

## Problem

Given two random variables  $X$  and  $Y$  over uniform distribution  $U(0, 1)$ . Find the PDF of  $Z = X + Y$ .

Since  $X, Y \sim U(0, 1)$ , the PDFs of  $X$  and  $Y$  are  $f_X(x) = 1$  for  $0 < x < 1$  and  $f_Y(y) = 1$  for  $0 < y < 1$ . (Properties of uniform distribution is listed in [wikipedia](#))

Then

$$\begin{aligned} f_Z(z) &= P(Z = z) = P(X + Y = z) = \sum_x P(X = x, Y = z - x) \\ &= \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx \end{aligned} \quad (1)$$

Since  $f_X$  and  $f_Y$  have nonzero values only in  $(0, 1)$ ,

$$0 < x < 1, \text{ and } 0 < z - x < 1 \text{ or } z - 1 < x < z \quad (2)$$

To combine these 2 conditions to find common ranges for  $x$  with nonzero  $f_X$  and  $f_Y$ , there are 2 cases:

1. When  $0 < z < 1$ , we need to choose 0 over  $z - 1$  on the left side and  $z$  over 1 on the right side. So

$$f_Z(z) = \int_0^z f_X(x) f_Y(z - x) dx = \int_0^z 1 dx = z \quad (3)$$

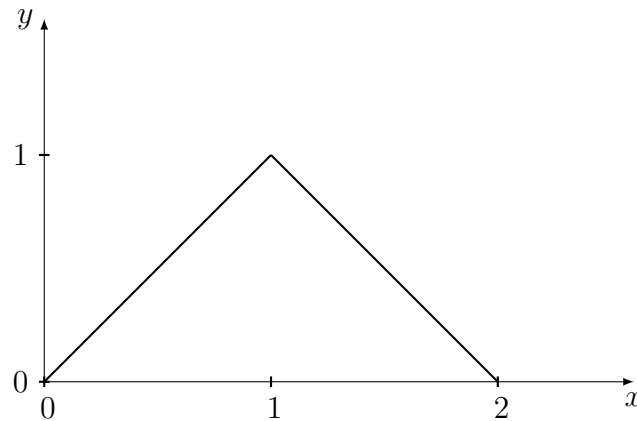
2. When  $1 < z < 2$ , we need to choose  $z - 1$  over 0 on the left side and 1 over  $z$  on the right side. So

$$f_Z(z) = \int_{z-1}^1 f_X(x) f_Y(z - x) dx = \int_{z-1}^1 1 dx = 2 - z \quad (4)$$

Therefore,

$$f_Z(z) = \begin{cases} z, & \text{if } 0 < z \leq 1 \\ 2 - z, & \text{if } 1 < z < 2 \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

The graph is 2 line segments



This [statisticalengineering.com post](#) and [stackoverflow discussion](#) discusses the same problem. For sum of  $n$  variables, see [Irwin-Hall Distribution](#).

This [post](#) discusses the exponential distribution case. If  $X, Y \sim \text{Exp}(\lambda)$ , the Equation (1) becomes

$$\begin{aligned}
 f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\
 &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\
 &= \lambda^2 \int_0^z e^{-\lambda z} dx \\
 &= \lambda^2 z e^{-\lambda z}
 \end{aligned} \tag{6}$$

where  $x$  and  $z-x$  have to be positive, which means  $0 < x < z$ .

More sum over other distributions are discussed [here](#).

Which one should we use to define  $((x-1)^- + 1)^+$ ?

$$(x)^+ \text{ or } (x)^- \text{ or } (x)^/ \text{ or } (x)^\sim \text{ or } (x)^*$$

See python code for graph. The function is

$$f(x) = \begin{cases} 0, & \text{when } x \leq 0 \\ x, & \text{when } 0 < x < 1 \\ 1, & \text{otherwise} \end{cases} \tag{7}$$