# EV of Min Pairs



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**Problem**

*Given three random variables $X$, $Y$, $Z$ over uniform distribution $U(0,1)$. Find the expectation of $W = \min(X+Y, Y+Z)$.*

$$
\begin{aligned}
\mathbb{E}(W) &= \mathbb{E}\Big[(X+Y)I(X<Z)\Big] + \mathbb{E}\Big[(Y+Z)I(Z<X)\Big] \\
&= \mathbb{E}\Big[X \cdot I(X<Z)\Big] + \mathbb{E}\Big[Y \cdot I(X<Z)\Big] + \mathbb{E}\Big[Y \cdot I(Z<X)\Big] + \mathbb{E}\Big[Z \cdot I(Z<X)\Big] \\
&= \mathbb{E}[Y] + 2\mathbb{E}\Big[X \cdot I(X<Z)\Big] \\
&= \frac{1}{2} + 2\int_0^1 \int_0^z x dx dz \\
&= \frac{1}{2} + \int_0^1 z^2 dz \\
&= \frac{1}{2} + \frac{1}{3} \\
&= \frac{5}{6}
\end{aligned}
$$

**Problem**

*Given five random variables $a, b, c, d, e$ over uniform distribution $U(0,1)$. Find the expectation of $Z = \min(a+b, b+c, c+d, d+e)$.*

Let $X = \min(a+b, b+c)$ and $Y = \min(c+d, d+e)$, Then

$$\mathbb{E}[Z] = \mathbb{E}\Big[\mathbb{E}\big[\min(X,Y)|c\big]\Big] = \int_0^1 \mathbb{E}\big[\min(X,Y)|c\big] dc$$

Simplify the integrand,

$$
\begin{aligned}
\mathbb{E}[min(X,Y)|c] &= \int_0^{1+c} P(\min(X,Y) \geqslant x|c) dx \\
&= \int_0^{1+c} P(X \geqslant x|c) \cdot P(Y \geqslant x|c) dx \\
&= \int_0^{1+c} P^2(X \geqslant x|c) dx
\end{aligned}
$$

Inside the integral,

$$\begin{aligned} P(X \geqslant x|c) &= P((a+b \geqslant x, b+c \geqslant x)|c) \\ &= \int_{(x-c)^+}^{1} \int_{(x-b)^+}^{1} dadb \\ &= \int_{(x-c)^+}^{1} \big(1-(x-b)^+\big)db \end{aligned}$$

**When $0 \leqslant x \leqslant c$**, the above integral becomes

$$\int_0^x \big(1-(x-b)\big)db + \int_x^1 db = 1 - \frac{x^2}{2}$$

**When $c \leqslant x \leqslant 1$**, the above integral becomes

$$\int_{x-c}^x \big(1-(x-b)\big)db + \int_x^1 db = 1 - x + c - \frac{c^2}{2}$$

**When $1 \leqslant x \leqslant 1+c$**, the above integral becomes

$$\int_{x-c}^1 \big(1-(x-b)\big)db = \frac{3}{2} - 2x + \frac{x^2}{2} + c - \frac{c^2}{2}$$

Now the expectation can be rewritten as

$$\mathbb{E}[Z] = \int_0^1 G(c)dc$$

where the integrand is

$$G(c) = \int_0^{1+c} S^2(x,c)dx$$

and

$$S(x,c) = \begin{cases} 1 - \frac{x^2}{2}, & \text{when } 0 \leqslant x \leqslant c \\ 1 - x + c - \frac{c^2}{2}, & \text{when } c \leqslant x \leqslant 1 \\ \frac{3}{2} - 2x + \frac{x^2}{2} + c - \frac{c^2}{2}, & \text{when } 1 \leqslant x \leqslant 1+c \end{cases}$$

**Solution 1: Integration by parts**

Note that

$$\begin{aligned} G(c) &= \int_0^{1+c} S^2(x,c)dx = xS^2(x,c)\Big|_0^{1+c} - \int_0^{1+c} 2xS(x,c)S_x^{'}(x,c)dx \\ &= \int_0^{1+c} -2xS(x,c)S_x^{'}(x,c)dx \end{aligned}$$

because $S$ is the survival function and is $0$ on the right boundary $1+c$. To derive the integrand,

$$S_x^{'}(x,c) = \begin{cases} -x, & \text{when } 0 \leqslant x \leqslant c \\ -1, & \text{when } c \leqslant x \leqslant 1 \\ -2+x, & \text{when } 1 \leqslant x \leqslant 1+c \end{cases}$$

then the integrand is

$$T(x,c) = -2xS(x,c)S_x^{'}(x,c) = \begin{cases} 2x^2 - x^4, \\ 2x - 2x^2 + x(2c-c^2), \\ -x^4 + 6x^3 - 8x^2 + (2x-x^2)(3+2c-c^2), \end{cases}$$

Now we compute each case.

When $0 \leqslant x \leqslant c$,

$$T_1(x,c) = 2x^2 - x^4$$

then

$$\begin{aligned} G_1(c) &= \int_0^c T_1(x,c)dx \\ &= \int_0^c (2x^2 - x^4)dx \\ &= \frac{2c^3}{3} - \frac{c^5}{5} \end{aligned}$$

and

$$E_1 = \int_0^1 G_1(c)dc = \frac{2}{12} - \frac{1}{30} = \frac{2}{15}$$

Similarly, when $c \leqslant x \leqslant 1$,

$$T_2(x,c) = 2x - 2x^2 + x(2c-c^2)$$

then

$$\begin{aligned} G_2(c) &= \int_c^1 T_2(x,c)dx \\ &= \int_c^1 \big[-2x^2 + x(2+2c-c^2)\big]dx \\ &= -\frac{2}{3} + \frac{2c^3}{3} + 2 + 2c - c^2 - c^2(2+2c-c^2) \quad = \frac{1}{3} + c - \frac{3c^2}{2} - \frac{c^3}{3} + \frac{c^4}{2} \end{aligned}$$

and

$$E_2 = \int_0^1 G_2(c)dc = \frac{1}{3} + \frac{1}{2} - \frac{1}{2} - \frac{1}{12} + \frac{1}{10} = \frac{7}{20}$$

When $1 \leqslant x \leqslant 1+c$,

$$T_3(x,c) = -x^4 + 6x^3 - 8x^2 + (2x - x^2)(3 + 2c - c^2)$$

then

$$\begin{aligned}
G_3(c) &= \int_1^{1+c} T_3(x,c)dx \\
&= \int_1^{1+c} \big[-x^4 + 6x^3 - 8x^2 + (2x - x^2)(3+2c-c^2)\big]dx \\
&= -\frac{1}{5}\big[(1+c)^5 - 1\big] + \frac{3}{2}\big[(1+c)^4 - 1\big] - \frac{8}{3}\big[(1+c)^3 - 1\big] \\
&\quad + \Big\{\big[(1+c)^2 - 1\big] - \frac{1}{3}\big[(1+c)^3 - 1\big]\Big\}(3 + 2c - c62) \\
&= \frac{41}{30} - \frac{1}{5}(1+c)^5 + \frac{3}{2}(1+c)^4 - \frac{8}{3}(1+c)^3 \\
&\quad + (-\frac{c^3}{3} + c)(3 + 2c - c^2) \\
&= \frac{41}{30} - \frac{1}{5}(1+c)^5 + \frac{3}{2}(1+c)^4 - \frac{8}{3}(1+c)^3 \\
&\quad + \frac{c^5}{3} - \frac{2c^4}{3} - 2c^3 + 2c^2 + 3c
\end{aligned}$$

and

$$\begin{aligned}
E_3 &= \int_0^1 G_3(c)dc = \frac{41}{30} - \frac{1}{30}[2^6 - 1] + \frac{3}{10}[2^5 - 1] - \frac{2}{3}[2^4 - 1] + \frac{1}{18} - \frac{2}{15} - \frac{1}{2} + \frac{2}{3} + \frac{3}{2} \\
&= \frac{1}{10} + \frac{1}{18} \\
&= \frac{7}{45}
\end{aligned}$$

Finally,

$$\mathbb{E}[Z] = E_1 + E_2 + E_3 = \frac{2}{15} + \frac{7}{20} + \frac{14}{90} = \frac{23}{36} = 0.63\overline{8}$$

The numbers are verified in sympy, see src/intro/problem1-1.py.

**Solution 2: Without integration by parts**

When $0 \leqslant x \leqslant c$,

$$S_1(x,c) = 1 - \frac{x^2}{2}$$

then

$$\begin{aligned} G_1(c) &= \int_0^c S_1^2(x,c)dx \\ &= \int_0^c (1 - x^2 + \frac{x^4}{4})dx \\ &= c - \frac{c^3}{3} + \frac{c^5}{20} \end{aligned}$$

and

$$\begin{aligned} E_1 &= \int_0^1 G_1(c)dc \\ &= \int_0^1 (c - \frac{c^3}{3} + \frac{c^5}{20})dc \\ &= \frac{1}{2} - \frac{1}{12} + \frac{1}{120} \\ &= \frac{17}{40} \end{aligned}$$

When $c \leqslant x \leqslant 1$,

$$S_2(x,c) = 1 + c - \frac{c^2}{2} - x$$

then

$$\begin{aligned} G_2(c) &= \int_c^1 S_2^2(x,c)dx \\ &= \int_c^1 [(1 + c - \frac{c^2}{2})^2 - 2x(1 + c - \frac{c^2}{2}) + x^2]dx \\ &= (1 + c - \frac{c^2}{2})^2 - (1 + c - \frac{c^2}{2}) + \frac{1}{3} - c(1 + c - \frac{c^2}{2})^2 + c^2(1 + c - \frac{c^2}{2}) - \frac{c^3}{3} \\ &= \frac{1}{3} - \frac{c^2}{2} - \frac{c^3}{3} + \frac{3c^4}{4} - \frac{c^5}{4} \end{aligned}$$

and

$$\begin{aligned} E_2 &= \int_0^1 G_2(c)dc \\ &= \frac{1}{3} - \frac{1}{6} - \frac{1}{12} + \frac{3}{20} - \frac{1}{24} \\ &= \frac{23}{120} \end{aligned}$$

When $1 \leqslant x \leqslant 1+c$,

$$S_3(x,c) = \frac{x^2}{2} - 2x + \frac{3}{2} + c - \frac{c^2}{2}$$

then

$$\begin{aligned}
G_3(c) &= \int_1^{1+c} S_3^2(x,c)dx \\
&= \int_1^{1+c} \Big[\frac{x^4}{4} - 2x^3 + 4x^2 + x^2(\frac{3}{2} + c - \frac{c^2}{2}) - 4x(\frac{3}{2} + c - \frac{c^2}{2}) + (\frac{3}{2} + c - \frac{c^2}{2})^2\Big]dx \\
&= \frac{1}{20}\big[(1+c)^5 - 1\big] - \frac{1}{2}\big[(1+c)^4 - 1\big] + \frac{4}{3}\big[(1+c)^3 - 1\big] \\
&\quad + \frac{1}{3}\big[(1+c)^3 - 1\big](\frac{3}{2} + c - \frac{c^2}{2}) - 2\big[(1+c)^2 - 1\big](\frac{3}{2} + c - \frac{c^2}{2}) \\
&\quad + c(\frac{3}{2} + c - \frac{c^2}{2})^2 \\
&= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} + (\frac{c^3}{3} - c^2 - 3c)(\frac{3}{2} + c - \frac{c^2}{2}) \\
&\quad + c(\frac{3}{2} + c - \frac{c^2}{2})^2 \\
&= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} - (\frac{c^3}{6} + \frac{3c}{2})(\frac{3}{2} + c - \frac{c^2}{2}) \\
&= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} + \frac{c^5}{12} - \frac{c^4}{6} + \frac{c^3}{2} - \frac{3c^2}{2} - \frac{9c}{4}
\end{aligned}$$

and

$$\begin{aligned}
E_3 &= \int_0^1 G_3(c)dc \\
&= -\frac{53}{60} + \frac{2^6 - 1}{120} - \frac{2^5 - 1}{10} + \frac{2^4 - 1}{3} + \frac{1}{72} - \frac{1}{30} + \frac{1}{8} - \frac{1}{2} - \frac{9}{8} \\
&= \frac{1}{45}
\end{aligned}$$

Combining $E_1, E_2$, and $E_3$,

$$\mathbb{E}[Z] = E_1 + E_2 + E_3 = \frac{17}{40} + \frac{23}{120} + \frac{1}{45} = \frac{23}{36} = 0.63\overline{8}$$

The numbers are verified in sympy, see src/intro/problem1-2.py.