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Problem

Given three random variables X, Y, Z over uniform distribution $U(0, 1)$. Find the expectation of $W = \min(X + Y, Y + Z)$.

$$\begin{aligned}
 \mathbb{E}(W) &= \mathbb{E}[(X + Y)I(X < Z)] + \mathbb{E}[(Y + Z)I(Z < X)] \\
 &= \mathbb{E}[X \cdot I(X < Z)] + \mathbb{E}[Y \cdot I(X < Z)] + \mathbb{E}[Y \cdot I(Z < X)] + \mathbb{E}[Z \cdot I(Z < X)] \\
 &= \mathbb{E}[Y] + 2\mathbb{E}[X \cdot I(X < Z)] \\
 &= \frac{1}{2} + 2 \int_0^1 \int_0^z x dx dz \\
 &= \frac{1}{2} + \int_0^1 z^2 dz \\
 &= \frac{1}{2} + \frac{1}{3} \\
 &= \frac{5}{6}
 \end{aligned}$$

Problem

Given five random variables a, b, c, d, e over uniform distribution $U(0, 1)$. Find the expectation of $Z = \min(a + b, b + c, c + d, d + e)$.

Let $X = \min(a + b, b + c)$ and $Y = \min(c + d, d + e)$, Then

$$\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[\min(X, Y)|c]] = \int_0^1 \mathbb{E}[\min(X, Y)|c] dc$$

Simplify the integrand,

$$\begin{aligned}\mathbb{E}[\min(X, Y)|c] &= \int_0^{1+c} P(\min(X, Y) \geq x|c)dx \\ &= \int_0^{1+c} P(X \geq x|c) \cdot P(Y \geq x|c)dx \\ &= \int_0^{1+c} P^2(X \geq x|c)dx\end{aligned}$$

Inside the integral,

$$\begin{aligned}P(X \geq x|c) &= P((a + b \geq x, b + c \geq x)|c) \\ &= \int_{(x-c)^+}^1 \int_{(x-b)^+}^1 dadb \\ &= \int_{(x-c)^+}^1 (1 - (x - b)^+)db\end{aligned}$$

When $0 \leq x \leq c$, the above integral becomes

$$\int_0^x (1 - (x - b))db + \int_x^1 db = 1 - \frac{x^2}{2}$$

When $c \leq x \leq 1$, the above integral becomes

$$\int_{x-c}^x (1 - (x - b))db + \int_x^1 db = 1 - x + c - \frac{c^2}{2}$$

When $1 \leq x \leq 1 + c$, the above integral becomes

$$\int_{x-c}^1 (1 - (x - b))db = \frac{3}{2} - 2x + \frac{x^2}{2} + c - \frac{c^2}{2}$$

Now the expectation can be rewritten as

$$\mathbb{E}[Z] = \int_0^{1+c} G(c)dc$$

where the integrand is

$$G(c) = \int_0^{1+c} S^2(x, c)dx$$

and

$$S(x, c) = \begin{cases} 1 - \frac{x^2}{2}, & \text{when } 0 \leq x \leq c \\ 1 - x + c - \frac{c^2}{2}, & \text{when } c \leq x \leq 1 \\ \frac{3}{2} - 2x + \frac{x^2}{2} + c - \frac{c^2}{2}, & \text{when } 1 \leq x \leq 1 + c \end{cases}$$

Solution 1: Integration by parts

Note that

$$\begin{aligned} G(c) &= \int_0^{1+c} S^2(x, c) dx = xS^2(x, c) \Big|_0^{1+c} - \int_0^{1+c} 2xS(x, c)S'_x(x, c) dx \\ &= \int_0^{1+c} -2xS(x, c)S'_x(x, c) dx \end{aligned}$$

To derive the integrand,

$$S'_x(x, c) = \begin{cases} -x, & \text{when } 0 \leq x \leq c \\ -1, & \text{when } c \leq x \leq 1 \\ -2 + x, & \text{when } 1 \leq x \leq 1 + c \end{cases}$$

then the integrand is

$$T(x, c) = -2xS(x, c)S'_x(x, c) = \begin{cases} 2x^2 - x^4, \\ 2x - 2x^2 + x(2c - c^2), \\ -x^4 + 6x^3 - 8x^2 + (2x - x^2)(3 + 2c - c^2), \end{cases}$$

Now we compute each case.

When $0 \leq x \leq c$,

$$T_1(x, c) = 2x^2 - x^4$$

then

$$\begin{aligned} G_1(c) &= \int_0^c T_1(x, c) dx \\ &= \int_0^c (2x^2 - x^4) dx \\ &= \frac{2c^3}{3} - \frac{c^5}{5} \end{aligned}$$

and

$$E_1 = \int_0^1 G_1(c) dc = \frac{2}{12} - \frac{1}{30} = \frac{2}{15}$$

Similarly, when $c \leq x \leq 1$,

$$T_2(x, c) = 2x - 2x^2 + x(2c - c^2)$$

then

$$\begin{aligned}
 G_2(c) &= \int_c^1 T_2(x, c) dx \\
 &= \int_c^1 [-2x^2 + x(2 + 2c - c^2)] dx \\
 &= -\frac{2}{3} + \frac{2c^3}{3} + 2 + 2c - c^2 - c^2(2 + 2c - c^2) = \frac{1}{3} + c - \frac{3c^2}{2} - \frac{c^3}{3} + \frac{c^4}{2}
 \end{aligned}$$

and

$$E_2 = \int_0^1 G_2(c) dc = \frac{1}{3} + \frac{1}{2} - \frac{1}{2} - \frac{1}{12} + \frac{1}{10} = \frac{7}{20}$$

When $1 \leq x \leq 1 + c$,

$$T_3(x, c) = -x^4 + 6x^3 - 8x^2 + (2x - x^2)(3 + 2c - c^2)$$

then

$$\begin{aligned}
 G_3(c) &= \int_1^{1+c} T_3(x, c) dx \\
 &= \int_1^{1+c} [-x^4 + 6x^3 - 8x^2 + (2x - x^2)(3 + 2c - c^2)] dx \\
 &= -\frac{1}{5}[(1+c)^5 - 1] + \frac{3}{2}[(1+c)^4 - 1] - \frac{8}{3}[(1+c)^3 - 1] \\
 &\quad + \{[(1+c)^2 - 1] - \frac{1}{3}[(1+c)^3 - 1]\}(3 + 2c - c^2) \\
 &= \frac{41}{30} - \frac{1}{5}(1+c)^5 + \frac{3}{2}(1+c)^4 - \frac{8}{3}(1+c)^3 \\
 &\quad + \left(-\frac{c^3}{3} + c\right)(3 + 2c - c^2) \\
 &= \frac{41}{30} - \frac{1}{5}(1+c)^5 + \frac{3}{2}(1+c)^4 - \frac{8}{3}(1+c)^3 \\
 &\quad + \frac{c^5}{3} - \frac{2c^4}{3} - 2c^3 + 2c^2 + 3c
 \end{aligned}$$

and

$$\begin{aligned}
 E_3 &= \int_0^1 G_3(c) dc = \frac{41}{30} - \frac{1}{30}[2^6 - 1] + \frac{3}{10}[2^5 - 1] - \frac{2}{3}[2^4 - 1] + \frac{1}{18} - \frac{2}{15} - \frac{1}{2} + \frac{2}{3} + \frac{3}{2} \\
 &= \frac{1}{10} + \frac{1}{18} \\
 &= \frac{14}{90}
 \end{aligned}$$

Finally,

$$\mathbb{E}[Z] = E_1 + E_2 + E_3 = \frac{2}{15} + \frac{7}{20} + \frac{14}{90} = \frac{23}{36} = 0.63\bar{8}$$

Solution 2: Without integration by parts

When $0 \leq x \leq c$,

$$S_1(x, c) = 1 - \frac{x^2}{2}$$

then

$$\begin{aligned} G_1(c) &= \int_0^c S_1^2(x, c) dx \\ &= \int_0^c \left(1 - x^2 + \frac{x^4}{4}\right) dx \\ &= c - \frac{c^3}{3} + \frac{c^5}{20} \end{aligned}$$

and

$$\begin{aligned} E_1 &= \int_0^1 G_1(c) dc \\ &= \int_0^1 \left(c - \frac{c^3}{3} + \frac{c^5}{20}\right) dc \\ &= \frac{1}{2} - \frac{1}{12} + \frac{1}{120} \\ &= \frac{17}{40} \end{aligned}$$

When $c \leq x \leq 1$,

$$S_2(x, c) = 1 + c - \frac{c^2}{2} - x$$

then

$$\begin{aligned} G_2(c) &= \int_c^1 S_2^2(x, c) dx \\ &= \int_c^1 \left[\left(1 + c - \frac{c^2}{2}\right)^2 - 2x\left(1 + c - \frac{c^2}{2}\right) + x^2\right] dx \\ &= \left(1 + c - \frac{c^2}{2}\right)^2 - \left(1 + c - \frac{c^2}{2}\right) + \frac{1}{3} - c\left(1 + c - \frac{c^2}{2}\right)^2 + c^2\left(1 + c - \frac{c^2}{2}\right) - \frac{c^3}{3} \\ &= \frac{1}{3} - \frac{c^2}{2} - \frac{c^3}{3} + \frac{3c^4}{4} - \frac{c^5}{4} \end{aligned}$$

and

$$\begin{aligned}
 E_2 &= \int_0^1 G_2(c)dc \\
 &= \frac{1}{3} - \frac{1}{6} - \frac{1}{12} + \frac{3}{20} - \frac{1}{24} \\
 &= \frac{23}{120}
 \end{aligned}$$

When $1 \leq x \leq 1+c$,

$$S_3(x, c) = \frac{x^2}{2} - 2x + \frac{3}{2} + c - \frac{c^2}{2}$$

then

$$\begin{aligned}
 G_3(c) &= \int_1^{1+c} S_3^2(x, c)dx \\
 &= \int_1^{1+c} \left[\frac{x^4}{4} - 2x^3 + 4x^2 + x^2 \left(\frac{3}{2} + c - \frac{c^2}{2} \right) - 4x \left(\frac{3}{2} + c - \frac{c^2}{2} \right) + \left(\frac{3}{2} + c - \frac{c^2}{2} \right)^2 \right] dx \\
 &= \frac{1}{20} [(1+c)^5 - 1] - \frac{1}{2} [(1+c)^4 - 1] + \frac{4}{3} [(1+c)^3 - 1] \\
 &\quad + \frac{1}{3} [(1+c)^3 - 1] \left(\frac{3}{2} + c - \frac{c^2}{2} \right) - 2[(1+c)^2 - 1] \left(\frac{3}{2} + c - \frac{c^2}{2} \right) \\
 &\quad + c \left(\frac{3}{2} + c - \frac{c^2}{2} \right)^2 \\
 &= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} + \left(\frac{c^3}{3} - c^2 - 3c \right) \left(\frac{3}{2} + c - \frac{c^2}{2} \right) \\
 &\quad + c \left(\frac{3}{2} + c - \frac{c^2}{2} \right)^2 \\
 &= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} - \left(\frac{c^3}{6} + \frac{3c}{2} \right) \left(\frac{3}{2} + c - \frac{c^2}{2} \right) \\
 &= -\frac{53}{60} + \frac{(1+c)^5}{20} - \frac{(1+c)^4}{2} + \frac{4(1+c)^3}{3} + \frac{c^5}{12} - \frac{c^4}{6} + \frac{c^3}{2} - \frac{3c^2}{2} - \frac{9c}{4}
 \end{aligned}$$

and

$$\begin{aligned}
 E_3 &= \int_0^1 G_3(c)dc \\
 &= -\frac{53}{60} + \frac{2^6 - 1}{120} - \frac{2^5 - 1}{10} + \frac{2^4 - 1}{3} + \frac{1}{72} - \frac{1}{30} + \frac{1}{8} - \frac{1}{2} - \frac{9}{8} \\
 &= \frac{1}{45}
 \end{aligned}$$

Combining E_1 , E_2 , and E_3 ,

$$\mathbb{E}[Z] = E_1 + E_2 + E_3 = \frac{17}{40} + \frac{23}{120} + \frac{1}{45} = \frac{23}{36} = 0.6\overline{38}$$