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Precise analytic approximations for the Bessel function $J_1(x)$

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A B S T R A C T

Precise and straightforward analytic approximations for the Bessel function $J_1(x)$ have been found. Power series and asymptotic expansions have been used to determine the parameters of the approximation, which is as a bridge between both expansions, and it is a combination of rational and trigonometric functions multiplied with fractional powers of x. Here, several improvements with respect to the so called Multipoint Quasirational Approximation technique have been performed. Two procedures have been used to determine the parameters of the approximations. The maximum absolute errors are in both cases smaller than 0.01. The zeros of the approximation are also very precise with less than 0.04 per cent for the first one. A second approximation has been also determined using two more parameters, and in this way the accuracy has been increased to less than 0.001.

Introduction

The Bessel functions $J_\nu(x)$ and in particular $J_1(x)$ are very important in several areas of Physics [1–7]. The power series of $J_1(x)$ is well known and its radius of convergence is infinite, that is, it is an entire function, however a lot of terms of this series are usually necessary to obtain a good approximation for large values of the variable x. Asymptotic expansions are also well known, which give good accuracy for very large values of x, but they fail for values of x in the intermediate region, which are important in the applications. Furthermore some polynomial approximation obtained by other authors are good only in an interval of the variable x [8]. Here an analytic approximation for $J_1(x)$ has been obtained, which gives good accuracy for every positive value of the variable x, as well as for the zeros of this function, which are usually very important in its applications. The procedure here used is an improvement of the multipoint quasi fractional approximation technique, MPQA, described in previous works [7–12]. Several applications of the method can be seen in Ref.[11], as for instance, Zeeman effect in 2-D [13], elliptic functions [14] and eigenvalues of an-harmonic potentials [10]. Power series as well as asymptotic expansions are used simultaneously, and a bridge function between both expansions has been found using rational functions combined with some elementary functions. Improvements to previous published approximations have also been found by using only rational functions with only even powers, as well as, a better choice of the auxiliary functions. In this way approximates with only 4 unknown parameters are good for most of the applications of this function. In this work the determination of the parameters has been done with two procedures. In the first procedure, the parameters are determined by using the coefficients of the power series together with the leading term of the asymptotic expansion. This leading term is also used in the second procedure, but the procedure is different with respect to the previous one, since now the power series is used only to determine the structure of the approximation, but the value of the coefficients in the expansions are not used, instead of that, the parameters are found by error minimization, as in mathematical statistic. Using this second procedure the largest maximum error decreases in about a factor two. The results will be presented in Sections "Parameter determination and results, MPQA technique" and "Parameters determination and results, using the statistic method" including several Figures, and the Sec tion "Conclusion" will be devoted to the Conclusion.

Outline of the procedure

The power series of $J_1(x)$ is [8]

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$$J_1(x) = \frac{x}{2}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+1)!}\left(\frac{x}{2}\right)^{2j} = \frac{x}{2}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \cdots\right) \tag{1}$$

The asymptotic expansion for large arguments is [8]

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}}\left[\left(1 + \frac{15}{128}\frac{1}{x^2} + \ldots\right)\cos\left(x - \frac{3\pi}{4}\right) + \left(\frac{3}{8x} - \frac{105}{1024}\frac{1}{x^3} + \ldots\right)\sin\left(x - \frac{3\pi}{4}\right)\right] \tag{2}$$

In this work the leading term of the asymptotic expansion is only considered, which is convenient to write as

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{3\pi}{4}4\right) = \sqrt{\frac{1}{\pi x}}(\sin x - \cos x) \tag{3}$$

The procedure to be followed is to build a bridge between the power series and the asymptotic expansion using rational functions combined with auxiliary functions. The additional condition to be imposed is that the approximation should have a Taylor expansions with only odd powers, no even powers. This condition is different to previous work in this matter [9], where both kind of powers appeared and a double number of parameters were needed in order to obtain the equality until a given power of the Taylor expansion. The efficiency now will be higher than previously, giving approximations with smaller errors, for a given number of parameters. The asymptotic expansion shows that there is a fractional power in x at the infinite, that is, at the infinite the approximation should be as $\frac{1}{\sqrt{x}}$, but for x = 0, the approximation should be as x, both conditions are accomplished by the function

$$\frac{x}{(1+\lambda^2x^2)^{\frac{3}{4}}} \tag{4}$$

which also produces only odd powers in its Taylor expansion. Our analysis shows that an adequate form for the approximant will be

$$\tilde{J}_1(x) = \frac{1}{(1+\lambda^2x^2)^{\frac{1}{4}}}\left[\frac{\sum_{i=0}^{n}p_ix^{2i}}{1+\sum_{j=1}^{n}q_jx^{2j}}\sin x + \frac{x}{(1+\lambda^2x^2)^{\frac{1}{2}}}\frac{\sum_{i=0}^{n}\tilde{p}_ix^{2i}}{1+\sum_{j=1}^{n}q_jx^{2j}}\cos x\right] \tag{5}$$

where the denominators of both rational functions have been chosen to be equals to avoid non linear algebraic equations in order to determine the parameters of the approximation. Now by imposing the conditions at the infinite, it is obtained

$$p_n = \frac{\lambda^{1/2}}{\sqrt{\pi}}q_n \quad ; \quad \tilde{p}_n = -\frac{\lambda^{3/2}}{\sqrt{\pi}}q_n \tag{6}$$

The case of n = 1 is the most important one, because it is very simple, and it gives right results for most of the applications. Then

$$\tilde{J}_1(x) = \frac{1}{(1+\lambda^2x^2)^{\frac{1}{4}}}\left[\frac{p_0+p_1x^2}{1+q_1x^2}\sin x + \frac{x}{(1+\lambda^2x^2)^{\frac{1}{2}}}\frac{\tilde{p}_0+\tilde{p}_1x^2}{1+q_1x^2}\cos x\right] \tag{7}$$

such that

$$p_1 = \frac{\lambda^{1/2}}{\sqrt{\pi}}q_1 \quad ; \quad \tilde{p}_1 = -\frac{\lambda^{3/2}}{\sqrt{\pi}}q_1 \tag{8}$$

The parameters to be determined are now $p_0, \tilde{p}_0$ and q_1, therefore only 3 terms of the Taylor expansion of $J_1(x)$ are needed.

Parameter determination and results, MPQA technique

To determine the parameters of the approximant, it is necessary to equalize three coefficients of the series

$$\begin{aligned}\frac{x}{2}(1+q_1x^2)(1+\lambda^2x^2)^{\frac{3}{4}}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} + \cdots\right) &= (1+\lambda^2x^2)^{\frac{1}{2}}(p_0+p_1x^2)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\\ &\quad + x(\tilde{p}_0+\tilde{p}_1x^2)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\end{aligned} \tag{9}$$

giving

$$\frac{1}{2} = p_0 + \tilde{p}_0 \tag{10}$$

$$\frac{1}{2}\left(q_1 + \frac{3}{4}\lambda^2 - \frac{1}{8}\right) = \frac{\sqrt{\lambda}q_1}{\sqrt{\pi}} - \frac{p_0}{6} + \frac{1}{2}\lambda^2p_0 - \frac{\lambda^{3/2}q_1}{\sqrt{\pi}} - \frac{\tilde{p}_0}{2} \tag{11}$$

$$\begin{aligned}&\frac{3}{8}\lambda^2q_1 - \frac{3\lambda^4}{64} - \frac{q_1}{16} - \frac{3}{64}\lambda^2 + \frac{1}{384}\\ &= -\frac{1}{6}\frac{\sqrt{\lambda}}{\sqrt{\pi}} + \frac{p_0}{120} + \frac{1}{2}\frac{\lambda^{\frac{5}{2}}q_1}{\sqrt{\pi}} - \frac{1}{12}\lambda^2p_0 - \frac{1}{8}\lambda^4p_0 + \frac{1}{2}\frac{\lambda^{\frac{3}{2}}q_1}{\sqrt{\pi}} + \frac{\tilde{p}_0}{24}\end{aligned} \tag{12}$$

from these Eqs. (10)–(12), it is obtained the values for q_1, p_0 and $\tilde{p}_0$ as

$$q_1 = -\frac{1}{8}\frac{a_{q_1}}{b_{q_1}} \tag{13}$$

where the parametres a_{q_1} and b_{q_1} are given by

$$a_{q_1} = \sqrt{\pi}(270\lambda^6 + 180\lambda^4 + 39\lambda^2 + 2) \tag{14}$$

$$\begin{aligned}b_{q_1} &= \sqrt{\lambda}(180\lambda^5 - 540\lambda^4 - 240\lambda^3 - 240\lambda^2 - 192\lambda + 32)\\ &\quad + \sqrt{\pi}(360\lambda^4 + 195\lambda^2 - 6)\end{aligned} \tag{15}$$

Furthermore

$$p_0 = \frac{15}{4}\frac{a_{p_0}}{c_{p_0}b_{p_0}} \tag{16}$$

where the parameters a_{p_0}, b_{p_0} and c_{p_0} are given by

$$\begin{aligned}a_{p_0} &= \sqrt{\lambda}(54\lambda^7 - 270\lambda^6 - 126\lambda^5 - 270\lambda^4 - 195\lambda^3 + 45\lambda^2\\ &\quad - 58\lambda + 10) + \sqrt{\pi}(189\lambda^6 + 270\lambda^4 - 51\lambda^2 - 2)\end{aligned} \tag{17}$$

$$\begin{aligned}b_{p_0} &= \sqrt{\lambda}(180\lambda^5 - 540\lambda^4 - 240\lambda^3 - 240\lambda^2 - 192\lambda + 32)\\ &\quad + \sqrt{\pi}(360\lambda^4 + 195\lambda^2 - 6)\end{aligned} \tag{18}$$

$$c_{p_0} = 3\lambda^2 + 2 \tag{19}$$

In a similar way $\tilde{p}_0$ is given by

$$\tilde{p}_0 = \frac{1}{4}\frac{a_{\tilde{p}_0}}{c_{\tilde{p}_0}b_{\tilde{p}_0}} \tag{20}$$

where

$$\begin{aligned}a_{\tilde{p}_0} &= \sqrt{\lambda}(270\lambda^7 + 810\lambda^6 + 1170\lambda^5 + 450\lambda^4 + 813\lambda^3 + 93\lambda^3\\ &\quad + 102\lambda - 22) - \sqrt{\pi}(675\lambda^6 + 495\lambda^4 + 21\lambda^2 - 6)\end{aligned} \tag{21}$$

$$\begin{aligned}b_{\tilde{p}_0} &= \sqrt{\lambda}(180\lambda^5 - 540\lambda^4 - 240\lambda^3 - 240\lambda^2 - 192\lambda + 32)\\ &\quad + \sqrt{\pi}(360\lambda^4 + 195\lambda^2 - 6)\end{aligned} \tag{22}$$

$$c_{\tilde{p}_0} = 3\lambda^2 + 2 \tag{23}$$

It is also convenient to define the parameters a, b and c, as

$$a_1 = -x^2\sqrt{\lambda}(270\lambda^6 + 180\lambda^4 + 39\lambda^2 + 2) \tag{24}$$

$$a_2 = \sqrt{\pi}(1890\lambda^4 + 810\lambda^2 - 30) \quad (25)$$

$$a_3 = \sqrt{\lambda}(540\lambda^5 - 2700\lambda^4 - 1620\lambda^3 - 900\lambda^2 - 870\lambda + 150) \quad (26)$$

$$a = \sin(x)\sqrt{1+\lambda^2x^2}(a_1 + a_2 + a_3) \quad (27)$$

$$b_1 = x^2\sqrt{\lambda}(270\lambda^7 + 180\lambda^5 + 39\lambda^3 + 2\lambda) \quad (28)$$

$$b_2 = -\sqrt{\pi}(450\lambda^4 + 30\lambda^2 - 6) \quad (29)$$

$$b_3 = \sqrt{\lambda}(180\lambda^5 + 540\lambda^4 + 660\lambda^3 - 60\lambda^2 + 102\lambda - 22) \quad (30)$$

$$b = x\cos(x)(b_1 + b_2 + b_3) \quad (31)$$

$$c_1 = \sqrt{\lambda}(1440\lambda^5 - 4320\lambda^4 - 1920\lambda^3 - 1920\lambda^2 - 1536\lambda + 256) + \sqrt{\pi}(2880\lambda^4 + 1560\lambda^2 - 48) \quad (32)$$

$$c_2 = \sqrt{\pi}(270\lambda^6 + 180\lambda^4 + 39\lambda^2 + 2)x^2 \quad (33)$$

$$c = (1+\lambda^2x^2)^{3/4}(c_1 - c_2) \quad (34)$$

The approximation $\widetilde{J}_1(x,\lambda)$ for $J_1(x)$ is

$$\widetilde{J}_1(x,\lambda) = \frac{a+b}{c} \quad (35)$$

An important point in this approximation is that the value of q_1 should be positive, because otherwise an undesirable pole would appear in $\widetilde{J}_1(x)$ for positive values of x. In order to avoid this problem a plot of q_1 as a function of λ is presented. (See Fig. 1)

In this way, it is clear that λ must be in the interval (0,1.1). Now we have to look for the best value of λ.

A plot of the maximum absolute error as a function of λ is shown in Fig. 2.

Now, λ is chosen by looking for the value that produces the smallest absolute error, which is for $\lambda = 0.3484$. Using this value in Eq. (33), the approximant for $\widetilde{J}_1(x)$ becomes

$$\widetilde{J}_1(x) = \frac{\sqrt{1+0.12138x^2}(46.68634 + 5.82514x^2)\sin x}{(57.70003 + 17.49211x^2)(1+0.12138x^2)^{\frac{3}{4}}} - \frac{x(17.83632 + 2.02948x^2)\cos x}{(57.70003 + 17.49211x^2)(1+0.12138x^2)^{\frac{3}{4}}} \quad (36)$$

In Fig. 3, the absolute error of this approximation is shown as a function of the variable x.

The maximum absolute error is 0,008 for $x_1 = 6.3$. However, outside of the region nearby x_1, the absolute error is less than 0,001. Absolute error must be consider, instead of relative errors, because of the zeros of $J_1(x)$, where the relative error would be infinite.

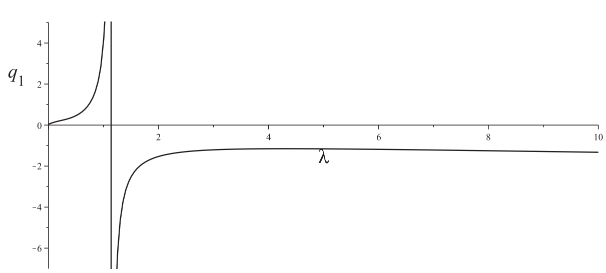

Fig. 1. Here q_1 is plotted as a function of λ, showing that q_1 is positive only for values of λ smaller than 1.139.

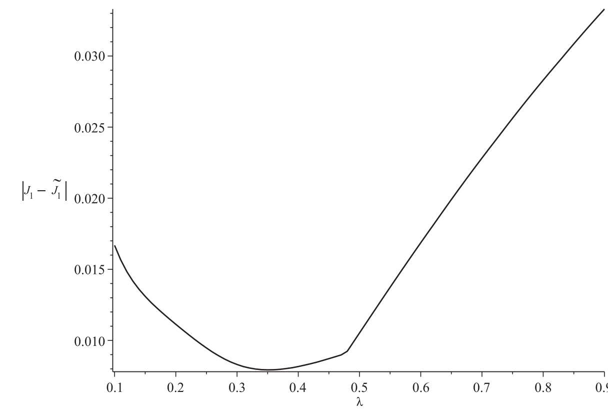

Fig. 2. Maximum absolute error as a function of the approximant determined only with the MPQA procedure.

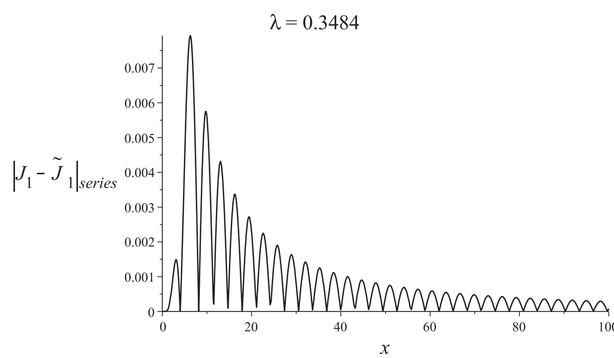

Fig. 3. Absolute error of the approximation as a function of variable x, with the parameters determined with the best value of $\lambda = 0.3484$.

Parameters determination and results, using the statistic method

The procedure now is as follows. The form of the approximation is that in Eq. (7), but the determination of the parameters is performed looking for the minimum square root error in the interval (0,100) for x, Thus a hundred of consecutive points are selected and the sum of the square errors is calculated. The parameters that minimize that sum are the selected parameters. In this way the new approximation $\hat{J}_1(x)$ for $J_1(x)$ is given by (See Fig. 4)

$$\widehat{J}_1(x) = \frac{(0.1601x^2 + 0.8660)\sin x}{(1+0.3489x^2)\sqrt[4]{1+0.4181x^2}} - \frac{x(0.1007x^2 + 0.3718)\cos x}{(1+0.4181x^2)^{3/4}(1+0.3489x^2)} \quad (37)$$

The Fig. 5 shows the absolute error for positive values of x. The maximum absolute error is 0.0038 at the point $x_2 = 6.6$.

However, outside of the region around x_2, the relative error is smaller than 0,001, which means that the approximation is very precise. In this calculation, as in the previous approximation, the absolute error is considered because the relative error will be infinite at the zeros of $J_1(x)$. Looking now for the relative errors of the zeros of the approximate function, these are shown in Table 1 plot of the relative errors of these zeros are shown in Fig. 6.

Finally an approximation with 9 parameters $\tilde{\tilde{J}}_1(x)$ has also been determined, which is

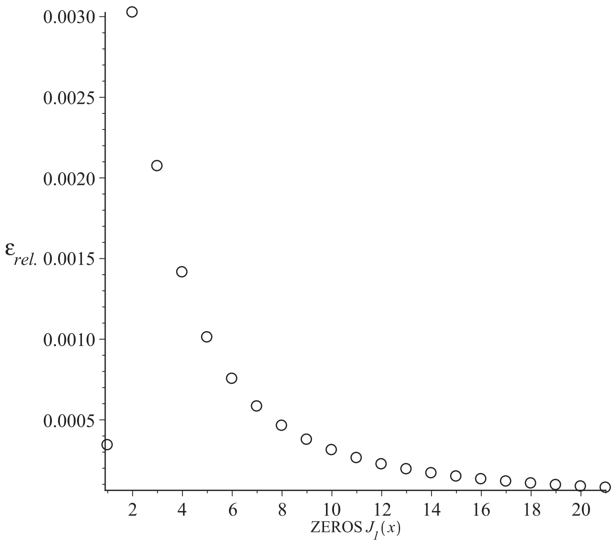

Fig. 4. Relative errors of zeros of $\tilde{J}_1(x)$, compared with those of $J_1(x)$, as a function of the cardinal number of corresponding zero.

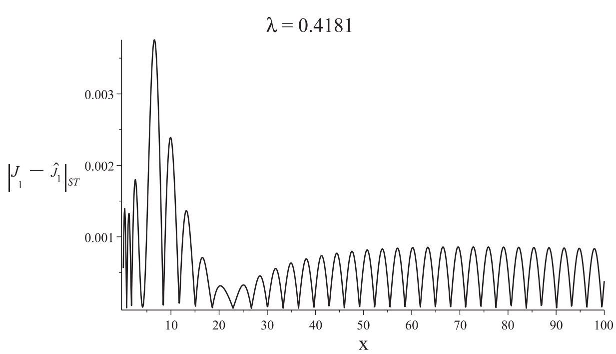

Fig. 5. Absolute error of the approximation as a function of the variable x, with the parameters determined with the statistical method and the form of approximant with the MPQA method.

$$\tilde{\tilde{J}}_1(x)=\frac{1}{2(1+\lambda^2x^2)^{1/4}}\left(\frac{p_0+p_1x^2+p_2x^4}{1+q_1x^2+q_2x^4}\sin x+\frac{x}{(1+\lambda^2x^2)^{1/2}}\frac{P_0+P_1x^2+P_2x^4}{1+q_1x^2+q_2x^4}\cos x\right) \tag{38}$$

where

$$P_2=-\frac{2\lambda^{3/2}}{\sqrt{\pi}}q_2 \quad ; \qquad p_2=\frac{2\lambda^{1/2}}{\sqrt{\pi}}q_2 \tag{39}$$

and

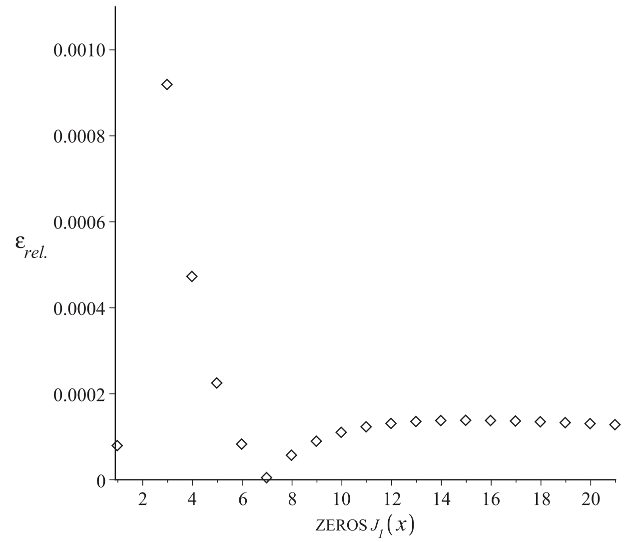

Fig. 6. Relative error of zeros of $\hat{J}_1(x)$, compared with those de $J_1(x)$, as a function of the cardinal number of corresponding zero.

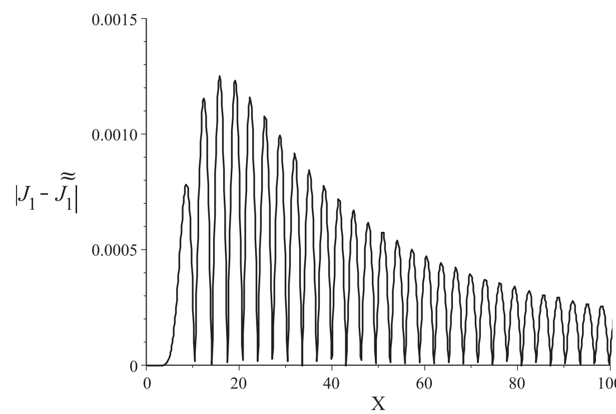

Fig. 7. Absolute error of the approximation as a function of variable x, with the parameter determined with the best value of $\lambda = 0.1$.

$$\begin{aligned} &P_0=-0.7763224930 \quad &&P_1=-0.03147133771\\ &p_0=1.776322448 &&p_1=0.2250803518\\ &q_1=0.4120981204 &&q_2=0.006571619275\\ & &&\lambda=0.1 \end{aligned} \tag{40}$$

A plot of the absolute error as a function of x is shown in Fig. 7, where it can be see that the maximum absolute error decreases in an order of magnitude compared with previous ones. Now the

Table 1
Relative errors at the zeros of the approximations, considering both approximations $\tilde{J}_1(x_n)$ and $\hat{J}_1(x_n)$.

Zero	$J_1(x_n)$	$\tilde{J}_1(x_n)$[a]	$\hat{J}_1(x_n)$[b]	$\lvert\frac{\tilde{x}-x}{x}\rvert^{a}, 10^{-4}$	$\lvert\frac{\hat{x}-x}{x}\rvert^{b}, 10^{-4}$
1	3.8317	3.8330	3.8314	3	0.8
2	7.0156	7.0368	7.0271	30	16
3	10.1735	10.1946	10.1827	21	9
4	13.3237	13.3425	13.3299	14	5
5	16.4706	16.4873	16.4742	10	2
10	32.1897	32.1997	32.1861	3	1
60	189.2790	189.2809	189.2671	0.1	0.6

[a] Approximation using series expansion.
[b] Approximation using statistical methods.

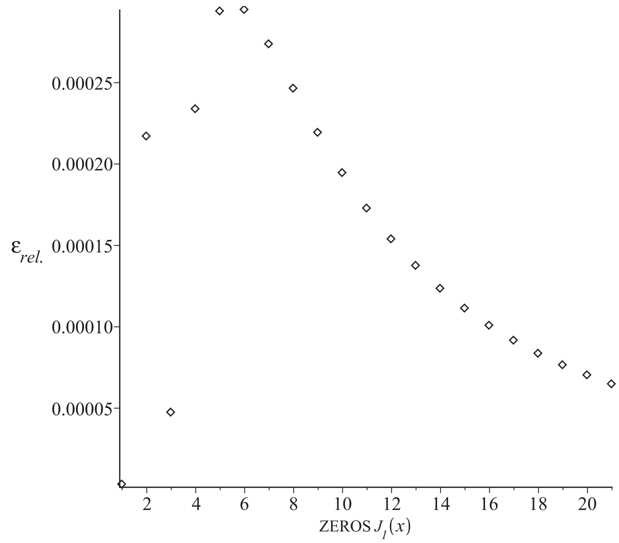

Fig. 8. Relative errors of zeros of $\tilde{\tilde{J}}_1(x)$, compared with those of $J_1(x)$, as a function of the cardinal number of corresponding zero.

maximum absolute error is about 0.0013. The accuracy increases with this approximation, and the absolute error becomes about 1/10 of those with the preceding approximations. The relative errors of zeros de $\tilde{\tilde{J}}_1(x)$ are shown in Fig. 8, in a similar way as it was done previously. Now the maximum of the relative errors decreases also in about one order of magnitude, since now is about 0.0003.

Conclusion

Precise analytic approximations for the Bessel function $J_1(x)$ have been found. The accuracy for positive values of the variable x, is less than 0.008 for the most simple approximation. Though the accuracy is high, however the parameters to determine are only six. Here as in Padé method rational functions are used, but these are combined with auxiliary functions as trigonometrical and fractional powers. To determine the form of the approximation, a join analysis of the power series and asymptotic expansion has to be carry out. Both expansions are simultaneously used to determine the parameters of the approximation in the first method here presented. However a second procedure has also been presented by using a minimizing method procedure. The determination of the parameters with this second procedure leads to an approximation where the maximum absolute error is about a half of the first one. The zeros of the approximation are also very precise.The relative errors for the first zeros are of order 10^{-4} and 10^{-5}, respectively, following the first and second procedure.A more precise approximation has also been determined. This have done by increasing the degree of the polynomial of the rational functions. The minimum increase in the degree of the polynomials is 2, because of all the conditions required with the present technique. This leads to a new approximation, where all the errors decrease in about one order of magnitude.It seems that the accuracy of the approximations here found are right for most of its applications in Physics, and they can be used to determine any value of $J_1(x)$,using only a simple pocket calculator.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, athttps://doi.org/10.1016/j.rinp.2018.01.071.

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