Computer Vision Solution Assignment - 1

Sai Kiran

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Exercise 1: Homogeneous Coordinates

(a)

Given $ax + by + c = 0$

Where $\ell = (a\ b\ c)^T$

In homogeneous coordinates, a 2D point is represented as

$$X = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{1}$$

where the 2D point (x, y) is represented as a 3D vector (x, y, w)
where w is the scaling factor.
The equation can be written in matrix form:

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2}$$

$X^T = \begin{bmatrix} x & y & 1 \end{bmatrix} \quad ; \quad \ell = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$X^T \cdot \ell = 0$

(b)

The two lines $\ell = (a\ b\ c)^T$ and $\ell' = (a'\ b'\ c')^T$

The scalar triple product

$$(a \times b) \cdot c = 0 \tag{3}$$

where

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{bmatrix} \tag{4}$$

Consider $a = \ell$, $b = \ell'$

$\ell \times \ell' = \begin{bmatrix} bc' - cb' & ca' - ac' & ab' - ba' \end{bmatrix}$
Since the vectors are in parallel b and b'

$$bc' - cb' = 0 \tag{5}$$

Similarly,

$$ca' - ac' = 0 \quad ; \quad ab' - ba' = 0 \tag{6}$$

$\ell \times \ell' = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$
$\Rightarrow$ the dot product of zero with vector ℓ is
$\ell' \cdot (\ell \times \ell') = \ell \cdot \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = 0$
$\ell \cdot (\ell \times \ell') = \ell \cdot \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = 0$
$\Rightarrow \ell \cdot (\ell \times \ell') = \ell' \cdot (\ell \times \ell') = 0$
We know that

$$X^T \cdot \ell = X \cdot \ell = 0 \tag{7}$$

By comparing above equations we get

$$X = \ell \times \ell' \tag{8}$$

(c)

Point X is represented as (x, y, w) as 3D vector similarly for X'
$X' = (x', y', w')$
To know that the line is passing through points

$$\ell = X \times X' \tag{9}$$

$= (x\ y\ w) \times (x'\ y'\ w')$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & w \\ x' & y' & w' \end{vmatrix}$
$= \begin{bmatrix} yw' - y'w & w'x - x'w & xy' - x'y \end{bmatrix}$
Since $\ell = ax + by + c$
Consider $a = yw' - y'w$ $b = w'x - x'w$ $c = xy' - x'y$
$ax + by + c = (yw' - y'w)x + (w'x - x'w)y + (xy' - x'y) = 0$
We know

$$(a \times b) \cdot c = 0 \tag{10}$$

Since $X \times X' = 0 \Rightarrow X \cdot (X \times X') = 0$
$\ell = X \times X'$

Exercise 2: Transformations in 2D

(a)

The matrix representatives of translation in 2D is

$$\text{Translation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Euclidean transformation = $\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ (Rotation + Translation)

Similarity transformation = $\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ (Scaling + Rotation + Translation)

Affine transformation = $\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$

Projective transformation = $\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$

(b)

The number of degrees of freedom for translation matrix is 2.

Degrees of freedom for Euclidean is 3.

Degrees of freedom for similarity is 4.

Degrees of freedom for affine is 6.

Degrees of freedom for projective is 8.

(c)

The number of degrees of freedom is less than number of elements in 3×3 matrix due to the fact that 3×3 matrix can represent a wider range of transformations like with projective transformation, affine transformation and more general linear transformations.

Exercise 3

(a)

Equation of line $ax + by + c = 0$

$\ell^T \cdot X = 0 \quad ; \quad \ell = (a\ b\ c)^T$

and X is the homogeneous coordinate for line & points.

$X' = HX \quad \rightarrow (1)$
The duality relationship between lines & points
$\ell' = H^{-T} \cdot X$
and similar for the line transformation for ℓ' & X'
$\ell' = H^{-T} \cdot X$
Applying first step

$$\ell' = H^{-T} \cdot (HX) \tag{12}$$
$$= (H^{-T} \cdot H)X \tag{13}$$

For any inverse transpose of matrix or scalar product with itself gives the identity matrix
$(I = H^{-T} \cdot H)$
$\ell' = I \cdot X$
$\ell' = X$
Similarly,

$$\ell = X' \tag{14}$$

Apply these two in equation (1)

$$\ell' = H \cdot \ell \tag{15}$$
$$\ell' = H^{-T} \cdot \ell \tag{16}$$

(b)

$$I = \frac{(\ell_1^T x_1)(\ell_2^T x_2)}{(\ell_1^T x_2)(\ell_2^T x_1)} \tag{17}$$

Lines $= \ell_1, \ell_2 \quad$ & points $= x_1 \& x_2$
Considering the lines & points which do not lie on the lines.
The original I

$$I_{original} = \frac{(\ell_1^T x_1)(\ell_2^T x_2)}{(\ell_1^T x_2)(\ell_2^T x_1)} \tag{18}$$

The projective transformation by matrix H to lines & points is
$\ell_1'(transformed) = H^{-T}\ell_1 \Rightarrow \ell_1 = H^T \ell_1'$
$\ell_2' = H^{-T}\ell_2 \Rightarrow \ell_2 = H^T \ell_2'$
$x_1' = Hx_1 \Rightarrow x_1 = H^{-1}x_1'$
$x_2' = Hx_2 \Rightarrow x_2 = H^{-1}x_2'$
After applying the transformation of lines & points in $I_{original}$ which becomes

$$I_{transformed} = \frac{(H\ell_1'^T H^{-1} x_1')(H\ell_2'^T H^{-1} x_2')}{(H\ell_1'^T H^{-1} x_2')(H\ell_2'^T H^{-1} x_1')} \tag{19}$$

Since $H \cdot H^{-T} = I$ (Identity matrix)

$$I_{transformed} = \frac{(\ell_1'^T x_1')(\ell_2'^T x_2')}{(\ell_1'^T x_2')(\ell_2'^T x_1')} \tag{20}$$

which is the same as the $I_{original}$ which concludes that I remains same under a projective transformation if it is a projective invariant.