Computer Vision Solution Assignment - 5

Sai Kiran

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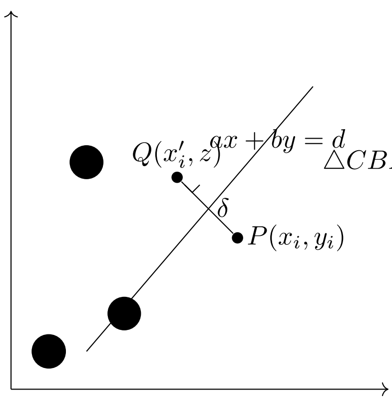
1) Total Least Squares Line Fitting

Given line: $ax + by + d = 0$ with $a^2 + b^2 = 1$



From the similar triangles, $\triangle CBA$ and $\triangle PRQ$:

$$\frac{|PB|}{|PQ|} = \frac{|CB|}{|CA|} \tag{1}$$

PQ is the distance:

$$|PQ| = \frac{|PQ| \times |CB|}{|CA|} \tag{2}$$

Distance $= |PQ| = \frac{|y_i - z| \times |b|}{\sqrt{a^2+b^2}}$ (in a right angle triangle)
Since $\sqrt{a^2 + b^2} = \sqrt{1} = 1$:
Distance $= |y_i - z| \times |b| = |y_i - z| \times |b| \quad \rightarrow (1)$
Let $Q(x_i, z)$ be a point on the line:
$ax_i + bz = d$
$\Rightarrow bz = d - ax_i$
$z = \frac{d - ax_i}{b}$
Substituting z in equation (1):

Distance $= \left|y_i - \frac{d-ax_i}{b}\right| \times |b|$
$= \frac{|by_i+ax_i-d|\times|b|}{|b|}$
$= |by_i + ax_i - d|$
$= |ax_i + by_i - d|$

2) Given the sum of squared distances between the point and line is:

$E = \sum_{i=1}^{n}(ax_i + by_i - d)^2$
Partial derivate: $\frac{\delta E}{\delta d} = 0$
$\frac{\delta}{\delta d}\left[\sum_{i=1}^{n}(ax_i + by_i - d)^2\right] = 0$
$\sum_{i=1}^{n} -2(ax_i + by_i - d) = 0$
$\sum_{i=1}^{n} -(ax_i + by_i - d) - \sum_{i=1}^{n}(ax_i + by_i - d) = 0$
$\sum_{i=1}^{n} -(ax_i + by_i - d) - \sum_{i=1}^{n}(ax_i + by_i - d) = 0$
$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} d = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i - nd$
$nd = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i$
$d = \frac{a}{n}\sum_{i=1}^{n} x_i + \frac{b}{n}\sum_{i=1}^{n} y_i$
$d = a\bar{x} + b\bar{y}$

3) Continuing:

$d = a\bar{x} + b\bar{y}$
$E = \sum_{i=1}^{n}(ax_i + by_i - (a\bar{x} + b\bar{y}))^2$
$= \sum_{i=1}^{n}(a(x_i - \bar{x}) + b(y_i - \bar{y}))^2$
$= \sum_{i=1}^{n}\left(\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x_i - \bar{x} \\ y_i - \bar{y} \end{bmatrix}\right)^2$

$$= \left\|\begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ x_2 - \bar{x} & y_2 - \bar{y} \\ x_3 - \bar{x} & y_3 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\right\|^2$$

$= \|UN\|^2$
$= (UN)^T(UN)$
$= N^TU^TUN$
$E = \begin{bmatrix} a & b \end{bmatrix} U^TU \begin{bmatrix} a \\ b \end{bmatrix}$

4) Finding the smallest eigenvalue:

$E = (UN)^T(UN)$
$= N^TU^TUN$

$$= \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} \sum_{i=1}^{n}(x_i - \bar{x})^2 & \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n}(y_i - \bar{y})^2 \end{bmatrix}$$

For the smallest eigenvalue:

$$\frac{dE}{dN} = 2(U^T U)N = 0 \tag{3}$$

$$(U^T U)N = 0 \tag{4}$$

Therefore, the solution for the total least squares line fitting is the eigenvector corresponding to the smallest eigenvalue of the matrix $U^T U$.