Normal Distribution $\mathcal{N}_p(\tilde{\mu}, q)$

Define $\theta \sim \mathcal{N}_p(\tilde{\mu}, q)$ as the normal distribution $\mathcal{N}(\mu, \sigma)$ that satisfies mode$(\theta) = \tilde{\mu}$ and $P(\theta \leqslant q) = p$. The values for the mean and standard deviation are solved analytically as $\mu = \tilde{\mu}$ and $\sigma = \frac{q-\mu}{\Phi^{-1}(p)}$, where Φ denotes the cumulative distribution function for a standard normal distribution, and Φ^{-1} denotes its quantile function. The distribution $\mathcal{N}_p(\tilde{\mu}, q)$ is well-defined for values (μ, q, p) that satisfy $\frac{q-\mu}{p-0.5} > 0$.

Since the normal distribution is completely specified by (μ, σ), quantities such as $P(\theta \leqslant \tilde{q})$ are also specified for any $\tilde{q}$. In particular, if $\theta \sim \mathcal{N}_p(\tilde{\mu}, q)$ then $P(\theta \leqslant \frac{q+\mu}{2}) = \Phi(\frac{\Phi^{-1}(p)}{2})$. Furthermore, $P(\theta \in (\mu, \frac{q+\mu}{2})) = |p - \Phi(\frac{\Phi^{-1}(p)}{2})|$.

Generalized Normal Distribution $\mathcal{GN}(\mu, \alpha, \beta)$

The pdf and cdf for a generalized normal distribution $\mathcal{GN}(\mu, \alpha, \beta)$ are

$$f(\theta) = \frac{\beta}{2\alpha\Gamma(1/\beta)} \exp\{-(\frac{|\theta-\mu|}{\alpha})^{\beta}\}$$
$$F(\theta) = \frac{1}{2} + \text{sign}(\theta-\mu)\frac{\gamma(\frac{1}{\beta}, \frac{|\theta-\mu|^{\beta}}{2\alpha^{\beta}})}{2\Gamma(\frac{1}{\beta})}$$

where μ is a location parameter, $\alpha > 0$ is a scale parameter, and $\beta > 0$ is a shape parameter, $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$, and $\gamma(x, a) = \int_0^{a} t^{x-1}e^{-t}dt$ (Nadarajah, 2005).

Generalized Normal Distribution $\mathcal{GN}_p(\tilde{\mu}, q, k)$

Define $\theta \sim \mathcal{GN}_p(\tilde{\mu}, q, k)$ as the generalized normal distribution $\mathcal{GN}(\mu, \alpha, \beta)$ that satisfies mode$(\theta) = \tilde{\mu}$, $P(\theta \leqslant q) = p$, and $P(\theta \in (q, \frac{q+\mu}{2})) = k \cdot |p - \Phi(\frac{\Phi^{-1}(p)}{2})|$. The mode is equal to $\mu = \tilde{\mu}$, and α and β are determined to minimize the function

$$(F_{\mu,\alpha,\beta}(q) - p)^2 + (F_{\mu,\alpha,\beta}(\frac{q+\mu}{2}) - k \cdot |p - \Phi(\frac{\Phi^{-1}(p)}{2})|)^2$$

with box-constrained optimization (Byrd et al., 1995).

Problem

Prove existence and uniqueness of a $\mathcal{GN}_p(\tilde{\mu}, q, k)$ distribution among well-defined choices of $(p, \tilde{\mu}, q, k)$.

References

Byrd, R. H., Lu, P., Nocedal, J., and Zhu, C. (1995). A limited memory algorithm for bound constrained optimization. *SIAM Journal on Scientific Computing* **16,** 1190–1208.

Nadarajah, S. (2005). A generalized normal distribution. *Journal of Applied Statistics* **32,** 685–694.

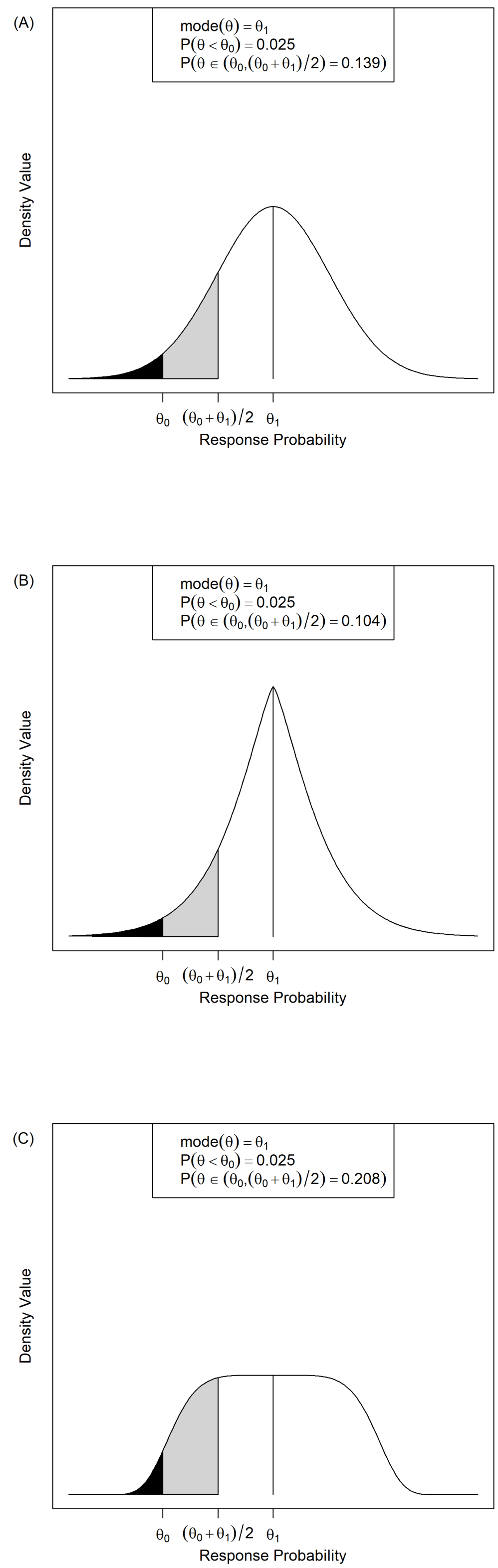

Figure 1. A, $\mathcal{GN}_{p=0.025}(\tilde{\mu} = \theta_1, q = \theta_0, \gamma = 1)$. B, $\mathcal{GN}_{p=0.025}(\tilde{\mu} = \theta_1, q = \theta_0, \gamma = 0.75)$, C, $\mathcal{GN}_{p=0.025}(\tilde{\mu} = \theta_1, q = \theta_0, \gamma = 1.5)$