Quantum Gravity

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We provide a brief historical overview on how canonical quantum gravity has evolved and given rise to the theory of loop quantum gravity. The constraints on the dynamics in this scheme is extensively discussed and it is investigated how loop quantum gravity proposes to deal with these constraints. Additionally, we see that this theory does not depend on any additional hypothesis other than well-established ones. Some applications are also discussed.

I. INTRODUCTION

Newton's theory of physics was long the dominating one for describing events in the Universe. Broadened descriptions came with Einstein's theory of relativity, arguing that space and time are linked together. Specifically, Einstein's theory of general relativity (GR) study effects on large scales. Quantum theory was introduced later and describes phenomena on small scales. Both of these theories are reliable in their respective scales. However, GR breaks down at the quantum level. Gravity is well understood in GR, arguing that gravity is an inherent property of spacetime itself. Formulating a theory that unifies GR with quantum theory thus has a key role in answering the question of how to describe spacetime [1].

The report is structured as follows: In Sec. II, we discuss our main investigation, starting with an overview of motivations for quantum gravity and particularly we discuss how canonical quantum gravity leads to the framework of loop quantum gravity (LQG). We also include a discussion of applications and a summary in Sec. III.

II. INVESTIGATION

The classical Einstein equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor, describe the connection between geometry and matter. However, quantum field theory (QFT) dictates that matter is quantized, meaning that the energy-momentum tensor becomes an operator on a Hilbert space. For this to be consistent with GR, there are two main possibilities. The first one is that geometry is also quantized, which can lead to $G_{\mu\nu}$ becoming an operator in a Hilbert space, or a sufficient formulation of quantum theory is provided. The second one is that geometry indeed stays classical and that the energy-momentum tensor $T_{\mu\nu}$ is an expectation value in a quantum state that depends on the classical geometry. Most researchers consider the first possibility to be the most plausible [2].

A problem with ordinary QFT is that GR results in non-renormalizable theory. Valuable insights can be gained from this scheme, but not at extreme scenarios, such as black holes and the Big Bang [3].

A. Motivation for Quantum Gravity

Singularities frequently appear in GR, most notably in the Big Bang and at the centers of black holes. This breakdown of theory suggests there to be a more complete framework that avoids the singularities [2]. Many times integrations in QFT lead to infinities, where regularization schemes can be used. It might, however, be reasonable to expect a fundamental theory to be intrinsically finite, and that a quantum theory of gravity could help shed further insight [2].

B. Canonical Quantum Gravity

As discussed above, there have been attempts to canonically quantize the theory of GR in hope of aiding towards getting a sufficient quantum theory of gravity. The basic theory for this was outlined by DeWitt in 1967, found in Ref. [4]. Here, DeWitt used familiar methods for quantization that have been successful in quantizing other fields and applied these to GR. In this paper, the quantization is based on the metric tensor $[g]$ being decomposed as

$$g_{\mu\nu} dx^\mu dx^\nu = (-\alpha^2 + \beta_k \beta^k) dt^2 + 2\beta_k dx^k dt + \gamma_{ij} dx^i dx^j, \tag{2}$$

where α and β_k are functions and γ_{ij} is the spatial metric, which raises and lowers indices of the given quantities.[1] The Latin indices correspond to the values $1, 2, 3$ and the Greek indices to $0, 1, 2, 3$. One can obtain the Lagrangian, found in Ref. [4], which depends on $\gamma \equiv \det(\gamma_{ij})$, the extrinsic curvature[2] K_{ij} and the

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[1] The functions α and β_k are commonly known as the *lapse* and *shift* functions, respectively. Ref. [4] does not use this naming.

[2] Extrinsic curvature is the curvature of a surface that is an embedding analyzed from the enveloping global manifold, whereas intrinsic curvature only depends on the spatial metric *in* the surface. These need not be related.

spatial scalar curvature ${}^{(3)}R$. From this Lagrangian one can obtain the conjugate momenta to α and β_i through functional derivatives. We get

$$\pi = \frac{\delta L}{\delta \dot{\alpha}} = 0, \ \pi^i = \frac{\delta L}{\delta \dot{\beta}_i} = 0, \tag{3}$$

where the dot denotes differentiation with respect to time. We notice that $\pi = \pi_i = 0$, expressing that the Lagrangian is independent of $\dot{\alpha}$ and $\dot{\beta}_i$. These are called *primary constraints* of the system. In particular, one can choose the gauge $\alpha = 1$ and $\beta_i = 0$, which after some computation gives the Hamiltonian $H = \int \mathcal{H} d^3x$, where

$$\mathcal{H} = \frac{1}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl})\pi^{ij}\pi^{kl} - \gamma^{1/2}({}^3R), \tag{4}$$

and π^{ij} denotes the conjugate momenta to γ_{ij}. Einstein's field equations can be obtained by taking the Poisson bracket of the dynamical variables with the Hamiltonian H. From this, one can also find the constraints $\mathcal{H} = 0$ and $\nabla_j \pi^{ij} = 0$, where ∇_j denotes the covariant differentiation with respect to the spatial metric γ_{ij}. In quantum theory, the given constraints become conditions on a state vector Ψ, such as $\mathcal{H}\Psi = 0$. Further information can be found in Ref. [4]. When quantizing GR, this constraint gets promoted to an operator. The resulting Hamiltonian constraint is known as the *Wheeler-DeWitt equation,* see Ref. [5].

The constraints on the system were very important factors to consider in order to develop the theory of canonical quantum gravity. Particularly, the constraint on $\mathcal{H}$ proved to be difficult to fulfill if one wanted to find a corresponding quantum operator, since the dependence on the canonical variables γ_{ij} and π^{ij} is complicated.

Before we move on, we introduce some concepts and notation. This discussion is constructed with information given in Ref. [5]. Let us introduce a three-dimensional set of three vector fields that are orthogonal $E^a_i, i = 1, 2, 3$. We can find the metric, denoted by q^{ab}, through $q^{ab} = E^a_i E^a_j \delta^{ij}$, where δ^{ij} is the Kronecker delta. What should be noted here is that we have flat space in coordinates i, j on the right-hand side and curved space with coordinates a, b on the left-hand side. We thus have two different types of indices. The quantity E^a_i is called a *triad.* We furthermore define *scalar densities.* These are quantities that transform as a scalar times the determinant of the metric. In particular, quantities that transform with the square root of the determinant are denoted by a tilde above the quantity, so that $\widetilde{F} = \sqrt{\det(q)}F$ [5].

In 1986, Ashtekar introduced new canonical variables, found in Ref. [6]. This formulation was partly based on SU(2) structure and Yang-Mills theory, which allowed for simplification of the constraints. The approach in this paper, however, suffered from various problems, some of which are discussed in Ref. [7]. In particular, the so-called Ashtekar connection variables were complex-valued, making it difficult to ensure that one recovers real GR. Secondly, the Hamiltonian constraint became polynomial only after rescaling it with the square root of the determinant of the spatial metric, i.e. the constraint became $\widetilde{\mathcal{H}} = \sqrt{\det(q)}\mathcal{H} = 0$. See Ref. [7].

We now present the approach from Ashtekar, inspired by Ref. [5]. One half of the Ashtekar variables is the so-called *densitized triads* $\widetilde{E}^a_i$, whereas the other part is variables that behave as an SU(2) Yang-Mills connection A^i_a. These constitute the configuration variables, $\widetilde{E}^a_i$ being the canonically conjugated momentum to A^i_a, so that

$$\{A^i_a(x), \widetilde{E}^b_j(y)\} = 8\pi G\beta\delta^a_b\delta^i_j\delta^3(x-y), \tag{5}$$

where G is Newton's constant and β is known as the *Barbero-Immirzi parameter.* The chosen value for this parameter has significant consequences for the theory, which will be discussed below. We can furthermore use the densitized triads to construct the spatial metric through $\widetilde{\widetilde{q}}^{ab} = \det(q)q^{ab} = \widetilde{E}^a_i\widetilde{E}^b_j\delta^{ij}$. The connection variables are related to the extrinsic curvature through $A^i_a = \Gamma^i_a + \beta K^i_a$, where $\Gamma^i_a = \Gamma_{ajk}\varepsilon^{jki}$ is the so-called *spin-connection* and $K^i_a = K_{ab}\widetilde{E}^{ai}/\det(q)$. Here the quantities Γ_{ajk} denote the connection. In this theory, one obtains a set of constraints, given as

$$G^i = \nabla_a \widetilde{E}^a_i = 0, \tag{6}$$

$$C_a = \widetilde{E}^b_i F^i_{ab} - A^i_a(\nabla_b \widetilde{E}^b_i) = 0, \tag{7}$$

where ∇_a denotes covariant differentiation with respect to the spatial metric and F^i_{ab} denotes the field strength tensor of the connection A^i_a. The first constraint is a sort-of Gauss' law, which generates the su(2) transformations in the variables. The other constraint is known as the *diffeomorphism constraint*[3]. Then one also has the Hamiltonian constraint

$$\begin{aligned} H = {} & \varepsilon_{ijk}\widetilde{E}^a_i\widetilde{E}^b_j F^k_{ab} \\ & + 2\frac{\beta^2+1}{\beta^2}(\widetilde{E}^a_i\widetilde{E}^b_j - \widetilde{E}^a_j\widetilde{E}^b_i)(A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b) = 0. \end{aligned} \tag{8}$$

Particularly, Ashtekar chose $\beta = i$, which caused the second term to vanish. This simplified the constraint, but caused some of the problems previously discussed [5].

Now, to move into the quantum realm, we introduce the so-called *coordinate representation*, which considers wavefunctions $\Psi(A^i_a)$ that depend on the configuration variable A^i_a. Similar representations have been used in quantizing the Maxwell and Yang-Mills theories. One then needs to promote the canonical variables to operators. In this case, it means that the connection becomes a multiplicative operator and the densitized triads become

[3] A diffeomorphism is a one-to-one map that, given a point p in a manifold, maps it to a new point $\Phi(p)$, while respecting the notion of differentiation. The constraint implies we require diffeomorphism symmetry in the system.

functional derivatives. Furthermore, when promoting the variables to operators, Poisson brackets become commutators, so that relation Eq. (5) becomes a commutator instead. Naturally, one has to promote the constraints to operator equations as well. This proved to be very difficult to do in this representation. Gauss' law imposes that the functions $\Psi(A)$ cannot be arbitrary and have to be gauge invariant functions of the connection. The second constraint imposes the states to be invariant under diffeomorphisms. The Hamiltonian constraint, however, has several problems. Aside from pure computational issues, such as those arising from the two triads, it was shown that there was difficulty in defining the product between operators so that an inner product could not be defined. This is a major problem, since one cannot compute expectation values nor transition probabilities. Some of the problems in quantizing the Hamiltonian constraint was handled by Thiemann in Ref. [7]. The difficulties of working in this representation led to the development and introduction of an alternative representation, known as the *loop representation*, which we discuss below [5].

The loop representation was motivated by the discovery of loop solutions to the Wheeler-DeWitt equation written in terms of the Ashtekar variables, see Ref. [8]. This led to the introduction of the loop representation of quantum GR by Rovelli and Smolin, see Ref. [9]. This representation is the basis of LQG.

Again, we introduce concepts mainly with inspiration from Ref. [5]. First, we introduce the concept of a *holonomy.* To draw a parallel to Maxwell's theory, we can find that, given a manifold, if one specifies the circulation of the vector potential along all possible curves on a manifold, one can implicitly obtain the fields. This result is gauge invariant. However, when dealing with the case of Yang-Mills theory, which is associated with the connection variables A^i_a, the gauge transformations are more complicated and this scheme does not given a gauge invariant result. One needs a different notion to help obtain the gauge-invariant information, which is provided by the so-called holonomies. From using the notion of parallel transport, which carries a quantity along a curve $\gamma^a(t)$ as "parallel to itself as possible", we can obtain the so-called parallel propagator, which is a matrix and can be found in Ref. [5]. If the curve along which one parallel transports is closed, then the propagator is called a holonomy. If one then takes the trace of the holonomy, one gets a scalar that happens to be invariant under gauge transformations. Without motivation, we later write down the expression for the holonomy in terms of a *path-ordered product* P, in which the factors get permutated such that the larger values of the time variables t_i appear first for the different factors $A_{a_i}(t_i)$. For example, if $t_2 > t_1 > t_3 \cdots t_n$, we get $P(A_{a_1}(t_1)A_{a_2}(t_2)\cdots A_{a_n}(t_n)) = A_{a_2}(t_2)A_{a_1}(t_1)\cdots A_{a_n}(t_n)$. If the variables commute then the path ordering has no effect, but if we do not have commutativity among the variables path ordering becomes important. A result, known as Giles' theorem, states that if one knows the trace of the holonomy along all possible loops in a manifold, then one can from these values reconstruct all gauge-invariant information present in the vector potential [5].

As discussed above, the Gauss' law constraint required that the wave functions are gauge invariant functions of the connection. However, as we know from the previous paragraph, this information can be retrieved from the traces of holonomies and can be taken to yield a basis of solutions for Gauss' law. With this, it is suggested that we can expand a state in this basis, giving

$$\Psi(A) = \sum_{\gamma} \Psi[\gamma] W_\gamma[A], \tag{9}$$

where the sum is a sum of all possible closed loops and $\Psi[\gamma]$ are coefficients and $W_\gamma[A]$ denote the traces of holonomies, which is given by

$$W_\gamma[A] = \mathrm{Tr}\Bigg(P\Bigg[\exp\Bigg(-\oint_\gamma \dot{\gamma}^a(s)A_a(s)ds\Bigg)\Bigg]\Bigg), \tag{10}$$

where A again denotes the connection. This is the trace of a *path-ordered exponential.* One can choose to work directly with the coefficients $\Psi[\gamma]$, which is called to work in the loop representation. The expansion we performed above is known as the *loop transform.* We have seen that this representation already have taken care of the Gauss' law constraint. The diffeomorphism constraint is easily managed, since one can simply consider functions of loops that are invariant under smooth deformations of the loops. These functions have been extensively studied and are known as *knot invariants.* Furthermore, the Hamiltonian constraint can be rewritten in terms of parallel transport operators, which in the loop representation prove to be natural. The constraints contain holonomies and volumes which promote to well defined operators in the loop representation. Although this gives a well-defined theory of quantum gravity without infinities, this does not, however, mean that it is an easy task to check if the theory has the correct physics in general, and indeed this has not been done as of yet. Despite this, there has been progress in certain circumstances [5]. Now, we present a non-technical discussion.

C. Loop Quantum Gravity

Particularly, the main ideas and assumptions that LQG rests on is that both GR and quantum mechanics are correct. It also only proposes a solution to quantizing gravity, not obtaining a grand unified theory. Furthermore, it does not require higher dimensions than the four spacetime dimensions. However, to merge GR with quantum mechanics, there is a necessity to abandon commonly held notions of spacetime. A consequence of taking GR to be correct is that ordinary QFT becomes inadequate, which is based on the assumption of the existence of a background spacetime, whereas GR tells us that there is no such background. Because of this, LQG only makes use of tools from quantum theory, and not necessarily QFT [10].

As mentioned, the basic variables in LQG are considered to be loops. This would imply that space is discretized. The loop states can be arranged in so-called *spin networks,* which will not be discussed further in this report. A non-technical approach is found in Ref. [1]. However, an interesting fact about the spin networks in LQG is that they are relational, meaning that they do not live in space, but rather generate space [1]. Since space itself is formed by loop states, the position of a loop state is relevant only with respect to other loops, and not with respect to the background. In quantum gravity, the volume V becomes an operator and the spectrum of V is discrete. Then, volume elements of space are given by eigenvalues of the volume operator. When two elements of space are adjacent, they are separated by a surface S. Denote the area of this surface by A. Similar to volume, this also becomes an operator. The eigenvalues for the area is found to be

$$A = 8\pi\beta\hbar G \sum_i \sqrt{j_i(j_i+1)}, \tag{11}$$

where β is the Barbero-Immirzi parameter, a constant of the theory, and j_i are multiplets of half integers [10].

Now, we discuss some applications, inspired by Chapter 8 in Ref. [10]. An application of this theory has been made in early universe cosmology. It has been possible to impose homogeneity and isotropy on the basis states and operators of the theory. The main results are: 1) An absence of singular behaviour at the Big Bang. 2) The evolution approximates the known Friedmann dynamics for large values of the scale factor $a(t)$, although they differ from it at small values. 3) The scale factor and volume of the Universe is quantized. 4) One can view the scale factor as a cosmological time parameter, leading to the result that the cosmological time is quantized. 5) Inflation in the early universe is predicted to be driven by quantum properties of the gravitational field.

The first two points are expected from a quantum theory of gravity, so as to give a consistent description of the Universe. The two subsequent points are characteristic properties of the theory as they reveal the quantization of geometry. The last point came as a surprise [10].

Another application is to black hole thermodynamics. It has been proved that the Einstein equations imply that the area of the event horizon of a black hole cannot decrease. It was suggested to associate an entropy S to a Schwarzschild black hole of surface area A through $S = a(k_B/\hbar G)A$, where a is a constant on the order of unity, k_B the Boltzmann constant, $\hbar$ the reduced Planck constant, and G Newton's constant. One has been able to derive this result from first principles in LQG [10].

Now, we discuss some problems with this framework. The main problem of LQG is to obtain GR in a suitably defined classical limit. Furthermore, it has not been possible to fully handle the difficulties which one runs into when quantizing the Hamiltonian constraint. Another problem is the suggestion that LQG is not locally Lorentz invariant. With the current understanding, we cannot answer if there are Lorentz violations, and in the case that they may exist, how severe they are. This is only a brief description of two problems with this scheme, for a more thorough discussion see Ref. [2].

There exist, naturally, other approaches than the ones we have discussed here. These include, but are not limited to, string theory, ordinary QFT, and supergravity. String theory is by many seen to be a promising candidate, the issue being how to handle the extra dimensions, which arise in this theory [2].

The main areas of interest of quantum gravity theories are cosmology and black holes. Most progress in describing these situations have been made in loop quantum cosmology, which was briefly discussed above. The current main question is how to describe spacetimes beyond a homogeneous and isotropic spacetime. No consensus has been reached as of 2016 [2].

III. SUMMARY

We have given a brief overview of the evolution of canonical quantum gravity into the theory of LQG. In particular, we have seen that this theory implies that space itself is composed of loops that generate space. Furthermore, it implies a quantization of geometry so that area and volume are being quantized. This theory was mainly developed from canonical quantum gravity to handle the difficulties in working with the given constraint for the dynamics of the theory. A difference between LQG and approaches such as string theory, is that LQG requires no extra dimensions. It is based on the idea of four spacetime dimensions being correct.

[1] L. Smolin, *Three roads to quantum gravity*, Science masters (Basic Books, New York, 2001).

[2] N. Bodendorfer, "An elementary introduction to loop quantum gravity," (2016), arXiv:1607.05129 [gr-qc].

[3] T. Thiemann, *Modern canonical quantum general relativity*, 1st ed. (Cambridge University Press, Cambridge, UK ; New York, 2008).

[4] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[5] R. Gambini and J. Pullin, *A first course in loop quantum gravity* (Oxford University Press, Oxford, 2011).

[6] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).

[7] T. Thiemann, Phys. Lett. B **380**, 257 (1996).

[8] T. Jacobson and L. Smolin, Nucl. Phys. B **299**, 295 (1988).

[9] C. Rovelli and L. Smolin, Nucl. Phys. B **331**, 80 (1990).

[10] C. Rovelli, *Quantum gravity* (Cambridge University Press, Cambridge, U.K., 2004).