Exercise - 1

(a) Given $ax + by + c = 0$

& $l = (a\ b\ c)^T$

In homogenous coordinates, a 2D point is represented $X = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

where 2D point (x, y) is represented as 3D vector (x, y, w)

where w is scaling factor

The equation can be written in matrix form

$$[x\ y\ 1]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$X^T = [x\ y\ 1] \ ; \ l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$X^T \cdot l = 0$$

(b) The two lines $l = (a\ b\ c)^T$ &

$l' = (a'\ b'\ c')^T$

the scaler triple product

$$(a \times b)^T \cdot c = 0$$

$$a \times b = [a_2 b_3 - a_3 b_2 \quad a_3 b_1 - a_1 b_3 \quad a_1 b_2 - a_2 b_1]$$

Consider

$a = l \quad ; \quad b = l'$.

$$l \times l' = [bc' - cb' \quad ca' - ac' \quad ab' - ba']$$

Since the vectors are in parallel b & b'

$bc' - cb' = 0$; same imples

for a & c

$ca' - ac' = 0$; $ab' - ba' = 0$

$$l \times l' = [0 \quad 0 \quad 0]$$

& the dot product of zero with vector l & l' with $l \times l'$ is zero

$$l \cdot (l \times l') = l \cdot [0\ 0\ 0] = 0$$

$$l' \cdot (l \times l') = l' \cdot [0\ 0\ 0] = 0$$

$$\therefore l \cdot (l \times l') = l'(l \times l') = 0$$

we know that

$$X^T \cdot l = X \cdot l^T = 0$$

By comparing above equations we get

$$X = l \times l'$$

(c) points x is represented as (x, y, w)
~~En~~ 3D vector similarly for x'

x' is (x', y', w')

to know that the line is passing through points

$$l = x \times x'$$

$$= (x \ y \ w) \times (x' \ y' \ w')$$

$$= \begin{vmatrix} i & j & k \\ x & y & w \\ x' & y' & w' \end{vmatrix}$$

$$= [yw' - y'w \quad w'x - x'w \quad xy' - x'y]$$

since $l = ax + by + c$

consider $a = yw' - y'w$

$b = w'x - x'w$

$c = xy' - x'y$

$$\therefore ax + by + c = (yw' - y'w)x + (w'x - x'w)y + (xy' - x'y) = 0$$

we know $(a \times b) \cdot c = 0$

Since $x \times x' = 0 \Rightarrow x \cdot (x \times x') = 0$

$$l = x \times x'$$

Exercise - 2 : Transformations in 2D.

(a) The matrix representatives of translation in 2D is

translation $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

Euclidean transformation (Rotation + translation) $= \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$

similarity transformation (scaling + rotation + transformation) $- \begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$

affine transformation $\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$

projective transformation $\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$

(b) The no. of degrees freedom for translation matrix is 2

Degrees of freedom for euclidean is 3

Degrees of freedom for similarity is 4

Degrees of freedom for affine is 6

Degrees of freedom for projective is 8

Degrees of freedom

(e) The number of degress of freedom is less than no. of elements in 3×3 matrix due to the 3×3 matrix can represent a wider range of transformations with projective transformation, affine transformation & more general linear transformations

Excersies -3

(a) Equation of line $ax+by+c=0$

$$l^T . x = 0 \quad ; \quad l = [a \; b \; c]^T$$

and x is the homogeneous co-ordinates for lines & points

$$x' = Hx \quad \rightarrow (1)$$

the duality relationship between lines & points

$$l = H^{-T} . x$$

and similar for the line transformation for l' & x'

$$l' = H^{-T} . x'$$

Applying first step

$$l' = H^{-T} . (Hx)$$

$$= (H^{-T} . H) x$$

For any inverse transpose of matrix ~~o~~ scalar product with itself gives the identity matrix

$(I = H^{-T} \cdot H)$

$$l' = I \cdot x$$

$$l' = x$$

similarly $l = x'$

apply these two in equation (1)

$$l = H \cdot l'$$

$$l' = H^{-T} \cdot l$$

(b) $$I = \frac{(l_1^T x_1)(l_2^T x_2)}{(l_1^T x_2)(l_2^T x_1)}$$

lines - l_1, l_2 & points, x_1 & x_2

Considering the lines & points which do not lie on the lines.

The original I

$$I_{original} = \frac{(l_1^T x_1)(l_2^T x_2)}{(l_1^T x_2)(l_2^T x_1)}$$

The projective transformation by matrix H to lines & points is

l_1' (transformed) $= H^{-T} l_1 \Rightarrow l_1 = H^{T} l_1'$

$l_2' = H^{-T} l_2 \Rightarrow l_2 = H^{T} l_2'$

$x_1' = H x_1 \Rightarrow x_1 = H^{-T} x_1'$

$x_2' = H x_2 \Rightarrow x_2 = H^{-T} x_2'$

after applying the transformation of lines & points in $I_{original}$ which becomes $I_{transformed}$

$$I_{transformed} = \frac{(H l_1'^{T} H^{-1} x_1')(H l_2'^{T} H^{-1} x_2')}{(H l_1'^{T} H^{-1} x_2')(H l_2'^{T} H^{-1} x_1')}$$

Since $H \cdot H^{-T} = I$ (Identity matrix)

$$I_{transformed} = \frac{(l_1'^{T} x_1')(l_2'^{T} x_2')}{(l_1'^{T} x_2')(l_2'^{T} x_1')}$$

which is the same as the $I_{original}$ which concludes that I remains same under a projective transformation & it is a projective invariant.