Interpolation

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1 Definition of interpolation

Let us focus on the case of approximating a given function by a polynomial of degree at most n. Then the *interpolation problem* can be stated as follows: Given $n + 1$ distinct points, x_0, x_1, ..., x_n called nodes and corresponding values f_0, f_1, $\ldots$, f_n, find a polynomial of degree at most n, $P_n(x)$, which satisfies (the interpolation property)

$$\begin{aligned} &P_n(x_0) = f_0 \\ &P_n(x_1) = f_1 \\ &\vdots \\ &P_n(x_n) = f_n. \end{aligned} \tag{1}$$

Let us represent such polynomial as $P_n(x) = a_0 + a_1x + \ + a_nx^n$. Then, the interpolation property means

$$\begin{aligned} &a_0 + a_1x_0 + \ + a_nx_0^n = f_0 \\ &a_0 + a_1x_1 + \ + a_nx_1^n = f_1 \\ &\vdots \\ &a_0 + a_1x_n + \ + a_nx_n^n = f_n \end{aligned} \tag{2}$$

This is a linear system of $n + 1$ equations in $n + 1$ unknowns (the polynomial coefficients $a_0, a_1, ..., a_n$). In matrix form:

$$\begin{bmatrix} x_0 & x_0 & x_0^2 & \cdots & x_0^n \\ x_1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{3}$$

Then our question is that does the linear system have a unique solution? The answer is yes. The reason is that the determinant is nonzero since we assume $x_0, x_1, \ldots, x_n$ are distinct.

In general, we often write the data as $(x_0, f_0), (x_1, f_1)$, etc. From the above, it looks like that we need to solve a linear system to find the interpolating polynomial $P_n(x)$. But it turns out that it is not necessary. We look at the following examples.

Example 1.1. *As an illustration let us consider interpolation by a linear polynomial, $P_1(x)$. Suppose we are given (x_0, f_0) and (x_1, f_1). We can write it in a the following form:*

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \tag{4}$$

Clearly, this polynomial has degree at most 1 and satisfies the interpolation property:

$$P_1(x_0) = f_0 \tag{5}$$

$$P_1(x_1) = f_1 \tag{6}$$

Example 1.2. *Given $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ let us construct $P_2(x)$, the polynomial of degree at most 2 which interpolates these points. The way we have written $P_1(x)$ in (3) is suggestive of how to explicitly write $P_2(x)$: $P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_1-x_1)} f_2$. If we define*

$$l_0^{(2)}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \tag{7}$$

$$l_1^{(2)}(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \tag{8}$$

$$l_2^{(2)}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \tag{9}$$

then we simply have

$$P_2(x) = l_0^{(2)}(x) f_0 + l_1^{(2)}(x) f_1 + l_2^{(2)}(x) f_2.$$

Note that each of the polynomials (7), (8) , *and* (9) *are exactly of degree 2 and they satisfy $l_j^{(2)}(x_k) = \delta_{jk}$. Therefore, it follows that $P_2(x)$ given by (9) satisfies the interpolation property*

$$P_2(x_0) = f_0 \tag{10}$$

$$P_2(x_1) = f_1 \tag{11}$$

$$P_2(x_2) = f_2 \tag{12}$$

We can now write down the polynomial (of degree at most n) which interpolates $n+1$ given values, $(x_0, f_0), \ldots, (x_n, f_n)$, where the interpolation nodes $x_0, \ldots, x_n$ are assumed distinct. Define

$$\begin{aligned} l_j^{(n)}(x) &= \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1}\cdots(x-x_n)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1}\cdots(x_j-x_n)} \\ &= \prod_{k=0, k\neq j}^{n} \frac{(x-x_k)}{(x_j-x_k)}, \qquad \text{for } j=0,1,\ldots,n \end{aligned}$$

These are called the *elementary Lagrange polynomials* of degree n. Note that $l_j^{(n)}(x_k) = \delta_{jk}$. Therefore

$$P_n(x) = l_0^{(n)}(x) f_0 + l_1^{(n)}(x) f_1 + \cdots + l_n^{(n)}(x) f_n = \sum_{j=0}^{n} l_j^{(n)}(x) f_j \tag{13}$$

interpolates the given data, i.e., it satisfies the interpolation property $P_n(x_j) = f_j$ for $j = 0, 1, 2, \ldots, n$. Relation (13) is called the Lagrange form of the interpolating polynomial.

The following result summarizes our discussion.

Theorem 1.3. *Given the $n+1$ values $(x_0, f_0), \ldots, (x_n, f_n)$, for $x_0, x_1, \ldots, x_n$ distinct. There is a unique polynomial of degree at most $n, P_n(x)$, such that $P_n(x_j) = f_j$ for $j = 0, 1, \ldots, n$.*

Proof. $P_n(x)$ in (13) is of degree at most n and interpolates the data. Uniqueness follows from the fundamental algebra : suppose there is another polynomial $Q_n(x)$ of degree at most n such that $Q_n(x_j) = f_j$ for $j = 0, 1, \ldots, n$. Consider $W(x) = P_n(x) - Q_n(x)$. This is a polynomial of degree at most n and $W(x_j) = P_n(x_j) - Q_n(x_j) = f_j - f_j = 0$ for $j = 0, 1, 2, \ldots, n$, which is impossible unless $W(x) \equiv 0$ which implies $Q_n = P_n$. □

2 Cauchy remainder

The general result about the interpolation error is the following theorem:

Theorem 2.1. *Let $f \in C^{n+1}[a, b], x_0, x_1, \ldots, x_n, x$ be contained in $[a, b]$, and $P_n(x)$ be the interpolation polynomial of degrees n of f at $x_0, \ldots, x_n$ then*

$$f(x) - P_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi(x)) (x-x_0)(x-x_1)\cdots(x-x_n) \tag{14}$$

where $\min\{x_0, \ldots, x_n, x\} < \xi(x) < \max\{x_0, \ldots, x_n, x\}$.

Proof. The right hand side of (14) is known as the Cauchy Remainder and the following proof is due to Cauchy. For x equal to one of the nodes x_j the result is trivially true. Take x fixed not equal to any of the nodes and define

$$\phi(t) = f(t) - P_n(t) - [f(x) - P_n(x)] \frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)} \tag{15}$$

Clearly, $\phi \in c^{n+1}[a, b]$ and vanishes at $t = x_0, x_1, \ldots, x_n, x$. That is, ϕ has at least $n+2$ zeros. Applying Rolle's Theorem $n+1$ times we conclude that there exists a point $\xi(x) \in (a, b)$ such that $\phi^{(n+1)}(\xi(x)) = 0$. Therefore,

$$0 = \phi^{(n+1)}(\xi(x)) = f^{(n+1)}(\xi(x)) - [f(x) - P_n(x)] \frac{(n+1)!}{(x - x_0)(x - x_1) \cdots (x - x_n)}$$

from which (14) follows. Note that the repeated application of Rolle's theorem implies that $\xi(x)$ is a number between $\min\{x_0, x_1, \ldots, x_n, x\}$ and $\max\{x_0, x_1, \ldots, x_n, x\}$.

□

Exercise 2.2. *For an interval [a, b], define* $h = (b - a)/n$ *for an integrer* $n > 0$. *Define evenly spaced node points by*

$$x_j = a + jh, \quad j = 0, 1, \ldots, n. \tag{16}$$

Thus, $x_0 = a, x_1 = a + h, \ldots, x_n = a + nh = b$. *Consider the polynomial*

$$\Psi_n(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$$

and show that

$$|\Psi_n(x)| \leq n! h^{n+1}, \quad a \leq x \leq b. \tag{17}$$