Solutions Lecture 9 (Section 10.3 and Chapter 11)

Make sure to import Numpy, Matplotlib and SciPy to be able to complete all the exercises.

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.optimize as optimize

# Display numerical values in NumPy arrays only up to three decimals,
# and suppress scientific notation
np.set_printoptions(precision=3, suppress=True)
```

Question 1

In this exercise we will implement the optimization problem

$$\begin{array}{ll} \min\limits_{w_0,\dots,w_{d-1},b} & \sum\limits_{i=0}^{d-1} w_i^2 \\ \text{s.t.} & y_i(w^T x_i + b) \geq 1 \quad \forall i \in \{0,\dots,m-1\} \end{array}$$

using the `minimize()` function from SciPy's `optimize` module.

For a general linear system of inequalities with variables $z = [z_0, \dots, z_{n-1}]$, and input data $r = [r_0, \dots, r_{m-1}]$ and matrix $A \in \mathbb{R}^{m \times n}$, you can add the system of linear inequalities

$$\sum_j a_{ij} z_j \geq r_i \quad \text{for } i = 0, \dots, m-1$$

using `constraints=optimize.LinearConstraint(A,lb=r)` as keyword argument in `minimize()`. See the documentation here.

a) Write a function `constr` which takes as input a matrix $X \in \mathbb{R}^{m \times d}$ whose rows respresent m historical data points $x_i \in \mathbb{R}^d$, and a one-dimensional array $y = [y_0, \dots, y_{m-1}] \in \{-1, 1\}^m$ representing the labels of the data. The function should return the constraint matrix A and the lower bound array r that model the inequality constraints $y_i(w^T x_i + b) \geq 1$ for $i \in \{0, \dots, m-1\}$ with the interpretation that $z = [w_0, \dots, w_{d-1}, b]$.

```python
def constr(X,y):
    m, d = np.shape(X)
    A = np.hstack((X,np.ones((m,1))))*y[:,None]
    r = np.ones(m)
    return A, r
```

Test your function on the input below.

```python
# Historical data
X = np.array([[3,3],[1.5,2.5],[1,2],[0.5,1.5],[2,2],[2,4],
                        [4,4],[2,6],[5,5.5],[7,6]]) # Data points

y = np.array([-1,-1,-1,-1,-1,-1,
                        1,1,1,1]) # Labels

A, r = constr(X,y)

print("Constraint matrix A:\n",A)
print("Lower bound array r:\n",r)
```

```
Constraint matrix A:
 [[-3.  -3.  -1. ]
 [-1.5 -2.5 -1. ]
 [-1.  -2.  -1. ]
 [-0.5 -1.5 -1. ]
 [-2.  -2.  -1. ]
 [-2.  -4.  -1. ]
 [ 4.   4.   1. ]
 [ 2.   6.   1. ]
 [ 5.   5.5  1. ]
 [ 7.   6.   1. ]]
Lower bound array r:
 [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
```

b) Write a function `separate()` which takes as input a matrix $X \in \mathbb{R}^{m \times d}$ whose rows respresent m historical data points $x_i \in \mathbb{R}^d$, and a one-dimensional array $y = [y_0, \ldots, y_{m-1}] \in \{-1, 1\}^m$ representing the labels of the data. It should output the solution to the minimization problem above, as an array $z = [w_0, \ldots, w_{d-1}, b]$, using `minimize()` with initial guess for w the average of all rows in X and $b = 0$. Use your function in part a) to add the linear constraints in the keyword argument `constraints`.

```python
import scipy.optimize as optimize

def objective(z):
    d = np.size(z) - 1
```

```python
    return np.sum(z[0:d]**2)

def separate(X,y):
    A, r = constr(X,y)
    guess = np.hstack((np.mean(X,axis=0),np.array([0])))
    result = optimize.minimize(objective,x0=guess,
                               constraints=optimize.LinearConstraint(A,lb=r))
    return result.x
```

Test your function on X and y as in part a).

```python
print(separate(X,y))
```

```
[ 1.  1. -7.]
```

Question 2

In this exercise we will implement the optimization problem

$$\min_{w_0,\dots,w_{d-1},b} \frac{1}{2}\sum_{i=0}^{d-1} w_i^2 + C\sum_{i=0}^{m-1} \max(0, 1 - y_i(w^T x_i + b)).$$

using the `minimize()` function from SciPy's `optimize` module.

a) Write a function `separate_C()` which takes as input a matrix $X \in \mathbb{R}^{m\times d}$ whose rows respresent m historical data points $x_i \in \mathbb{R}^d$, a one-dimensional array $y = [y_0, \dots, y_{m-1}] \in \{-1, 1\}^m$ representing the labels of the data, and a number $C \in \mathbb{R}_{>0}$. It should return the solution to the minimization problem above, as an array $z = [w_0, \dots, w_{d-1}, b]$, using `minimize()` with initial guess for w the average of all rows in X (similar as in the previous exercise) and $b = 1$, and the `'Nelder-Mead'` method. Hint: The function `np.maximum()` might be handy in your solution.

```python
def objective(z,X,y,C):
    d = np.size(z) - 1

    w = z[0:d]
    # First term in objective
    term1 = (1/2)*np.sum(w**2)

    # Second term in objective
    term2 = np.sum(np.maximum(0,(1 - y*((X @ w) + z[d]))))
    return term1 + C*term2


def separate_C(X,y,C):
    m, d = np.shape(X)
    guess = np.hstack((np.mean(X,axis=0),np.array([1])))
```

```
    result = optimize.minimize(objective,x0=guess,
                               args=(X,y,C),method='Nelder-Mead')
    return result.x
```

Test your function with the matrix X and y from Question 1, and $C = 0.3$.

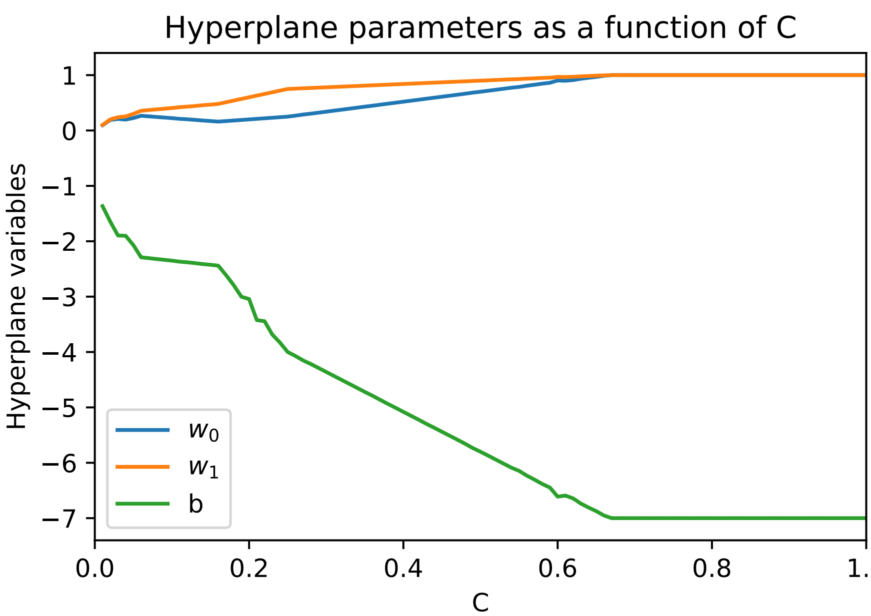
```
# Historical data
X = np.array([[3,3],[1.5,2.5],[1,2],[0.5,1.5],[2,2],[2,4],
                        [4,4],[2,6],[5,5.5],[7,6]]) # Data points

y = np.array([-1,-1,-1,-1,-1,-1,
                        1,1,1,1]) # Labels

# Parameter C
C = 0.3

print(separate_C(X,y,C))
```

```
[ 0.34  0.78 -4.36]
```

b) Execute your function in part a) for values $C \in \{0.01, 0.2, \ldots, 1\}$ on the same data as in part a) and plot the values of $w_0 = w_0(C), w_1 = w_1(C)$ and $b = b(C)$ in a figure with C on the x-axis and the values of the three variables on the y-axis. You may use a for-loop. Your figure should look roughly like this. Note that the coefficiens w_0, w_1 converge to 1, and b to -7. This was indeed the solution found in Question 1.

```
# Values of C
C = np.arange(0.01,1.01,0.01)

# Input size
m, d = np.shape(X)

hyperplane = np.zeros((np.size(C),d+1))
count = 0
for c in C:
    hyperplane[count] = separate_C(X,y,c)
    count += 1

# Create figure()
plt.figure

# Set x-axis range
plt.xlim(0,np.max(C))

# Set x-axis label
plt.xlabel('C')
```

```
plt.ylabel('Hyperplane variables')

# Plot the three lines
labels = ['$w_0$', '$w_1$', 'b']

for i in range(d+1):
    plt.plot(C,hyperplane[:,i],label=labels[i])

# Create legend
plt.legend()

# Create title
plt.title("Hyperplane parameters as a function of C")

# Show plot
plt.show()
```



Question 3

Write a function `sum_k` that takes as input two numbers k and n. It should return the integer points $x = [x_0, \ldots, x_{n-1}] \in \{0, 1, 2, \ldots, k\}^n$ for which $\sum_{i=0}^{n-1} x_i = k$. You may return the integer points as you like (in a list, array, or tuple) and use a `for`-loop.

```python
import itertools

def sum_k(k,n):
    a = np.arange(0,k+1)

    sum_equals_k = []
    for p in itertools.product(a, repeat=n):
        if(np.sum(p) == k):
            sum_equals_k.append(p)
    return sum_equals_k
```

Your function should give the following output on the input below.

```python
k = 5
n = 3

# Integer points returned as tuples
print(sum_k(k,n))
```

```
[(0, 0, 5), (0, 1, 4), (0, 2, 3), (0, 3, 2), (0, 4, 1), (0, 5, 0), (1, 0, 4), (1, 1, 3), (1,
```