A Comparative Study on Failure Data Analysis with Extended Weibull Distribution

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Abstract

We generally work with 2-parameter Weibull distribution. Marshall and Olkin's approach of inserting an additional parameter into a family of distributions yielded 3-parameter Weibull distribution (1997). As a particular instance, it has the two-parameter Weibull distribution. One advantage of this distribution is that the hazard-rate might be growing, decreasing, or first increasing, then reducing, then increasing again. We investigate model characterization using the Weibull Probability Plot (WPP). The WPP for the actual data set might be concave, convex, or S-shaped. A technique for estimating parameters based on WPP is described. Furthermore, the maximum likelihood estimator is shown. To demonstrate the technique and application, an example is provided.

Acknowledgement

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1 Introduction

The Weibull distribution, or wabl in short, is a continuous probability distribution used in probability theory and statistics. Although it was initially recognized by Maurice René Fréchet and first used by *Rosin & Rammler (1933)*[4] to define a particle size distribution, it was named after Swedish mathematician Waloddi Weibull who explained it in detail in 1951.

The probability density function of 2-parameter Weibull random variable is X is

$$f(x;\eta,\beta) = \begin{cases} \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}\exp\left(-\left(\frac{x}{\eta}\right)^{\beta}\right) & ; x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

Analysis of lifespan data has made considerable use of the Weibull distribution. In addition to the typical 2- and 3-parameter Weibull distributions, many alternative models built on the typical Weibull distributions are also used. The 3-parameter Weibull models are particularly interesting since they carry the fewest parameters when compared to other types of Weibull models and are more adaptable than the typical Weibull distributions for modeling lifespan data.

As given by [3], if $\bar{F}$ is the survival function of one-parameter exponential family with parameter β, then the survival function of 2-parameter exponential family, where the newly added parameter is ν , is generated by

$$\bar{F}(x;\nu,\beta) = \frac{\nu\bar{F}(x)}{1-(1-\nu)\bar{F}(x)} \tag{1.2}$$

The article [1] discusses Extended Weibull distribution, which was created via the intriguing method of adding a new parameter to an existing model. Simulating data from a three-parameter Weibull distribution is done and the parameters are estimated using appropriate parameter estimation methods and assess how flexible the model is when modeling failure data.

2 Extended Weibull Distribution

By adding a parameter to the family of 2-parameter Weibull distributions, the extended Weibull distribution is created. The survival function of 3-parameter Weibull distribution is

$$\bar{F}\left(t;\beta,\eta,\nu\right)=\frac{\nu\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}{1-\left(1-\nu\right)\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)} \tag{2.1}$$

with $\nu>0, t\geq 0$.

Hence, the cumulative distribution function (CDF) is given by

$$F\left(t\right)=1-\bar{F}\left(t;\beta,\eta,\nu\right)=\frac{1-\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}{1-\left(1-\nu\right)\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)} \tag{2.2}$$

The common 2-parameter Weibull distribution is a special case of the above distribution with $\nu=1$.

The probability density function (pdf) associated with (2.2) is given by

$$f\left(t\right)=\frac{\left(\frac{\nu\beta}{\eta}\right)\left(\frac{t}{\beta}\right)^{\beta-1}\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)}{\left\{1-\left(1-\nu\right)\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)\right\}^{2}} \tag{2.3}$$

and the corresponding hazard-rate function (HRF) is

$$h\left(t\right)=\frac{\left(\frac{\beta}{\eta}\right)\left(\frac{t}{\beta}\right)^{\beta-1}}{\left\{1-\left(1-\nu\right)\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)\right\}} \tag{2.4}$$

2.1 Model Analysis

This family of distributions provided by Eq. 1.2 has the following intriguing trait, as shown by the [3]. It is geometric-minimum (maximum) stable. In other words, if N is geometrically distributed on $\{1, 2, ...\}$ and$X_i;\ (i = 1, 2, ..., N)$ is a sequence of independent, identically distributed random variables with distribution from this family, then the minimum (maximum) of $X_i;\ i \leq N$ also has a distribution in the family.

2.1.1 Moments

The kth moment of the random variable from this distribution is given by

$$\begin{aligned} E\left(T^k\right) &= \int_0^\infty t^k f\left(t\right) dt \\ &= k \int_0^\infty t^{k-1} \bar{F}\left(t\right) dt \\ &= \nu k \eta^k \int_0^\infty \frac{\exp\left(-\left(\frac{t}{\eta}\right)^\beta\right)}{1-\left(1-\nu\right)\exp\left(-\left(\frac{t}{\eta}\right)^\beta\right)} \left(\frac{t}{\eta}\right)^{k-1} d\left(\frac{t}{\eta}\right) \\ &= \frac{\nu k \eta^k}{\beta} \int_0^1 \frac{\left(-\ln\left(y\right)\right)^{\frac{k}{\beta}-1}}{1-\left(1-\nu\right)y} dy,\ k > 0 \end{aligned} \tag{2.5}$$

The moments must be computed numerically because they cannot be obtained in closed form. However, using various software tools, it is simple to calculate the moments based on 2.5.

When $k = \beta$, then we have

$$E\left(T^\beta\right) = \frac{\eta^\beta}{1-\frac{1}{\nu}} \ln\left(\nu\right) \tag{2.6}$$

The first moment for various combinations of β and ν, and $\eta = 1$ are tabulated in [3].

2.1.2 Characterization of pdf

1. For $\nu \leq 1$ and $\beta > 1$ $f\left(t\right)$ is unimodal.

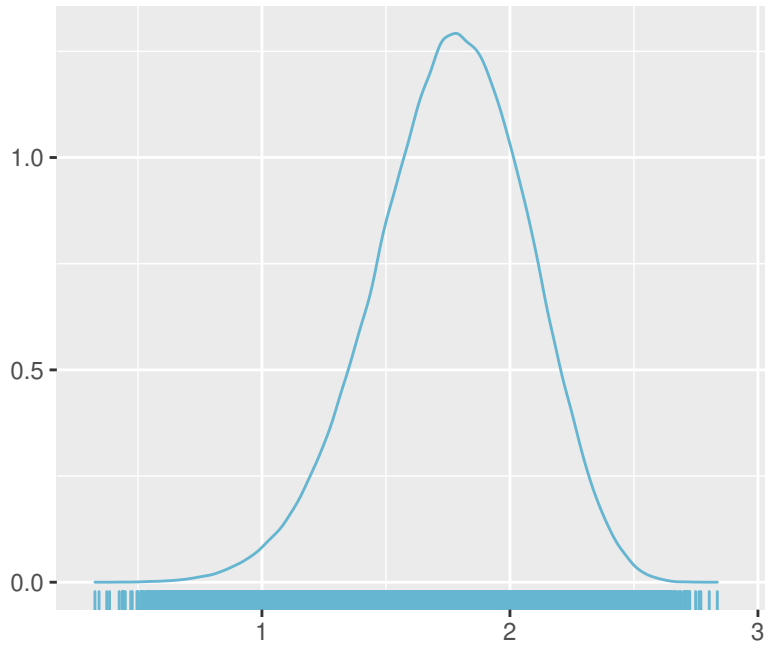

Figure 1: Density plot for $\beta = 7.5, \eta = 2, \nu = 0.5$

2. For $\nu \leq 1$ and $\beta \leq 1$ $f(t)$ is decreasing in t.

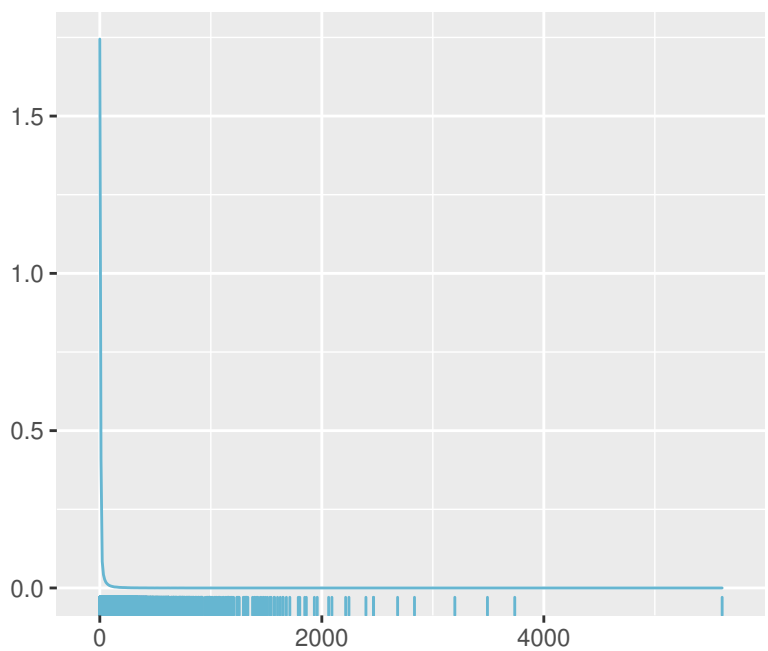

Figure 2: Density plot for $\beta = 0.3, \eta = 1, \nu = 1$

3. For $\nu > 1$ and $\beta < 1$ $f(t)$ increases initially and then decreases eventually in t.

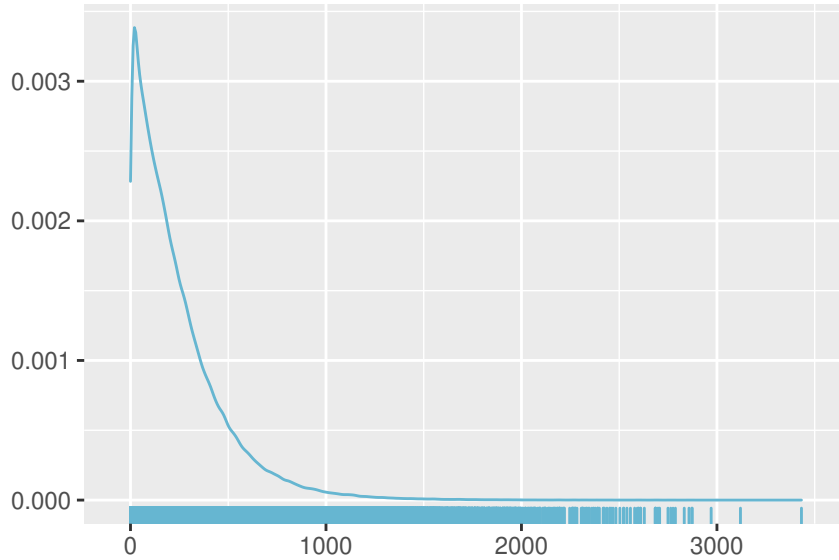

Figure 3: Density plot for $\beta = 0.69, \eta = 82.98, \nu = 4.4$

4. For $\nu > 1$ and $\beta > 1$ $f(t)$ is unimodal.

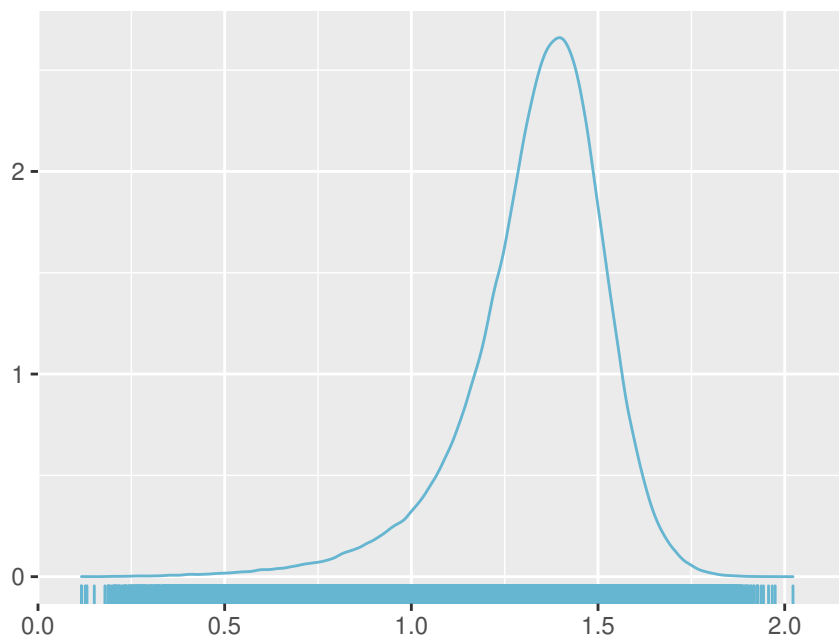

Figure 4: Density plot for $\beta = 4, \eta = 1, \nu = 30$

2.2 WPP for the Extended Weibull Distribution

The Weibull Probability Plot (WPP) for a distribution starts with the Weibull transformation

$$y = y(x) = \ln(-\ln(1 - F(t))) \text{ and } x = \ln(t)$$

where $F(t)$ is the CDF. The WPP of a typical 2-parameter Weibull distribution is a straight line, but the plot of a typical 3-parameter Weibull distribution is convex.

If the survival function is given by 2.1; from the Weibull transformation in 2.4 we have

$$\begin{aligned} y &= \ln\left[-\left\{\ln\nu - \ln\left(\exp\left(\left(\frac{t}{\eta}\right)^{\beta}\right) - (1-\nu)\right)\right\}\right] \\ &= \ln\left[-\left\{\ln\nu - \ln\left(\exp\left(\left(\frac{e^x}{\eta}\right)^{\beta}\right) - (1-\nu)\right)\right\}\right] \end{aligned} \tag{2.7}$$

This is a smooth curve and let denote it by C . As $x \to -\infty$ from 2.7 using the facts that as $u \to 0$, $\ln(1-u) \approx -u$ and $\exp(u) \approx 1 + u$, we can see that

$$\begin{aligned} y &= \ln\left[-\left\{\ln\nu - \ln\left(\exp\left(\left(\frac{e^x}{\eta}\right)^{\beta}\right) - (1-\nu)\right)\right\}\right] \\ &= \ln\left[-\left\{\ln\frac{1}{\dfrac{\exp\left(\left(\frac{e^x}{\eta}\right)^{\beta}\right)}{\nu} - \left(\dfrac{1}{\nu} - 1\right)}\right\}\right] \\ &= \ln\left[\left\{\ln\left(\frac{\exp\left(\left(\frac{e^x}{\eta}\right)^{\beta}\right)}{\nu} - \left(\frac{1}{\nu} - 1\right)\right)\right\}\right] \\ &= \ln\left[\left\{\ln\left(1 - \left(-\frac{\exp\left(\left(\frac{e^x}{\eta}\right)^{\beta}\right)}{\nu} + \frac{1}{\nu}\right)\right)\right\}\right] \end{aligned}$$

$$
\begin{aligned}
&\approx \ln\left\{\left(\frac{\exp\left(\left(\frac{e^x}{\eta}\right)^\beta\right)}{\nu} - \frac{1}{\nu}\right)\right\} \\
&\approx \ln\left(\frac{\left(\frac{e^x}{\eta}\right)^\beta}{\nu}\right) \\
&= \beta \ln\left(\frac{e^x}{\eta}\right) - \ln\nu = \beta\left(x - \ln\eta\right) - \ln\nu \qquad (2.8)
\end{aligned}
$$

This is a straight line and let L_a denote it. This implies that C approaches to the asymptote L_a when x becomes smaller.

Let C and L_a intercept x-axis at X_0 and X_x, respectively. When $y = 0$, 2.8 reduces to

$$
\eta^\beta = \frac{\exp\left(\beta X_x\right)}{\nu} \qquad (2.9)
$$

Similarly from 2.7 for $y = 0$ we have

$$
\exp\left[\left(\frac{e^{X_0}}{\eta}\right)^\beta\right] = 1 + \nu\left(e - 1\right) \qquad (2.10)
$$

Note that, 2.9 and 2.10 will be applied to parameter estimation which will be described later. Also, $X_0 > 0,\ Y_0 < 0$ or $X_0 < 0,\ Y_0 > 0$; where X_0 and Y_0 are the x-intercept and y-intercept respectively of the model plot. However, in the real applicable range, we can only see part of the whole plot. That means the part that is concave ($\nu > 1$), convex ($\nu < 1$), or likely S-shaped.

3 Parameter Estimation

Following is a discussion of the graphical technique and maximum likelihood estimate (MLE).

3.1 Graphical Approach

As mentioned in [1], WPP is the first step in graphical modeling of a given data collection. The graphical technique can be carried out in the following phases for a specific batch of n data.

1. Sort the data in ascending order as $t_1, t_2, ..., t_i, .., t_n$

2. Obtain x_i and y_i, $1 \leq i \leq n$, as follows:

$$x_i = \ln(t_i) \text{ and } y_i = \ln\left(-\ln\left(1 - \frac{i}{n+1}\right)\right)$$

   One can also use other forms, for example, $\frac{i-0.3}{n+0.4}$ instead of $\frac{i}{n+1}$ to compute the empirical CDF. If the data set $(t_1, t_2, ..., t_i, .., t_n)$ contains censored data, the empirical CDF is not given by $\frac{i}{n+1}$ #2 have to be modified. Refer to *Kaplan-Meier Method* of [2] for description in detail on the modifications.

3. Plot y_i vs. x_i to generate WPP. The extended Weibull may be a suitable model for the data if the fitting plot is smoothly convex, concave, or most likely S-shaped, and it approaches a straight line as x decreases.

4. Fit a straight line L_a to the fitting plot's left side. Its slope produces the parameter β.

5. X_0 and X_x should be read from the fitting curve. Equations 2.9 and 2.10 are then combined to produce η and ν.

6. Observe the y-coordinate (Y_0) at the point where the y-axis and C meet, calculate Y_0 by 2.7using the aforementioned estimated values of (β, η, ν) when $y = 0$. Check to see if the reading from the fitting plot agrees with the calculated Y_0 by Eq. 2.7. If not, adjust L_a as necessary and repeat Steps 4-6.

7. If the constraint mentioned in #6 is not satisfied, other models should be tried.

A rudimentary approach that is based on detailed observation is the graphic approach. For model selection, the plot's characterization is useful, nevertheless. Additionally, more sophisticated techniques based on rigorous statistical tests should be used to determine whether the model under consideration is adequate in order to provide reliable estimations.

3.2 MLE Method

Numerous statistical techniques are available to estimate model parameters. These include the quantile (or percentile) estimate, the method of moments, and MLE. The bias correction for the least squares estimator of the Weibull shape parameter [5], the SCEM-UA algorithm [6], and other approaches to parameter estimation are also options. The method of moments is not applicable to the extended Weibull because the moments are not analytically tractable. One can use the quantile estimation. One can get the parameter estimates by considering the lowest and upper quantile estimates as well as the median estimate. However, this approach is generally less effective than the MLE.

When given a set of data, which may be complete or censored, the MLE of the model parameters is effective. Let's say there were N test items, and k of them failed. Let $t_1 \leq t_2 \leq ... \leq t_k$ be the times to failure of the k failed items, for example, when considering the Type-II censored scenario. The log-likelihood function used in [1] is

$$\begin{aligned} l(\beta,\eta,\nu;t) = {} & k\ln(\nu\beta) + (\beta-1)\sum_{i=1}^{k}\ln t_i - \beta k\ln\eta - \sum_{i=1}^{k}\left(\frac{t_i}{\eta}\right)^{\beta} \\ & - 2\sum_{i=1}^{k}\ln\left\{1-(1-\nu)\exp\left(-\left(\frac{t_i}{\eta}\right)^{\beta}\right)\right\} + (N-k)\ln\nu \\ & - (N-k)\left(\frac{t_k}{\eta}\right)^{\beta} - (N-k)\ln\left\{1-(1-\nu)\exp\left(-\left(\frac{t_k}{\eta}\right)^{\beta}\right)\right\} \end{aligned} \tag{3.1}$$

By taking the partial derivatives of $l(\beta,\eta,\nu;t)$ with respect to β, η and ν, and letting them be equal to zero, we have the following three equations:

$$\begin{aligned} \frac{\partial l(\beta,\eta,\nu;t)}{\partial\beta} = {} & \frac{k}{\beta} + \sum_{i=1}^{k}\ln t_i - k\ln\eta - \sum_{i=1}^{k}\frac{\left(\frac{t_i}{\eta}\right)^{\beta}\ln\left(\frac{t_i}{\eta}\right)\left\{1+(1-\nu)\exp\left(-\left(\frac{t_i}{\eta}\right)^{\beta}\right)\right\}}{\left\{1-(1-\nu)\exp\left(-\left(\frac{t_i}{\eta}\right)^{\beta}\right)\right\}} \\ & - \frac{\left(\frac{t_k}{\eta}\right)^{\beta}\ln\left(\frac{t_k}{\eta}\right)(N-k)}{\left\{1-(1-\nu)\exp\left(-\left(\frac{t_k}{\eta}\right)^{\beta}\right)\right\}} = 0 \end{aligned} \tag{3.2}$$

$$\begin{aligned} \frac{\partial l(\beta,\eta,\nu;t)}{\partial\eta} = {} & -\frac{k\beta}{\eta} + \frac{\beta}{\eta^{\beta+1}}\sum_{i=1}^{k}(t_i)^{\beta} + 2\sum_{i=1}^{k}\frac{(1-\nu)\left(\frac{\beta}{\eta}\right)\left(\frac{t_i}{\eta}\right)^{\beta}}{\left\{\exp\left(\left(\frac{t_i}{\eta}\right)^{\beta}\right)-(1-\nu)\right\}} \\ & + \frac{\frac{\beta}{\eta^{\beta+1}}t_k^{\beta}(N-k)}{\left\{1-(1-\nu)\exp\left(-\left(\frac{t_k}{\eta}\right)^{\beta}\right)\right\}} = 0 \end{aligned} \tag{3.3}$$

$$\frac{\partial l\left(\beta,\eta,\nu;t\right)}{\partial\nu} = \frac{N}{\nu}+\frac{\beta}{\eta^{\beta+1}}\sum_{i=1}^{k}\left(t_i\right)^{\beta}-2\sum_{i=1}^{k}\frac{1}{\left\{\exp\left(\left(\frac{t_i}{\eta}\right)^{\beta}\right)-\left(1-\nu\right)\right\}} - \frac{\left(N-k\right)}{\left\{\exp\left(\left(\frac{t_k}{\eta}\right)^{\beta}\right)-\left(1-\nu\right)\right\}}=0 \tag{3.4}$$

By resolving these equations, the estimates for these three parameters can be determined. They are nonlinear equations have no analytical answers; statistical software or other specialized software tools should be used to obtain numerical answers.

Discussing the presence and uniqueness of the solutions is an intriguing problem related to the MLE of model parameters; for examples, see [8] and [7] . The discussion on the existence and uniqueness of the solution to the equations is based on Mittal and Dahiya (1989) [7]. The process from 3.3 to 3.4 will be laborious. It needs more research.

4 Comparative Study

We have considered 4 sets of parameters (β, η, ν). For each set we have simulated 3×10^5 samples from Extended-Weibull(β, η, ν) using *Inverse Transform* technique. This is followed by a comparative study between the accuracy of the estimates using *Graphical method* and *MLE method*.

4.1 Graphical Approach

Here $\hat{Y}_0$ represents the the y-intercept of the curve

$$y = \ln\left[-\left\{\ln\hat{\nu} - \ln\left(\exp\left(\left(\frac{e^x}{\hat{\eta}}\right)^{\hat{\beta}}\right) - (1-\hat{\nu})\right)\right\}\right] \tag{4.1}$$

True Parameter (β, η, ν)	X_0	X_x	$\hat{\beta}$	$\hat{\eta}$	$\hat{\nu}$	Y_0	$\hat{Y}_0$	**Estimated Parameter** $\left(\hat{\beta}, \hat{\eta}, \hat{\nu}\right)$
$(0.698, 82.98, 4.4)$	5.51	4.93	0.97	3.68	29.92	-4.53	-4.53	$(0.97, 3.68, 29.92)$
$(0.3, 1, 1)$	0.003	56.07	0.12	-0.51	-0.36	0.001	-	$(0.12, -, -)$
$(4, 1, 30)$	0.34	-0.09	6.45	1.18	-0.12	-2.88	-	$(6.45, 1, 18, -)$
$(7.5, 2, 0.5)$	0.63	0.15	11.12	2.07	-1.44	-4.54	-	$(11.12, 2.07, -)$

Table 1: Estimates after performing Graphical Approach

From Table 1., it is quite clear that performance of the graphical method is not satisfactory. This method either failed to have good estimates of the parameters, as in first case or is unable to estimate the model parameters, as in rest of the cases. In last three cases after some iterations, $\hat{\nu}$ has become negative, resulting in non-real values of $\hat{Y}_0$ due to the term $\ln\hat{\nu}$ in 4.1, which stops any further iterations to occur. Hence, Graphical approach is a crude method in estimating the model parameters using the simulated data.

The WPP for the four sets of simulated data along with the asymptote L_a (in red) to the curve C (in blue) are shown below.

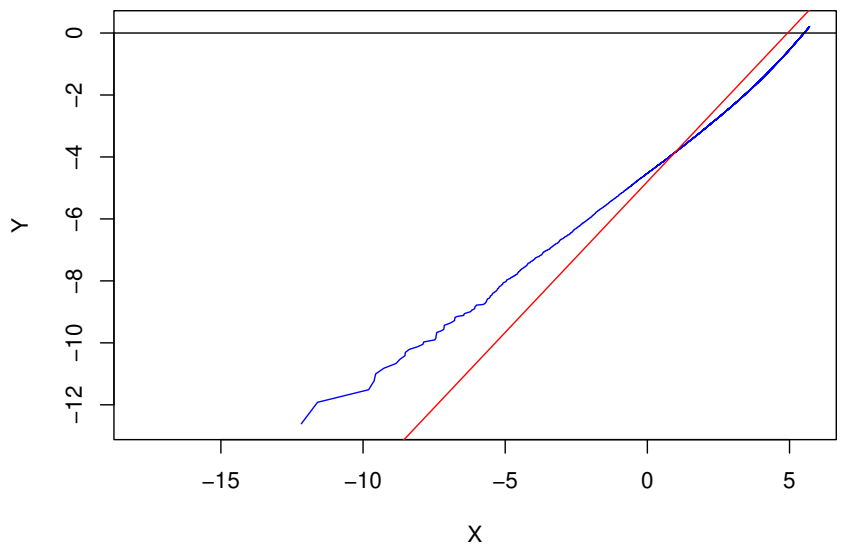

Figure 5: For $(\beta, \eta, \nu) = (0.698, 82.98, 4.4)$

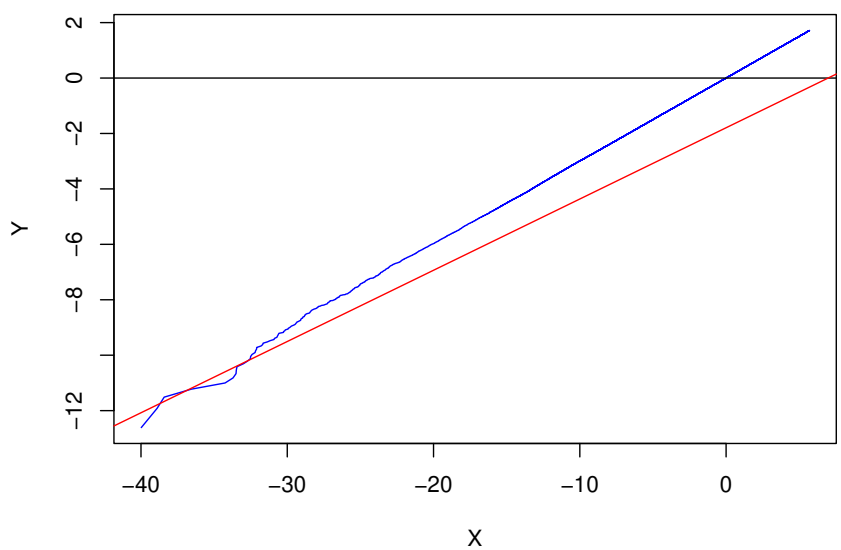

Figure 6: For $(\beta, \eta, \nu) = (0.3, 1, 1)$

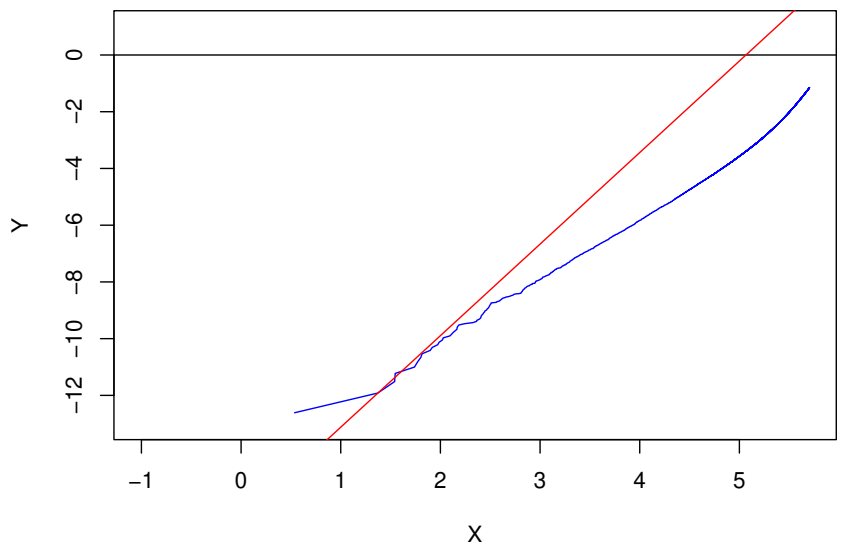

Figure 7: For $(\beta, \eta, \nu) = (4, 1, 30)$

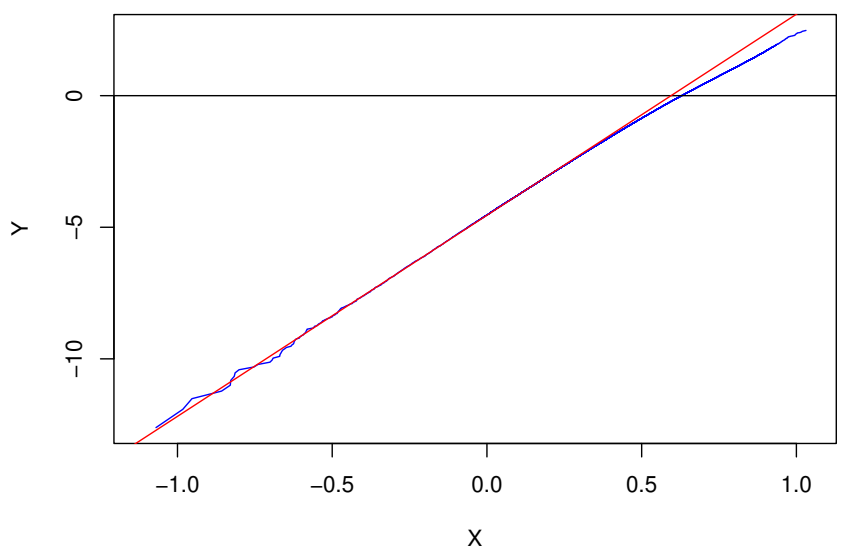

Figure 8: For $(\beta, \eta, \nu) = (7.5, 2, 0.5)$

4.2 MLE approach

We have considered 4 sets of parameters (β, η, ν). For each set we have simulated 3×10^5 samples 50 many times from Extended-Weibull(β, η, ν) using *Inverse Transform* technique. The MLE of the true parameters are taken to be the corresponding means i.e. if $\hat{\beta}_1, \hat{\beta}_2, ..., \hat{\beta}_{50}$ are the 50 MLE's of β then

$$\hat{\beta}_{MLE} = \frac{\sum_{i=1}^{50} \hat{\beta}_i}{50}$$

Similarly, $\hat{\eta}_{MLE}$ and $\hat{\nu}_{MLE}$ can be obtained. The following table represents the maximum likelihood estimates of the true parameters along with their standard errors.

True Parameters (β, η, ν)	$\hat{\beta}$	s.e. $\left(\hat{\beta}\right)$	$\hat{\eta}$	s.e. $(\hat{\eta})$	$\hat{\nu}$	s.e. $(\hat{\nu})$	MLE
$(0.698, 82.98, 4.4)$	0.697	0.0058	82.726	2.3222	4.418	0.1417	$(0.69, 82.72, 4.42)$
$(0.3, 1, 1)$	0.299	0.0013	0.999	0.0376	1.001	0.0188	$(0.29, 0.99, 1.00)$
$(4, 1, 30)$	3.999	0.0354	1.000	0.0053	29.983	1.1022	$(3.99, 0.99, 29.89)$
$(7.5, 2, 0.5)$	6.0358	2.0598	1.7558	0.3428	6.061	7.8256	$(6.03, 1.76, 6.06)$

Table 2: MLE's and their standard errors

Table 2 depicts that the MLE approach gives quite accurate estimates of the true parameters in most of the scenarios. In comparison to Graphical method, MLE performs way better.

5 Summary

In this project, we have successfully reproduced various estimation methods (Graphical Method and MLE Method) for estimating model parameters of Extended Weilbull distribution from the paper [1]. The main purpose of the project was to estimate different parameters using simulated data and compare the performance of these two methods under different sets of parameters. Though Graphical method does not perform well on some sets of parameters as it results in absurd estimates, MLE yields good estimates in most of the situations. Hence MLE method is more reliable than crude Graphical method.

6 Contribution

- ***Sankhadeep Mitra:*** Finding the paper (50%), Writing code (50%), Presentation(35%),Preparing report (25%)
- ***Sampriti Dutta:*** Finding the paper (50%), Writing code (25%),Preparing report(25%), Presentation (35%)
- ***Soumya Paul:*** Provided theoretical inputs,Writing code (25%), Preparing report(50%), Presentation (35%)

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