Q1.

①

Lax-Wendroff Method.

$$U_j^{n+1} = U_j^n - \frac{V}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{V^2}{2}(U_{j+1}^n + U_{j-1}^n - 2U_j^n)$$

periodic boundary condition

$$U_{j+n} = U_j \qquad j=1,2,\cdots$$

So

$$U_2^{n+1} = U_2^n - \frac{V}{2}(U_3^n - U_{n+1}^n) + \frac{V^2}{2}(U_3^n + U_n^n - 2U_2^n)$$

$$U_{n+1}^{n+1} = U_{n+1}^n - \frac{V}{2}(U_2^n - U_{n+1}^n) + \frac{V^2}{2}(U_2^n + U_n^n - 2U_{n+1}^n)$$

Up-wind Method.

$$U_j^{n+1} = U_j^n - V(U_j^n - U_{j-1}^n)$$

$$U_2^{n+1} = U_2^n - V(U_2^n - U_{n+1}^n)$$

② to satisfy the CFL condition

$$U_{j+p} \le U_j - \frac{k}{h} \le U_{j+q} -$$

$$-q \le V \le -p$$

For Lax-Wendroff method.

$$p = -1 \quad q = 1$$

So $\quad 0 \le V \le 1$

For up-wind method.

$$p = -1 \quad q = 0$$

$$0 \le V \le 1.$$

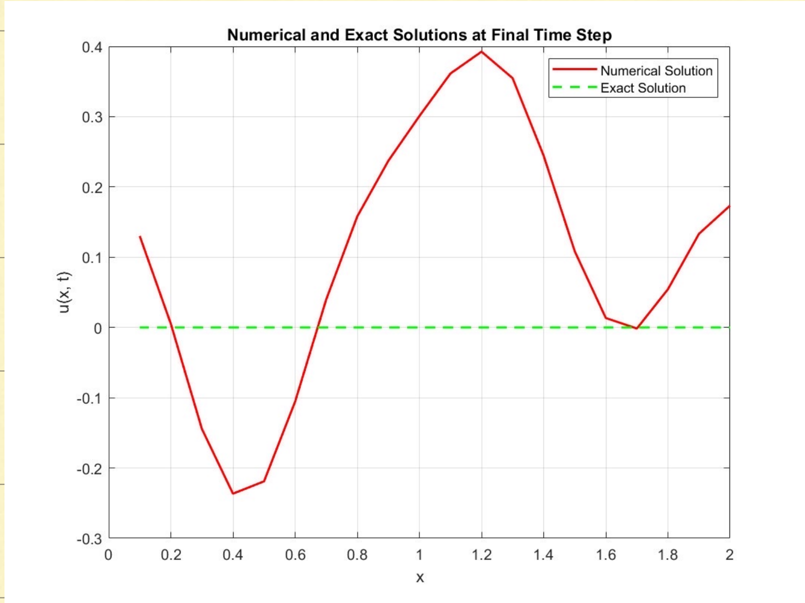
③ If $\quad U(x,0) = H(x-0.2)$
$\quad -H(x-0.4)$

For Lax-Wendroff method

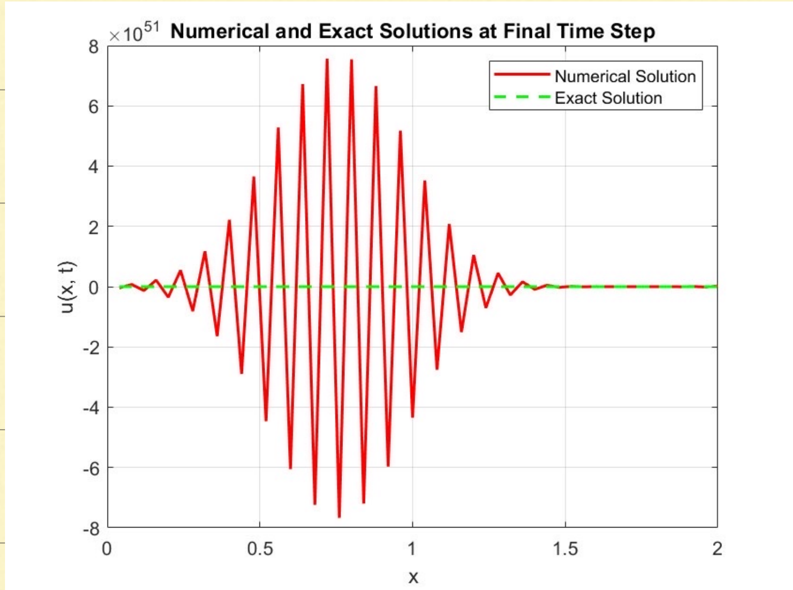
$$V = \frac{k}{h} = \frac{5}{K} \Big/ \frac{2}{N} = \frac{5}{2} \cdot \frac{N}{K}$$

If $0 < V < 1$ $\quad N=20 \quad K=200$

solution is.

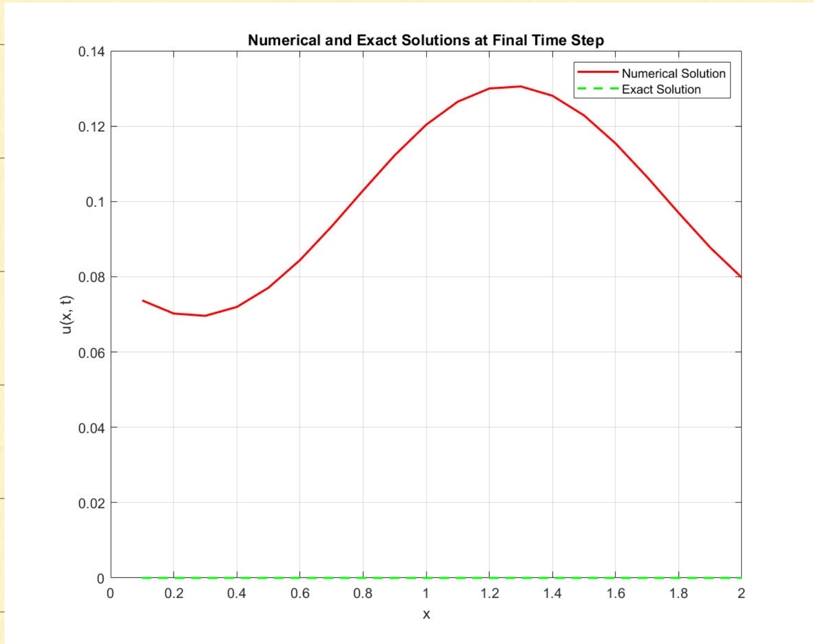


If $V > 1$ $\quad N=50, K=50$

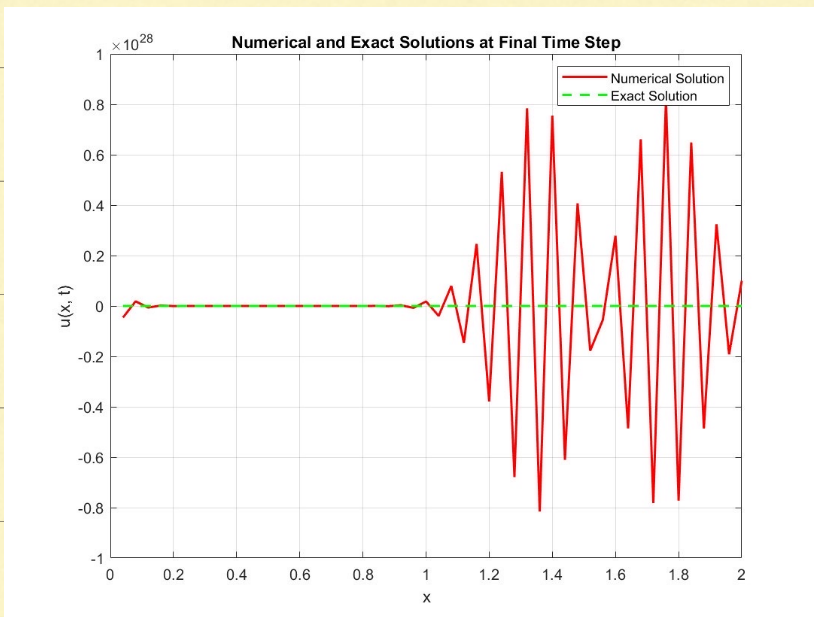


For up-wind method

$N=20 \quad K=200 \quad 0<V<1$



$N=K=50, \quad V>1.$ solution is.



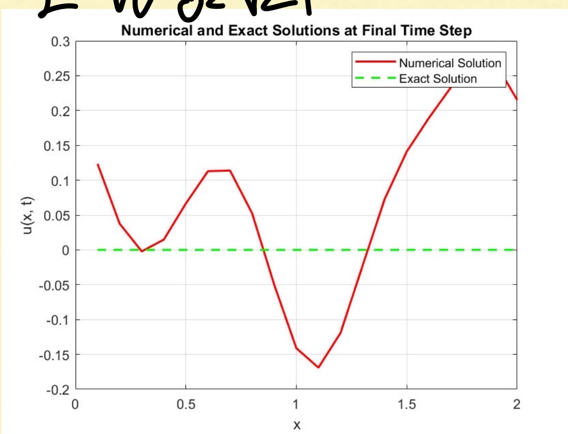
If $u(x,0) = e^{-10(4x-1)^2}$

similarly to above, we have.

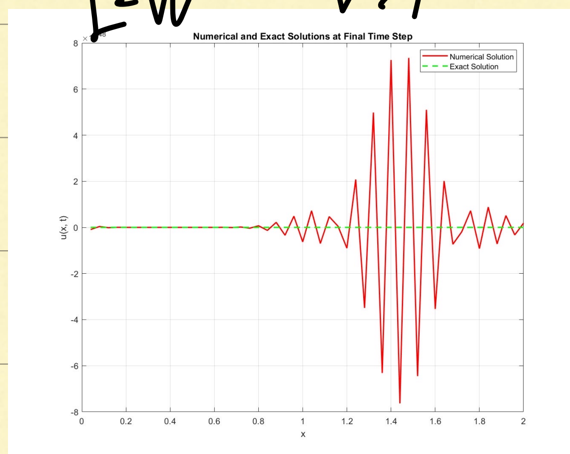
$K=200$, $N=20$ for $0<V<1$

$K=N=50$ for $V>1$. the solution

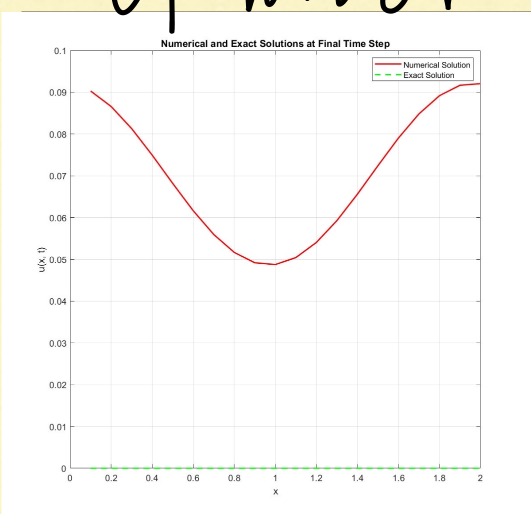
L-W $0<V<1$



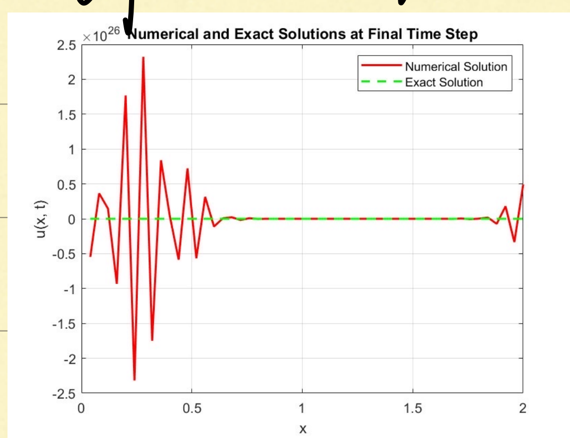
L-W $V>1$



Up-wind $0<V<1$



up wind $V>1$



For both initial condition, up-wind method has less error.

Q2.

The file CN and TRBFD are used to solve the Problem.

The two methods are unconditional stability So A-stability and L-stability.

For the accuracy of order.

$$e_{n-2} = C(h^q + k^p)$$

$$e_{n-1} = C(h^q + (\tfrac{k}{2})^p) \quad e_n = C(h^q + (\tfrac{k}{4})^p)$$

$$e_{n-2} - e_{n-1} = C\left(\left(\frac{k}{2}\right)^p - \left(\frac{k}{4}\right)^p\right)$$

$$\frac{e_{n-2} - e_{n-1}}{e_{n-1} - e_n} = \left(\frac{1}{2}\right)^p$$

So $$p = \frac{\log \Delta e_n - \log \Delta e_{n-1}}{\log k_n - \log k_{n-1}}$$

similarly, we can find

$$\tilde{e}_{n-2} = C(h^q + k^p)$$

$$\tilde{e}_{n-1} = C\left(k^p + \left(\frac{h}{2}\right)^q\right)$$

$$\tilde{e}_{n-2} = C\left(k^p + \left(\frac{h}{4}\right)^q\right)$$

So $$\frac{\tilde{e}_{n-2}-\tilde{e}_{n-1}}{\tilde{e}_{n-1}-\tilde{e}_{n}}=\left(\frac{1}{2}\right)^{q}$$

$$q=\frac{\log\Delta\tilde{e}_{n}-\log\Delta\tilde{e}_{n-1}}{\log\left(\frac{1}{2}\right)}$$

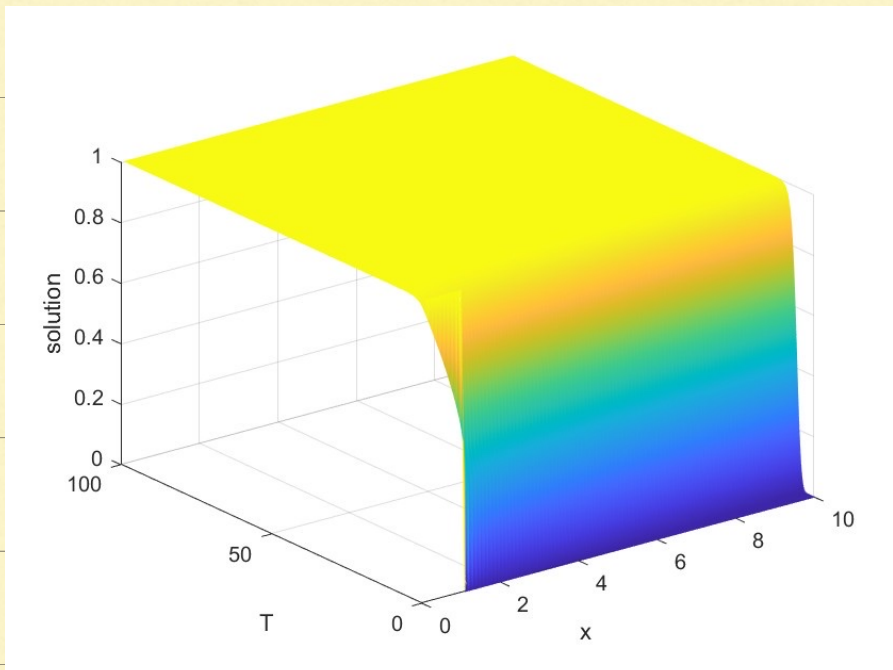
The fine accuracy and accuracy are used to calculate the order. for P, q

$K=100$ $N=1000$, we have $P=2$ for both method.

choose $K=1000$ $N=100$ we have $q=2$ for both method.

Q3.

(a) The file sol is used to solve this problem. choose $L=10$. $N=100$. the solution is.

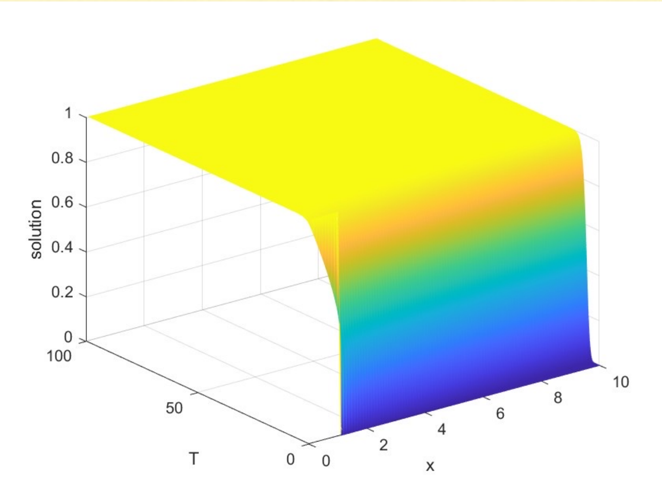


(b)

Choose initial condition

$$u_0(x) = 1 - H(x-1).$$

Solution is

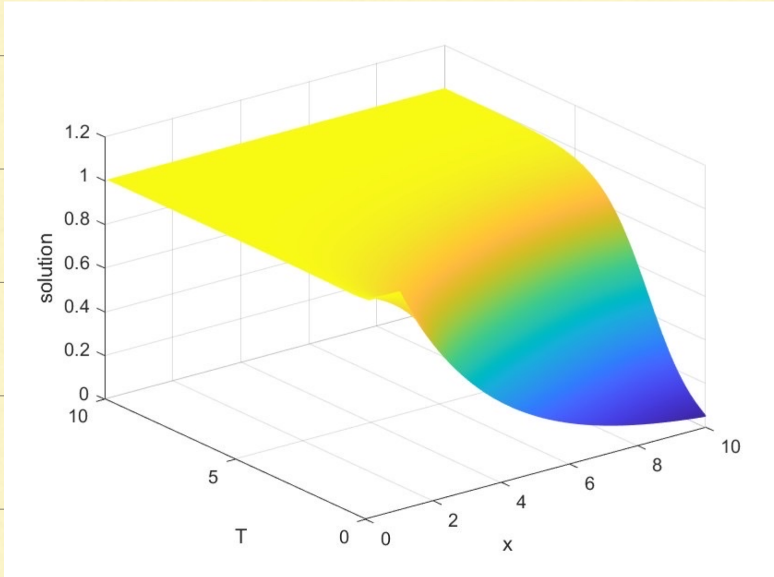


Choose initial condition

$$u_0(x) = \begin{cases} 1 & 0 < x < 1 \\ e^{-\frac{x-1}{3}} & x \geq 1 \end{cases}$$

run sol2 file $L=10$, $N=100$

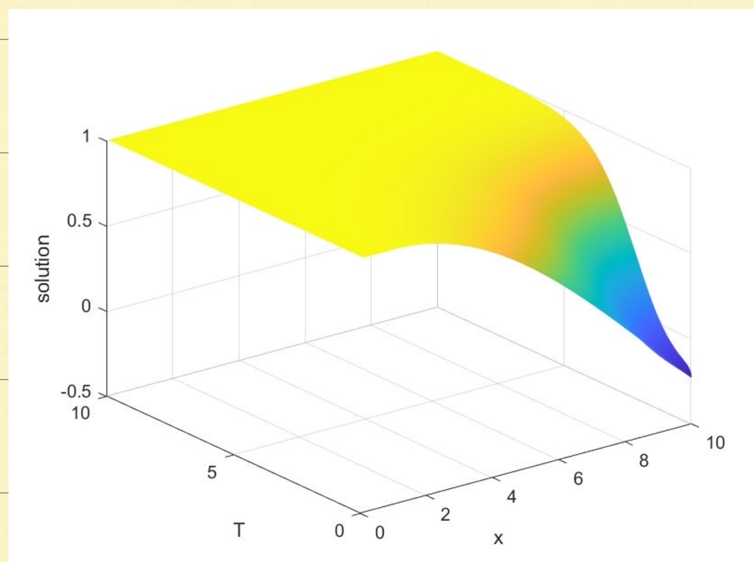
solution is



choose initinal condition

$$u(x,0) = \begin{cases} 1 & 0 < x < 1 \\ \cos\left(\frac{x-1}{5}\right) & x \geq 1 \end{cases}$$

solution is



For above solution, our
t from 0 to 100,
we check the solution
after $t=60$ and
compare with $u \equiv 1$,
run the file steadystatecheck
we find the maximum
error are 0.001, 0, 0.
So steady state solution
is $u \equiv 1$.

Q4.

a. for u_1

$$\frac{du_1}{dt} = a\frac{u_0+u_2-2u_1}{h^2} - c\frac{u_2-u_0}{2h}.$$

for u_N

$$\frac{du_N}{dt} = a\frac{u_{N-1}+u_{N+1}-2u_N}{h^2} - c\frac{u_{N+1}-u_{N-1}}{2h}.$$

$$u_0 = u_{N+1}$$

$$-3u_0+4u_1-u_2 = u_{N-1}-4u_N+3u_{N+1}$$

so $\quad 6u_{N+1} = 4u_N+4u_1-u_2-u_{N-1}$

$$u_0 = u_{N+1} = \frac{2u_N}{3} + \frac{2}{3}u_1 - \frac{u_2}{6} - \frac{u_{N-1}}{6}$$

$$\frac{du_1}{dt} = \frac{a}{h^2}\left(\frac{5}{6}u_2 + \frac{2}{3}u_N - \frac{4}{3}u_1 - \frac{u_{N-1}}{6}\right)$$
$$- \frac{c}{2h}\left(\frac{7}{6}u_2 - \frac{2}{3}u_N - \frac{2}{3}u_1 + \frac{u_{N-1}}{6}\right)$$

$$\frac{du_N}{dt} = \frac{a}{h^2}\left(\frac{5}{6}u_{N-1} - \frac{4}{3}u_N + \frac{2}{3}u_1 - \frac{u_2}{6}\right)$$
$$- \frac{c}{2h}\left(\frac{2}{3}u_N + \frac{2}{3}u_1 - \frac{u_2}{6} - \frac{7u_{N-1}}{6}\right)$$

for $2 \le j \le N-1$

$$\frac{du_j}{dt} = \frac{a}{h^2}(u_{j-1} + u_{j+1} - 2u_j) - \frac{c}{2h}(u_{j+1} - u_{j-1})$$

(b) For N,

$$A_1 = -2 \times \mathrm{diag}(\mathrm{ones}(N,1)) \cdot \frac{a}{h^2}$$

$$+ \mathrm{diag}(\mathrm{ones}(N-1,1), 1) \cdot \frac{a}{h^2}$$

$$+ \mathrm{diag}(\mathrm{ones}(N-1,1), -1) \cdot \frac{a}{h^2}.$$

then replace the first row

with $\frac{a}{h^2}\left(-\frac{4}{3}, \frac{5}{6}, \cdots, -\frac{1}{6}, \frac{2}{3}\right)$

replace the last row

with $\frac{a}{h^2}\left(\frac{2}{3}, -\frac{1}{6}, \cdots, \frac{5}{6}, -\frac{4}{3}\right)$

$$A_2 = \mathrm{diag}(\mathrm{zeros}(N,1))$$

$$+ \mathrm{diag}(-1 \times \mathrm{ones}(N-1,1), -1) \cdot \frac{c}{2h}$$

$$+ \mathrm{diag}(\mathrm{ones}(N-1,1), 1) \cdot \frac{c}{2h}.$$

then replace the first row

with. $\frac{c}{2h}\left(-\frac{2}{3}, \frac{7}{6}, \cdots, \frac{1}{6}, -\frac{2}{3}\right)$

the replace the last row

with $\frac{c}{2h}\left(\frac{2}{3}, -\frac{1}{6}, \cdots, -\frac{7}{6}, \frac{2}{3}\right)$

So $F(u) = (A_1 + A_2)\begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_N \end{pmatrix}.$

(c). $k = \frac{t}{K}$.

$$F = I - \frac{k}{2}A$$

$$B = I + \frac{k}{2}A.$$

$$F U_{j+1}^{n+1} = B U_j^n$$

$$\left(I - \frac{k}{2}A\right) U_{j+1}^{n+1} = \left(I + \frac{k}{2}A\right) U_j^n.$$

$$U_{j+1}^{n+1} = \left(I - \frac{k}{2}A\right)^{-1} \left(I + \frac{k}{2}A\right) U_j^n.$$