Basic concepts in matrix algebra

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Contents

This appendix supplies some general matrix algebra that is useful in these notes. Proof of these, as well as additional explanation, can be found in e.g. Searle [1982], Searle et al. [1992] and/or Knight [2008]. And Schaeffer's EuropeNotes p. 194.

The matrix V is the square variance-covariance matrix $V = ZGZ' + D$.

Denoting the derivative of the matrix:

$$\mathbf{V}_i = \frac{\partial \mathbf{V}}{\partial \sigma_i^2} = \begin{cases} \mathbf{D}_n & \text{when } \sigma_i^2 = \sigma_e^2 \\ \mathbf{ZGZ}' & \text{when } \sigma_i^2 = \sigma_g^2 \end{cases} \tag{1}$$

The derivative of the first quadratic from with respect to X:

$$\frac{\partial (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{\partial \mathbf{X}} = -2\mathbf{X}'\mathbf{V}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \tag{2}$$

The derivative of the *inverse* matrix

$$\frac{\partial \mathbf{V}^{-1}}{\partial \sigma_i^2} = -\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1} \tag{3}$$

The derivative of the *trace* of the matrix

$$\frac{\partial \operatorname{Tr}(\mathbf{V})}{\partial \sigma_i^2} = \operatorname{Tr}(\mathbf{V}_i) \tag{4}$$

The derivative of the logarithm of the matrix' determinant:

$$\frac{\partial \ln |\mathbf{V}|}{\partial \sigma_i^2} = \operatorname{Tr}(\mathbf{V}^{-1}\mathbf{V}_i) \tag{5}$$

The derivative of the first and second quadratic forms, when taken with respect to σ_i^2:

$$\frac{\partial (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{\partial \sigma_i^2} = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}_i(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \tag{6a}$$

$$\frac{\partial (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{\partial \sigma_i^2} = -(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \tag{6b}$$

and the derivative of the logarithm of the determinant of the second quadratic form:

$$\frac{\partial \ln|\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|}{\partial \sigma_i^2} = \mathrm{Tr}((\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{X}) \tag{6c}$$

but also [Knight, 2008, p. 21]:

$$\begin{aligned}
&\frac{\partial \ln|\mathbf{V}|}{\partial \sigma_i^2} + \frac{\partial \ln|\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|}{\partial \sigma_i^2} \\
&= \mathrm{Tr}(\mathbf{V}^{-1}\mathbf{V}_i) - \mathrm{Tr}((\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{X}) \\
&= \mathrm{Tr}(\mathbf{V}^{-1}\mathbf{V}_i - (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{X}) \\
&= \mathrm{Tr}(\mathbf{P}\mathbf{V}_i).
\end{aligned}$$

A *projection matrix*, P can be defined as

$$\mathbf{P} = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1} \tag{7}$$

% and has following nice properties:

$$\mathbf{P}\mathbf{X} = 0 \tag{8}$$

$$\mathbf{P}\mathbf{y} = \mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}), \text{ where} \tag{9}$$

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \tag{10}$$

$$\hat{\mathbf{P}}\mathbf{y} = \hat{\mathbf{e}} \tag{11}$$

The derivative of the projection matrix:

$$\frac{\partial \mathbf{P}}{\partial \sigma_i^2} = -\mathbf{P}\mathbf{V}_i\mathbf{P} \tag{12}$$

For traces, we have:

$$\mathrm{Tr}(\mathbf{P}) = \mathbf{y}'\mathbf{P}\mathbf{P}\mathbf{y} \tag{13a}$$

or

$$\mathrm{Tr}(\mathbf{P}\mathbf{V}_i) = \mathbf{y}'\mathbf{P}\mathbf{V}_i\mathbf{P}\mathbf{y}. \tag{13b}$$

%

Some forms

In (**??**) on page **??**, the form $\mathbf{ZGZ}'$ is used. The following is the result if we consider a very simple example with two animals, where the first has two observations and the second only one.

$$\begin{aligned}
&\mathbf{y} = \mu + \mathbf{Z}\mathbf{g} + \mathbf{e} \\
&\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \\
\mathbf{ZGZ}' &= \begin{bmatrix} g_{11} & g_{11} & g_{12} \\ g_{11} & g_{11} & g_{12} \\ g_{21} & g_{21} & g_{22} \end{bmatrix} & \mathbf{ZZ}' = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{14}$$

1 Derivation of mean and variance of quadratic forms

If we assume that $\mathbf{y} \sim N(\mu, \mathbf{V})$ then [Searle et al., 1992, p. 466]

$$\begin{aligned}
\mathrm{E}(\mathbf{y}'\mathbf{Q}\mathbf{y}) &= \mathrm{Tr}(\mathbf{Q}\mathbf{V}) + \mu'\mathbf{Q}\mu \\
\mathrm{Var}(\mathbf{y}'\mathbf{Q}\mathbf{y}) &= 2\,\mathrm{Tr}(\mathbf{Q}\mathbf{V}\mathbf{Q}\mathbf{V}) + 4\mu'\mathbf{Q}\mathbf{V}\mathbf{Q}\mu
\end{aligned}$$

The quadratic form $\mathbf{y}'\mathbf{Q}\mathbf{y}$ has a non-central χ^2 distribution with $rank(\mathbf{Q})$ degrees of freedom and non-central parameter $\frac{1}{2}\mu'\mathbf{Q}\mu$, if and only if $\mathbf{Q}\mathbf{V}$ is idempotent (Theorem S2, p. 467, Searle et al., 1992). When $\mathbf{Q}\mathbf{V}$ is idempotent, then $\mathbf{Q}\mathbf{V}\mathbf{Q}\mathbf{V} = \mathbf{Q}\mathbf{V} \Rightarrow \mathrm{Tr}(\mathbf{Q}\mathbf{V}\mathbf{Q}\mathbf{V}) = \mathrm{Tr}(\mathbf{Q}\mathbf{V})$, and $\mathrm{Tr}(\mathbf{Q}\mathbf{V}) = rank(\mathbf{Q}\mathbf{V})$.

Because $\mu = 0$, $\mathbf{V} = \mathbf{I}$, and $\mathbf{Q} = \mathbf{V}_0^{\frac{1}{2}}\mathbf{M}\mathbf{V}_0^{\frac{1}{2}}$ (where $\mathbf{M} = \mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}$), the expectation and variance of the quadratic form are

$$\begin{aligned}
\mathrm{E}(\mathbf{y}'\mathbf{Q}\mathbf{y}) &= \mathrm{Tr}(\mathbf{Q}) \\
&= \mathrm{Tr}\left(\frac{1}{2}\mathbf{V}_0^{\frac{1}{2}}\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{V}_0^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{V}_0^{\frac{1}{2}}\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{V}_0^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(\mathbf{y}'\mathbf{Q}\mathbf{y}) &= 2\,\mathrm{Tr}(\mathbf{Q}\mathbf{Q}) \\
&= 2\,\mathrm{Tr}\left(\left[\frac{1}{2}\mathbf{V}_0^{\frac{1}{2}}\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{V}_0^{\frac{1}{2}}\right]\left[\frac{1}{2}\mathbf{V}_0^{\frac{1}{2}}\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{V}_0^{\frac{1}{2}}\right]\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\left[\mathbf{V}_0^{\frac{1}{2}}\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{V}_0^{\frac{1}{2}}\right]\left[\mathbf{V}_0^{\frac{1}{2}}\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{V}_0^{\frac{1}{2}}\right]\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{V}_0^{\frac{1}{2}}\mathbf{V}_0^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\mathbf{V}_0^{-1}\mathbf{Z}\mathbf{G}_i\mathbf{Z}'\right).
\end{aligned}$$

We can further elaborate that $\mathbf{y}'\mathbf{Q}\mathbf{y}$ is χ^2 distributed with $\mathrm{Tr}(\mathbf{Q})$ degrees of freedom under the conditions above.

References

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