COL 352	**Introduction to Automata and Theory of Computation**	**Holi 2022**

Problem Set 2

Out: February 18 — *Due:* **February 26**

Please attempt all problems. Read the instructions for problem sets posted on the announcement channel in MS Teams (and also here) carefully. Turn in your solutions via Gradescope by 11 pm on the due date.

1. Prove that $L_1 = \{bin(p) : p \text{ is a prime number}\}$ is not a regular language.

2. The n-th Fibonacci number is defined as $F_1 = 1, F_2 = 1$, and for all $n \geq 3, F_n = F_{n-1} + F_{n-2}$. Consider the language over $\Sigma = \{a\}$

$$L_2 = \{a^m \mid m = F_n\}$$

   Is L_2 regular? Justify your answer.

3. If A is any language, let $A_{\frac{1}{2}-}$ denote the set of all first halves of strings in A so that

$$A_{\frac{1}{2}-} = \{x \mid \text{for some } y, |x| = |y| \text{ and } xy \in A\}$$

   Show that if A is regular, then so is $A_{\frac{1}{2}-}$.

4. If A is any language, let $A_{\frac{1}{3}-\frac{1}{3}}$ denote the set of strings in A with the middle-third removed so that

$$A_{\frac{1}{3}-\frac{1}{3}} = \{xz \mid \text{for some } y, |x| = |y| = |z| \text{ and } xyz \in A\}$$

   Show that if A is regular, then $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular.

5. A 2-NFA A is a 5-tuple $A = (Q, S, t, F, \Delta)$ where Q is the set of states, S the set of start states, t is an accept state, the transition function

$$\Delta : Q \times (\Sigma \cup \{\#, \$\}) \to 2^{Q \times (\{L,R\})}$$

   Assume that whenever M accepts, it does so my moving the head (pointer) all the way to the right endmarker \$ and entering accept state t. In the subsequent two questions, we will try to prove that 2-NFAs accept only regular languages.

(a) Let $x = a_1 \ldots a_n \in \Sigma^*, a_i \in \Sigma$, $1 \leq i \leq n$. Let $a_0 = \#, a_{n+1} = \$$. Argue that x is not accepted by A if and only if there exist sets $W_i \subseteq Q, 0 \leq i \leq n+1$ such that the following hold:

- $S \subseteq W_0$
- If $u \in W_i$, $0 \leq i \leq n$, and $(v, R) \in \Delta(u, a_i)$, then $v \in W_{i+1}$
- If $u \in W_i$, $1 \leq i \leq n+1$, and $(v, L) \in \Delta(u, a_i)$, then $v \in W_{i-1}$ and

- $t \notin W_{n+1}$.

(b) Using the previous part, show that $L(A)$ is regular.

6. Let $M = (Q, \Sigma, q_0, \delta, F)$ be a DFA and let h be a state of M called its "home". A synchronizing sequence for M and h is a string $s \in \Sigma^*$ where $\hat{\delta}(q, s) = h$ for every $q \in Q$. Say that M is synchronizable if it has a synchronizing sequence for some state h. Prove that if M is a k-state synchronizable DFA, then it has a synchronizing sequence of length at most k^3. Can you improve upon this bound?

7. For every string $x \in \{0,1\}^+$ consider the number

$$0.x = x[1] \cdot \frac{1}{2} + x[2] \cdot \frac{1}{2^2} + \cdots + x[|x|] \cdot \frac{1}{2^{|x|}}$$

where $|x|$ is the length of x. For a real number $\theta \in [0,1]$ let

$$L_\theta = \{x : 0.x \leq \theta\}$$

Prove that L_θ is regular if and only if θ is rational.