MAST90026 Computational Differential Equations: Week 9

Jesse Collis
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The University of Melbourne

Wave equation

Consider $u_{tt} = c^2 u_{xx}$. Define $v = u_t, w = -cu_x$ then

$$v_t = u_{tt} = c^2 u_{xx} = -cw_x,$$

and

$$v_t + cw_x = 0, \qquad w_t + cv_x = 0.$$

So if we let

$$\mathbf{u} = \left[\begin{array}{c} v \\ w \end{array} \right],$$

$$\mathbf{u}_t + \left[\begin{array}{cc} 0 & c \\ c & 0 \end{array} \right] \mathbf{u}_x = 0,$$

$$\mathbf{u}_t + A\mathbf{u}_x = 0, \qquad \text{(both linear)}.$$

The system is hyperbolic if A is diagonalisable with real eigenvalues, $\det(A - I\lambda) = 0$. e.g.:

$$A = \left[\begin{array}{cc} 0 & c \\ c & 0 \end{array} \right], \quad \Longrightarrow \quad \lambda = \pm c.$$

Advection equation

Initial boundary value problem(IBVP):

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad x \in (0, 1)$$

Initial condition: $\quad u(x, 0) = f(x)$

Boundary conditions: $\begin{cases} u(0, t) = g_0(t) \text{ if } c > 0 \\ u(1, t) = g_1(t) \text{ if } c < 0 \end{cases}$

Advection equation: solution

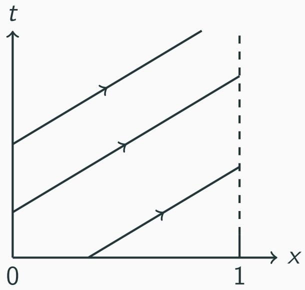
$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = \left(\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}\right)dt$$

If $\frac{dx}{dt} = c \Rightarrow x = ct + \xi$ Characteristics

$$\Downarrow$$

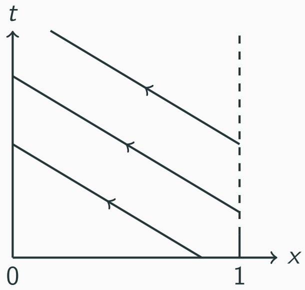
$$du = 0, \Rightarrow \underbrace{u(x,t) = f(\xi) = f(x - ct)}_{\text{General solution}}$$

Advection equation: solution when $c > 0$



$$u(x,t) = \begin{cases} f(x-ct), & \text{if } x - ct > 0 \\ g_0(t - x/c), & \text{if } x - ct < 0 \end{cases}$$

Advection equation: solution when $c < 0$



$$u(x,t) = \begin{cases} f(x - ct), & \text{if } x - ct < 1 \\ g_1(t - x/c), & \text{if } x - ct > 1 \end{cases}$$

Advection equation: Method of line

Consider $u_t + cu_x = 0$ with periodic BCs $u(0,t) = u(1,t)$.

BCs are $u_{N+1} = u_1$, and we need to solve for $u_2, \ldots, u_{N+1}$. Then using our MoL equations from before:

$$
\begin{aligned}
\dot{u}_j(t) &= -\tfrac{c}{2h}(u_{j+1} - u_{j-1}), \\
\dot{u}_2(t) &= -\tfrac{c}{2h}(u_3 - u_1) = -\tfrac{c}{2h}(u_3 - u_{N+1}), \qquad &&\text{(not solving for } u_1\text{)}, \\
\dot{u}_{N+1}(t) &= -\tfrac{c}{2h}(u_{N+2} - u_N) = -\tfrac{c}{2h}(u_2 - u_N), \qquad &&\text{(periodic BCs)}.
\end{aligned}
$$

with initial condition $\dot{u}_j(0) = f(x_j)$.

Advection equation: Method of line in matrix form

Matrix form:

$$\dot{\mathbf{u}} = A\mathbf{u},$$

where the matrix A is given by:

$$A = -\frac{c}{2h}\begin{bmatrix} 0 & 1 & \cdots & & & 0 & -1 \\ -1 & 0 & 1 & \cdots & & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 1 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Advection equation: eigenvalue of matrix

A is skew-symmetric: $A^T = -A$! Its eigenvalues are imaginary.

$$\lambda_p = -\frac{ic}{h}\sin(2\pi ph), \qquad p = 1, \ldots, N.$$

Starts small, goes up and down. $\lambda_p \in [-ic/h, ic/h]$.

For stability, we need to find time step, k, such that RAS for time stepper includes all λ_p.

For FTCS, there is no k which makes $|w^n|$ bounded as $n \to \infty$. FTCS is *never stable for the advection equation*.

Advection equation: leap frog method

Leap frog method:

$$\dot{u} = \frac{u^{n+1} - u^{n-1}}{2k}.$$

Its RAS is $|\mathfrak{Im}(z)| \leq 1$. Leapfrog will be stable provided

$$\lambda_k \in [-ick/h, ick/h] \subset [-i, i],$$

i.e. $\left|\frac{ck}{h}\right| < 1$.

Courant Number: $\frac{ck}{h} = \nu$. Condition on stability is $\nu < 1$.

Fully discrete leapfrog method (CTCS).

$$u_j^{n+1} = u_j^{n-1} - \tfrac{ck}{h}(u_{j+1}^n - u_{j-1}^n).$$

This is an explicit method, $O(h^2, k^2)$. Three time levels means we need a starting procedure and more memory.

Advection equation: Lax-Friedrichs Method (LxF)

For FTCS we had:

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n).$$

If we replace u_j^n with the average $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$ we get LxF:

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n).$$

To do stability analysis, need to write as

$$\dot{\mathbf{u}} = B\mathbf{u}$$

Advection equation: Lax-Friedrichs Method (LxF)

$\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$ can be rewritten as $u_j^n + \frac{1}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$, which is a diffusive (central difference of u_{xx}) term.

I.e., method adds artificial/numerical diffusion to stabilise the solution.

Approximation of time derivative at j:

$$u_j^{n+1} = u_j^n + \frac{1}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) - \tfrac{\nu}{2}(u_{j+1}^n - u_{j-1}^n)$$

In matrix form

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{k} = \tfrac{1}{2k}A_+\mathbf{u} - \tfrac{c}{2h}A_-\mathbf{u} = B\mathbf{u}.$$

where A_+ is symmetric and A_- is skew-symmetric.

Advection equation: Lax-Friedrichs Method (LxF)

Eigenvalues of B: $-\frac{ic}{h}\sin(2\pi ph) - \frac{1}{k}(1 - cos(2\pi ph)) = \lambda_p$.

At time step k, $k\lambda_p = -i\nu \sin(2\pi ph) - (1 - cos(2\pi ph)$ which lie on an ellipse.

Since p is a real number,

$$k\lambda_p = -\underbrace{i\nu \sin(2\pi ph)}_{\text{imaginary}} - \underbrace{(1 - cos(2\pi ph))}_{\text{real}}.$$

So $k\lambda_p$ in RAS if $|k\lambda_p + 1| < 1$, i.e. $\nu^2 \sin^2(2\pi ph) + \cos^2(2\pi ph) < 1$ $\Rightarrow |\nu| < 1$.

LxF is stable iff $|\nu| < 1$. But it is only 1st order in time, LTE $= \mathcal{O}(k, h^2)$.

Advection equation: Lax-Wendroff Method (LxW)

To get a 2nd order method, use Taylor series expansion on PDE

$$u(x,t+k) = u(x,t) + ku_t(x,t) + \tfrac{1}{2}k^2 u_{tt}(x,t) + O(k^3),$$

$$u_t = -cu_x, \qquad u_{tt} = c^2 u_{xx},$$

$$\implies \quad u(x,t+k) = u(x,t) - kc \underbrace{u_x}_{\text{CD}} + \frac{k^2c^2}{2} \underbrace{u_{xx}}_{\text{CD}} + O(k^3),$$

$$\implies \quad u_j^{n+1} = u_j^n - \tfrac{\nu}{2}(u_{j+1}^n - u_{j-1}^n) + \tfrac{\nu^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

$\dot{\mathbf{u}} = B\mathbf{u}$ evolves the system and:

$$k\lambda_p = -i\left(\tfrac{ck}{h}\right)\sin(\pi ph) + \left(\tfrac{ck}{h}\right)^2(\cos(\pi ph) - 1), \qquad p = 1, \dots, N.$$

It is stable if $|\nu| \leq 1$ and LTE $= \mathcal{O}(k^2, h^2)$.

Advection equation: upwind method

The advection equation has a direction:

- If $c > 0$, information comes from the left.
- If $c < 0$, information comes from the right.

Use 1-sided FD, depending on sign of c. This gives us "upwind methods": use FTBS for $c > 0$ and use FTFS for $c < 0$.

$$u_j^{n+1} = \begin{cases} u_j^n - \nu(u_j^n - u_{j-1}^n), & \text{if } c > 0 \\ u_j^n - \nu(u_{j+1}^n - u_j^n), & \text{if } c < 0 \end{cases}$$

Advection equation: stability of upwind method

FTBS stable for $0 < \nu < 1$ ($c > 0$ only).

FTFS stable for $-1 < \nu < 0$ ($c < 0$ only).

FTBS and FTFS have LTE $= O(k, h)$.

Even though lower accuracy than Lax-Wendroff Method it can often perform better if discontinuities are present in solution.

Advection equation: Beam-Warming method

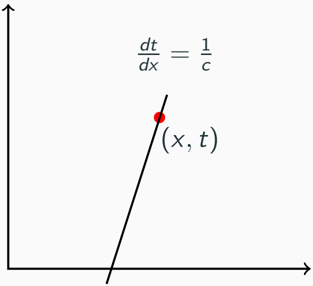
To get 2nd order method, use 1-sided 2nd order FD in Lax-Wendroff method,

$$u_j^{n+1} = \begin{cases} u_j^n - \frac{\nu}{2}(3u_j^n - 4u_{j-1}^n + u_{j-2}^n) + \frac{\nu^2}{2}(u_j^n - 2u_{j-1}^n + u_{j-2}^n), & \text{if } c > 0 \\ u_j^n - \frac{\nu}{2}(-3u_j^n + 4u_{j+1}^n - u_{j+2}^n) + \frac{\nu^2}{2}(u_j^n - 2u_{j+1}^n + u_{j+2}^n), & \text{if } c < 0 \end{cases}$$

Stable when $0 \leq \nu \leq 2$ if $c > 0$ and $-2 \leq \nu \leq 0$ if $c \leq 0$.

Advection equation: analytical domain of dependence

Domain of dependence of analytical solution: $D \subseteq \mathbb{R}$ is the smallest region such that $u(x, t)$ depends only on the values of $f(x)$ where $x \in D$.



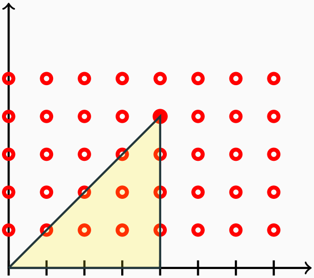
Analytic

We therefore have $D = \{x - ct\}$; i.e., solution at $u(x, t)$ only depends on $f(x - ct)$ (because all information travels along a characteristic with no diffusion).

Advection equation: numerical domain of dependence

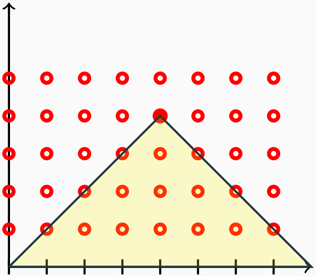
Domain of dependence of numerical solution: $D_h \subseteq \mathbb{R}$ is the smallest region such that u_j^n depends only on the values of $f(x_j)$ where $x_j \in D_h$.

Upwind scheme



Numerical

LxW scheme



Numerical

Advection equation: numerical domain of dependence

Define $r = k/h$. From the numerical schemes we can calculate

$$D_h = \left(X + p\frac{T}{r}, X + q\frac{T}{r}\right).$$

LxW has $p = -1, q = 1$ so domain of dependence is

$$D_h = \left(X - \frac{T}{r}, X + \frac{T}{r}\right).$$

UW ($c > 0$) has $p = -1, q = 0$ so domain of dependence is

$$D_h = \left(X - \frac{T}{r}, X\right).$$

Advection equation: CFL condition

CFL (Courant-Friedrichs-Lewy) condition: Necessary condition for convergence as $h, k \to 0$ of a numerical method for an advection equation is that

$$D \subseteq D_h$$

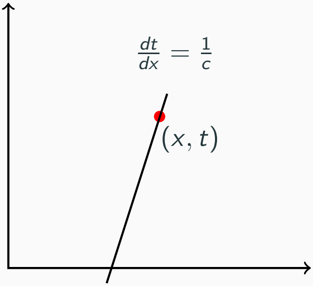
If $D \nsubseteq D_h$ then information from the initial condition would not propagate forwards to the correct location.

For UW and LxW we have

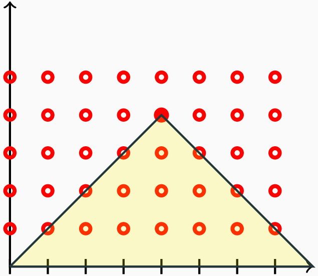
$$\begin{aligned} & X - cT \in [X + pT/r, X + qT/r] \\ \Rightarrow \quad & pT/r \leq -cT \leq qT/r \\ \Rightarrow \quad & -q \leq cr = \nu \leq -p \end{aligned}$$

I.e., CFL gives same condition as stability.

Visualisation of CFL condition

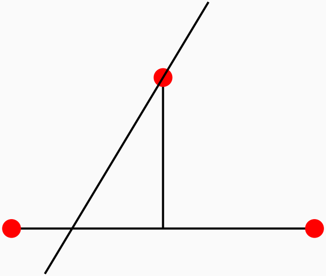


Analytic



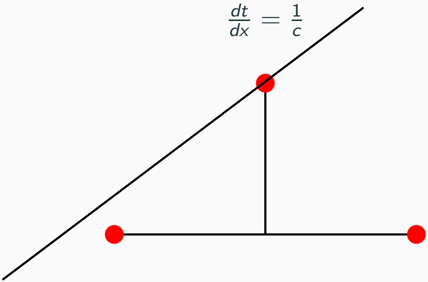
Numerical

Advection equation: CFL condition for LxW scheme



$\nu < 1$

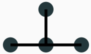
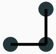
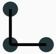
Stable



$\nu > 1$

Unstable

Advection equation: examples of CFL condition

LxF		$p = -1, q = 1$	$-1 \leq \nu \leq 1$
LxW		$p = -1, q = 1$	$-1 \leq \nu \leq 1$
UW		$p = -1, q = 0$	$0 \leq \nu \leq 1$
UW		$p = 0, q = 1$	$-1 \leq \nu \leq 0$
BW		$p = -2, q = 0$	$0 \leq \nu \leq 2$
BW		$p = 0, q = 2$	$-2 \leq \nu \leq 0$

But not sufficient

FTCS  $\Rightarrow |\nu| \leq 1$ from CFL condition but not stable for any ν

End of week 9!