School of Mathematics and Statistics
MAST90026 Computational Differential Equations
2024

Assignment 2: Elliptic PDEs
Due: 11AM Wednesday, May 1st.

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1 Elliptic equations by Finite Differences (4 marks)

a. Write code to solve the Poisson equation

$$-\nabla^2 u = f(x,y)$$

on a general rectangle (with sides parallel to either the x or y axes) with inhomogeneous Dirichlet boundary conditions. Test your code for the case where the Poisson equation has the exact solution $u(x,y) = \exp(x+y)$ on $\Omega = (0,1)\times(0,2)$. The right hand side function f and inhomogeneous Dirichlet boundary condition g are given by u.

b. Modify your code in (a) to solve the following diffusion reaction equation

$$-\nabla^2 u + (x^2+y^2)u = f(x,y),$$

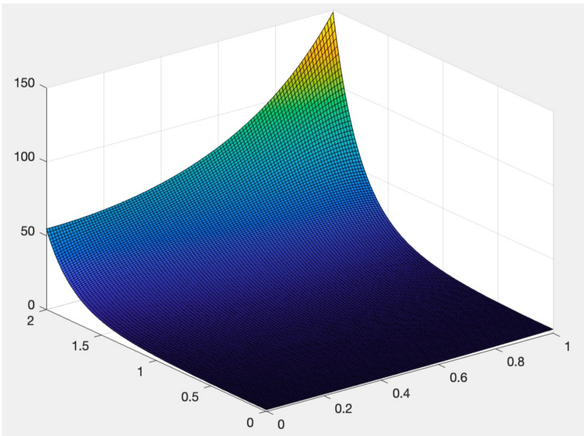
with analytical solution

$$u = \exp(x^2+y^2).$$

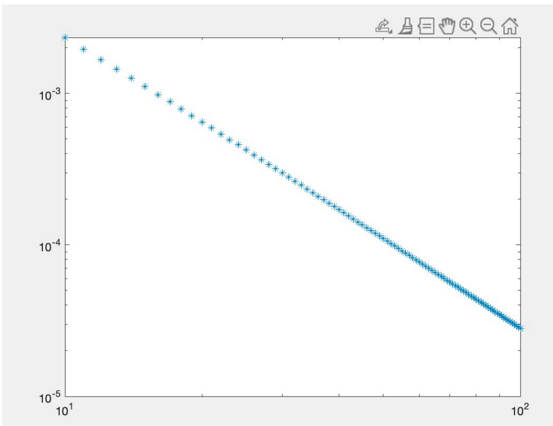
on $\Omega = (0,1)\times(0,2)$. The right hand side function f and Dirichlet boundary condition g are given by u.

For both (a) and (b), discuss how the maximum grid error behaves with Δx and Δy. You may find a `loglog` plot to be helpful.

(a) The file FDM2D.m is to solve the Poisson equation $-\nabla^2 u = f(x,y)$ on a general rectangle with inhomogeneous Dirichlet boundary condition with $u(x,y) = \exp(x+y)$, $a=0$, $b=1$, $c=0$, $d=2$, $n=100$, $m=200$. The figure of the solutions:



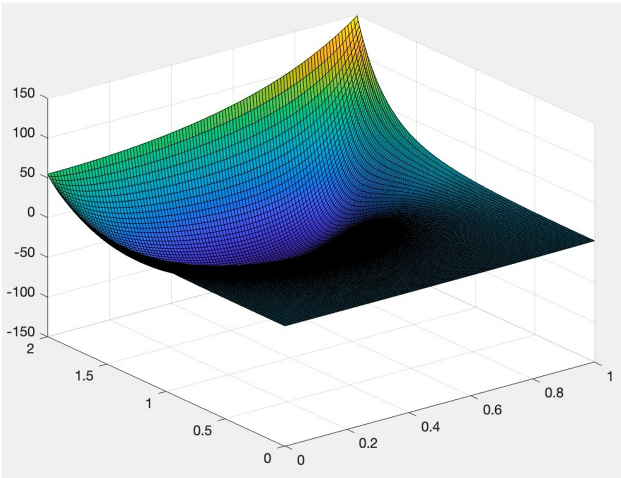
The file Err.m is used to test the maximal error of the exact solution and the numerical solution of Poisson equation. And plot the error:



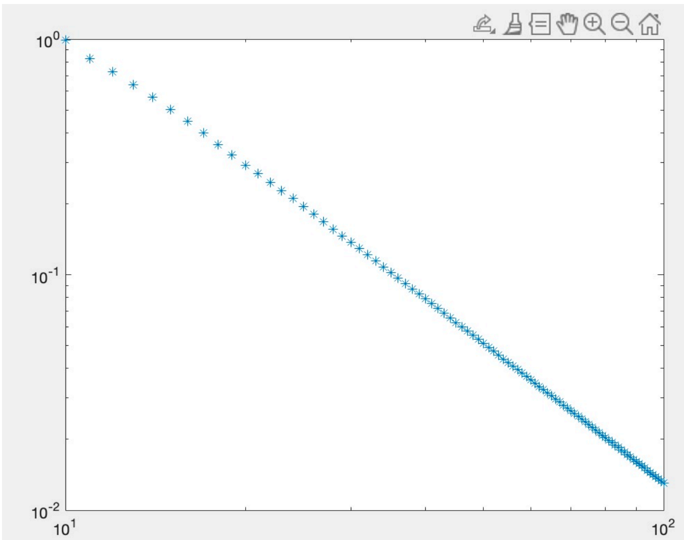
we can find that the maximum grid error decrease as Δx Δy decrease.

(b) The file FDM2DMOD.m is used to solve the diffusion reaction equation $-\nabla^2 u + (x^2+y^2)u = f(x,y)$ with non-homogeneous Dirichlet boundary condition g are given by u with $u = \exp(x^2+y^2)$ $a=0$, $b=1$, $c=0$, $d=2$, $n=100$, $m=200$.

The figure of the solutions:



The file Err.m is used to test the maximal error of the numerical solution and the exact solution and I provide the plot:



we can find that the maximum grid error decrease as Δx Δy decrease.

2 Finite element methods in 2D (7 marks)

In this question you are asked to solve the model elliptic problem using linear finite elements on a conforming triangular mesh.

a. Write Matlab functions with the following definitions

```
function F = elem_load(P1, P2, P3, f, quad_order)
function M = elem_mass(P1, P2, P3, q, quad_order)
function K = elem_stiff(P1, P2, P3, D, quad_order)
```

which calculate the element load vector, element mass matrix and element stiffness matrix respectively. The inputs `P1`, `P2`, `P3`, `f`, `q` and `D` are exactly as defined in the week 6 and 7 lab resources. The additional input `quad_order` may only take on the value of 1 or 2. If `quad_order` is 1 then you should use linearly accurate quadrature using the triangle vertices and if `quad_order` is 2 you should use the 3 point quadratically accurate quadrature rule as defined in class.

The file elem_load.m calculate the element load vector.

The file elem_mass.m calculate the element mass matrix

The file elem_stiff.m calculate the element stiffness matrix.

b. write a Matlab function with the following declaration

```
function u = FEM_Elliptic_2D_Dirichlet(node, elem, Dfunc, qFunc, fFunc, gFunc, quad_order)
```

with the inputs defined exactly as given in the week 6 and 7 lab resources and `quad_order` as defined in (a). This function should solve the model elliptic equation with Dirichlet boundary conditions. We assume a rectangular domain with the rectangle sides parallel to either the x or y axes.

The file FEM_Elliptic_2D_Dirichlet.m solve the model elliptic equation with Dirichlet boundary conditions.

c. Test your code from (b) for the Poisson equation with exact solution $u = \sin(x)\cos(4y)$ on the unit square with a uniform mesh constructed using the Matlab functions `meshgrid` and `delaunay`. The governing equation and Dirichlet boundary conditions are defined by the exact solution. Perform a grid refinement analysis (i.e., increasing the number of points in both the x and y directions) for both linearly and quadratically accurate quadrature rules. For simplicity you may use the ∞−norm to calculate the error. Discuss the order of the error and any differences you observe between the quadrature schemes.

Now, let's run file uniformmesh.m and

① Let $n=50$ quad_order $=1$, it shows the maximum grid error is 1.8259×10^{4}

② Let $n=50$ quad_order $=2$, it shows the maximum grid error is 1.1274×10^{4}

we find that for quad-order 2, the maximum grid error is smaller than that of qual-order 1.

d. Repeat the test of (b) on an unstructured mesh using the problem in (c). Use the matrices contained within `unstructured_mesh`.

file uniformrefine.m is used to refine the unstructured element.

Run unstructured_mesh.mat as provided.

Then run unstructured_analysis.m

① Let quad_order = 1, we have error = 0.0029, $P \approx 1.5422$

② Let quad_order = 2, we have error = 0.0029, $P \approx 1.5084$

As we can see the order for quad_order is smaller.

3 Finite element method for 2D nonlinear equation (4 marks)

Consider the following nonlinear equation

$$-\nabla^2 u + u^2 = f(x, y) \tag{1}$$

with inhomogeneous Dirichlet boundary condition g. We consider a 2D problem on the unit square $\Omega = (0,1) \times (0,1)$. We choose f and g such that the exact solution is $u = \exp(x + 2y)$.

a. Given a current approximation u_k, derive the linearised elliptic equation at u_{k+1} using quasilinearisation.

(a) $u_{k+1}^2 = u_k^2 + 2u_k(u_{k+1} - u_k)$

$\therefore \; -\nabla^2 u_{k+1} + u_{k+1}^2 = f$

$\Rightarrow \; -\nabla u_{k+1}^2 + 2u_k u_{k+1} = f + u_k^2$

b. Solve the linearised elliptic equation and iterate until your solution converges by modifying the finite element method code that you wrote in Question 2(b). Use any `node` and `elem` definitions that you like but note that your choice will affect the convergence. In order to iterate you will need an initial guess of the solution. DO NOT choose the analytical solution as the initial guess otherwise you will get zero marks for this question part.

$f(x,y) = -\nabla^2 u + u^2 = -5e^{x+2y} + e^{2x+4y}$ for 2(b)

4 Galerkin equations with Robin boundary conditions (3 marks)

Consider the model elliptic equation over 2D domain Ω subject to the Robin boundary condition

$$a(x,y)u(x,y) + b(x,y)\boldsymbol{n}\cdot\nabla u = g(x,y)$$

defined on the boundary $\delta\Omega$ where $b \neq 0$ anywhere on $\delta\Omega$.

Derive the Galerkin equations for a given basis $\{\phi_i\}$ where $i = 1, ...n$. Hint: you should not make any assumptions about the basis aside from the fact that $\phi_i \in H^1(\Omega)\ \forall i$. You should not submit any code for this question.

$$-\nabla\cdot(D\nabla u) + qu = f$$

$$-\int_\Omega \nabla\cdot(D\nabla u)v\, dA + \int_\Omega quv\, dA = \int_\Omega fv\, dA$$

By diverges theorem,

$$\int_\Omega D\cdot(D\nabla u)v\, dA = \int_{\partial\Omega}(D\nabla u\cdot\vec{n})v\, ds - \int_\Omega D\nabla u\cdot\nabla v\, dA$$

$$\Rightarrow \int_\Omega D\nabla u\cdot\nabla v\, dA + \int_\Omega quv\, dA = \int_\Omega fv\, dA + \int_{\partial\Omega} D\nabla u\cdot\vec{n}\, v\, ds$$

$$= \int_\Omega fv\, dA + \int_{\partial\Omega} D\,\frac{g-au}{b}\, v\, ds$$

Let $u = \sum_{j=1}^{n} u_j\phi_j$

$$\sum_{j=1}^{n} u_j\left[\int_\Omega D\nabla\phi_j\ \nabla\phi_i + q\phi_i\phi_j\, dA + \int_{\partial\Omega}\frac{Da}{b}\phi_j\phi_i\, ds\right]$$

$$= \int_\Omega f\phi_i\, dA + \int_{\partial\Omega}\frac{Dg}{b}\phi_i\, ds \qquad \text{for } i = 1, 2, \cdots n$$

5 Finite element methods using rectangular elements (2 marks)

In this question we will derive formulae for the stiffness matrix and load vector on rectangular elements.

a. Consider a rectangle defined by points $\{P_1, P_2, P_3, P_4\}$ where $P_i = (x_i, y_i)$, $i = 1, ..., 4$.
Derive the quantities J_R and b_R of the affine map F_R, i.e.,

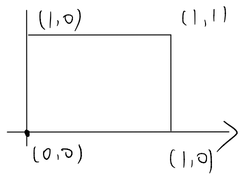
$$(x, y) = F_R(\xi, \eta) \qquad \Leftrightarrow \qquad \begin{bmatrix} x \\ y \end{bmatrix} = J_R \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \vec{b}_R$$

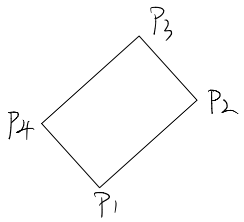
so that $F_R(0, 0) = P_1$, $F_R(1, 0) = P_2$, $F_R(1, 1) = P_3$ and $F_R(0, 1) = P_4$.
Submit a simple piece of code with the declaration

```
function [JR, bR] = rectangle_map(P1, P2, P3, P4)
```

that returns J_R and $\vec{b}_R$ for a given set of points.

(a)





$P_1 = (x_1, y_1)$
$P_2 = (x_2, y_2)$
$P_3 = (x_3, y_3)$
$P_4 = (x_4, y_4)$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} a &= x_2 - x_1 \\ c &= y_2 - y_1 \\ b &= x_4 - x_1 \\ d &= y_4 - y_1 \end{aligned}$$

$$J_R = \begin{pmatrix} x_2 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_4 - y_2 \end{pmatrix} \qquad b_R = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

see the file rectangle_map.m

b. Using the nodal basis functions for a rectangular bilinear element as given in the lecture slides, write down analytic expressions for the element stiffness matrix and element load vector that only involve integrals over the unit square.

$$\nabla_S N_1 = \begin{pmatrix} \eta - 1 \\ \xi - 1 \end{pmatrix} \qquad \nabla_S N_2 = \begin{pmatrix} 1 - \eta \\ -\xi \end{pmatrix} \qquad \nabla_S N_3 = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \qquad \nabla_S N_4 = \begin{pmatrix} -\eta \\ 1 - \xi \end{pmatrix}$$

Load vector:

$$\int_{T_R} f(\vec{x}(s))\, N_j(\vec{s})\, |\det J_R|\, d\xi\, d\eta \quad \text{for } 1 \le j \le 4$$

$$|\det J_R| = (x_2 - x_1)(y_4 - y_1) - (x_4 - x_1)(y_2 - y_1)$$

Stiff matrix:

$$K_T = \left[\int_T D \nabla \phi_i \nabla \phi_j \, dA \right]_{1 \le i, j \le 4}$$

$$\int_T D \nabla\phi_i \cdot \nabla\phi_j \, dA = \int_{T_R} D(\vec{x}(\vec{s}))\, (J_R^{-T} \nabla_{\vec{s}} N_i(\vec{s})) \cdot (J_R^{-T} \nabla_{\vec{s}} N_j(s))\, |\det J_R|\, d\xi\, d\eta \quad \text{for } \begin{aligned} 1 \le i \le 4 \\ 1 \le j \le 4 \end{aligned}$$

c. For the case where $D(x,y)=1$, $q(x,y)=0$, $f(x,y)=2$, $P_1=(1,1)$, $P_2=(2,3)$, $P_3=(1,7/2)$ and $P_4=(0,3/2)$, evaluate the element stiffness matrix and element load vector. Your final answer should be fully evaluated, i.e., contain only numbers. Note you may want to do this calculation with the aid of MATLAB/other computer software because the calculations are fairly lengthy, but your final answer must be exact.

If $D=1$ $f=2$ $J=\begin{pmatrix} x_2-x_1 & x_4-x_1 \\ y_2-y_1 & y_4-y_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & \frac{1}{2} \end{pmatrix}$

For load vector. $|\det J_R| = \frac{5}{2}$

$\int_{TR} 5 N_j(s)\, d\varepsilon d\eta$

$\int_{TR} 5 N_1(s)\, d\varepsilon d\eta = 5\int_0^1\int_0^1 (1-\varepsilon)(1-\eta)\, d\varepsilon d\eta = \frac{5}{4}$

$\int_{TR} 5 N_2(s)\, d\varepsilon d\eta = 5\int_0^1\int_0^1 \varepsilon(1-\eta)\, d\varepsilon d\eta$

$= \int_{TR} 5N_4(s)\, d\varepsilon d\eta$

$= 5\int_0^1\int_0^1 \eta(1-\varepsilon)\, d\varepsilon d\eta$

$= \frac{5}{4}$

$5\int_{TR} N_3(s)\, d\varepsilon d\eta = \frac{5}{4}$ $\therefore$ Load vector $= \begin{pmatrix} \frac{5}{4} \\ \frac{5}{4} \\ \frac{5}{4} \\ \frac{5}{4} \end{pmatrix}$

for stiff matrix:

$(J_R^T) = \begin{pmatrix} 1 & 2 \\ -1 & \frac{1}{2} \end{pmatrix}$ $J_R^{-T} = \frac{2}{5}\begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix}$

$J_R^{-T}\nabla_S N_1 = \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} \eta-1 \\ \varepsilon-1 \end{pmatrix} = \begin{pmatrix} \frac{\eta}{5}+\frac{3}{5}-\frac{4}{5}\varepsilon \\ \frac{2}{5}\eta+\frac{2}{5}\varepsilon-\frac{4}{5} \end{pmatrix}$

$J_R^{-T}\nabla_S N_2 = \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} 1-\eta \\ -\varepsilon \end{pmatrix} = \begin{pmatrix} \frac{1}{5}-\frac{\eta}{5}+\frac{4}{5}\varepsilon \\ \frac{2}{5}-\frac{2}{5}\eta-\frac{2}{5}\varepsilon \end{pmatrix}$

$J_R^{-T}\nabla_S N_3 = \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} \eta \\ \varepsilon \end{pmatrix} = \begin{pmatrix} \frac{\eta-4\varepsilon}{5} \\ \frac{2\eta+2\varepsilon}{5} \end{pmatrix}$

$J_R^{-T}\nabla_S N_1 = \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} -\eta \\ 1-\varepsilon \end{pmatrix} = \begin{pmatrix} -\frac{\eta}{5}+\frac{4}{5}\varepsilon-\frac{4}{5} \\ -\frac{2}{5}\eta+\frac{2}{5}-\frac{2}{5}\varepsilon \end{pmatrix}$