Q1.

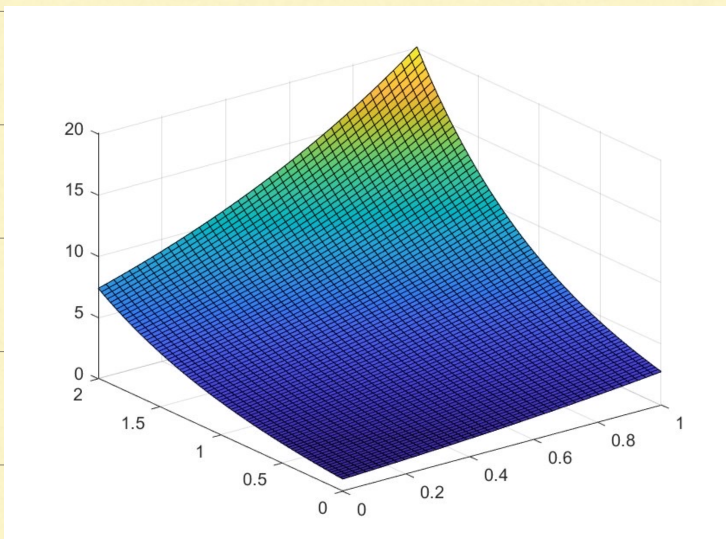
a) For this equation

$$-\nabla^2 u = f(x,y)$$

If $u = e^{x+y}$, then $f(x,y) = -2e^{x+y}$

Let $a=0$ $b=1$ $c=0$ $d=2$

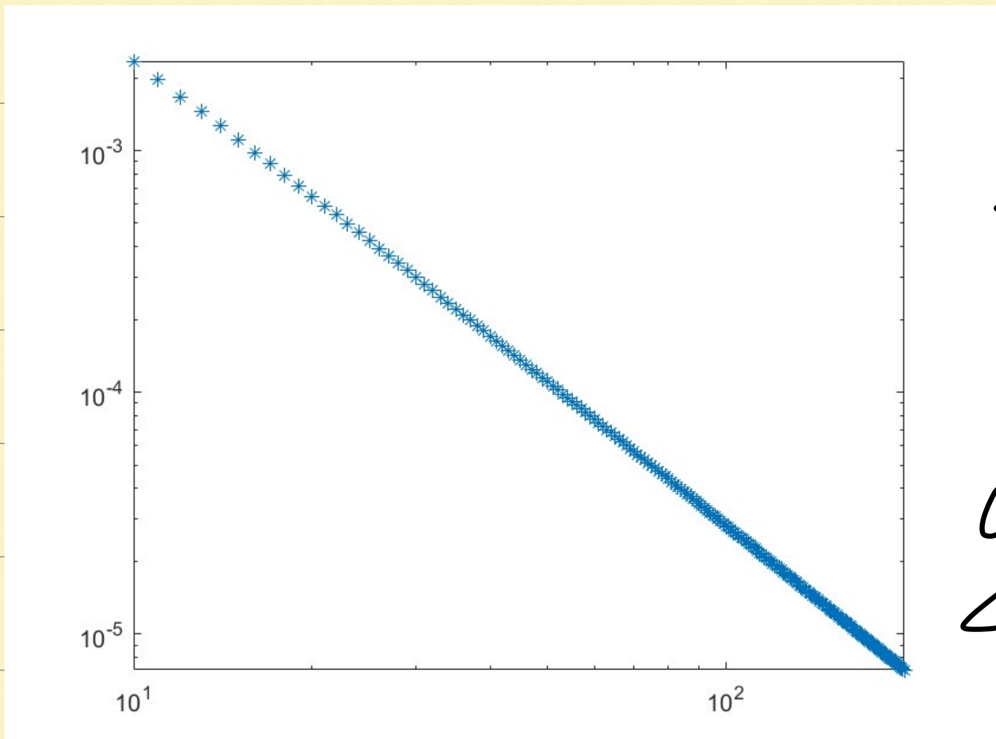
$n=40$, $m=80$, Run the file FDM2D will obtain the solution



Next, run the file Errr,
we plot the maximum grid error behaves with $\Delta x, \Delta y$

$\Delta x = \frac{1}{N}$ $\Delta y = \frac{2}{N}$, Let $N=200$

we plot the log-log plot of error with N.



we can find the maximum grid error decrease as $\Delta x, \Delta y$ decrease.

b) $u = e^{x^2+y^2}$

$q(x,y) = x^2+y^2$.

$f(x,y) = -\nabla^2 u + q(x,y)u$

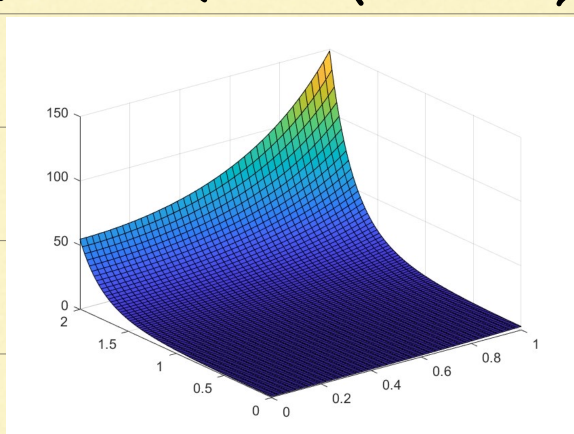
$= (-3x^2-3y^2-4)e^{x^2+y^2}$

Let $a=0$ $b=1$ $c=0$ $d=2$

$n=40$ $m=80$.

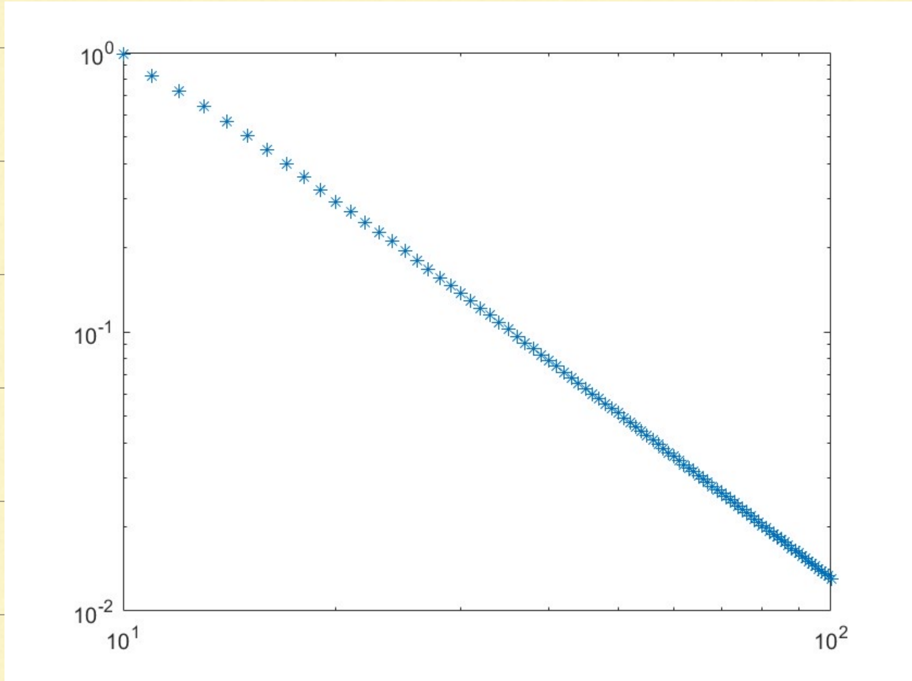
Run file FDM2DMOD.

the solution is.



Next, run the Err file.

Set N=100, plot the maximum grid error and N



It is also that maximal grid error decrease as $\Delta x, \Delta y$ decrease.

Q2.

(a) and (b)

The matlab code are provided in the zip file.

(c) For the uniformmesh.

Run file uniformmesh and. Let quad_order =1

For $n=50$, it will show the maximum grid error is. 1.6546×10^{-4}.

Also

We have order

$$p \approx \frac{\ln \frac{\|E(N)\|}{\|E(2N)\|}}{\ln 2}$$

Let $N=50$, $2N=100$

the file uniformmesh

return $p \approx 2$.

Again, run uniformmesh

Let quad_order=2

we have when $N=50$

maximum grid error is

1.1274×10^{-4}

order $p \approx 2$.

we find the order of error are both 2.

and we find the for quad-order 2 method, the maximum grid error is smaller than quad_order1.

d. First run the unstructured_mesh file generate the element and node.

Remark: the uniformrefine

file is used to refine the unstructured element

Run the unstructured_analysis file Let quad_order =1 we have

maximum grid error. is 0.0054

$p \approx 1.7146$.

Again, Let quad_order=2

maximum grid error is 0.0029 and $p \approx 1.5084$.

we find. the maximum grid error is smaller for quad_order = 2 . and the order for quad_order=2 is also smaller.

Q3.

(a).

$$u_{k+1}^2 = u_k^2 + 2u_k(u_{k+1} - u_k)$$

So

$$-\nabla^2 u_{k+1} + u_{k+1}^2 = f$$

$$\Rightarrow -\nabla^2 u_{k+1} + 2u_k u_{k+1} = f + u_k^2.$$

b. $f(x,y) = -\nabla^2 u + u^2$

$$= -5e^{x+2y} + e^{2x+4y}$$

For this question,

Run the test file.

use uniformmesh on $(0,1)\times(0,1)$,

Let $n=100$, we can

find the maximum grid error

is 7.9858×10^{-4},

iteration number is 6.

for the initial guess with

$U_0=[0]_{n\times n}$.

Q4.

$$-\nabla\cdot(D\nabla u)+qu=f$$

$$-\int_\Omega \nabla\cdot(D\nabla u)v\,dA$$

$$+\int_\Omega quv\,dA=\int_\Omega fv\,dA$$

By diverges theorem

$$\int_\Omega \nabla\cdot(D\nabla u)v\,dA$$

$$=\int_{\partial\Omega}(D\nabla u\cdot\vec{n})v\,dS$$

$$-\int_\Omega D\nabla u\cdot\nabla v\,dA$$

$\Rightarrow$

$$\int_\Omega D\nabla u\cdot\nabla v\,dA+\int_\Omega q u v\,dA$$

$$=\int_\Omega f v\,dA+\int_{\partial\Omega} D\nabla u\cdot\vec{n}\,v\,ds$$

$$=\int_\Omega f v\,dA+\int_{\partial\Omega} D\,\frac{g-au}{b}\,v\,ds.$$

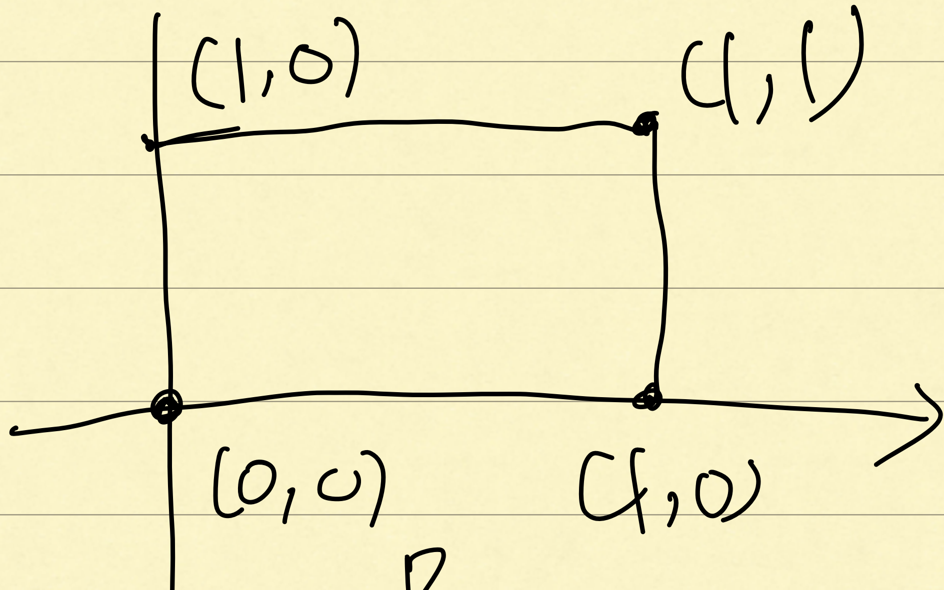
Let $u=\sum_{j=1}^{n} u_j\phi_j$

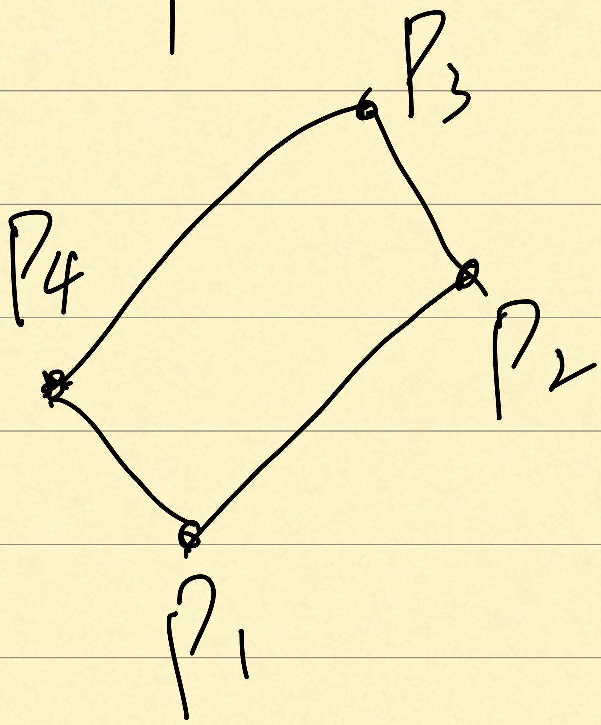
$$\sum_{j=1}^{n} u_j\left[\int_\Omega D\nabla\phi_j\cdot\nabla\phi_i+q\,\phi_i\phi_j\,dA+\int_{\partial\Omega}\frac{Da}{b}\phi_j\phi_i\,ds\right]$$

$$=\int_\Omega f\phi_i\,dA+\int_{\partial\Omega}\frac{Dg}{b}\phi_i\,ds$$

for $i=1,2,\cdots n$.

Q5. (a)





$P_1 = (x_1, y_1)$

$P_2 = (x_2, y_2)$

$P_3 = (x_3, y_3)$

$P_4 = (x_4, y_4)$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\Longrightarrow \quad a = x_2 - x_1$$

$$c = y_2 - y_1$$

$$b = x_4 - x_1$$

$$d = y_4 - y_1$$

$$J_R = \begin{pmatrix} x_2 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_4 - y_1 \end{pmatrix} \quad b_R = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

File rectangle_map show this.

2.

$$\nabla_s N_1 = \begin{pmatrix} \eta-1 \\ \xi-1 \end{pmatrix}$$

$$\nabla_s N_2 = \begin{pmatrix} 1-\eta \\ -\xi \end{pmatrix}$$

$$\nabla_s N_3 = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$$

$$\nabla_s N_4 = \begin{pmatrix} -\eta \\ 1-\xi \end{pmatrix}.$$

Load vector.

$$\int_{T_R}\int f(\vec{x}(s))N_j(\vec{s})\,|\det J_R|\,d\xi\,d\eta$$

for $1 \le j \le 4$.

$$|\det J_R| = (x_2 - x_1)(y_4 - y_1) - (x_4 - x_1)(y_2 - y_1)$$

Stiff matrix.

$$K_T = \left[\int_T D\nabla\phi_i \cdot \nabla\phi_j \, dA\right]_{1\le i,j\le 4}$$

$$\int_T D\nabla\phi_i : \nabla\phi_j \, dA$$

$$= \int_{T_R} D(\vec{x}(\vec{s}))\left(J_R^{-T}\nabla_{\vec{s}} N_i(\vec{s})\right) \cdot \left(J_R^{-T}\nabla_{\vec{s}} N_j(\vec{s})\right)|\det J_R| \, d\xi \, d\eta$$

for

$1 \le i \le 4$

$1 \le j \le 4.$

(c) If $D=1$ $f=2$.

$$J=\begin{pmatrix} x_2-x_1 & x_4-x_1 \\ y_2-y_1 & y_4-y_1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -1 \\ 2 & \frac{1}{2} \end{pmatrix}.$$

For load vector.

$$|\det J_R| = \frac{5}{2}$$

$$\int_{T_R} 5\, N_j(s)\, d\xi\, d\eta$$

$$\int_{T_R} 5 N_1(s)\, d\xi d\eta = 5\int_0^1\int_0^1 (1-\xi)(1-\eta)\, d\xi d\eta$$

$$= \frac{5}{4}$$

$$\int_{T_R} 5 N_2(s)\, d\xi d\eta$$

$$= 5\int_0^1\int_0^1 \xi(1-\eta)\, d\xi d\eta$$

$$= \int_{T_R} 5 N_4(s)\, d\xi d\eta$$

$$= 5\int_0^1\int_0^1 \eta(1-\xi)\, d\xi d\eta$$

$$= \frac{5}{4}$$

$$5\int_{T_R} N_3(s)\, d\xi d\eta = \frac{5}{4}$$

So

Load vector $\begin{pmatrix} \frac{5}{4} \\ \frac{5}{4} \\ \frac{5}{4} \\ \frac{5}{4} \end{pmatrix}$.

For stiff matrix.

$$\left(J_R^T\right) = \begin{pmatrix} 1 & 2 \\ -1 & \frac{1}{2} \end{pmatrix}$$

$$J_R^{-T} = \frac{2}{5}\begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix}.$$

$$J_R^{-T} \cdot \nabla_s N_1$$

$$= \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} \eta - 1 \\ \xi - 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\eta}{5} + \frac{3}{5} - \frac{4}{5}\xi \\ \frac{2}{5}\eta + \frac{2}{5}\xi - \frac{4}{5} \end{pmatrix}$$

$$J_R^{-T} \nabla_s N_2$$

$$= \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 - \eta \\ -\xi \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} - \frac{\eta}{5} + \frac{4}{5}\xi \\ \frac{2}{5} - \frac{2}{5}\eta - \frac{2}{5}\xi \end{pmatrix}.$$

$$J_R^{-T} D_S N_3$$

$$= \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\eta - 4\xi}{5} \\ \frac{2\eta + 2\xi}{5} \end{pmatrix}$$

$$J_R^{-T} D_S N_4 = \begin{pmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} -\eta \\ 1-\xi \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\eta}{5} + \frac{4}{5}\xi - \frac{4}{5} \\ -\frac{2}{5}\eta + \frac{2}{5} - \frac{2}{5}\xi \end{pmatrix}.$$