Computer Vision Solution Assignment - 2

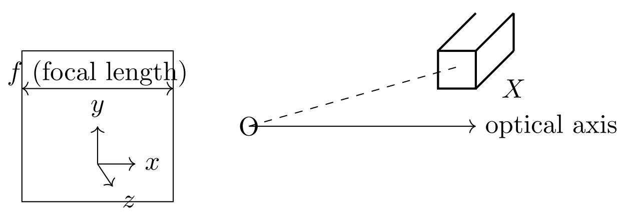
Sai Kiran

September 2023

Exercise 1

1) The image of the 3D objects after the mapping from a camera model with the pinhole camera which is called the pinhole camera model.

The figure below shows the simple camera model.



where f - focal length
O - camera origin
P - principal point

The point X which is co-ordinate in a 3D object is

$$X_c = \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T \tag{1}$$

The point projected on the pinhole camera

$$x_p = \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{bmatrix} x_p & y_p \end{bmatrix}^T \tag{2}$$

By using the similar triangles, we can get the co-ordinates of projected equation.

From the 3D object to image, the projected

$$\frac{x_c}{z_c} = \frac{x_p}{f} \tag{3}$$

$$\frac{1}{x_p} = \frac{1}{f \cdot x_c} \cdot z_c \tag{4}$$

$$x_p = f \cdot \frac{x_c}{z_c} \tag{5}$$

and in same way

$$\frac{y_c}{y_p} = \frac{z_c}{f} \tag{6}$$

$$\frac{1}{y_p} = \frac{z_c}{f \cdot y_c} \tag{7}$$

$$y_p = f \cdot \frac{y_c}{z_c} \tag{8}$$

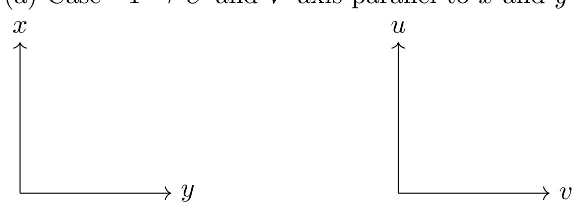
Therefore

$$(x_p, y_p)^T = \begin{bmatrix} f\frac{x_c}{z_c} & f\frac{y_c}{z_c} \end{bmatrix}^T \tag{9}$$

Exercise 2

Given the pixel coordinates $p = [u, v]^T$ and pixel co-ordinates of the principal point are (u_0, v_0) and the perspective projection co-ordinates are $(x_p, y_p)^T = X_p$

(a) Case - 1 $\rightarrow$ U and V axis parallel to x and y axis.



From the perspective projection, we know that the focal lengths of respective axis are f_u and f_v.

$$(x_p, y_p)^T = \begin{bmatrix} f\frac{x_c}{z_c} & f\frac{y_c}{z_c} \end{bmatrix}^T \tag{10}$$

$$u = f\frac{x_c}{z_c} + u_0 \tag{11}$$

$$v = f\frac{y_c}{z_c} + v_0 \tag{12}$$

The equations in homogeneous co-ordinate gives

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix} \tag{13}$$

$$K = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

The perspective projection is

$$u = f_u \frac{x_c}{z_c} + u_0 \tag{15}$$

$$v = f_v \frac{y_c}{z_c} + v_0 \tag{16}$$

Exercise 3

The homogeneous coordinates of p is

$$\bar{p} = (u, v, 1) \tag{17}$$

We know that

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} \tag{18}$$

Applying transformation on both sides for homogeneous co-ordinates

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} \tag{19}$$

where $K_{3\times3} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = K \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} \tag{20}$$

$$\bar{p} = K \cdot X_c \tag{21}$$

article amsmath, amssymb, amsthm mathtools bm

Exercise 4: Camera Projection Matrix

Imaged points are often expressed in an arbitrary frame of reference called the world coordinate frame. The mapping from the world frame to the camera coordinate frame is a rigid transformation consisting of a 3D rotation **R** and translation **t**:

$$\mathbf{x}_c = \mathbf{R}\mathbf{x}_w + \mathbf{t} \tag{22}$$

Solution:

We need to find the camera projection matrix $\mathbf{P}$ that projects a point from world coordinates $\mathbf{x}_w$ to pixel coordinates.

First, let's express this transformation using homogeneous coordinates. The transformation from world coordinates to camera coordinates can be written as:

$$\begin{pmatrix} \mathbf{x}_c \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} \mathbf{x}_w \\ 1 \end{pmatrix} \tag{23}$$

From Exercise 3, we know that the relationship between the the camera coordinaand the and the pixel coordinates is given by the intrinsic calibration matrix $\mathbf{K}$:

$$\tilde{\mathbf{p}} = \mathbf{K}\mathbf{x}_c \tag{24}$$

where $\tilde{\mathbf{p}}$ is the homogeneous representation of the pixel coordinates $\mathbf{p}$.

Combining these two transformations, we get the following:

$$\begin{aligned} \tilde{\mathbf{p}} &= \mathbf{K}\mathbf{x}_c && (25) \\ &= \mathbf{K}(\mathbf{R}\mathbf{x}_w + \mathbf{t}) && (26) \\ &= \mathbf{K}\mathbf{R}\mathbf{x}_w + \mathbf{K}\mathbf{t} && (27) \end{aligned}$$

In homogeneous coordinates, this can be written as:

$$\tilde{\mathbf{p}} = \begin{bmatrix} \mathbf{K}\mathbf{R} & \mathbf{K}\mathbf{t} \end{bmatrix} \begin{pmatrix} \mathbf{x}_w \\ 1 \end{pmatrix} \tag{28}$$

Therefore, the 3×4 camera projection matrix $\mathbf{P}$ is:

$$\mathbf{P} = \begin{bmatrix} \mathbf{K}\mathbf{R} & \mathbf{K}\mathbf{t} \end{bmatrix} \tag{29}$$

In full form, if $\mathbf{K} = \begin{bmatrix} f_x & 0 & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$, and $\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$, then:

$$\mathbf{P} = \begin{bmatrix} f_x r_{11} + u_0 r_{31} & f_x r_{12} + u_0 r_{32} & f_x r_{13} + u_0 r_{33} & f_x t_x + u_0 t_z \\ f_y r_{21} + v_0 r_{31} & f_y r_{22} + v_0 r_{32} & f_y r_{23} + v_0 r_{33} & f_y t_y + v_0 t_z \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \tag{30}$$

0.1 Exercise 5: Rotation Matrix

A rigid coordinate transformation can be represented with a rotation matrix $\mathbf{R}$ and a translation vector $\mathbf{t}$, which transform a point $\mathbf{x}$ to $\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$. Now, let the 3×3 matrix $\mathbf{R}$ be a 3D rotation matrix, which rotates a vector $\mathbf{x}$ by the angle θ about the axis $\mathbf{u}$ (a unit vector). According to the Rodrigues formula, we have the following.

$$\mathbf{R}\mathbf{x} = \cos\theta\mathbf{x} + \sin\theta(\mathbf{u} \times \mathbf{x}) + (1 - \cos\theta)\mathbf{u} \cdot \mathbf{x})\mathbf{u} \tag{31}$$

(a) Geometric Justification for the Rodrigues formula

To derive the Rodrigues formula, we will decompose the vector $\mathbf{x}$ into components parallel and perpendicular to the rotation axis $\mathbf{u}$:

$$\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp} \tag{32}$$

where $\mathbf{x}_{\parallel} = (\mathbf{u} \cdot \mathbf{x})\mathbf{u}$ is the component of $\mathbf{x}$ parallel to $\mathbf{u}$, and $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\parallel}$ is the component perpendicular to $\mathbf{u}$.

When rotating the vector $\mathbf{x}$ around the axis $\mathbf{u}$ by angle θ:

1. The parallel component $\mathbf{x}_{\parallel}$ remains unchanged, as it lies along the rotation axis. 2. The perpendicular component $\mathbf{x}_{\perp}$ rotates in a plane perpendicular to $\mathbf{u}$.

Let us define two orthogonal unit vectors in this plane: - $\mathbf{v}_1 = \mathbf{x}_{\perp}/\|\mathbf{x}_{\perp}\|$ (unit vector in direction of $\mathbf{x}_{\perp}$) - $\mathbf{v}_2 = \mathbf{u} \times \mathbf{v}_1$ (unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}_1$)

The rotation of $\mathbf{x}_{\perp}$ by angle θ in this plane can be expressed as

$$\begin{aligned} \mathbf{R}\mathbf{x}_{\perp} &= \|\mathbf{x}_{\perp}\|(\cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2) && (33) \\ &= \cos\theta\mathbf{x}_{\perp} + \sin\theta\|\mathbf{x}_{\perp}\|\mathbf{v}_2 && (34) \end{aligned}$$

Given that $\mathbf{v}_2 = \mathbf{u} \times \mathbf{v}_1$ and $\mathbf{v}_1 = \mathbf{x}_{\perp}/\|\mathbf{x}_{\perp}\|$, we have:

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{u} \times \frac{\mathbf{x}_{\perp}}{\|\mathbf{x}_{\perp}\|} && (35) \\ \|\mathbf{x}_{\perp}\|\mathbf{v}_2 &= \mathbf{u} \times \mathbf{x}_{\perp} && (36) \end{aligned}$$

Since $\mathbf{u}$ is perpendicular to $\mathbf{x}_{\perp}$, we have $\mathbf{u} \times \mathbf{x}_{\perp} = \mathbf{u} \times \mathbf{x}$, giving us:

$$\mathbf{R}\mathbf{x}_{\perp} = \cos\theta\mathbf{x}_{\perp} + \sin\theta(\mathbf{u} \times \mathbf{x}) \tag{37}$$

Combining the rotation of both components:

$$\begin{aligned} \mathbf{R}\mathbf{x} &= \mathbf{R}\mathbf{x}_{\parallel} + \mathbf{R}\mathbf{x}_{\perp} && (38) \\ &= \mathbf{x}_{\parallel} + \cos\theta\mathbf{x}_{\perp} + \sin\theta(\mathbf{u} \times \mathbf{x}) && (39) \end{aligned}$$

Substituting $\mathbf{x}_\parallel = (\mathbf{u}\cdot\mathbf{x})\mathbf{u}$ and $\mathbf{x}_\perp = \mathbf{x} - \mathbf{x}_\parallel = \mathbf{x} - (\mathbf{u}\cdot\mathbf{x})\mathbf{u}$:

$$\begin{aligned}
\mathbf{R}\mathbf{x} &= (\mathbf{u}\cdot\mathbf{x})\mathbf{u} + \cos\theta[\mathbf{x} - (\mathbf{u}\cdot\mathbf{x})\mathbf{u}] + \sin\theta(\mathbf{u}\times\mathbf{x}) && (40)\\
&= (\mathbf{u}\cdot\mathbf{x})\mathbf{u} + \cos\theta\mathbf{x} - \cos\theta(\mathbf{u}\cdot\mathbf{x})\mathbf{u} + \sin\theta(\mathbf{u}\times\mathbf{x}) && (41)\\
&= \cos\theta\mathbf{x} + \sin\theta(\mathbf{u}\times\mathbf{x}) + (1-\cos\theta)(\mathbf{u}\cdot\mathbf{x})\mathbf{u} && (42)
\end{aligned}$$

This is the Rodrigues rotation formula.

(b) Expressions for the Elements of R

Let $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\mathbf{x} = (x_1, x_2, x_3)^T$.
The cross product $\mathbf{u}\times\mathbf{x}$ is given by:

$$\mathbf{u}\times\mathbf{x} = \begin{pmatrix} u_2x_3 - u_3x_2 \\ u_3x_1 - u_1x_3 \\ u_1x_2 - u_2x_1 \end{pmatrix} \tag{43}$$

The scalar product $\mathbf{u}\cdot\mathbf{x}$ is:

$$\mathbf{u}\cdot\mathbf{x} = u_1x_1 + u_2x_2 + u_3x_3 \tag{44}$$

Substituting these into the Rodrigues formula:

$$\begin{aligned}
\mathbf{R}\mathbf{x} &= \cos\theta\mathbf{x} + \sin\theta(\mathbf{u}\times\mathbf{x}) + (1-\cos\theta)(\mathbf{u}\cdot\mathbf{x})\mathbf{u} && (45)\\
&= \cos\theta\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \sin\theta\begin{pmatrix} u_2x_3 - u_3x_2 \\ u_3x_1 - u_1x_3 \\ u_1x_2 - u_2x_1 \end{pmatrix} + (1-\cos\theta)(u_1x_1 + u_2x_2 + u_3x_3)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} && (46)
\end{aligned}$$

We can organize this to determine the rotation matrix $\mathbf{R}$ by comparing with $\mathbf{R}\mathbf{x} = \begin{pmatrix} R_{11}x_1 + R_{12}x_2 + R_{13}x_3 \\ R_{21}x_1 + R_{22}x_2 + R_{23}x_3 \\ R_{31}x_1 + R_{32}x_2 + R_{33}x_3 \end{pmatrix}$

Expanding the components of $\mathbf{R}\mathbf{x}$:

First component:

$$\begin{aligned}
[\mathbf{R}\mathbf{x}]_1 &= \cos\theta\cdot x_1 + \sin\theta(u_2x_3 - u_3x_2) + (1-\cos\theta)u_1(u_1x_1 + u_2x_2 + u_3x_3) && (47)\\
&= \cos\theta\cdot x_1 + \sin\theta(u_2x_3 - u_3x_2) + (1-\cos\theta)(u_1^2x_1 + u_1u_2x_2 + u_1u_3x_3) && (48)
\end{aligned}$$

Rearranging to group terms with x_1, x_2, and x_3:

$$[\mathbf{R}\mathbf{x}]_1 = [\cos\theta + (1-\cos\theta)u_1^2]x_1 + [(1-\cos\theta)u_1u_2 - \sin\theta u_3]x_2 + [(1-\cos\theta)u_1u_3 + \sin\theta u_2]x_3 \tag{49}$$

Therefore:

$$R_{11} = \cos\theta + (1-\cos\theta)u_1^2 \tag{50}$$
$$R_{12} = (1-\cos\theta)u_1u_2 - \sin\theta u_3 \tag{51}$$
$$R_{13} = (1-\cos\theta)u_1u_3 + \sin\theta u_2 \tag{52}$$

Second component:

$$[\mathbf{Rx}]_2 = \cos\theta \cdot x_2 + \sin\theta(u_3x_1 - u_1x_3) + (1-\cos\theta)u_2(u_1x_1 + u_2x_2 + u_3x_3) \tag{53}$$
$$= \cos\theta \cdot x_2 + \sin\theta(u_3x_1 - u_1x_3) + (1-\cos\theta)(u_1u_2x_1 + u_2^2x_2 + u_2u_3x_3) \tag{54}$$

Rearranging:

$$[\mathbf{Rx}]_2 = [(1-\cos\theta)u_1u_2 + \sin\theta u_3]x_1 + [\cos\theta + (1-\cos\theta)u_2^2]x_2 + [(1-\cos\theta)u_2u_3 - \sin\theta u_1]x_3 \tag{55}$$

Therefore:

$$R_{21} = (1-\cos\theta)u_1u_2 + \sin\theta u_3 \tag{56}$$
$$R_{22} = \cos\theta + (1-\cos\theta)u_2^2 \tag{57}$$
$$R_{23} = (1-\cos\theta)u_2u_3 - \sin\theta u_1 \tag{58}$$

Third component:

$$[\mathbf{Rx}]_3 = \cos\theta \cdot x_3 + \sin\theta(u_1x_2 - u_2x_1) + (1-\cos\theta)u_3(u_1x_1 + u_2x_2 + u_3x_3) \tag{59}$$
$$= \cos\theta \cdot x_3 + \sin\theta(u_1x_2 - u_2x_1) + (1-\cos\theta)(u_1u_3x_1 + u_2u_3x_2 + u_3^2x_3) \tag{60}$$

Rearranging:

$$[\mathbf{Rx}]_3 = [(1-\cos\theta)u_1u_3 - \sin\theta u_2]x_1 + [(1-\cos\theta)u_2u_3 + \sin\theta u_1]x_2 + [\cos\theta + (1-\cos\theta)u_3^2]x_3 \tag{61}$$

Therefore:

$$R_{31} = (1-\cos\theta)u_1u_3 - \sin\theta u_2 \tag{62}$$
$$R_{32} = (1-\cos\theta)u_2u_3 + \sin\theta u_1 \tag{63}$$
$$R_{33} = \cos\theta + (1-\cos\theta)u_3^2 \tag{64}$$

Combining all elements, the rotation matrix is:

$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)u_1^2 & (1-\cos\theta)u_1u_2 - \sin\theta u_3 & (1-\cos\theta)u_1u_3 + \sin\theta u_2 \\ (1-\cos\theta)u_1u_2 + \sin\theta u_3 & \cos\theta + (1-\cos\theta)u_2^2 & (1-\cos\theta)u_2u_3 - \sin\theta u_1 \\ (1-\cos\theta)u_1u_3 - \sin\theta u_2 & (1-\cos\theta)u_2u_3 + \sin\theta u_1 & \cos\theta + (1-\cos\theta)u_3^2 \end{bmatrix} \tag{65}$$

This can also be expressed in a more compact form using the cross-product matrix of $\mathbf{u}$:

Let $[\mathbf{u}]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$ be the skew-symmetric matrix corresponding to the cross product with $\mathbf{u}$.

Then:

$$\mathbf{R} = \cos\theta \mathbf{I} + \sin\theta [\mathbf{u}]_\times + (1 - \cos\theta)\mathbf{u}\mathbf{u}^T \tag{66}$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\mathbf{u}\mathbf{u}^T$ is the outer product of $\mathbf{u}$ with itself.