Q1) 1.5 marks

Integration by parts on the BVP leads to

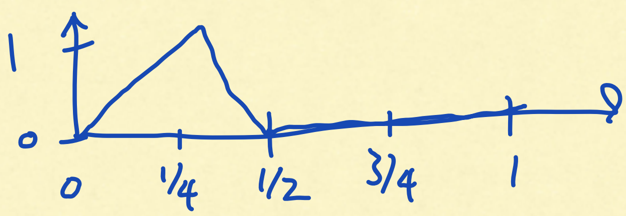
$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 v\,dx$$

as $v(0) = v(1) = 0$.

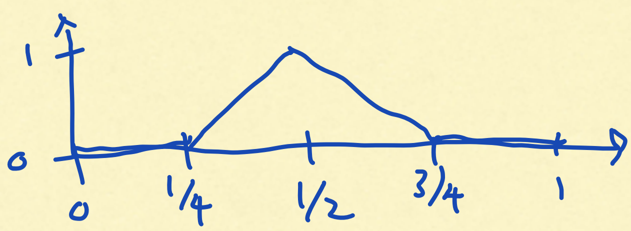
Now $V = \{\phi_2, \phi_3, \phi_4\}$, I've relabelled from 2-4 to be consistent with lecture notes.

where

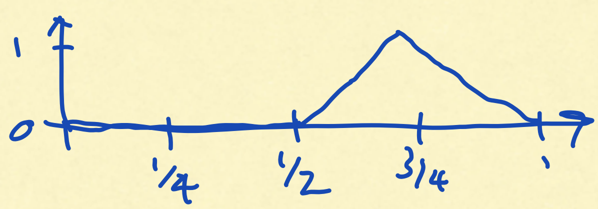
ϕ_2:



ϕ_3:



ϕ_4:



E_1 E_2 E_3 E_4

We thus have

$$\int_0^1 u' \phi_i' \, dx + \int_0^1 u \phi_i \, dx = \int_0^1 \phi_i \, dx \qquad i = \{2, 3, 4\}$$

Now $u = u_h = \sum_{j=2}^{4} \alpha_j \phi_j$ (due to homogeneous BC's)

$$\int_0^1 \sum_{j=2}^{4} \alpha_j \phi_j' \phi_i' \, dx + \int_0^1 \sum_{j=2}^{4} \alpha_j \phi_j \phi_i \, dx = \int_0^1 \phi_i \, dx$$

Now based on where the hat functions overlap we can write the linear system as

$$\alpha_2 \int_0^1 \phi_2'^2 \, dx + \alpha_3 \int_0^1 \phi_2' \phi_3' \, dx + \alpha_2 \int_0^1 \phi_2^2 \, dx + \alpha_3 \int_0^1 \phi_2 \phi_3 \, dx = \int_0^1 \phi_2 \, dx$$

$$\alpha_2 \int_0^1 \phi_2' \phi_3' \, dx + \alpha_3 \int_0^1 \phi_3'^2 \, dx + \alpha_4 \int_0^1 \phi_3' \phi_4' \, dx + \alpha_2 \int_0^1 \phi_2 \phi_3 \, dx + \alpha_3 \int_0^1 \phi_3^2 \, dx + \alpha_4 \int_0^1 \phi_3 \phi_4 \, dx = \int_0^1 \phi_3 \, dx$$

$$\alpha_3 \int_0^1 \phi_3' \phi_4' \, dx + \alpha_4 \int_0^1 \phi_4'^2 \, dx + \alpha_3 \int_0^1 \phi_3 \phi_4 \, dx + \alpha_4 \int \phi_4^2 \, dx = \int_0^1 \phi_4 \, dx$$

Splitting into an integration over elements we have

$$\alpha_2 \int_{E_1} \phi_2'^2 \, dx + \alpha_2 \int_{E_2} \phi_2'^2 \, dx + \alpha_3 \int_{E_2} \phi_2' \phi_3' \, dx$$
$$+ \alpha_2 \int_{E_1} \phi_2^2 \, dx + \alpha_2 \int_{E_2} \phi_2^2 \, dx + \alpha_3 \int_{E_2} \phi_2 \phi_3 \, dx = \int_{E_1} \phi_2 \, dx + \int_{E_2} \phi_2 \, dx$$

$$\alpha_2 \int_{E_2} \phi_2' \phi_3' \, dx + \alpha_3 \int_{E_2} \phi_3'^2 \, dx + \alpha_3 \int_{E_3} \phi_3'^2 \, dx + \alpha_4 \int_{E_3} \phi_3' \phi_4' \, dx$$
$$+ \alpha_2 \int_{E_2} \phi_2 \phi_3 \, dx + \alpha_3 \int_{E_2} \phi_3^2 \, dx + \alpha_3 \int_{E_3} \phi_3^2 \, dx + \alpha_4 \int_{E_3} \phi_3 \phi_4 \, dx$$
$$= \int_{E_2} \phi_3 \, dx + \int_{E_3} \phi_3 \, dx$$

$$\alpha_3 \int_{E_3} \phi_3' \phi_4' \, dx + \alpha_4 \int_{E_3} \phi_4'^2 \, dx + \alpha_4 \int_{E_4} \phi_4'^2 \, dx$$
$$+ \alpha_3 \int_{E_3} \phi_3 \phi_4 \, dx + \alpha_4 \int_{E_3} \phi_4^2 \, dx + \alpha_4 \int_{E_4} \phi_4^2 \, dx = \int_{E_3} \phi_4 \, dx + \int_{E_4} \phi_4 \, dx$$

Putting it all together into Matrix form gives

$$\underline{\underline{A}} \, \underline{\alpha} = \underline{f}$$

where $\underline{\underline{A}} = \underline{\underline{K}} + \underline{\underline{M}}$

and $$\underline{\underline{K}} = \underline{\underline{K}}_{\epsilon_1} + \underline{\underline{K}}_{\epsilon_2} + \underline{\underline{K}}_{\epsilon_3} + \underline{\underline{K}}_{\epsilon_4}$$

$$\underline{\underline{M}} = \underline{\underline{M}}_{\epsilon_1} + \underline{\underline{M}}_{\epsilon_2} + \underline{\underline{M}}_{\epsilon_3} + \underline{\underline{M}}_{\epsilon_4}$$

$$\underline{f} = \underline{f}_{\epsilon_1} + \underline{f}_{\epsilon_2} + \underline{f}_{\epsilon_3} + \underline{f}_{\epsilon_4}$$

$$\underline{\underline{K}}_{\epsilon_1} = \begin{bmatrix} \int_{\epsilon_1} \phi_2' \, dx & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\underline{K}}_{\epsilon_2} = \begin{bmatrix} \int_{\epsilon_2} \phi_2'^2 \, dx & \int_{\epsilon_2} \phi_2' \phi_3' \, dx & 0 \\ \int_{\epsilon_2} \phi_2' \phi_3' \, dx & \int_{\epsilon_2} \phi_3'^2 \, dx & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{K}}_{\epsilon_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \int_{\epsilon_3} \phi_3' \, dx & \int_{\epsilon_3} \phi_3' \phi_4' \, dx \\ 0 & \int_{\epsilon_3} \phi_3' \phi_4' \, dx & \int_{\epsilon_3} \phi_4'^2 \, dx \end{bmatrix} \qquad \underline{\underline{K}}_{\epsilon_4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \int_{\epsilon_4} \phi_4'^2 \, dx \end{bmatrix}$$

$$\underline{\underline{M}}_{\epsilon_1} = \begin{bmatrix} \int_{\epsilon_1} \phi_2^2 \, dx & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\underline{M}}_{\epsilon_2} = \begin{bmatrix} \int_{\epsilon_2} \phi_2^2 \, dx & \int_{\epsilon_2} \phi_2 \phi_3 \, dx & 0 \\ \int_{\epsilon_2} \phi_2 \phi_3 \, dx & \int_{\epsilon_2} \phi_3^2 \, dx & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{M}}_{\epsilon_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \int_{\epsilon_3} \phi_3 \, dx & \int_{\epsilon_3} \phi_3 \phi_4 \, dx \\ 0 & \int_{\epsilon_3} \phi_3 \phi_4 \, dx & \int_{\epsilon_3} \phi_4^2 \, dx \end{bmatrix} \qquad \underline{\underline{M}}_{\epsilon_4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \int_{\epsilon_4} \phi_4^2 \, dx \end{bmatrix}$$

$$\underline{f}_{\epsilon_1} = \begin{bmatrix} \int_{\epsilon_1} \phi_2 \, dx \\ 0 \\ 0 \end{bmatrix} \quad \underline{f}_{\epsilon_2} = \begin{bmatrix} \int_{\epsilon_2} \phi_2 \, dx \\ \int_{\epsilon_2} \phi_3 \, dx \\ 0 \end{bmatrix} \quad \underline{f}_{\epsilon_3} = \begin{bmatrix} 0 \\ \int_{\epsilon_3} \phi_3 \, dx \\ \int_{\epsilon_3} \phi_4 \, dx \end{bmatrix} \quad \underline{f}_{\epsilon_4} = \begin{bmatrix} 0 \\ 0 \\ \int_{\epsilon_4} \phi_4 \, dx \end{bmatrix}$$

Using the <u>trapezoidal rule</u> the integrals evaluate as

$$\int_{\epsilon_i} \phi_i'^2 \, dx = \frac{1}{h} \int_0^1 N_1'^2 \, d\xi = \frac{1}{h}$$

$$\int_{\epsilon_i} \phi_{i+1}' \, dx = \frac{1}{h} \int_0^1 N_2'^2 \, d\xi = \frac{1}{h}$$

$$\int_{\epsilon_i} \phi_i' \phi_{i+1}' \, dx = \frac{1}{h} \int_0^1 N_1' N_2' \, d\xi = \frac{-1}{h}$$

$$\int_{\epsilon_i} \phi_i^2 \, dx = h \int_0^1 N_1^2 \, d\xi \approx \frac{h}{2} \left(N_1^2(0) + N_1^2(1) \right) = \frac{h}{2}$$

$$\int_{\epsilon_i} \phi_{i+1}^2 \, dx = h \int_0^1 N_2^2 \, d\xi \approx \frac{h}{2}$$

$$\int_{\epsilon_i} \phi_i \phi_{i+1} \, dx \approx 0$$

$$\int_{\epsilon_i} \phi_i \, dx = \frac{h}{2}$$

$$\int_{\epsilon_i} \phi_{i+1} \, dx = \frac{h}{2}$$

Putting it all together

$$\underline{\underline{K}}_{\epsilon_1} = \frac{1}{h}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\underline{K}}_{\epsilon_2} = \frac{1}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\underline{K}}_{\epsilon_3} = \frac{1}{h}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underline{\underline{K}}_{\epsilon_4} = \frac{1}{h}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{\underline{M}}_{\epsilon_1} = \frac{h}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\underline{M}}_{\epsilon_2} = \frac{h}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{M}}_{\epsilon_3} = \frac{h}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{\underline{M}}_{\epsilon_4} = \frac{h}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{f}_{\epsilon_1} = \frac{h}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{f}_{\epsilon_2} = \frac{h}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \underline{f}_{\epsilon_3} = \frac{h}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \underline{f}_{\epsilon_4} = \frac{h}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \underline{\underline{K}} = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\underline{\underline{M}} = h\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Your result may look different to this if you used a quadratically accurate quadrature scheme. Either is fine!

$$\underline{f} = h\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Rearranging we see that this system is

$$\left(\frac{1}{h^2}\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\underline{\alpha} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

which is exactly the same as the central finite difference scheme on this grid!

Q2) 1 marks

Starting with

$$-(D(x)u')' + q(x)u = f(x)$$

we make the residual orthogonal to all functions in H' with respect to the inner product

$$\langle f, g\rangle = \int_a^b f g \, dx$$

$$\langle -(D(x)u')', v\rangle + \langle q(x)u, v\rangle = \langle f, v\rangle$$

Upon integration by parts we have

$$\langle D(x)u', v'\rangle + \langle q(x)u, v\rangle = \langle f, v\rangle + D(b)u'(b)v(b) - D(a)u'(a)v(a)$$

Now since we know $u(b)$, we restrict our test space to have $v(b)=0$

We also have $u'(a)=u(a)-\alpha$

$$\Rightarrow \langle D(x)u',v'\rangle+\langle q(x)u,v\rangle=\langle f,v\rangle-D(a)(u(a)-\alpha)v(a)$$

let $u=\phi_0+\sum_{i=1}^{N}c_i\phi_i$ where $\phi_0(b)=\beta$, $\phi_0(a)=0$

$\phi_i(b)=0$ for $i=1,\dots,N$

Note the ϕ_i's for $i=1,\dots,N$ are the trial and test spaces.

The Galerkin equations are therefore

$$\langle D(x)\sum_{i=1}^{N}c_i\phi_i',\phi_j'\rangle+\langle q(x)\sum_{i=1}^{N}c_i\phi_i,\phi_j\rangle+D(a)\sum_{i=1}^{N}c_i\phi_i(a)\phi_j(a)$$

$$=\langle f,\phi_j\rangle+D(a)\alpha\phi_j(a)-\langle D(x)\phi_0',\phi_j'\rangle-\langle q(x)\phi_0,\phi_j\rangle$$

for $j=1,\dots,N$

Q3) 2.5 marks