Question 1 (7 marks)

let $u(x_j, t_n) = u_j^n$

where $j \in \{0, \ldots, N\}$

the periodic boundary condition then implies that $u_j^n = u_{j+N}^n \quad \forall j, n$

Lax-Wendroff

Our discretized equation is

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\nu^2}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Now because of the periodic BC we don't need both $j=0$ & $j=N$, so we let $j = \{1, \ldots, N\}$.

Note as $c=1$, $\nu = \frac{k}{h} = \frac{\Delta t}{\Delta x}$

i) The two equations modified by the boundary condition are

$$u_1^{n+1} = u_1^n - \frac{\nu}{2}\left(u_2^n - u_N^n\right) + \frac{\nu^2}{2}\left(u_2^n - 2u_1^n + u_N^n\right)$$

$$u_N^{n+1} = u_N^n - \frac{\nu}{2}\left(u_1^n - u_{N-1}^n\right) + \frac{\nu^2}{2}\left(u^n - 2u_N^n + u_{N-1}^n\right)$$

ii) Because $k, h > 0$ the CFL condition is $\nu < 1$

Upwind Method

i) Our discretised equation is

$$u_j^{n+1} = u_j^n - \nu\left(u_j^n - u_{j-1}^n\right)$$

where

$$u_N^{n+1} = u_N^n - \nu\left(u_N^n - u_1^n\right)$$

ii) Again the CFL condition is $\nu < 1$

Some Observations

1) when the CFL condition is not satisfied the method is unstable as expected.

2) To maintain the sharp edges of the square pulse we have to use an extremely high spatial discretisation ($\sim 100{,}000$ for me)

3) As expected, LxW displays Gibbs phenomena on the square pulse

Question 2 (? marks)

After implementing we find that

Crank-Nicolson has $p=2$ & $q=2$

TR-BDF2 has $p=2$ & $q=2$

Marks are awarded for correct code, explanation of error formula, and correct values of p & q for each method.

Upon investigating, there is an issue with using the point $x=1/2$ for the error as this point always produces $u=0$ to machine precision. This does not effect your written work or or implementation of the algorithms however I will remain flexible with the grading & award marks based on intelligent observations even if you only analyzed the point $x=1/2$.

Question 3

See code for implementation.

We can see there is a steady state solution of $u=1$, however we don't approach this for all initial conditions, sometimes the solution grows unbounded

Question 4

a) Firstly, the periodic boundary conditions give

$$u_{N+1} = u_0$$

and

$$\frac{1}{2h}\left(-3u_0 + 4u_1 - u_2\right) = \frac{1}{2h}\left(u_{N-1} - 4u_N + 3u_{N+1}\right)$$

from which we can solve for u_0 & u_{N+1} to find

$$u_0 = u_{N+1} = \frac{1}{6}\left(4u_1 - u_2 + 4u_N - u_{N-1}\right)$$

Spatially discretising then leads to

$$\frac{du_j}{dt} = \frac{a}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) - \frac{c}{2h}\left(u_{j+1} - u_{j-1}\right)$$

Substitution of u_0 & u_{N+1} into the above equation when $j = 1$ & N respectively gives

$$\frac{du_1}{dt} = \frac{a}{6h^2}\left(-8u_1 + 5u_2 + 4u_N - u_{N-1}\right)$$
$$- \frac{c}{12h}\left(-4u_1 + 7u_2 - 4u_N + u_{N-1}\right)$$

$$\frac{du_N}{dt} = \frac{a}{6h^2}\left(4u_1 - u_2 - 8u_N + 5u_{N-1}\right)$$
$$- \frac{c}{12h}\left(4u_1 - u_2 + 4u_N - 7u_{N-1}\right)$$

b) Using the results from (a) we can write

$$\frac{d\underline{u}}{dt} = \underline{f}(\underline{u}) = A\underline{u}$$

where $A = \frac{a}{h^2} A^{(2)} - \frac{c}{2h} A^{(1)}$

$$A^{(2)} = \begin{bmatrix} -\frac{8}{6} & \frac{5}{6} & & & -\frac{1}{6} & \frac{4}{6} \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ \frac{4}{6} & -\frac{1}{6} & & & \frac{5}{6} & -\frac{8}{6} \end{bmatrix}$$

$$A^{(1)} = \begin{bmatrix} -\frac{4}{6} & \frac{7}{6} & & & & \frac{1}{6} & -\frac{4}{6} \\ -1 & 0 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 0 & 1 \\ \frac{4}{6} & -\frac{1}{6} & & & & -\frac{7}{6} & \frac{4}{6} \end{bmatrix}$$

c) Crank-Nicolson gives

$$\frac{\underline{u}^{N+1} - \underline{u}^{N}}{k} = \frac{1}{2}\left(A\underline{u}^{N+1} + A\underline{u}^{N}\right)$$

$$\Rightarrow \left(I - \frac{k}{2}A\right)\underline{u}^{N+1} = \left(I + \frac{k}{2}A\right)\underline{u}^{N}$$

which is our fully discrete system.

d) This exactly follows the example derivation given in the Canvas announcement