# Computer Vision Solution Assignment - 6

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## Exercise 1

### (a) The least squares error is given by

$$E = \sum_{i=1}^{n} \|x_i' - Mx_i - t\|^2 \tag{1}$$

$$= \sum_{i=1}^{n} \left\| \begin{pmatrix} x_i' \\ y_i' \end{pmatrix} - \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right\|^2 \tag{2}$$

Gradient w.r.t to $M$:

$$\frac{\partial E}{\partial m_{ij}} = \sum_{i=1}^{n} 2(x_i' - Mx_i - t)(-x_{ij}) \tag{3}$$

Gradient w.r.t to $t$:

$$\frac{\partial E}{\partial t_k} = \sum_{i=1}^{n} 2(x_i' - Mx_i - t)_k \tag{4}$$

Gradient w.r.t to $(m_1, m_2, m_3, m_4, t_1, t_2)$:

$$\frac{\partial E}{\partial m_1} = \sum_{i=1}^{n} 2(x_i' - m_1 x_i - m_2 y_i - t_1)(-x_i) \tag{5}$$

$$\frac{\partial E}{\partial m_2} = \sum_{i=1}^{n} 2(x_i' - m_1 x_i - m_2 y_i - t_1)(-y_i) \tag{6}$$

$$\frac{\partial E}{\partial m_3} = \sum_{i=1}^{n} 2(y_i' - m_3 x_i - m_4 y_i - t_2)(-x_i) \tag{7}$$

$$\frac{\partial E}{\partial m_4} = \sum_{i=1}^{n} 2(y_i' - m_3 x_i - m_4 y_i - t_2)(-y_i) \tag{8}$$

$$\frac{\partial E}{\partial t_1} = \sum_{i=1}^{n} 2(x_i' - m_1 x_i - m_2 y_i - t_1)(-1) \tag{9}$$

$$\frac{\partial E}{\partial t_2} = \sum_{i=1}^{n} 2(y_i' - m_3 x_i - m_4 y_i - t_2)(-1) \tag{10}$$

**(b) Gradient w.r.t to $M$**

$$\frac{\partial E}{\partial m_{ij}} = \sum_{i=1}^{n} 2(x_i' - Mx_i - t)(-x_{ij}) \tag{11}$$

Therefore, the $h_m$ is the matrix containing all the gradients for all elements of $M$:

$$h_m = [m_1 \ m_2 \ m_3 \ m_4] \tag{12}$$

Gradient equations for $M$ in matrix form:

$$\frac{\partial E}{\partial m} = J_m^T \cdot R_m \tag{13}$$

Where $J_m$ is the Jacobian matrix (6×n matrix) to $M$ of transformed points, and $R_m$ is (6×1 Matrix).

Gradient w.r.t to $t$:

$$\frac{\partial E}{\partial t_k} = \sum_{i=1}^{n} 2(x_i' - Mx_i - t)_k \tag{14}$$

Combining both:

$$\left[\frac{\partial E}{\partial m_{ij}}; \frac{\partial E}{\partial t_k}\right] = \left[J_m^T; J_t^T\right] \cdot [R_m; R_t] \tag{15}$$

$$S = \left[J_m^T; J_t^T\right] \tag{16}$$
$$U = [R_m; R_t] \tag{17}$$
$$h = S^{-1} \cdot U \tag{18}$$
$$(Sh = U) \tag{19}$$

**(c)** $h = S^{-1} \cdot U$

Given:

$$(0,0) \to (1,2) \tag{20}$$
$$(1,0) \to (3,2) \tag{21}$$
$$(0,1) \to (1,4) \tag{22}$$

General affine transformation matrix:

$$\begin{bmatrix} m_1 & m_2 & t_1 \\ m_3 & m_4 & t_2 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Correspondence $(0,0) \to (1,2)$:

$$m_1 \times 0 + m_2 \times 0 + t_1 = 1 \Rightarrow t_1 = 1 \tag{24}$$
$$m_3 \times 0 + m_4 \times 0 + t_2 = 2 \Rightarrow t_2 = 2 \tag{25}$$

$$\Rightarrow \begin{bmatrix} m_1 & m_2 & 1 \\ m_3 & m_4 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

Correspondence $(1,0) \to (3,2)$:

$$m_1 \times 1 + m_2 \times 0 + t_1 = 3 \tag{27}$$
$$m_1 + 1 = 3 \Rightarrow m_1 = 2 \tag{28}$$
$$m_3 \times 1 + m_4 \times 0 + t_2 = 2 \tag{29}$$
$$m_3 + 2 = 2 \Rightarrow m_3 = 0 \tag{30}$$

Correspondence $(0,1) \to (1,4)$:

$$m_1 \times 0 + m_2 \times 1 + t_1 = 1 \Rightarrow m_2 = 0 \tag{31}$$
$$m_3 \times 0 + m_4 \times 1 + t_2 = 4 \Rightarrow m_4 = 2 \tag{32}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

is the affine transformation matrix.

## Exercise 2

### (a) Compute the vectors $v' = x'_2 - x'_1$ and $v = x_2 - x_1$

$$v = x_2 - x_1 \tag{34}$$

$$v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}; \quad v' = \begin{pmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{pmatrix} \tag{35}$$

$$= s \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \tag{36}$$

For $\cos\theta = \frac{v' \cdot v}{\|v'\| \cdot \|v\|}$:

$$\cos\theta = \frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2} \cdot \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{37}$$

$$\theta = \cos^{-1}\left(\frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2} \cdot \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\right) \tag{38}$$

### (b) The scale factor $s$ is

$$s = \frac{\|v'\|}{\|v\|} = \frac{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{39}$$

### (c) $x' = sRx + t$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = s \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \tag{40}$$

$$x' = s\cos\theta \cdot x - s\sin\theta \cdot y + t_x \tag{41}$$

$$\Rightarrow t_x = x' - s\cos\theta \cdot x + s\sin\theta \cdot y \tag{42}$$

$$y' = s\sin\theta \cdot x + s\cos\theta \cdot y + t_y \tag{43}$$

$$\Rightarrow t_y = y' - s\sin\theta \cdot x - s\cos\theta \cdot y \tag{44}$$

### (d) Given $\left\{\left(\frac{1}{2}, 0\right) \to (0,0)\right\}$ and $\left\{\left(0, \frac{1}{2}\right) \to (-1,-1)\right\}$

$$(x_1, y_1) = \left(\frac{1}{2}, 0\right), \quad (x_2, y_2) = \left(0, \frac{1}{2}\right) \tag{45}$$

$$(x'_1, y'_1) = (0,0), \quad (x'_2, y'_2) = (-1,-1) \tag{46}$$

$$v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{47}$$

$$v' = \begin{pmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{48}$$

$$\theta = \cos^{-1}\left(\frac{(-1)(-\frac{1}{2}) + (-1)(\frac{1}{2})}{\sqrt{(-1)^2 + (-1)^2} \cdot \sqrt{(-\frac{1}{2})^2 + (\frac{1}{2})^2}}\right) \tag{49}$$

$$= \cos^{-1}\left(\frac{0}{\sqrt{2} \cdot \frac{1}{\sqrt{2}}}\right) \tag{50}$$

$$= \cos^{-1}(0) = \frac{\pi}{2} \tag{51}$$

$$s = \frac{\|v'\|}{\|v\|} = \frac{\sqrt{(-1)^2 + (-1)^2}}{\sqrt{(-\frac{1}{2})^2 + (\frac{1}{2})^2}} = \frac{\sqrt{2}}{\frac{1}{\sqrt{2}}} = 2 \tag{52}$$

$$t_x = x_1' - s\cos\theta \cdot x_1 + s\sin\theta \cdot y_1 \tag{53}$$

$$= 0 - 2\cos\frac{\pi}{2} \cdot \frac{1}{2} + 2\sin\frac{\pi}{2} \cdot 0 \tag{54}$$

$$= 0 - 2(0) \cdot \frac{1}{2} + 2(1) \cdot 0 = 0 \tag{55}$$

$$t_y = y_1' - s\sin\theta \cdot x_1 - s\cos\theta \cdot y_1 \tag{56}$$

$$= 0 - 2\sin\frac{\pi}{2} \cdot \frac{1}{2} - 2\cos\frac{\pi}{2} \cdot 0 \tag{57}$$

$$= 0 - 2(1) \cdot \frac{1}{2} + 2(0) \cdot 0 = -1 \tag{58}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = 2\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{59}$$

$$= 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{60}$$