PDF of RV Summation

Problem

Given two random variables X and Y over uniform distribution $U(0,1)$. Find the PDF of $Z = X + Y$.

Since $X, Y \sim U(0,1)$, the PDFs of X and Y are $f_X(x) = 1$ for $0 < x < 1$ and $f_Y(y) = 1$ for $0 < y < 1$. (Properties of uniform distribution is listed in wikipedia)

Then

$$\begin{aligned} f_Z(z) = P(Z = z) = P(X + Y = z) = \sum_x P(X = x, Y = z - x) \\ = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx \end{aligned} \tag{1}$$

Since f_X and f_Y have nonzero values only in $(0,1)$,

$$0 < x < 1,\ and\ 0 < z - x < 1\ or\ z - 1 < x < z \tag{2}$$

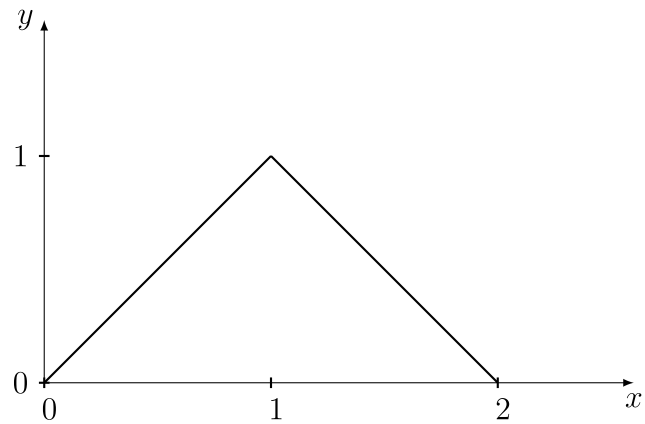
To combine these 2 conditions to find common ranges for x with nonzero f_X and f_Y, there are 2 cases:

1. When $0 < z < 1$, we need to choose 0 over $z - 1$ on the left side and z over 1 on the right side. So
$$f_Z(z) = \int_0^z f_X(x) f_Y(z - x) dx = \int_0^z 1 dx = z \tag{3}$$
2. When $1 < z < 2$, we need to choose $z - 1$ over 0 on the left side and 1 over z on the right side. So
$$f_Z(z) = \int_{z-1}^1 f_X(x) f_Y(z - x) dx = \int_{z-1}^1 1 dx = 2 - z \tag{4}$$

Therefore,

$$f_Z(z) = \begin{cases} z, & \text{if } 0 < z \leqslant 1 \\ 2 - z, & \text{if } 1 < z < 2 \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

The graph looks like



This statisticalengineering.com post and stackoverflow discussion discusses the same problem. For sum of n variables, see Irwin-Hall Distribution.
This post discusses the exponential distribution case. If $X, Y \sim Exp(\lambda)$, the Equation (1) becomes

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \lambda^2 \int_0^z e^{-\lambda z} dx \\ &= \lambda^2 z e^{-\lambda z} \end{aligned} \tag{6}$$

where x and $z-x$ have to be positive, which means $0 < x < z$.
More sum over other distributions are discussed here.