

## Solutions Lecture 4 (Chapter 6)

Make sure to import Numpy and the `optimize` module from SciPy.

```
import numpy as np
import scipy.optimize as optimize
```

### Question 1

Execute `fsolve()` with initial guesses in  $[-10, -9.5, -9, \dots, 9, 9.5, 10]$  to find roots of the function  $f(x) = x \cdot \cos(x)$ . You may use a for-loop to iterate over the initial guesses. Store the found roots in a list called `roots`.

The output list `roots` should look like this. Recall that `fsolve()` returns an array with one element (which you can access by indexing it at position 0).

`[-10.995574287564276, 1.5707963267948954, -7.853981633974491, -7.853981633974483, -7.853981633974491, 1.5707963267948954, 7.853981633974491, 7.853981633974483, 10.995574287564276]`

```
def f(x):
    return x*np.cos(x)

guess = np.arange(-10,10.5,0.5)

roots = []
for i in guess:
    x_root = optimize.fsolve(f,x0=i)[0]
    roots.append(x_root)

print(roots)
```

`[-10.995574287564276, 1.5707963267948954, -7.853981633974491, -7.853981633974483, -7.853981633974491, 1.5707963267948954, 7.853981633974491, 7.853981633974483, 10.995574287564276]`

### Question 2

We will write a function that can compute a root of a polynomial function.

- a) Write a function that takes as input a number  $x$  and an array of coefficients  $a = [a_0, \dots, a_{n-1}]$ . It should return the function  $f(x) = \sum_{i=0}^{n-1} a_i x^i$ .

```
def f(x,a):
    n = np.size(a)
    return np.sum(a*(x**np.arange(0,n)))
```

The derivative of  $f$  is given by  $f'(x) = \sum_{i=1}^{n-1} i \cdot a_i x^{i-1}$ .

- b) Write a function `f_deriv` that takes as input a number  $x$  and an array of coefficients  $a = [a_0, \dots, a_{n-1}]$ . It should return the derivative  $f'(x)$  of  $f$  in the point  $x$ .

```
def f_deriv(x,a):
    n = np.size(a)
    return np.sum(np.arange(1,n)*a[1:]*x**np.arange(0,n-1))
```

- c) Write a function `root_polynomial` that takes as input an array of coefficients  $a = [a_0, \dots, a_{n-1}]$  and an initial guess  $r$ . It should output a root of the function  $f$  if one is found, and a message saying no root was found otherwise. Use `root_scalar()` with Newton's method and initial guess  $r$ . The output property `converged` of `root_scalar()` tells you whether or not Newton's method has found a root. Also, you will need to use the `args` keyword argument.

Your functions should give the following output on the input below.

```
# First polynomial
a = np.array([1.5, 0, -2.5, 0.3, 1])
r = 1.5

result = root_polynomial(a,r)
print(result)

# Second polynomial (this function has no root)
a2 = np.array([1.5, 0, -2.5, 0.3, 3])
r2 = 1.5

result2 = root_polynomial(a2,r2)
print(result2)
```

The found root is  $x = -1.5180670079327394$

No root was found; it might be that no root exists.

```
def root_polynomial(a,r):
    x = optimize.root_scalar(f, args=(a), x0=r, method='newton',
                             fprime=f_deriv)

    if x.converged == True:
        return f"The found root is x = {x.root}"
    else:
        return "No root was found; it might be that no root exists."
```

```

# First polynomial
a = np.array([1.5, 0, -2.5, 0.3, 1])
r = 1.5

result = root_polynomial(a,r)
print(result)

# Second polynomial (this function has no root)
a2 = np.array([1.5, 0, -2.5, 0.3, 3])
r2 = 1.5

result2 = root_polynomial(a2,r2)
print(result2)

```

The found root is  $x = -1.5180670079327394$   
 No root was found; it might be that no root exists.

### Question 3

In this problem, we try to solve the 2-by-2 nonlinear system of equations  $f(x) = 0$  introduced by Bloggs where

$$f(x) = \begin{bmatrix} f_0(x) \\ f_1(x) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2 + 1 \\ x_1 - \cos(\frac{\pi}{2}x_2) \end{bmatrix}$$

with Jacobian

$$J(f) = \begin{bmatrix} 2x_1 & -1 \\ 1 & \frac{\pi}{2} \sin(\frac{\pi}{2}x_2) \end{bmatrix}.$$

The problem has solutions  $(-1, 2)$ ,  $(0, 1)$ , and  $(-1/\sqrt{2}, 3/2)$ . The goal of this exercise is to show that the `root()` function might, or might not, converge depending on the initial guess that we choose.

a) Implement  $f$  and its Jacobian as Python functions called `f` and `jac_f`.

```

def f(x):
    return np.array([x[0]**2 - x[1] + 1, x[0] - np.cos(np.pi / 2 * x[1])])

def jac_f(x):
    return np.array([[2 * x[0], -1], [1, np.pi / 2 * np.sin(np.pi / 2 * x[1])]])

```

For initial guesses, we will take points on the circle with radius 2 centered around  $(0, 0)$ . This circle can be parameterized by  $[2 \cos(\rho), 2 \sin(\rho)]$  for  $\rho \in [0, 2\pi]$ .

- b) Write a function `circle()` that takes as input a one-dimensional array  $\rho = [\rho_0, \dots, \rho_{k-1}]$  and outputs the points  $[2 \cos(\rho_i), 2 \sin(\rho_i)]$  on the rows of a  $k \times 2$  array. Do not use for-loops.

```
def circle(rho):
    x = np.array([np.cos(rho), np.sin(rho)]).T
    return 2 * x
```

- c) For  $k = 50$  evenly spaced values of  $\rho \in [0, 2\pi]$ , use `root()` to find a root of the system  $f(x) = 0$  with initial guess  $[2 \cos(\rho), 2 \sin(\rho)]$ , Jacobian information, and the 'hybr' method. For every value of  $\rho$  the output should be a short message including 1) the value of  $\rho$  and 2) the root that was found, or a message that no root was found. Have a look at the output message of the function `root()` to see how you can determine whether the method found a root or not. For  $k = 50$  evenly spaced points in  $[0, 2\pi]$  your output should look like this. As you can see, all of the three roots of the system are sometimes found, and also sometimes no root is found.

```
For guess = [2. 0.], found root is [1.5781495e-12 1.0000000e+00]
For guess = [1.98358003 0.25575432], found root is [-2.57655834e-14 1.00000000e+00]
For guess = [1.93458973 0.50730917], found root is [1.23867782e-13 1.00000000e+00]
For guess = [1.85383351 0.75053401], found root is [1.57396466e-13 1.00000000e+00]
For guess = [1.74263741 0.9814351 ], found root is [9.05935849e-13 1.00000000e+00]
For guess = [1.60282724 1.19622106], found root is [1.38409487e-15 1.00000000e+00]
For guess = [1.4366987 1.3913651], found root is [-1.32700583e-12 1.00000000e+00]
For guess = [1.2469796 1.56366296], found root is [-3.80159246e-13 1.00000000e+00]
For guess = [1.03678514 1.71028553], found root is [8.04962149e-11 1.00000000e+00]
For guess = [0.80956669 1.82882525], found root is [1.54091134e-17 1.00000000e+00]
For guess = [0.56905517 1.91733571], found root is [-7.72638622e-15 1.00000000e+00]
For guess = [0.31919979 1.97436357], found root is [-5.90449013e-12 1.00000000e+00]
For guess = [0.06410316 1.99897243], found root is [-0.70710678 1.5      ]
For guess = [-0.19204605 1.99075823], found root is [-0.70710678 1.5      ]
For guess = [-0.44504187 1.94985582], found root is [-1. 2.]
For guess = [-0.69073011 1.87693684], found root is [-1. 2.]
For guess = [-0.92507658 1.77319861], found root is [-1. 2.]
For guess = [-1.14423332 1.64034451], found root is [-0.70710678 1.5      ]
For guess = [-1.34460178 1.48055599], found root is [-0.70710678 1.5      ]
For guess = [-1.52289192 1.29645679], found root is [-0.70710678 1.5      ]
For guess = [-1.67617621 1.0910698 ], found root is [-0.70710678 1.5      ]
For guess = [-1.80193774 0.86776748], found root is [-0.70710678 1.5      ]
For guess = [-1.89811149 0.63021644], found root is [-1. 2.]
For guess = [-1.96311831 0.38231726], found root is [4.60067409e-13 1.00000000e+00]
For guess = [-1.99589079 0.12814044], found root is [4.60088962e-14 1.00000000e+00]
For guess = [-1.99589079 -0.12814044], found root is [-4.39476516e-13 1.00000000e+00]
For guess = [-1.96311831 -0.38231726], found root is [7.42968452e-17 1.00000000e+00]
For guess = [-1.89811149 -0.63021644], found root is [8.20547063e-14 1.00000000e+00]
For guess = [-1.80193774 -0.86776748], found root is [-5.80092403e-14 1.00000000e+00]
```

```

For guess = [-1.67617621 -1.0910698 ], found root is [4.97457028e-14 1.00000000e+00]
For guess = [-1.52289192 -1.29645679], found root is [-2.90995034e-14 1.00000000e+00]
For guess = [-1.34460178 -1.48055599], found root is [2.29128908e-16 1.00000000e+00]
For guess = [-1.14423332 -1.64034451], found root is [-3.51617808e-12 1.00000000e+00]
For guess = [-0.92507658 -1.77319861], found root is [2.52638402e-13 1.00000000e+00]
For guess = [-0.69073011 -1.87693684], found root is [1.66343153e-11 1.00000000e+00]
For guess = [-0.44504187 -1.94985582], found root is [-0.70710678 1.5      ]
For guess = [-0.19204605 -1.99075823], found root is [-0.70710678 1.5      ]
For guess = [ 0.06410316 -1.99897243], found root is [-0.70710678 1.5      ]
For guess = [ 0.31919979 -1.97436357], no root was found
For guess = [ 0.56905517 -1.91733571], no root was found
For guess = [ 0.80956669 -1.82882525], no root was found
For guess = [ 1.03678514 -1.71028553], found root is [-0.70710678 1.5      ]
For guess = [ 1.2469796  -1.56366296], found root is [-2.26536179e-16 1.00000000e+00]
For guess = [ 1.4366987  -1.3913651], found root is [9.29740721e-15 1.00000000e+00]
For guess = [ 1.60282724 -1.19622106], found root is [9.29961312e-13 1.00000000e+00]
For guess = [ 1.74263741 -0.9814351 ], no root was found
For guess = [ 1.85383351 -0.75053401], found root is [-7.74845403e-14 1.00000000e+00]
For guess = [ 1.93458973 -0.50730917], found root is [1.66798808e-12 1.00000000e+00]
For guess = [ 1.98358003 -0.25575432], found root is [-9.27679759e-14 1.00000000e+00]
For guess = [ 2.0000000e+00 -4.8985872e-16], found root is [1.57814971e-12 1.00000000e+00]

```

```

# Evenly spaced points in the interval [0,2*pi]
k = 50
rho = np.linspace(0,2*np.pi,k)

# Define initial guess in k x 2 matrix (no for-loop needed here)
guess = circle(rho)

# Loop over values in rho
for i in guess:
    result = optimize.root(f, x0=i, jac=jac_f, method='hybr')
    if result.success == True:
        print(f"For guess = {i}, found root is {result.x}")
    else:
        print(f"For guess = {i}, no root was found")

```

```

For guess = [2. 0.], found root is [1.5781495e-12 1.0000000e+00]
For guess = [1.98358003 0.25575432], found root is [-2.57655834e-14 1.00000000e+00]
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For guess = [ 2.0000000e+00 -4.8985872e-16], found root is [1.57814971e-12 1.00000000e+00]

```

#### Question 4

Let  $A \in \mathbb{R}^{n \times n}$  be a two-dimensional array,  $b \in \mathbb{R}^n$  a one-dimensional array and

$c \in \mathbb{R}$  a scalar. Consider the function

$$f(x) = f(x_0, \dots, x_{n-1}) = x^T A x + b^T x + c = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{ij} x_i x_j + \sum_{i=0}^{n-1} b_i x_i + c$$

that takes as input a one-dimensional array  $x$ . The gradient of this function is given by  $\nabla f(x) = Ax + b$ .

Write a function `quadratic_minimizer` that takes as input  $A, b$  and  $c$ . It should compute a (local) minimizer of the function  $f$  using `minimize()` with gradient information, the 'CG' method, and initial guess  $x = [1, 1, \dots, 1]$ .

It should give the following output on the given input.

```
# Example usage
A = np.array([[3, 0, 0], [0, 4, 0], [0, 0, 5]])
b = np.array([-1, -4, 1])
c = 6

result = quadratic_minimizer(A, b, c)
print("Optimal solution:", result.x)
print("Function value at optimal point:", result.fun)
```

```
Optimal solution: [ 0.33333333  1.          -0.2         ]
Function value at optimal point: 6.0
```

```
def f(x, A, b, c):
    return x.T @ (A @ x) + b @ x + c

def gradient_f(x, A, b, c):
    return A @ x + b

def quadratic_minimizer(A, b, c):
    n = np.shape(A)[0]
    guess = np.ones(n)
    result = optimize.minimize(f, guess, args=(A, b, c),
                              jac=gradient_f, method='CG')

    return result

# Example usage
A = np.array([[3, 0, 0], [0, 4, 0], [0, 0, 5]])
b = np.array([-1, -4, 1])
c = 6

result = quadratic_minimizer(A, b, c)
print("Optimal solution:", result.x)
print("Function value at optimal point:", result.fun)
```

```
Optimal solution: [ 0.33333333  1.          -0.2         ]
```

Function value at optimal point: 6.0

Because the matrix  $A$  is symmetric and has positive eigenvalues, the function  $f$  is convex, meaning that an optimal solution can be found at the point where the gradient is the all-zeros vector, that is,  $x^* = -A^{-1}b$ . This is indeed the solution that we found.

```
eig_A, _ = np.linalg.eig(A)

print(f"The eigenvalues of A are {eig_A}")

# Solution to gradient = 0
x = -1*(np.linalg.inv(A) @ b)

print("Point where gradient is zero:", x)
print("Function value at point where gradient is zero:", f(x,A,b,c))
```

The eigenvalues of A are [3. 4. 5.]

Point where gradient is zero: [ 0.33333333 1. -0.2 ]

Function value at point where gradient is zero: 6.0

### Question 5

Write a function `capacitated_norm()` taking as input vectors  $a, b, w \in \mathbb{R}_{\geq 0}^n$  and a capacity value  $W$ , that returns the solution to the problem

$$\begin{aligned} \max \quad & \sqrt{\sum_{i=0}^{n-1} x_i^2} \\ \text{s.t.} \quad & \sum_{i=0}^{n-1} w_i x_i \leq W \\ & a_i \leq x_i \leq b_i \text{ for } i = 0, \dots, n-1 \end{aligned}$$

using `minimize()` with the 'trust-constr' method. Take  $a$  as the initial guess. You may assume that  $\sum_i w_i a_i \leq W$ , so that this problem always has a solution. Hint: You may want to have a look at the `constraints` keyword argument in the documentation of `minimize()` to see how you can input additional arguments to a constraint.

```
def f(x):
    return - np.linalg.norm(x)

def constraint(x,w,W):
    return W - np.sum(w*x)

def capacitated_norm(a,b,w,W):
    n = np.size(w)
```



```

    cons = {'type' : 'ineq', 'fun' : constraint, 'args' : (w,W)}
    intervals = [(a[i],b[i]) for i in range(n)]
    result = optimize.minimize(f, constraints=cons, bounds=intervals,
                              x0=a, method='trust-constr')

    return result.x

n = 7
a = np.array([0,0,1,4,1,0,7])
b = np.array([1,4,3,6,8,1,9])
w = np.array([0,6,5,4,6,5,3])
W = 2*np.sum(a)

print(capacitated_norm(a,b,w,W))

[ 0.97618582 -0.89189189  0.25675676  3.40540541  0.10810811 -0.74324324
  6.55405406]

def f(x):
    return - np.linalg.norm(x)

def constraint(x,w,W):
    return W - np.sum(w*x)

def capacitated_norm(a,b,w,W):
    n = np.size(w)
    cons = {'type' : 'ineq', 'fun' : constraint, 'args' : (w,W)}
    intervals = [(a[i],b[i]) for i in range(n)]
    result = optimize.minimize(f, constraints=cons, bounds=intervals,
                              x0=a, method='trust-constr')

    return result.x, result.success

# Alternatively, if you don't want to use the `args` keyword argument in cons

def capacitated_norm(a,b,w,W):
    n = np.size(w)

    def constraint_first(x):
        return constraint(x,w,W) # Constraint now only depends on first input x

    cons = {'type' : 'ineq', 'fun' : constraint_first}
    intervals = [(a[i],b[i]) for i in range(n)]
    result = optimize.minimize(f, constraints=cons, bounds=intervals,
                              x0=a, method='trust-constr')

    return result.x

n = 7

```

```
a = np.array([0,0,1,4,1,0,7])
b = np.array([1,4,3,6,8,1,9])
w = np.array([0,6,5,4,6,5,3])
W = 2*np.sum(a)

print(capacitated_norm(a,b,w,W))
```

[ 0.97618582 -0.89189189 0.25675676 3.40540541 0.10810811 -0.74324324  
 6.55405406]