Department of Mathematics and Statistics
MAST90026 Computational Differential Equations
2024

Assignment 3: Parabolic/hyperbolic PDEs
Due: 11AM Wednesday 15th May.

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Question 1

① Lax - Wendroff Method

$$u_j^{n+1} = u_j^n - \frac{V}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{V^2}{2}(u_{j+1}^n + u_{j-1}^n - 2u_j^n)$$

$u_{j+n} = u_j \quad j = 1, 2, 3 \cdots$ boundary condition

$$\therefore u_2^{n+1} = u_2^n - \frac{V}{2}(u_3^n - u_{n+1}^n) + \frac{V^2}{2}(u_3^n + u_n^n - 2u_2^n)$$

$$u_{n+1}^{n+1} = u_{n+1}^n - \frac{V}{2}(u_2^n - u_{n+1}^n) + \frac{V^2}{2}(u_2^n + u_n^n - 2u_{n+1}^n)$$

up-wind method.

$$u_j^{n+1} = u_j^n - V(u_j^n - u_{j-1}^n)$$

$$u_2^{n+1} = u_2^n - V(u_2^n - u_{n+1}^n)$$

② To satisfy the CFL condition

$$u_{j+p} \le u_j - \frac{k}{h} \le u_{j+q}$$

$$-q \le V \le -p$$

for Lax-wendroff method: $p = -1 \quad q = 1$ so $0 \le V \le 1$

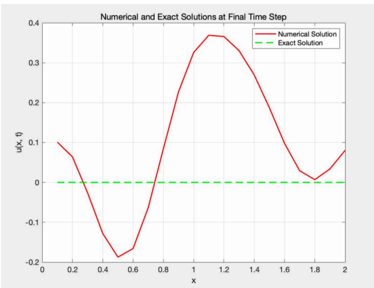
for up-wind method: $p = -1 \quad q = 0 \quad 0 \le V \le 1$

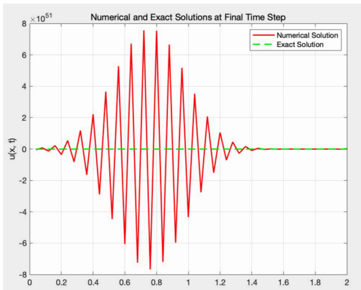
③ If $u(x, 0) = H(x - 0.2) - H(x - 0.4)$

for Lax-Wendroff method $V = \frac{k}{h} = \frac{5}{k} / \frac{2}{N} = \frac{5}{2} \cdot \frac{N}{k}$

run file LWM.m

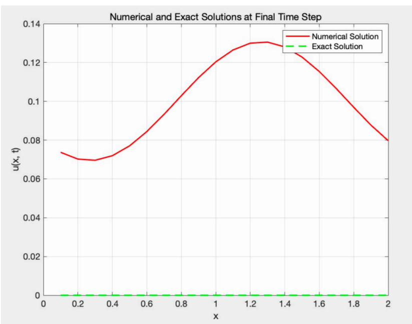
if $N = 20 \quad k = 200$ — if $N = 50, \ k = 50$

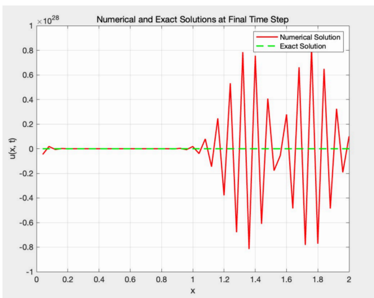




For up-wind method, $N = 20, k = 200, 0 < V < 1$ — $N = 50 \quad k = 50 \quad V > 1$

run file UP.m



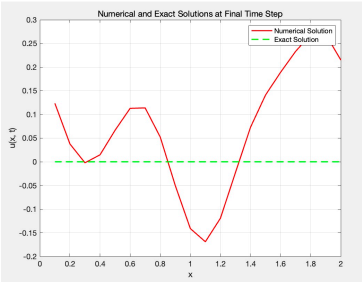


If $u(x,0) = e^{-10(4x-1)^2}$, similarly to above, we have $K=200$, $N=20$ for $0<v<1$

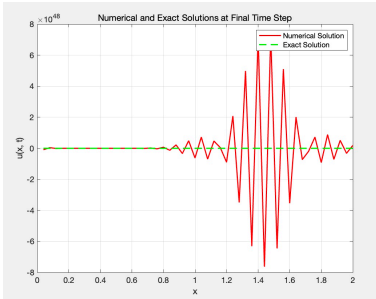
$K=50$, $N=50$ for $v>1$, the solution =

Run file LWM1.m

Lax-Wendroff method

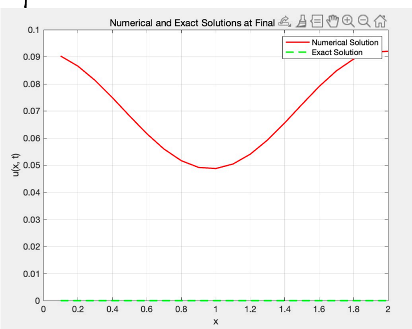


Lax-Wendroff method

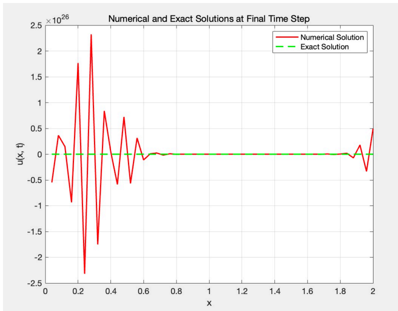


Run file UP1.m

up-wind method



up-wind method



For both initial condition, up-wind method has less error.

Question 2.

The file CN.m is used to solve the problem.
The file TRBFD.m is used to solve the problem.

The 2 methods are unconditional stability so A-stability and L-stability.
For the acceracy of order.

$e_{n-2} = C(h^q + k^p)$

$e_{n-1} = C(h^q + (\frac{k}{2})^p)$ $e_n = C(h^q + (\frac{k}{4})^p)$

$e_{n-2} - e_{n-1} = C((\frac{k}{2})^p - (\frac{k}{4})^p)$

$\frac{e_{n-2} - e_{n-1}}{e_{n-1} - e_n} = (\frac{1}{2})^p$

so $p = \frac{\log \Delta e_n - \log \Delta e_{n-1}}{\log k_n - \log k_{n-1}}$

similarly, we can find :

$\tilde{e}_{n-2} = C(h^q + k^p)$

$\tilde{e}_{n-1} = C(k^p + (\frac{h}{2})^q)$

$\tilde{e}_{n-2} = C(k^p + (\frac{h}{4})^q)$

so $\frac{\tilde{e}_{n-2} - \tilde{e}_{n-1}}{\tilde{e}_{n-1} - \tilde{e}_n} = (\frac{1}{2})^q$ $q = \frac{\log \Delta \tilde{e}_n - \log \Delta \tilde{e}_{n-1}}{\log(\frac{1}{2})}$

The file accuracy.m and accuracy2.m are used to calculate the order for p.q.

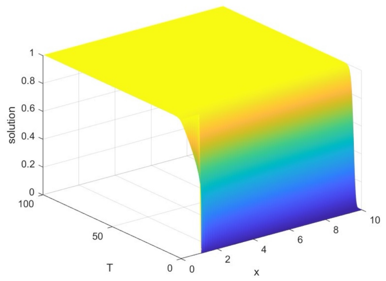
$k=100$ $N=1000$ we have $p=2$

for both method choose $k=1000$ $N=100$ $q=2$

Question 3.

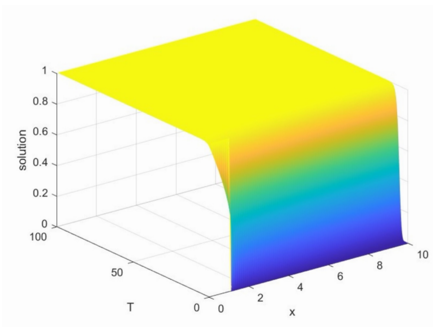
(a) The file sol.m is used to solve this problem.

choose $L=10$ $N=100$ The solution is:



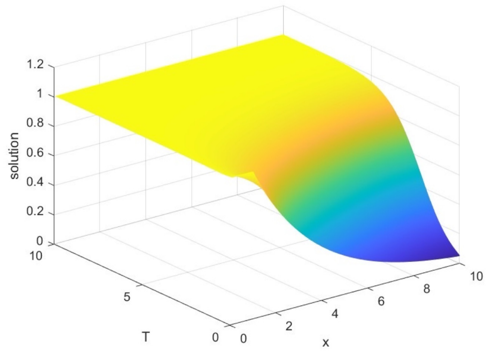
(b) choose initial condition

$u_0(x) = 1 - H(x-1)$ The solution is:

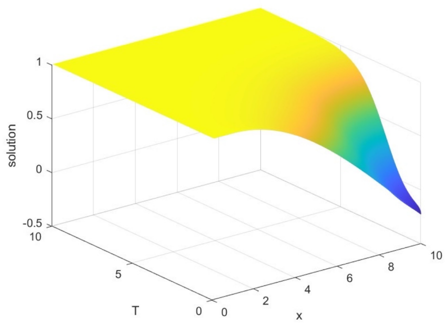


choose initial condition $u_0(x) = \begin{cases} 1 & 0<x<1 \\ e^{-\frac{x-1}{3}} & x \geq 1 \end{cases}$

run file sol2.m $L=10$, $N=100$



choose initial condition, $u(x,0) = \begin{cases} 1 & 0<x<1 \\ \cos(-\frac{x-1}{5}) & x \geq 1 \end{cases}$



For above solution, our t from 0 to 100, we check the solution after $t=60$ and compare with $u=1$, run the file steadystatecheck.m we find the maximum error are 0.001, 0, 0. so steadystate solution is $u=1$.

Question 4.

(a) for u_1 $\quad \frac{du_1}{dt} = a\frac{u_0+u_2-2u_1}{h^2} - c\frac{u_2-u_0}{2h}$

for u_N $\quad \frac{du_N}{dt} = a\frac{u_{N-1}+u_{N+1}-2u_N}{h^2} - c\frac{u_{N+1}-u_{N-1}}{2h}$

$u_0 = u_{N+1}$

$-3u_0+4u_1-2u_2 = u_{N+1}-4u_N+3u_{N+1}$

so $6u_{N+1} = 4u_N+4u_1-u_2-u_{N+1}$

$u_0 = u_{N+1} = \frac{2u_N}{5} + \frac{2}{3}u_1 - \frac{u_2}{6} - \frac{u_{N-1}}{6}$

$\frac{du_1}{dt} = \frac{a}{h^2}\left(\frac{5}{6}u_2+\frac{2}{3}u_N-\frac{4}{3}u_1-\frac{u_{N-1}}{6}\right)$

$\quad - \frac{c}{2h}\left(\frac{7}{6}u_2-\frac{2}{3}u_N-\frac{2}{3}u_1+\frac{u_{N-1}}{6}\right)$

$\frac{du_N}{dt} = \frac{a}{h^2}\left(\frac{5}{6}u_{N-1}-\frac{4}{3}u_N+\frac{2}{3}u_1-\frac{u_2}{6}\right)$

$\quad - \frac{c}{2h}\left(\frac{2}{3}u_N+\frac{2}{3}u_1-\frac{u_2}{6}-\frac{7u_{N-1}}{6}\right)$

for $2 \le j \le N-1$

$\frac{du_j}{dt} = \frac{a}{h^2}(u_{j-1}+u_{j+1}-2u_j) - \frac{c}{2h}(u_{j+1}-u_{j-1})$

(b) For N,

$A_1 = -2\times \text{diag}(\text{ones}(N,1))\cdot\frac{a}{h^2} + \text{diag}(\text{ones}(N-1,1),-1)\cdot\frac{a}{h^2}$

$\quad + \text{diag}(\text{ones}(N-1,1),-1)\cdot\frac{a}{h^2}$

then replace the first row with $\frac{a}{h^2}\left(-\frac{4}{3}, \frac{5}{6}, \cdots, -\frac{1}{6}, \frac{2}{3}\right)$

replace the last row with $\frac{a}{h^2}\left(\frac{2}{3}, -\frac{1}{6}, \cdots, \frac{5}{6}, -\frac{4}{3}\right)$

$A_2 = \text{diag}(\text{zeros}(N,1)) + \text{diag}(-1\times\text{ones}(N-1,1),-1)\cdot\frac{c}{2h}$

$\quad + \text{diag}(\text{ones}(N-1,1),1)\cdot\frac{c}{2h}$

then replace the first row with $\frac{c}{2h}\left(-\frac{2}{3}, \frac{7}{6}, \cdots \frac{1}{6}, -\frac{2}{3}\right)$

the replace the last row with $\frac{c}{2h}\left(\frac{2}{3}, -\frac{1}{6}, \cdots, -\frac{7}{6}, \frac{2}{3}\right)$

so $F(u) = (A_1+A_2)\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$

(c) $k = \frac{t}{k}$

$F = I - \frac{k}{2}A$

$B = I + \frac{k}{2}A$

$F u_{j+1}^{n+1} = B u_j^n$

$(I - \frac{k}{2}A) u_{j+1}^{n+1} = (I + \frac{k}{2}A) u_j^n$

$u_{j+1}^{n+1} = (I - \frac{k}{2}A)^{-1} (I + \frac{k}{2}A) u_j^n$