1a) 0.5 marks

let $Au^k = \lambda^k u^k$

for $j = 2, 3, \ldots, N-1$

$\Rightarrow \dfrac{1}{h^2}\left(\sin\dfrac{k\pi(j-1)}{N+1} - 2\sin\dfrac{k\pi j}{N+1} + \sin\dfrac{k\pi(j+1)}{N+1}\right)$

$= \lambda^k \sin\dfrac{k\pi j}{N+1}$

$\Rightarrow \dfrac{2}{h^2}\sin\dfrac{k\pi j}{N+1}\left(\cos\dfrac{k\pi}{N+1} - 1\right) = \lambda^k \sin\dfrac{k\pi j}{N+1}$ (trig angle addition identities)

$\Rightarrow \lambda^k = \dfrac{2}{h^2}\left(\cos\dfrac{k\pi}{N+1} - 1\right)$

b) 0.75 marks
Now A is an $N \times N$ symmetric matrix with N distinct eigenvalues so we can diagonalise

$A = QDQ^T$ where Q is an orthogonal matrix and

$$D = \frac{2}{h^2} \begin{bmatrix} \cos\frac{\pi}{N+1} - 1 & 0 & & \\ 0 & \cos\frac{2\pi}{N+1} - 1 & 0 & \\ & & \ddots & \\ & & 0 & \cos\frac{N\pi}{N+1} - 1 \end{bmatrix}$$

Therefore $\quad A^{-1} = Q\, D^{-1}\, Q^{T}$

$$\| A^{-1} \|_2 = \| Q D^{-1} Q^{T} \|_2 = \| D^{-1} \|_2 \quad \text{because}$$

Q is orthogonal

Because D is diagonal

$$D^{-1} = \frac{h^2}{2} \begin{bmatrix} \frac{1}{\cos\frac{\pi}{N+1} - 1} & & & \\ & \frac{1}{\cos\frac{2\pi}{N+1} - 1} & & \\ & & \ddots & \\ & & & \frac{1}{\cos\frac{N\pi}{N+1} - 1} \end{bmatrix}$$

Again because D^{-1} is diagonal,

$$\|D^{-1}\|_2 = \max_i |D_{ii}^{-1}| = \frac{h^2}{2}\left|\frac{1}{\cos\frac{\pi}{N+1}-1}\right|$$

Now $h = \frac{1}{N+1}$, since $x \in (0,1)$

So $\|D^{-1}\|_2 = \frac{h^2}{2}\left|\frac{1}{\cos h\pi - 1}\right|$

Taking a Taylor series around $h = 0$ gives

$$\|A^{-1}\|_2 = \|D^{-1}\|_2 = \frac{h^2}{2}\left|-\frac{2}{h^2\pi^2} + O(1)\right| = \frac{1}{\pi^2} + O(h^2)$$

$$\Rightarrow \lim_{h \to 0} \|A^{-1}\| = \frac{1}{\pi^2}$$

2a) $u' = 1 + \frac{1}{\varepsilon}\frac{e^{x/\varepsilon}}{e^{1/\varepsilon}-1}$ & $u'' = \frac{1}{\varepsilon^2}\frac{e^{x/\varepsilon}}{e^{1/\varepsilon}-1}$

$$\Rightarrow \varepsilon u'' - u' = \frac{1}{\varepsilon}\frac{e^{x/\varepsilon}}{e^{1/\varepsilon}-1} - 1 - \frac{1}{\varepsilon}\frac{e^{x/\varepsilon}}{e^{1/\varepsilon}-1} = -1$$

additionally

$$u(0) = 1 + 0 + \frac{e^{0/\varepsilon} - 1}{e^{1/\varepsilon} - 1} = 1$$

$$u(1) = 1 + 1 + \frac{e^{1/\varepsilon} - 1}{e^{1/\varepsilon} - 1} = 3$$

Therefore solution solves the BVP

b)

i) 1.5 marks

ii) 1.5 marks

c) 0.5 marks

The central finite difference scheme has $O(h^2)$ accuracy and the central-upwind scheme has $O(h)$ accuracy. However as ε gets smaller the central upwind scheme begins to outperform the central scheme due to the rapid variation in the true solution around $x = 1$.