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Problem

Given three random variables X, Y, Z over uniform distribution $U(0,1)$. Find the expectation of $W = \min(X+Y, Y+Z)$.

$$\begin{aligned}
\mathbb{E}(W) &= \mathbb{E}\Big[(X+Y)I(X<Z)\Big] + \mathbb{E}\Big[(Y+Z)I(Z<X)\Big] \\
&= \mathbb{E}\Big[X \cdot I(X<Z)\Big] + \mathbb{E}\Big[Y \cdot I(X<Z)\Big] + \mathbb{E}\Big[Y \cdot I(Z<X)\Big] + \mathbb{E}\Big[Z \cdot I(Z<X)\Big] \\
&= \mathbb{E}[Y] + 2\mathbb{E}\Big[X \cdot I(X<Z)\Big] \\
&= \frac{1}{2} + 2\int_0^1 \int_0^z x dx dz \\
&= \frac{1}{2} + \int_0^1 z^2 dz \\
&= \frac{1}{2} + \frac{1}{3} \\
&= \frac{5}{6}
\end{aligned}$$

Problem

Given five random variables a, b, c, d, e over uniform distribution $U(0,1)$. Find the expectation of $Z = \min(a+b, b+c, c+d, d+e)$.

Let $X = \min(a+b, b+c)$ and $Y = \min(c+d, d+e)$, Then

$$\mathbb{E}[Z] = \mathbb{E}\Big[\mathbb{E}\big[\min(X,Y)|c\big]\Big] = \int_0^1 \mathbb{E}\big[\min(X,Y)|c\big] dc$$

Simplify the integrand,

$$\begin{aligned}\mathbb{E}[min(X,Y)|c] &= \int_0^{1+c} P(\min(X,Y) \geqslant x|c)dx \\ &= \int_0^{1+c} P(X \geqslant x|c) \cdot P(Y \geqslant x|c)dx \\ &= \int_0^{1+c} P^2(X \geqslant x|c)dx\end{aligned}$$

Inside the integral,

$$\begin{aligned}P(X \geqslant x|c) &= P((a+b \geqslant x, b+c \geqslant x)|c) \\ &= \int_{(x-c)^+}^1 \int_{(x-b)^+}^1 dadb \\ &= \int_{(x-c)^+}^1 \big(1-(x-b)^+\big)db\end{aligned}$$

When $0 \leqslant x \leqslant c$, the above integral becomes

$$\int_0^x \big(1-(x-b)\big)db + \int_x^1 db = 1 - \frac{x^2}{2}$$

When $c \leqslant x \leqslant 1$, the above integral becomes

$$\int_{x-c}^x \big(1-(x-b)\big)db + \int_x^1 db = 1 - x + c - \frac{c^2}{2}$$

When $1 \leqslant x \leqslant 1+c$, the above integral becomes

$$\int_{x-c}^1 \big(1-(x-b)\big)db = \frac{3}{2} - 2x + \frac{x^2}{2} + c - \frac{c^2}{2}$$

Now the expectation can be rewritten as

$$\mathbb{E}[Z] = \int_0^1 G(c)dc$$

where the integrand is

$$G(c) = \int_0^{1+c} S^2(x,c)dx$$

and

$$S(x,c) = \begin{cases} 1 - \frac{x^2}{2}, & \text{when } 0 \leqslant x \leqslant c \\ 1 - x + c - \frac{c^2}{2}, & \text{when } c \leqslant x \leqslant 1 \\ \frac{3}{2} - 2x + \frac{x^2}{2} + c - \frac{c^2}{2}, & \text{when } 1 \leqslant x \leqslant 1+c \end{cases}$$

Solution 1: Integration by parts

Note that

$$
\begin{aligned}
G(c) &= \int_0^{1+c} S^2(x,c)dx = xS^2(x,c)\Big|_0^{1+c} - \int_0^{1+c} 2xS(x,c)S_x^{'}(x,c)dx \\
&= \int_0^{1+c} -2xS(x,c)S_x^{'}(x,c)dx
\end{aligned}
$$

because S is the survival function and is 0 on the right boundary $1+c$. To derive the integrand,

$$
S_x^{'}(x,c) = \begin{cases} -x, & \text{when } 0 \leqslant x \leqslant c \\ -1, & \text{when } c \leqslant x \leqslant 1 \\ -2+x, & \text{when } 1 \leqslant x \leqslant 1+c \end{cases}
$$

then the integrand is

$$
T(x,c) = -2xS(x,c)S_x^{'}(x,c) = \begin{cases} 2x^2 - x^4, \\ 2x - 2x^2 + x(2c - c^2), \\ -x^4 + 6x^3 - 8x^2 + (2x - x^2)(3 + 2c - c^2), \end{cases}
$$

Now we compute each case.

When $0 \leqslant x \leqslant c$,

$$
T_1(x,c) = 2x^2 - x^4
$$

then

$$
\begin{aligned}
G_1(c) &= \int_0^c T_1(x,c)dx \\
&= \int_0^c (2x^2 - x^4)dx \\
&= \frac{2c^3}{3} - \frac{c^5}{5}
\end{aligned}
$$

and

$$
E_1 = \int_0^1 G_1(c)dc = \frac{2}{12} - \frac{1}{30} = \frac{2}{15}
$$

Similarly, when $c \leqslant x \leqslant 1$,

$$
T_2(x,c) = 2x - 2x^2 + x(2c - c^2)
$$

then

$$\begin{aligned} G_2(c) &= \int_c^1 T_2(x,c)dx \\ &= \int_c^1 \left[-2x^2 + x(2+2c-c^2) \right] dx \\ &= -\frac{2}{3} + \frac{2c^3}{3} + 2 + 2c - c^2 - c^2(2+2c-c^2) \quad = \frac{1}{3} + c - \frac{3c^2}{2} - \frac{c^3}{3} + \frac{c^4}{2} \end{aligned}$$

and

$$E_2 = \int_0^1 G_2(c)dc = \frac{1}{3} + \frac{1}{2} - \frac{1}{2} - \frac{1}{12} + \frac{1}{10} = \frac{7}{20}$$

When $1 \leqslant x \leqslant 1+c$,

$$T_3(x,c) = -x^4 + 6x^3 - 8x^2 + (2x - x^2)(3 + 2c - c^2)$$

then

$$\begin{aligned} G_3(c) &= \int_1^{1+c} T_3(x,c)dx \\ &= \int_1^{1+c} \left[-x^4 + 6x^3 - 8x^2 + (2x-x^2)(3+2c-c^2) \right] dx \\ &= -\frac{1}{5}\left[(1+c)^5 - 1\right] + \frac{3}{2}\left[(1+c)^4 - 1\right] - \frac{8}{3}\left[(1+c)^3 - 1\right] \\ &\quad + \left\{ \left[(1+c)^2 - 1\right] - \frac{1}{3}\left[(1+c)^3 - 1\right] \right\}(3 + 2c - c62) \\ &= \frac{41}{30} - \frac{1}{5}(1+c)^5 + \frac{3}{2}(1+c)^4 - \frac{8}{3}(1+c)^3 \\ &\quad + (-\frac{c^3}{3} + c)(3 + 2c - c^2) \\ &= \frac{41}{30} - \frac{1}{5}(1+c)^5 + \frac{3}{2}(1+c)^4 - \frac{8}{3}(1+c)^3 \\ &\quad + \frac{c^5}{3} - \frac{2c^4}{3} - 2c^3 + 2c^2 + 3c \end{aligned}$$

and

$$\begin{aligned} E_3 &= \int_0^1 G_3(c)dc = \frac{41}{30} - \frac{1}{30}[2^6 - 1] + \frac{3}{10}[2^5 - 1] - \frac{2}{3}[2^4 - 1] + \frac{1}{18} - \frac{2}{15} - \frac{1}{2} + \frac{2}{3} + \frac{3}{2} \\ &= \frac{1}{10} + \frac{1}{18} \\ &= \frac{7}{45} \end{aligned}$$

Finally,

$$\mathbb{E}[Z] = E_1 + E_2 + E_3 = \frac{2}{15} + \frac{7}{20} + \frac{14}{90} = \frac{23}{36} = 0.63\overline{8}$$

The numbers are verified in sympy, see src/intro/problem1-1.py.

Solution 2: Without integration by parts

When $0 \leqslant x \leqslant c$,

$$S_1(x, c) = 1 - \frac{x^2}{2}$$

then

$$\begin{aligned} G_1(c) &= \int_0^c S_1^2(x, c)dx \\ &= \int_0^c (1 - x^2 + \frac{x^4}{4})dx \\ &= c - \frac{c^3}{3} + \frac{c^5}{20} \end{aligned}$$

and

$$\begin{aligned} E_1 &= \int_0^1 G_1(c)dc \\ &= \int_0^1 (c - \frac{c^3}{3} + \frac{c^5}{20})dc \\ &= \frac{1}{2} - \frac{1}{12} + \frac{1}{120} \\ &= \frac{17}{40} \end{aligned}$$

When $c \leqslant x \leqslant 1$,

$$S_2(x, c) = 1 + c - \frac{c^2}{2} - x$$

then

$$\begin{aligned} G_2(c) &= \int_c^1 S_2^2(x, c)dx \\ &= \int_c^1 [(1 + c - \frac{c^2}{2})^2 - 2x(1 + c - \frac{c^2}{2}) + x^2]dx \\ &= (1 + c - \frac{c^2}{2})^2 - (1 + c - \frac{c^2}{2}) + \frac{1}{3} - c(1 + c - \frac{c^2}{2})^2 + c^2(1 + c - \frac{c^2}{2}) - \frac{c^3}{3} \\ &= \frac{1}{3} - \frac{c^2}{2} - \frac{c^3}{3} + \frac{3c^4}{4} - \frac{c^5}{4} \end{aligned}$$

and

$$\begin{aligned}E_2 &= \int_0^1 G_2(c)dc \\ &= \frac{1}{3} - \frac{1}{6} - \frac{1}{12} + \frac{3}{20} - \frac{1}{24} \\ &= \frac{23}{120}\end{aligned}$$

When $1 \leqslant x \leqslant 1+c$,

$$S_3(x,c) = \frac{x^2}{2} - 2x + \frac{3}{2} + c - \frac{c^2}{2}$$

then

$$\begin{aligned}G_3(c) &= \int_1^{1+c} S_3^2(x,c)dx \\ &= \int_1^{1+c} \Big[\frac{x^4}{4} - 2x^3 + 4x^2 + x^2(\frac{3}{2} + c - \frac{c^2}{2}) - 4x(\frac{3}{2} + c - \frac{c^2}{2}) + (\frac{3}{2} + c - \frac{c^2}{2})^2\Big]dx \\ &= \frac{1}{20}\big[(1+c)^5 - 1\big] - \frac{1}{2}\big[(1+c)^4 - 1\big] + \frac{4}{3}\big[(1+c)^3 - 1\big] \\ &\quad + \frac{1}{3}\big[(1+c)^3 - 1\big](\frac{3}{2} + c - \frac{c^2}{2}) - 2\big[(1+c)^2 - 1\big](\frac{3}{2} + c - \frac{c^2}{2}) \\ &\quad + c(\frac{3}{2} + c - \frac{c^2}{2})^2 \\ &= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} + (\frac{c^3}{3} - c^2 - 3c)(\frac{3}{2} + c - \frac{c^2}{2}) \\ &\quad + c(\frac{3}{2} + c - \frac{c^2}{2})^2 \\ &= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} - (\frac{c^3}{6} + \frac{3c}{2})(\frac{3}{2} + c - \frac{c^2}{2}) \\ &= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} + \frac{c^5}{12} - \frac{c^4}{6} + \frac{c^3}{2} - \frac{3c^2}{2} - \frac{9c}{4}\end{aligned}$$

and

$$\begin{aligned}E_3 &= \int_0^1 G_3(c)dc \\ &= -\frac{53}{60} + \frac{2^6 - 1}{120} - \frac{2^5 - 1}{10} + \frac{2^4 - 1}{3} + \frac{1}{72} - \frac{1}{30} + \frac{1}{8} - \frac{1}{2} - \frac{9}{8} \\ &= \frac{1}{45}\end{aligned}$$

Combining E_1, E_2, and E_3,

$$\mathbb{E}[Z] = E_1 + E_2 + E_3 = \frac{17}{40} + \frac{23}{120} + \frac{1}{45} = \frac{23}{36} = 0.63\overline{8}$$

The numbers are verified in sympy, see src/intro/problem1-2.py.