Name:

andrewID:

Please read all of the following information.

Write your name and andrewID clearly in the designated locations above.

Do not open the test until permission is given. You have the full class period to complete your answers.

Put all notes, calculators, and electronic devices away. None of these are permitted.

Grading will be done with Gradescope, so please write answers only on the pages assigned to the appropriate question. If you need to put work on a different page, please write a note on the page where the answer appears.

For full credit, your work must be shown clearly and legibly, with appropriate justifications.

You do not have to complete every part of every problem perfectly to do well. Be sure to give yourself time to work on every problem.

Good luck!

1. Let $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(a) Express A^{-1} as a product of elimination matrices, or explain why it doesn't exist.

(b) What is the rank of A?

(c) Find all solutions to $A\mathbf{x} = \mathbf{0}$ or explain why none exist.

(d) Find all solutions to $A\mathbf{x} = (4, 0, 1, 3)$ or explain why none exist.

(e) Express A^T as a product of elimination matrices.

(f) Suppose $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^4$, where $\mathbf{x} = 2\mathbf{v} - \mathbf{w}$. Suppose $A\mathbf{v} = (0, 3, 1, 2)$ and $A\mathbf{w} = (2, 4, 4, 1)$. What is $A\mathbf{x}$?

Solution:

(a) Since A is the product of four elimination matrices $A = E_{14}PE_3E_{32}$, and we know the forms of inverse matrices for elimination matrices,

$$\begin{aligned} A^{-1} &= E_{32}^{-1}E_3^{-1}P^{-1}E_{14}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(b) Since A is invertible, its rank is 4.

(c) Since A is invertible, $A\mathbf{x} = \mathbf{0}$ has only one solution, $\mathbf{x} = (0, 0, 0, 0)$.

(d) Since A is invertible, $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^4$, $\mathbf{x} = A^{-1}\mathbf{b}$. So if $A\mathbf{x} = (4, 0, 1, 3)$, then

$$
\begin{aligned}
\mathbf{x} &= A^{-1}(4,0,1,3) \\
&= E_{32}^{-1}E_3^{-1}P^{-1}E_{14}^{-1}(4,0,1,3) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 1 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}
\end{aligned}
$$

(e) Since A is the product of four elimination matrices $A = E_{14}PE_3E_{32}$, and $(AB)^T = B^TA^T$,

$$
\begin{aligned}
A^T &= E_{32}^TE_3^TP^TE_{14}^T \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

(f)

$$
\begin{aligned}
A\mathbf{x} &= A(2\mathbf{v} - \mathbf{w}) && \text{substitution} \\
&= 2(A\mathbf{v}) - A\mathbf{w} && \text{linearity} \\
&= 2(0,3,1,2) - (2,4,4,1) && \text{substitution} \\
&= (-2,2,-2,3) && \text{vector arithmetic}
\end{aligned}
$$

2. Let A be a 3×5 matrix where

- he first three columns $\{A_{*1}, A_{*2}, A_{*3}\}$ form a linearly independent set of vectors,
- $A_{*4} = -A_{*1} + 2A_{*3}$, and
- $A_{*5} = 2A_{*1} - A_{*2} + 3A_{*3}$.

(a) Find the reduced row echelon form for A.

(b) What is the rank of A?

(c) What are the dimensions for the four fundamental subspaces of A?

(d) Find a basis for the row space of A.

(e) Find a basis for the null space of A.

(f) Find a basis for the column space of A.

(g) Suppose $A(5, 0, -3, 2, 3) = (9, 2, 4)$. Describe all solutions to $A\mathbf{x} = (9, 2, 4)$

(h) Write an expression for P, the matrix where for $\mathbf{b} \in \mathbb{R}^5$, $P\mathbf{b}$ is the projection of $\mathbf{b}$ onto $N(A)$. (You do not need to multiply matrices or otherwise simplify your expression.)

Solution:

(a) Since row operations preserve dependence relations among columns, the first three columns will be pivot columns, and the RREF is

$$R = \begin{pmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

(b) The rank of A is 3.

(c) Since $m = 3$, $n = 5$, and $r = 3$,

- $\dim C(A^T) = r = 3$
- $\dim C(A) = r = 3$
- $\dim N(A) = n - r = 5 - 3 = 2$
- $\dim N(A^T) = m - r = 3 - 3 = 0$

(d) The pivot rows of R form basis for the row space of A:

$$\{(1, 0, 0, -1, 2), (0, 1, 0, 0, -1), (0, 0, 1, 2, 3)\}$$

(e) A basis for the null space of A is given by the columns of the matrix

$$N = \begin{pmatrix} -F \\ I \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. a basis is

$$\{(1, 0, -2, 1, 0), (-2, 1, -3, 0, 1)\}$$

(f) Since A has full row rank, the column space of A is all of $\mathbb{R}^3$. Thus any basis for $\mathbb{R}^3$ will work, for example the standard basis

$$\{(1,0,0),(0,1,0),(0,0,1)\}.$$

(g) Suppose $A(5,0,-3,2,3)=(9,2,4)$. Describe
All solutions to $A\mathbf{x}=(9,2,4)$ are of the form $\mathbf{x}_p+\mathbf{x}_n$ where $\mathbf{x}_p$ is a particular solution, and $\mathbf{x}_n$ can be any element of the null space of A. Since we have one particular solution $\mathbf{x}_p=(5,0,-3,2,3)$, and a basis for the null space (from Part (e)), $A\mathbf{x}=(9,2,4)$ if and only if

$$\mathbf{x}=(5,0,-3,2,3)+a(1,0,-2,1,0)+b(-2,1,-3,0,1),$$

where a and b are real numbers.

(h) Since the columns of N are a basis for $N(A)$,

$$P=N(N^TN)^{-1}N^T.$$

3. Suppose $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^5$, and let $V = \text{span}\{\mathbf{v}, \mathbf{w}\}$ and $W = \text{span}\{2\mathbf{v}, -\mathbf{w}\}$.

(a) Define the span of a set of vectors.

(b) Prove that if $\mathbf{x} \in W$ then $\mathbf{x} \in V$.

(c) Define the orthogonal complement of a vector space in $\mathbb{R}^n$.

(d) Suppose $\mathbf{y}$ is orthogonal to $\mathbf{v}$ and $\mathbf{y}$ is orthogonal to $\mathbf{w}$. Show that $\mathbf{y} \in V^\perp$.

(e) Suppose $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$. What is the dimension of V in this case? Find a basis for V in this case.

Solution:

(a) The span of a set of vectors is the set of all linear combinations of those vectors.

(b) If $\mathbf{x} \in W = \text{span}\{2\mathbf{v}, -\mathbf{w}\}$, then $\mathbf{x} = a(2\mathbf{v}) + b(-\mathbf{w})$ for some scalars a and b. Thus $\mathbf{x} = (2a)\mathbf{v} + (-b)\mathbf{w}$. Since $\mathbf{x}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$, it is in $\text{span}\{\mathbf{v}, \mathbf{w}\}$, i.e. $\mathbf{x} \in V$.

(c) The orthogonal complement of a vector space V in $\mathbb{R}^n$ is the set of all vectors that are orthogonal to every vector in V, i.e

$$V^\perp = \{\mathbf{w} : \mathbf{w}^T\mathbf{v} = 0 \quad \forall \mathbf{v} \in V\}.$$

(d) Since $V = \text{span}\{\mathbf{v}, \mathbf{w}\}$, let $a\mathbf{v} + b\mathbf{w}$ be an arbitrary element of V. Then, using linearity of the dot product,

$$\begin{aligned}
\mathbf{y}^T(a\mathbf{v} + b\mathbf{w}) &= a(\mathbf{y}^T\mathbf{v}) + b(\mathbf{y}^T\mathbf{w}) && \text{linearity of dot product} \\
&= a \cdot 0 + b \cdot 0 && \text{def of orthogonality} \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Since $\mathbf{y}^T(a\mathbf{v} + b\mathbf{w}) = 0$, $\mathbf{y} \perp (a\mathbf{v} + b\mathbf{w})$, i.e. $\mathbf{y}$ is orthogonal to any vector in V, so $\mathbf{y} \in V^\perp$.

(e) If $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$, then all vectors in V are scalar multiples of $\mathbf{v}$:

$$a\mathbf{v} + b\mathbf{w} = a\mathbf{v} + b(c\mathbf{v}) = (a + bc)\mathbf{v}.$$

Thus in this case $\mathbf{v}$ is one-dimensional, with basis $\{\mathbf{v}\}$.