Department of Mathematics and Statistics
MAST90026 Computational Differential Equations
2024

Assignment 3: Parabolic/hyperbolic PDEs
Due: 11AM Wednesday 15th May.

This assignment is worth 20% of the total assessment in this subject. You should submit copies of MATLAB programs (include all files necessary for the programs to run) and sufficient relevant output in PDF form. All hand written working should be scanned and converted to a PDF. Your PDF should also include any tables, figures and comments or explanations of your results.

All files should be compressed into a single zip file *with your student ID number in the file name.*

1 Solving the advection equation (7 marks)

Solve the advection equation

$$u_t + u_x = 0, \quad 0 \leq x \leq 2, 0 \leq t \leq 5,$$

subject to the periodic boundary condition $u(0,t) = u(2,t)$ with initial conditions

a. square pulse:
$$u(x,0) = H(x-0.2) - H(x-0.4).$$

b. Gaussian wavepacket:
$$u(x,0) = \exp(-10(4x-1)^2).$$

Solve using the following methods as defined in the Week 9 lecture slides:

- Lax-Wendroff Method
- Upwind Method

For each method:

i. Explicitly write out any of the discretised equations which are modified by the periodic boundary condition.

ii. State the condition on ν to satisfy the CFL condition.

iii. Solve for two values of ν, one that satisfies the CFL condition and one that does not. In each case submit a plot of u vs x at $t = 5$.

Write a short paragraph discussing the performance of each method with the two initial conditions. Which method do you think is most suited to each initial condition?

2 Finite difference methods for the heat equation (7 marks)

The 1-D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1,\ t \ge 0,$$

where

$$\partial_x u(0,t) = \partial_x u(1,t) = 0,$$
$$u(x,0) = \cos(\pi x),$$

has the exact solution $u(x,t) = \cos(\pi x)\, e^{-\pi^2 t}$.
Solve this system using:

a. Crank-Nicolson finite differencing; and

b. TR-BDF2 method (see below).

For the Neumann boundary conditions, use the Ghost-point method as discussed earlier in the course. By choosing various time steplengths $\Delta t = k$ and space step lengths $\Delta x = h$, illustrate the *stability* and *accuracy* of each method.
To track the error, it is sufficient to look at the point $(x,t) = (1/2, 1)$.

Use the following method to estimate the order in space and time of each method:
If we assume that the error due to the size of the space steps is independent of the error due to the time step lengths i.e.

$$e_{total} = e_{time}(k) + e_{space}(h) = O(h^q, k^p), \tag{1}$$

then we note that the order of the error due to the time steps may be found using

$$p = \frac{\log(\Delta e_n) - \log(\Delta e_{n-1})}{\log(k_n) - \log(k_{n-1})},$$

where Δe_n is the change in error between the result with time step k_n and the result with time step k_{n-1} (i.e. $\Delta e_n = e_n - e_{n-1}$) when $\Delta x = h$ is held fixed. Typically, $k_n = k_{n-1}/2$ i.e. we run a series of calculations halving k each time.

Explain why this formula works, assuming Equation (1) holds. Derive a similar formula for q.

TR-BDF2

The TR-BDF2 method comes from applying the 2-stage implicit RK method instead of the implicit trapezoid rule used to obtain the Crank-Nicolson method. It is

$$u^* = u^n + \frac{k}{4}\left[f(t_n, u^n) + f(t_n + k/2, u^*)\right],$$

$$u^{n+1} = \frac{1}{3}\left[4u^* - u^n + kf(t_n + k, u^{n+1})\right].$$

3 Fisher's equation (3 marks)

Use the Method of Lines (use ODE45) with Central Difference discretization to solve Fisher's equation

$$u_t = u_{xx} + u(1-u),$$

on $[0, 10]$ subject to mixed BCs

$$u(0,t) = 1;\ \partial_x u(10,t) = 0.$$

a. Solve the case where the initial condition is

$$u_0(x) = 1 - H(x-1)$$

where $H(x)$ is the Heaviside step function.

b. Experiment with other initial conditions and submit plots for at least 3 different cases to the initial condition in (a). Based on your observations find a steady state solution to Fisher's equations with the given boundary conditions. Do you always approach this steady state?

4 Advection diffusion equation (3 marks)

Consider the advection diffusion equation

$$u_t = a u_{xx} - c u_x$$

subject to the periodic boundary conditions

$$u(0,t) = u(1,t) \qquad \text{and} \qquad \partial_x u(0,t) = \partial_x u(1,t)$$

a. Consider a grid of N evenly spaced internal points, $u_j(t)$ and with $u_0(t) = u(0,t)$ and $u_{N+1}(t) = u(1,t)$. Using 2nd order accurate central differencing, derive the semi-discrete (i.e., discretise in space but not in time) equations for $u_1(t)$, $u_j(t)$ where $j = 2, ..., N-1$ and $u_N(t)$. Hint: for the periodic Neumann condition, use the 3 point formulae $\partial_x u(0,t) \approx (-3u_0(t)+4u_1(t)-u_2(t))/(2h)$ and $\partial_x u(1,t) \approx (u_{N-1}(t) - 4u_N(t) + 3u_{N+1}(t))/(2h)$.

b. Write your system of equations in (a) in the form $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ where $\mathbf{u} = [u_1(t), ..., u_N(t)]^T$ and define $\mathbf{f}(\mathbf{u})$.

c. Using the Crank-Nicholson method to handle the time derivative, write down a fully discrete system of equations to solve this advection-diffusion equation.

d. Prove the local truncation error of your scheme in (c) is of the form $\tau = O(h^p, k^q)$ and in so doing find the values of p and q. You may ignore the effect of the boundary conditions by only examining the spatial points $j = 2, ..., N-1$.

Note: this question does not require you to do any coding.