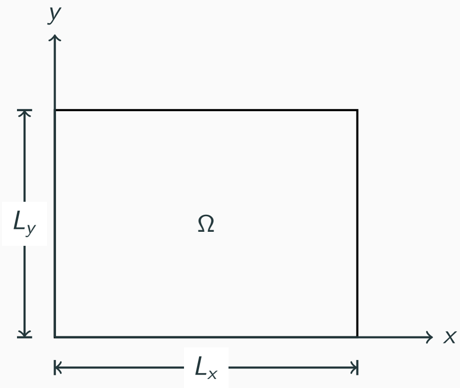
- How to handle the curved boundaries?
- How to define the mesh in 2D?

To avoid the problem, considering a rectangle aligned with the axes. In addition, we consider Poisson equation by assuming $D = 1$.

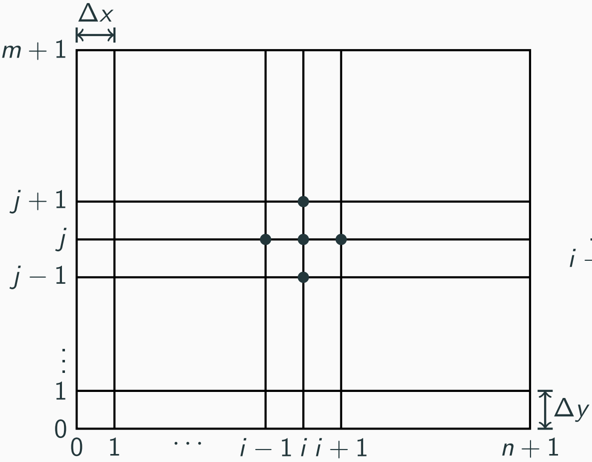
# Poisson equation in 2D

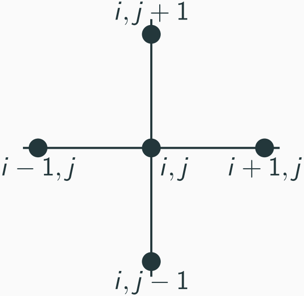
$$
\begin{aligned}
-\nabla^2 u(x,y) &= f(x,y), \qquad && \text{in } \Omega, \\
u &= g_D, && \text{on } \partial\Omega.
\end{aligned}
$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$



# Discretization of domain




$\Delta x = \frac{L_x}{n+1}, \quad \Delta y = \frac{L_y}{m+1}, \quad x_i = i\Delta x, \quad y_j = j\Delta y$

# Approximation of differential operator

Use central finite difference on each direction:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

$$\left.\frac{\partial^2 v}{\partial y^2}\right|_{i,j} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

Five point stencil:

$$-\nabla^2 u_{ij} \approx -\frac{1}{\Delta x^2}\left(u_{i-1,j} - 2u_{ij} + u_{i+1,j}\right) - \frac{1}{\Delta y^2}\left(u_{i,j-1} - 2u_{ij} + u_{i,j+1}\right)$$

In particular, $\Delta x = \Delta y = h$:

$$-\nabla^2 u_{ij} \approx -\frac{1}{h^2}\left(u_{i-1,j} + u_{i,j-1} - 4u_{ij} + u_{i+1,j} + u_{i,j+1}\right)$$

# Equations

$-u_{xx} - u_{yy} = f$ suggests:

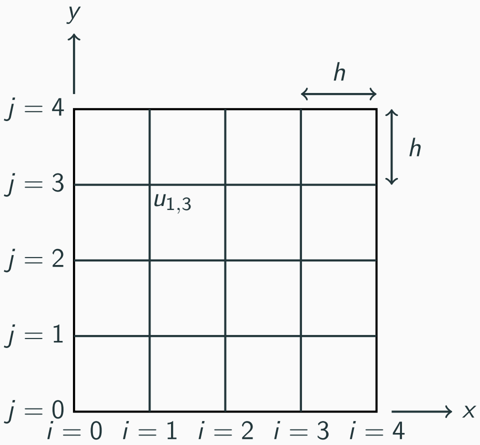
$$-\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{i,j}$$

$$u_{0,j} = g_D(x_0, y_j), u_{n+1,j} = g_D(x_{n+1}, y_j) \quad 1 \leq j \leq m$$
$$u_{i,0} = g_D(x_i, y_0), u_{i,m+1} = g_D(x_i, y_{m+1}) \quad 1 \leq i \leq n$$
$$\Rightarrow \quad A\vec{u} = \vec{f}$$

# Example of Poisson equation in 2D

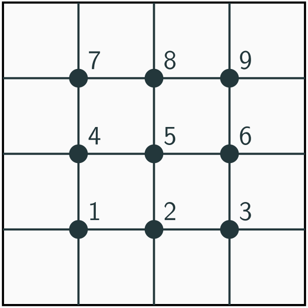
$m = n = 3$

$\Delta x = \Delta y = h$

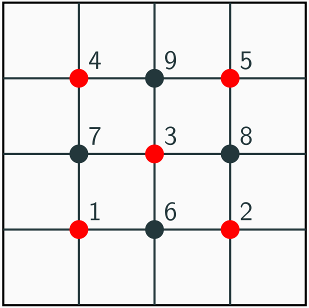


# Numbering scheme

Natural rowwise ordering




Red-black ordering



# Equation at internal points – natural rowwise ordering

Eqn.1: $\quad -\frac{1}{h^2}\left(-4u_1 + u_2 + u_4\right) = f_{1,1} + \frac{u_{0,1}+u_{1,0}}{h^2}$

Eqn.2: $\quad -\frac{1}{h^2}\left(u_1 - 4u_2 + u_3 + u_5\right) = f_{2,1} + \frac{u_{2,0}}{h^2}$

Eqn.3: $\quad -\frac{1}{h^2}\left(u_2 - 4u_3 + u_6\right) = f_{3,1} + \frac{u_{3,0}+u_{4,1}}{h^2}$

Eqn.4: $\quad -\frac{1}{h^2}\left(u_1 - 4u_4 + u_5 + u_7\right) = f_{1,2} + \frac{u_{0,2}}{h^2}$

Eqn.5: $\quad -\frac{1}{h^2}\left(u_2 + u_4 - 4u_5 + u_6 + u_8\right) = f_{2,2}$

Eqn.6: $\quad -\frac{1}{h^2}\left(u_3 + u_5 - 4u_6 + u_9\right) = f_{3,2} + \frac{u_{4,2}}{h^2}$

Eqn.7: $\quad -\frac{1}{h^2}\left(u_4 - 4u_7 + u_8\right) = f_{13} + \frac{u_{0,3}+u_{1,4}}{h^2}$

Eqn.8: $\quad -\frac{1}{h^2}\left(u_5 + u_7 - 4u_8 + u_9\right) = f_{2,3} + \frac{u_{2,4}}{h^2}$

Eqn.9: $\quad -\frac{1}{h^2}\left(u_6 + u_8 - 4u_9\right) = f_{33} + \frac{u_{3,4}+u_{4,3}}{h^2}.$

# Example: matrix and vector – natural rowwise ordering

$$
A = \frac{1}{h^2}\left[\begin{array}{ccc|ccc|ccc}
4 & -1 & & -1 & & & & & \\
-1 & 4 & -1 & & -1 & & & & \\
 & -1 & 4 & & & -1 & & & \\
\hline
-1 & & & 4 & -1 & & -1 & & \\
 & -1 & & -1 & 4 & -1 & & -1 & \\
 & & -1 & & -1 & 4 & & & -1 \\
\hline
 & & & -1 & & & 4 & -1 & \\
 & & & & -1 & & -1 & 4 & -1 \\
 & & & & & -1 & & -1 & 4
\end{array}\right]
$$

Assume $g_D = 0$, we have

$$
f = (f_{1,1}, f_{2,1}, f_{3,1}, f_{1,2}, f_{2,2}, f_{3,2}, f_{1,3}, f_{2,3}, f_{3,3})^T
$$

# General finite difference method

$$-\frac{1}{\Delta y^2}u_{i,j-1}-\frac{1}{\Delta x^2}u_{i-1,j}+2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)u_{i,j}-\frac{1}{\Delta x^2}u_{i+1,j}-\frac{1}{\Delta y^2}u_{i,j+1}=f_{i,j}.$$

Let $\vec{u}_j = \left(u_{1,j}, u_{2,j}, \cdots, u_{n,j}\right)^T$ for $1 \le j \le m$.

Rewrite as

$$D\vec{u}_{j-1} + C\vec{u}_j + D\vec{u}_{j+1} = \vec{f}_j;$$

where

$$C = \begin{bmatrix} 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) & -\frac{1}{\Delta x^2} & & & \\ -\frac{1}{\Delta x^2} & 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) & -\frac{1}{\Delta x^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{\Delta x^2} & 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) & -\frac{1}{\Delta x^2} \\ & & & -\frac{1}{\Delta x^2} & 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) \end{bmatrix}$$

# General finite difference method (Continue)

$$D = \begin{bmatrix} -\frac{1}{\Delta y^2} & & & & \\ & -\frac{1}{\Delta y^2} & & & \\ & & \ddots & & \\ & & & -\frac{1}{\Delta y^2} & \\ & & & & -\frac{1}{\Delta y^2} \end{bmatrix}, \vec{f_j} = \begin{pmatrix} f_{1,j} + \frac{1}{\Delta x^2} g_D(x_0, y_j) \\ f_{2,j} \\ \vdots \\ f_{n-1,j} \\ f_{n,j} + \frac{1}{\Delta x^2} g_D(x_{n+1}, y_j). \end{pmatrix}$$

$C, D : (n \times n)$

# Linear system

$$A\vec{u} = \begin{bmatrix} C & D & & & \\ D & C & D & & \\ & \ddots & \ddots & \ddots & \\ & & D & C & D \\ & & & D & C \end{bmatrix} \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_{m-1} \\ \vec{u}_m \end{pmatrix} = \begin{pmatrix} \vec{f}_1 - D\vec{u}_0 \\ \vec{f}_2 \\ \vdots \\ \vec{f}_{m-1} \\ \vec{f}_m - D\vec{u}_{m+1} \end{pmatrix}.$$

A: Block $(m \times m)$ tridiagonal matrix.

# Implementation of 2D FDM

Kronecker product: If $S$ is a $m \times n$ matrix and $T$ is a $p \times q$ matrix, then the Kronecker product of $S$ and $T$ is $mp \times nq$ given by

$$S \otimes T = \begin{pmatrix} s_{11}T & \cdots & s_{1n}T \\ \vdots & & \\ s_{m1}T & \cdots & s_{mn}T \end{pmatrix}.$$

# Implementation of 2D FDM (Continue)

Using Kronecker product, we can see that

$$A = \begin{bmatrix} C & & & & \\ & C & & & \\ & \ddots & \ddots & \ddots & \\ & & & C & \\ & & & & C \end{bmatrix} + \begin{bmatrix} O & D & & & \\ D & O & D & & \\ & \ddots & \ddots & \ddots & \\ & & D & O & D \\ & & & D & O \end{bmatrix} = I_m \otimes C + E \otimes D;$$

where $I_m$ is the $m \times m$ identities matrix and

$$E = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

In Matlab, you can use the function kron(S, T) to compute $S \otimes T$.

# Truncation error

$$\underbrace{\begin{aligned} &-\frac{u\left(x_{i+1}, y_j\right) - 2u\left(x_i, y_j\right) + u\left(x_{i-1}, y_j\right)}{\Delta x^2} \\ &-\frac{u\left(x_i, y_{j+1}\right) - 2u\left(x_i, y_j\right) + u\left(x_i, y_{j-1}\right)}{\Delta y^2} = f\left(x_i, y_j\right) \\ &-\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}\left(x_i + \theta_i^x \Delta x, y_j\right) - \frac{\Delta y^2}{12}\frac{\partial^4 u}{\partial y^4}\left(x_i, y_j + \theta_j^y \Delta y\right) \end{aligned}}_{\tau_{i,j}}$$

For $u \in \mathcal{C}^4$, $\tau_{i,j} \sim \mathcal{O}\left(\Delta x^2, \Delta y^2\right) \quad$ for all $i, j$

# Convergence

Original scheme

$$A\vec{u} = \vec{f}.$$

Consistency

$$A\hat{u} = \vec{f} + \vec{\tau}.$$

Error equation

$$\mathbf{E} = -A^{-1}\vec{\tau}.$$

Taking norms

$$\begin{aligned} \|\mathbf{E}\| &= \| - A^{-1}\vec{\tau}\| \\ &\leq \|A^{-1}\| \|\vec{\tau}\| \\ &\leq \|A^{-1}\| Ch^2. \end{aligned}$$

# Recap of FD Methods

Easy for simple geometry e.g. rectangles.

Convergence theory can be tricky.

Produces large sparse matrices, even in 2D.

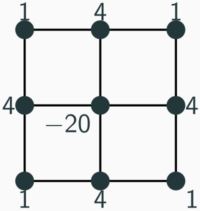
# Other topics

Curved Regions also make finite difference methods messy, but possible. See for example 'Immersed Interface Method' Ref: Li & Leveque 1994.

Nonlinear elliptic PDEs can be tackled by quasi-linearisation or by Newton's method and its variants

For easy problems ($-\nabla^2 u = f$) on a square there are special 'fast Poisson solvers' that don't generalize to the variable coefficient case, or other coordinate systems.

# 9-point stenicl




Finite difference scheme:

$$\nabla_9^2 u_{ij} = \frac{1}{6h^2}[4u_{i-1,j} + 4u_{i+1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{i,j}].$$

# Truncation error of 9-point stencil

Compute:

$$\nabla_9^2 u(x_i, y_j) = \nabla^2 u + \underbrace{\frac{1}{12} h^2 (u_{xxxx} + 2u_{xxyy} + u_{yyyy}) + O(h^4)}_{\tau_{ij}},$$

Reformulate:

$$\begin{aligned} \tau_{ij} =& \frac{1}{12} h^2 \nabla^2 (\nabla^2 u) + O(h^4) \\ =& -\frac{1}{12} h^2 \nabla^2 f + O(h^4). \end{aligned}$$

Observation: if we know $f$ analytically, we can evaluate the first term of $\tau_{ij}$.

Example: $f = 0$ then error is $O(h^4)$, $\nabla_9^2 u = 0 \to A_9 \vec{u} = \vec{b}$, $\tau = O(h^4)$.

# 4th Order method for Poisson equation

For Poisson equation:

$$-\nabla_9^2 u = f + \frac{1}{12} h^2 \nabla^2 f + O(h^4),$$

Consider different problem:

$$-\nabla_9^2 u = f + \frac{1}{12} h^2 \nabla^2 f = \tilde{f},$$

then

$$-\nabla^2 u + \frac{1}{12} h^2 \nabla^2 \tilde{f} + O(h^4) = \tilde{f},$$

therefore

$$-\nabla^2 u + \frac{1}{12} h^2 \nabla^2 f + O(h^4) = f + \frac{1}{12} h^2 \nabla^2 f,$$

so

$$-\nabla^2 u + O(h^4) = f,$$

i.e. we've solved $-\nabla^2 u = f$ to $O(h^4)$.

# Unknown right hand side

If we don't know $f$ analytically, but just at grid points. Just form

$$
\begin{aligned}
\tilde{f}_{i,j} =& f_{i,j} + \frac{1}{12} h^2 \nabla_5^2 f_{i,j} \\
=& f_{i,j} + \frac{1}{12} h^2 (\nabla^2 f + O(h^2)) \\
=& f_{i,j} + \frac{1}{12} h^2 \nabla^2 f_{i,j} + O(h^4),
\end{aligned}
$$

# Method of deferred corrections

Solve $A_5\vec{u} = \vec{f}$ which gives $\vec{u}$ with an error $O(h^2)$. We know

$$\tau_{ij} = \frac{1}{12}h^2(u_{xxxx} + u_{yyyy}) + O(h^4).$$

Then, since $A_5\hat{u} - \vec{f} = \tau$ and $A_5\vec{u} - \vec{f} = \vec{0}$,

$$A_5(\vec{u} - \hat{u}) = -\vec{\tau} \Rightarrow A_5 E = -\tau.$$

We now attempt to use $\vec{u}$ to estimate $\tau$. For example we can use central differences to get $u_{xxxx}$ and $u_{yyyy}$ to $O(h^2)$. Then we obtain

$$\hat{\tau} = \vec{\tau} + O(h^4),$$

and solve

$$A_5 E = -\hat{\tau},$$

then update

$$\tilde{\vec{u}} = \hat{\vec{u}} - E,$$

which has improved our estimate to $O(h^4)$.

**End of week 5!**