THE GAUSSIAN INTEGRAL

KEITH CONRAD

Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x, \quad J = \int_0^{\infty} e^{-x^2}\,\mathrm{d}x, \quad \text{and } K = \int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x.$$

These numbers are positive, and $J = I/(2\sqrt{2})$ and $K = I/\sqrt{2\pi}$.

Theorem. *With notation as above, $I = \sqrt{2\pi}$, or equivalently $J = \sqrt{\pi}/2$, or equivalently $K = 1$.*

We will give multiple proofs of this result. (Other lists of proofs are in [3] and [8].) The theorem is subtle because there is no simple antiderivative for $e^{-\frac{1}{2}x^2}$ (or e^{-x^2} or $e^{-\pi x^2}$). For comparison, $\int_0^\infty xe^{-\frac{1}{2}x^2}\,\mathrm{d}x$ can be computed using the antiderivative $-e^{-\frac{1}{2}x^2}$: this integral is 1.

1. First Proof: Polar coordinates

The most widely known proof uses multivariable calculus: express J^2 as a double integral and then pass to polar coordinates:

$$J^2 = \int_0^{\infty} e^{-x^2}\,\mathrm{d}x \int_0^{\infty} e^{-y^2}\,\mathrm{d}y = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$

This is a double integral over the first quadrant, which we will compute by using polar coordinates. In polar coordinates, the first quadrant is $\{(r,\theta) : r \geq 0 \text{ and } 0 \leq \theta \leq \pi/2\}$. Writing $x^2 + y^2 = r^2$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$,

$$\begin{aligned} J^2 &= \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{\infty} re^{-r^2}\,\mathrm{d}r \cdot \int_0^{\pi/2} \mathrm{d}\theta \\ &= -\frac{1}{2}e^{-r^2}\Big|_0^{\infty} \cdot \frac{\pi}{2} \\ &= \frac{1}{2}\cdot\frac{\pi}{2} \\ &= \frac{\pi}{4}. \end{aligned}$$

Taking square roots, $J = \sqrt{\pi}/2$. This method is due to Poisson [8, p. 3].

2. Second Proof: Another change of variables

Our next proof uses another change of variables to compute J^2, but this will only rely on single-variable calculus. As before, we have

$$J^2 = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y,$$

but instead of using polar coordinates we make a change of variables $x = yt$ with $\mathrm{d}x = y\,\mathrm{d}t$, so

$$J^2 = \int_0^\infty \int_0^\infty e^{-y^2(t^2+1)} y \,\mathrm{d}t\,\mathrm{d}y = \int_0^\infty \left(\int_0^\infty y e^{-y^2(t^2+1)}\,\mathrm{d}y \right) \mathrm{d}t.$$

Since $\int_0^\infty y e^{-ay^2}\,\mathrm{d}y = \frac{1}{2a}$ for $a > 0$, we have

$$J^2 = \int_0^\infty \frac{\mathrm{d}t}{2(t^2+1)} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

so $J = \sqrt{\pi}/2$. This approach is due to Laplace [6, pp. 94–96] and historically precedes the more familiar technique in the first proof above. We will see in our seventh proof that this was not Laplace's first method.

3. Third Proof: Differentiating under the integral sign

For $t > 0$, set

$$A(t) = \left(\int_0^t e^{-x^2}\,\mathrm{d}x \right)^2.$$

The integral we want to calculate is $A(\infty) = J^2$ and then take a square root.

Differentiating $A(t)$ with respect to t,

$$A'(t) = 2 \int_0^t e^{-x^2}\,\mathrm{d}x \cdot e^{-t^2} = 2e^{-t^2} \int_0^t e^{-x^2}\,\mathrm{d}x.$$

Let $x = ty$, so

$$A'(t) = 2e^{-t^2} \int_0^1 t e^{-t^2y^2}\,\mathrm{d}y = \int_0^1 2t e^{-(1+y^2)t^2}\,\mathrm{d}y.$$

The function under the integral sign is easily antidifferentiated *with respect to* t:

$$A'(t) = \int_0^1 -\frac{\partial}{\partial t} \frac{e^{-(1+y^2)t^2}}{1+y^2}\,\mathrm{d}y = -\frac{d}{dt} \int_0^1 \frac{e^{-(1+y^2)t^2}}{1+y^2}\,\mathrm{d}y.$$

Letting

$$B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x,$$

we have $A'(t) = -B'(t)$ for all $t > 0$, so there is a constant C such that

$$A(t) = -B(t) + C \tag{3.1}$$

for all $t > 0$. To find C, we let $t \to 0^+$ in (3.1). The left side tends to $(\int_0^0 e^{-x^2}\,\mathrm{d}x)^2 = 0$ while the right side tends to $-\int_0^1 \mathrm{d}x/(1+x^2) + C = -\pi/4 + C$. Thus $C = \pi/4$, so (3.1) becomes

$$\left(\int_0^t e^{-x^2}\,\mathrm{d}x \right)^2 = \frac{\pi}{4} - \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x.$$

Letting $t \to \infty$ in this equation, we obtain $J^2 = \pi/4$, so $J = \sqrt{\pi}/2$.

A comparison of this proof with the first proof is in [17].

4. Fourth Proof: A volume integral

Our next proof is due to T. P. Jameson [4] (and rediscovered by A. L. Delgado [2]). Revolve the curve $z = e^{-\frac{1}{2}x^2}$ in the xz-plane around the z-axis to produce the "bell surface" $z = e^{-\frac{1}{2}(x^2+y^2)}$ in $\mathbf{R}^3$. We will compute the volume V of the region below this surface and above the xy-plane in two ways.

First we compute the volume by horizontal slices, which are discs: $V = \int_0^1 A(z)\,\mathrm{d}z$ where $A(z)$ is the area of the disc formed by slicing the surface at height z. Writing the radius of this disc at height z as $r(z)$, we have $A(z) = \pi r(z)^2$ and we need to find $r(z)$. An equation of the surface at height z is $z = e^{-\frac{1}{2}r(z)^2}$, so $r(z)^2 = -2\log z$. Therefore $A(z) = -2\pi \log z$, so

$$V = \int_0^1 -2\pi \log z \,\mathrm{d}z = -2\pi \left(z\log z - z\right)\Big|_0^1 = -2\pi(-1 - \lim_{z\to 0^+} z\log z).$$

By L'Hopital's rule $\lim_{z\to 0^+} z\log z = 0$, so $V = 2\pi$. (A calculation of V with cylinders instead of discs is in [10].)

For a second method of computing V, let's use slices in planes $y =$ constant. We get $V = \int_{-\infty}^{\infty} A(y)\,\mathrm{d}y$, where $A(y_0)$ is the area under the surface lying in the plane of points $y = y_0$. Since

$$A(y) = \int_{-\infty}^{\infty} z(y)\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,\mathrm{d}x = e^{-\frac{1}{2}y^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x = e^{-\frac{1}{2}y^2} I,$$

we have $V = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} I\,\mathrm{d}y = I^2$. Comparing the two formulas for V, we have $2\pi = I^2$, so $I = \sqrt{2\pi}$.

5. Fifth Proof: The Γ-function

For any integer $n \geq 0$, we have $n! = \int_0^{\infty} t^n e^{-t}\,\mathrm{d}t$. For $x > 0$ we define

$$\Gamma(x) = \int_0^{\infty} t^x e^{-t}\,\frac{\mathrm{d}t}{t},$$

so $\Gamma(n) = (n-1)!$ when $n \geq 1$. Using integration by parts, $\Gamma(x+1) = x\Gamma(x)$. One of the basic properties of the Γ-function [13, pp. 193–194] is

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t. \tag{5.1}$$

Set $x = y = 1/2$:

$$\Gamma\left(\frac{1}{2}\right)^2 = \int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)}}.$$

Note

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} \sqrt{t}e^{-t}\,\frac{\mathrm{d}t}{t} = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t = \int_0^{\infty} \frac{e^{-x^2}}{x} 2x\,\mathrm{d}x = 2\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = 2J,$$

so $4J^2 = \int_0^1 \mathrm{d}t/\sqrt{t(1-t)}$. With the substitution $t = \sin^2\theta$,

$$4J^2 = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\,\mathrm{d}\theta}{\sin\theta\cos\theta} = 2\frac{\pi}{2} = \pi,$$

so $J = \sqrt{\pi}/2$. Equivalently, $\Gamma(1/2) = \sqrt{\pi}$. Any method that proves $\Gamma(1/2) = \sqrt{\pi}$ is also a method that calculates $\int_0^{\infty} e^{-x^2}\,\mathrm{d}x$.

6. Sixth Proof: Asymptotic Estimates

We will show $J = \sqrt{\pi}/2$ by a technique whose steps are based on [14, p. 371].

For $x \geq 0$, power series expansions show $1 + x \leq e^x \leq 1/(1-x)$. Reciprocating and replacing x with x^2, we get

$$1 - x^2 \leq e^{-x^2} \leq \frac{1}{1+x^2}. \tag{6.1}$$

for all $x \in \mathbf{R}$.

For any positive integer n, raise the terms in (6.1) to the nth power and integrate from 0 to 1:

$$\int_0^1 (1-x^2)^n \, dx \leq \int_0^1 e^{-nx^2} \, dx \leq \int_0^1 \frac{dx}{(1+x^2)^n}.$$

Under the changes of variables $x = \sin\theta$ on the left, $x = y/\sqrt{n}$ in the middle, and $x = \tan\theta$ on the right,

$$\int_0^{\pi/2} (\cos\theta)^{2n+1} \, d\theta \leq \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} \, dy \leq \int_0^{\pi/4} (\cos\theta)^{2n-2} \, d\theta. \tag{6.2}$$

Set $I_k = \int_0^{\pi/2} (\cos\theta)^k \, d\theta$, so $I_0 = \pi/2$, $I_1 = 1$, and (6.2) implies

$$\sqrt{n} I_{2n+1} \leq \int_0^{\sqrt{n}} e^{-y^2} \, dy \leq \sqrt{n} I_{2n-2}. \tag{6.3}$$

We will show that as $k \to \infty$, $kI_k^2 \to \pi/2$. Then

$$\sqrt{n} I_{2n+1} = \frac{\sqrt{n}}{\sqrt{2n+1}} \sqrt{2n+1} I_{2n+1} \to \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}$$

and

$$\sqrt{n} I_{2n-2} = \frac{\sqrt{n}}{\sqrt{2n-2}} \sqrt{2n-2} I_{2n-2} \to \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2},$$

so by (6.3) $\int_0^{\sqrt{n}} e^{-y^2} \, dy \to \sqrt{\pi}/2$. Thus $J = \sqrt{\pi}/2$.

To show $kI_k^2 \to \pi/2$, first we compute several values of I_k explicitly by a recursion. Using integration by parts,

$$I_k = \int_0^{\pi/2} (\cos\theta)^k \, d\theta = \int_0^{\pi/2} (\cos\theta)^{k-1} \cos\theta \, d\theta = (k-1)(I_{k-2} - I_k),$$

so

$$I_k = \frac{k-1}{k} I_{k-2}. \tag{6.4}$$

Using (6.4) and the initial values $I_0 = \pi/2$ and $I_1 = 1$, the first few values of I_k are computed and listed in Table 1.

k	I_k	k	I_k
0	$\pi/2$	1	1
2	$(1/2)(\pi/2)$	3	$2/3$
4	$(3/8)(\pi/2)$	5	$8/15$
6	$(15/48)(\pi/2)$	7	$48/105$

Table 1

From Table 1 we see that

$$I_{2n}I_{2n+1} = \frac{1}{2n+1}\frac{\pi}{2} \tag{6.5}$$

for $0 \leq n \leq 3$, and this can be proved for all n by induction using (6.4). Since $0 \leq \cos\theta \leq 1$ for $\theta \in [0, \pi/2]$, we have $I_k \leq I_{k-1} \leq I_{k-2} = \frac{k}{k-1}I_k$ by (6.4), so $I_{k-1} \sim I_k$ as $k \to \infty$. Therefore (6.5) implies

$$I_{2n}^2 \sim \frac{1}{2n}\frac{\pi}{2} \Longrightarrow (2n)I_{2n}^2 \to \frac{\pi}{2}$$

as $n \to \infty$. Then

$$(2n+1)I_{2n+1}^2 \sim (2n)I_{2n}^2 \to \frac{\pi}{2}$$

as $n \to \infty$, so $kI_k^2 \to \pi/2$ as $k \to \infty$. This completes our proof that $J = \sqrt{\pi}/2$.

Remark 6.1. This proof is closely related to the fifth proof using the Γ-function. Indeed, by (5.1)

$$\frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2}+\frac{1}{2})} = \int_0^1 t^{(k+1)/2+1}(1-t)^{1/2-1}\,\mathrm{d}t,$$

and with the change of variables $t = (\cos\theta)^2$ for $0 \leq \theta \leq \pi/2$, the integral on the right is equal to $2\int_0^{\pi/2}(\cos\theta)^k\,\mathrm{d}\theta = 2I_k$, so (6.5) is the same as

$$\begin{aligned}
\frac{\pi}{2(2n+1)} &= I_{2n}I_{2n+1} \\
&= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+2}{2})}\frac{\Gamma(\frac{2n+2}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+3}{2})} \\
&= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})^2}{4\Gamma(\frac{2n+1}{2}+1)} \\
&= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})^2}{4\frac{2n+1}{2}\Gamma(\frac{2n+1}{2})} \\
&= \frac{\Gamma(\frac{1}{2})^2}{2(2n+1)},
\end{aligned}$$

or equivalently $\Gamma(1/2)^2 = \pi$. We saw in the fifth proof that $\Gamma(1/2) = \sqrt{\pi}$ if and only if $J = \sqrt{\pi}/2$.

7. Seventh Proof: The original proof

The original proof that $J = \sqrt{\pi}/2$ is due to Laplace [7] in 1774. (An English translation of Laplace's article is mentioned in the bibliographic citation for [7], with preliminary comments on that article in [15].) He wanted to compute

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{-\log x}}. \tag{7.1}$$

Setting $y = \sqrt{-\log x}$, this integral is $2\int_0^\infty e^{-y^2}\,\mathrm{d}y = 2J$, so we expect (7.1) to be $\sqrt{\pi}$.

Laplace's starting point for evaluating (7.1) was a formula of Euler:

$$\int_0^1 \frac{x^r\,\mathrm{d}x}{\sqrt{1-x^{2s}}}\int_0^1 \frac{x^{s+r}\,\mathrm{d}x}{\sqrt{1-x^{2s}}} = \frac{1}{s(r+1)}\frac{\pi}{2} \tag{7.2}$$

for positive r and s. (Laplace himself said this formula held "whatever be" r or s, but if $s < 0$ then the number under the square root is negative.) Accepting (7.2), let $r \to 0$ in it to get

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{2s}}} \int_0^1 \frac{x^s\,\mathrm{d}x}{\sqrt{1-x^{2s}}} = \frac{1}{s}\frac{\pi}{2}. \tag{7.3}$$

Now let $s \to 0$ in (7.3). Then $1 - x^{2s} \sim -2s\log x$ by L'Hopital's rule, so (7.3) becomes

$$\left(\int_0^1 \frac{\mathrm{d}x}{\sqrt{-\log x}}\right)^2 = \pi.$$

Thus (7.1) is $\sqrt{\pi}$.

Euler's formula (7.2) looks mysterious, but we have met it before. In the formula let $x^s = \cos\theta$ with $0 \leq \theta \leq \pi/2$. Then $x = (\cos\theta)^{1/s}$, and after some calculations (7.2) turns into

$$\int_0^{\pi/2} (\cos\theta)^{(r+1)/s-1}\,\mathrm{d}\theta \int_0^{\pi/2} (\cos\theta)^{(r+1)/s}\,\mathrm{d}\theta = \frac{1}{(r+1)/s}\frac{\pi}{2}. \tag{7.4}$$

We used the integral $I_k = \int_0^{\pi/2} (\cos\theta)^k\,\mathrm{d}\theta$ before when k is a nonnegative integer. This notation makes sense when k is any positive real number, and then (7.4) assumes the form $I_\alpha I_{\alpha+1} = \frac{1}{\alpha+1}\frac{\pi}{2}$ for $\alpha = (r+1)/s-1$, which is (6.5) with a possibly nonintegral index. Letting $r = 0$ and $s = 1/(2n+1)$ in (7.4) recovers (6.5). Letting $s \to 0$ in (7.3) corresponds to letting $n \to \infty$ in (6.5), so the 6th proof is in essence a more detailed version of Laplace's 1774 argument.

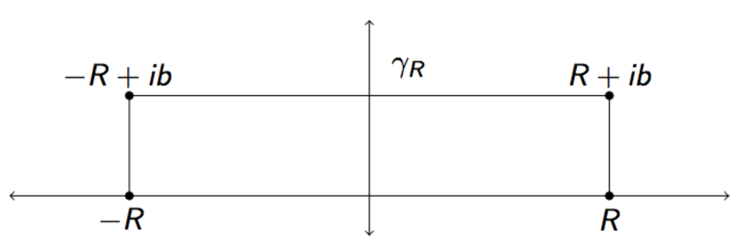
8. Eighth Proof: Contour Integration

We will calculate $\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x$ using contour integrals and the residue theorem. However, we can't just integrate $e^{-z^2/2}$, as this function has no poles. For a long time nobody knew how to handle this integral using contour integration. For instance, in 1914 Watson [16, p. 79] wrote at the end of his book "Cauchy's theorem cannot be employed to evaluate all definite integrals; thus $\int_0^\infty e^{-x^2}\,\mathrm{d}x$ has not been evaluated except by other methods." In the 1940s several contour integral solutions were published using awkward contours such as parallelograms [9], [11, Sect. 5] (see [1, Exer. 9, p. 113] for a recent appearance). Our approach will follow Kneser [5, p. 121] (see also [12, pp. 413–414] or [18]), using a rectangular contour and the function

$$\frac{e^{-z^2/2}}{1 - e^{-\sqrt{\pi}(1+i)z}}.$$

This function comes out of nowhere, so our first task is to motivate the introduction of this function.

We seek a meromorphic function $f(z)$ to integrate around the rectangular contour γ_R in the figure below, with vertices at $-R$, R, $R+ib$, and $-R+ib$, where b will be fixed and we let $R \to \infty$.



Suppose $f(z) \to 0$ along the right and left sides of γ_R uniformly as $R \to \infty$. Then by applying the residue theorem and letting $R \to \infty$, we would obtain (if the integrals converge)

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \int_{\infty}^{-\infty} f(x+ib)\,\mathrm{d}x = 2\pi i \sum_a \mathrm{Res}_{z=a} f(z),$$

where the sum is over poles of $f(z)$ with imaginary part between 0 and b. This is equivalent to

$$\int_{-\infty}^{\infty} (f(x) - f(x+ib))\,\mathrm{d}x = 2\pi i \sum_a \mathrm{Res}_{z=a} f(z).$$

Therefore we want $f(z)$ to satisfy

$$f(z) - f(z+ib) = e^{-z^2/2}, \tag{8.1}$$

where $f(z)$ and b need to be determined.

Let's try $f(z) = e^{-z^2/2}/d(z)$, for an unknown denominator $d(z)$ whose zeros are poles of $f(z)$. We want $f(z)$ to satisfy

$$f(z) - f(z+\tau) = e^{-z^2/2} \tag{8.2}$$

for some τ (which will *not* be purely imaginary, so (8.1) doesn't quite work, but (8.1) is only motivation). Substituting $e^{-z^2/2}/d(z)$ for $f(z)$ in (8.2) gives us

$$e^{-z^2/2}\left(\frac{1}{d(z)} - \frac{e^{-\tau z - \tau^2/2}}{d(z+\tau)}\right) = e^{-z^2/2}. \tag{8.3}$$

Suppose $d(z+\tau) = d(z)$. Then (8.3) implies

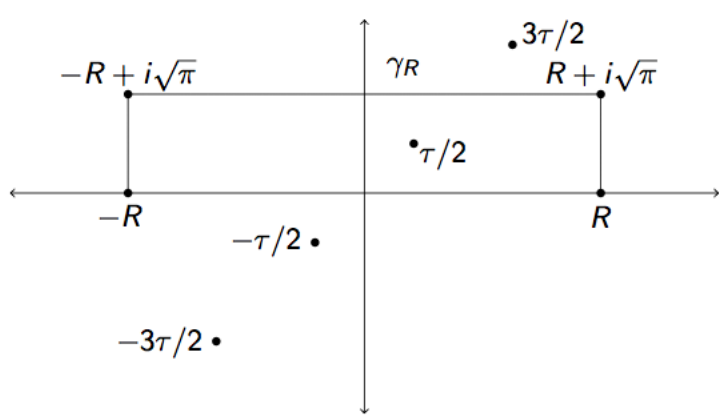
$$d(z) = 1 - e^{-\tau z - \tau^2/2},$$

and with this definition of $d(z)$, $f(z)$ satisfies (8.2) if and only if $e^{\tau^2} = 1$, or equivalently $\tau^2 \in 2\pi i \mathbf{Z}$. The simplest nonzero solution is $\tau = \sqrt{\pi}(1+i)$. From now on this is the value of τ, so $e^{-\tau^2/2} = e^{-i\pi} = -1$ and then

$$f(z) = \frac{e^{-z^2/2}}{d(z)} = \frac{e^{-z^2/2}}{1 + e^{-\tau z}},$$

which is Kneser's function mentioned earlier. This function satisfies (8.2) and we henceforth ignore the motivation (8.1). Poles of $f(z)$ are at odd integral multiples of $\tau/2$.

We will integrate this $f(z)$ around the rectangular contour γ_R below, whose height is $\mathrm{Im}(\tau)$.



The poles of $f(z)$ nearest the origin are plotted in the figure; they lie along the line $y = x$. The only pole of $f(z)$ inside γ_R (for $R > \sqrt{\pi}/2$) is at $\tau/2$, so by the residue theorem

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}_{z=\tau/2} f(z) = 2\pi i \frac{e^{-\tau^2/8}}{(-\tau)e^{-\tau^2/2}} = \frac{2\pi i e^{3\tau^2/8}}{-\sqrt{\pi}(1+i)} = \sqrt{2\pi}.$$

As $R \to \infty$, the value of $|f(z)|$ tends to 0 uniformly along the left and right sides of γ_R, so

$$\begin{aligned}\sqrt{2\pi} &= \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \int_{\infty+i\sqrt{\pi}}^{-\infty+i\sqrt{\pi}} f(z)\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} f(x+i\sqrt{\pi})\,\mathrm{d}x.\end{aligned}$$

In the second integral, write $i\sqrt{\pi}$ as $\tau - \pi$ and use (real) translation invariance of $\mathrm{d}x$ to obtain

$$\begin{aligned}\sqrt{2\pi} &= \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} f(x+\tau)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (f(x) - f(x+\tau))\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x \quad \text{by (8.2).}\end{aligned}$$

9. Ninth Proof: Stirling's Formula

Besides the integral formula $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x = \sqrt{2\pi}$ that we have been discussing, another place in mathematics where $\sqrt{2\pi}$ appears is in Stirling's formula:

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n} \quad \text{as } n \to \infty.$$

In 1730 De Moivre proved $n! \sim C(n^n/e^n)\sqrt{n}$ for some positive number C without being able to determine C. Stirling soon thereafter showed $C = \sqrt{2\pi}$ and wound up having the whole formula named after him. We will show that determining that the constant C in Stirling's formula is $\sqrt{2\pi}$ is equivalent to showing that $J = \sqrt{\pi}/2$ (or, equivalently, that $I = \sqrt{2\pi}$).

Applying (6.4) repeatedly,

$$\begin{aligned}I_{2n} &= \frac{2n-1}{2n} I_{2n-2} \\ &= \frac{(2n-1)(2n-3)}{(2n)(2n-2)} I_{2n-4} \\ &\;\;\vdots \\ &= \frac{(2n-1)(2n-3)(2n-5)\cdots(5)(3)(1)}{(2n)(2n-2)(2n-4)\cdots(6)(4)(2)} I_0.\end{aligned}$$

Inserting $(2n-2)(2n-4)(2n-6)\cdots(6)(4)(2)$ in the top and bottom,

$$I_{2n} = \frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\cdots(6)(5)(4)(3)(2)(1)}{(2n)((2n-2)(2n-4)\cdots(6)(4)(2))^2}\frac{\pi}{2} = \frac{(2n-1)!}{2n(2^{n-1}(n-1)!)^2}\frac{\pi}{2}.$$

Applying De Moivre's asymptotic formula $n! \sim C(n/e)^n\sqrt{n}$, ,

$$I_{2n} \sim \frac{C((2n-1)/e)^{2n-1}\sqrt{2n-1}}{2n(2^{n-1}C((n-1)/e)^{n-1}\sqrt{n-1})^2}\frac{\pi}{2} = \frac{(2n-1)^{2n}\frac{1}{2n-1}\sqrt{2n-1}}{2n\cdot 2^{2(n-1)}Ce(n-1)^{2n}\frac{1}{(n-1)^2}(n-1)}\frac{\pi}{2}$$

as $n \to \infty$. For any $a \in \mathbf{R}$, $(1 + a/n)^n \to e^a$ as $n \to \infty$, so $(n + a)^n \sim e^a n^n$. Substituting this into the above formula with $a = -1$ and n replaced by $2n$,

$$I_{2n} \sim \frac{e^{-1}(2n)^{2n}\frac{1}{\sqrt{2n}}}{2n \cdot 2^{2(n-1)}Ce(e^{-1}n^n)^2\frac{1}{n^2}n}\frac{\pi}{2} = \frac{\pi}{C\sqrt{2n}}. \tag{9.1}$$

Since $I_{k-1} \sim I_k$, the outer terms in (6.3) are both asymptotic to $\sqrt{n}I_{2n} \sim \pi/(C\sqrt{2})$ by (9.1). Therefore

$$\int_0^{\sqrt{n}} e^{-y^2}\,\mathrm{d}y \to \frac{\pi}{C\sqrt{2}}$$

as $n \to \infty$, so $J = \pi/(C\sqrt{2})$. Therefore $C = \sqrt{2\pi}$ if and only if $J = \sqrt{\pi}/2$.

10. Tenth Proof: Fourier transforms

For a continuous function $f\colon \mathbf{R} \to \mathbf{C}$ that is rapidly decreasing at $\pm\infty$, its Fourier transform is the function $\mathcal{F}f\colon \mathbf{R} \to \mathbf{C}$ defined by

$$(\mathcal{F}f)(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\,\mathrm{d}x.$$

For example, $(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$.

Here are three properties of the Fourier transform.

- If f is differentiable, then after using differentiation under the integral sign on the Fourier transform of f we obtain
$$(\mathcal{F}f)'(y) = \int_{-\infty}^{\infty} -ixf(x)e^{-ixy}\,\mathrm{d}x = -i(\mathcal{F}(xf(x)))(y).$$
- Using integration by parts on the Fourier transform of f, with $u = f(x)$ and $\mathrm{d}v = e^{-ixy}\,\mathrm{d}x$, we obtain
$$\mathcal{F}(f')(y) = iy(\mathcal{F}f)(y).$$
- If we apply the Fourier transform twice then we recover the original function up to interior and exterior scaling:

$$(\mathcal{F}^2 f)(x) = 2\pi f(-x). \tag{10.1}$$

Let's show the appearance of 2π in (10.1) is equivalent to the evaluation of I as $\sqrt{2\pi}$.

Fixing $a > 0$, set $f(x) = e^{-ax^2}$, so

$$f'(x) = -2axf(x).$$

Applying the Fourier transform to both sides of this equation implies $iy(\mathcal{F}f)(y) = -2a\frac{1}{-i}(\mathcal{F}f)'(y)$, which simplifies to $(\mathcal{F}f)'(y) = -\frac{1}{2a}y(\mathcal{F}f)(y)$. The general solution of $g'(y) = -\frac{1}{2a}yg(y)$ is $g(y) = Ce^{-y^2/(4a)}$, so

$$(\mathcal{F}f)(y) = Ce^{-y^2/(4a)}$$

for some constant C. Letting $a = \frac{1}{2}$, so $f(x) = e^{-x^2/2}$, we obtain

$$(\mathcal{F}f)(y) = Ce^{-y^2/2} = Cf(y).$$

Setting $y = 0$, the left side is $(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = I$, so $I = Cf(0) = C$.

Applying the Fourier transform to both sides of the equation $(\mathcal{F}f)(y) = Cf(y)$, we get $2\pi f(-x) = C(\mathcal{F}f)(x) = C^2 f(x)$. At $x = 0$ this becomes $2\pi = C^2$, so $I = C = \pm\sqrt{2\pi}$. Since $I > 0$, the number I is $\sqrt{2\pi}$. If we didn't know the constant on the right side of (10.1) were 2π, whatever its value is would wind up being C^2, so saying 2π appears on the right side of (10.1) is equivalent to saying $I = \sqrt{2\pi}$.

References

[1] C. A. Berenstein and R. Gay, *Complex Variables*, Springer-Verlag, New York, 1991.
[2] A. L. Delgado, "A Calculation of $\int_0^\infty e^{-x^2}\,dx$", The College Math. J. **34** (2003), 321–323.
[3] H. Iwasawa, "Gaussian Integral Puzzle", Math. Intelligencer **31** (2009), 38–41.
[4] T. P. Jameson, "The Probability Integral by Volume of Revolution," Mathematical Gazette **78** (1994), 339–340.
[5] H. Kneser, *Funktionentheorie*, Vandenhoeck and Ruprecht, 1958.
[6] P. S. Laplace, *Théorie Analytique des Probabilités*, Courcier, 1812.
[7] P. S. Laplace, "Mémoire sur la probabilité des causes par les évènemens," Oeuvres Complétes **8**, 27–65. (English trans. by S. Stigler as "Memoir on the Probability of Causes of Events", Statistical Science **1** (1986), 364–378.)
[8] P. M. Lee, `http://www.york.ac.uk/depts/maths/histstat/normal_history.pdf`.
[9] L. Mirsky, *The Probability Integral*, Math. Gazette **33** (1949), 279. Online at `www.jstor.org/stable/3611303`.
[10] C. P. Nicholas and R. C. Yates, "The Probability Integral," Amer. Math. Monthly **57** (1950), 412–413.
[11] G. Polya, "Remarks on Computing the Probability Integral in One and Two Dimensions," pp. 63–78 in *Berkeley Symp. on Math. Statist. and Prob.*, Univ. California Press, 1949.
[12] R. Remmert, *Theory of Complex Functions*, Springer-Verlag, 1991.
[13] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, 1976.
[14] M. Spivak, *Calculus*, W. A. Benjamin, 1967.
[15] S. Stigler, "Laplace's 1774 Memoir on Inverse Probability", Statistical Science **1** (1986), 359–363.
[16] G. N. Watson, *Complex Integration and Cauchy's Theorem*, Cambridge Univ. Press, Cambridge, 1914.
[17] `http://gowers.wordpress.com/2007/10/04/when-are-two-proofs-essentially-the-same/#comment-239`.
[18] `http://math.stackexchange.com/questions/34767/int-infty-infty-e-x2-dx-with-complex-analysis`.